symmetry
MDPI

Article

On Some Important Class of Dynamic Hilbert's-Type Inequalities on Time Scales

Hassan M. El-Owaidy [1], Ahmed A. El-Deeb [1,*], Samer D. Makharesh [1], Dumitru Baleanu [2,3,4] and Clemente Cesarano [5,*]

[1] Department of Mathematics, Faculty of Science, Al-Azhar University, Nasr City 11884, Cairo, Egypt; elowaidy@azhar.edu.eg (H.M.E.-O.); samerderhaem.211@azhar.edu.eg (S.D.M.)
[2] Institute of Space Science, 077125 Bucharest, Romania; dumitru@cankaya.edu.tr
[3] Department of Mathematics, Cankaya University, 06530 Ankara, Turkey
[4] Department of Medical Research, China Medical University, Taichung 40447, Taiwan
[5] Section of Mathematics, International Telematic University Uninettuno, Corso Vittorio Emanuele II, 39, 00186 Rome, Italy
* Correspondence: ahmedeldeeb@azhar.edu.eg (A.A.E.-D.); c.cesarano@uninettunouniversity.net (C.C.)

Abstract: In this important work, we discuss some novel Hilbert-type dynamic inequalities on time scales. The inequalities investigated here generalize several known dynamic inequalities on time scales and unify and extend some continuous inequalities and their corresponding discrete analogues. Our results will be proved by using some algebraic inequalities, Hölder inequality, and Jensen's inequality on time scales.

Keywords: dynamic Hilbert's inequality; dynamic inequality; time scales calculus

check for updates

Citation: El-Owaidy, H.M.; El-Deeb, A.A.; Makharesh, S.D.; Baleanu, D.; Cesarano, C. On Some Important Class of Dynamic Hilbert's-Type Inequalities on Time Scales. *Symmetry* **2022**, *14*, 1395. https://doi.org/10.3390/sym14071395

Academic Editor: Wei-Shih Du

Received: 8 June 2022
Accepted: 29 June 2022
Published: 7 July 2022

1. Introduction

The celebrated Hardy-Hilbert's integral inequality [1] is

$$\int_0^{+\infty}\int_0^{+\infty}\frac{F(\vartheta)g(\varsigma)}{\vartheta+\varsigma}\mathrm{d}\vartheta\mathrm{d}\varsigma \le \frac{\pi}{\sin\frac{\pi}{p}}\left[\int_0^{+\infty}F^p(\vartheta)\mathrm{d}\vartheta\right]^{1/p}\left[\int_0^{+\infty}g^q(\varsigma)\mathrm{d}\varsigma\right]^{1/q}, \tag{1}$$

where $p > 1, q = p/p-1$. Putting $p = q = 2$, we get:

$$\int_0^{+\infty}\int_0^{+\infty}\frac{F(\vartheta)g(\varsigma)}{\vartheta+\varsigma}\mathrm{d}\vartheta\mathrm{d}\varsigma \le \pi\left[\int_0^{+\infty}F^2(\vartheta)\mathrm{d}\vartheta\right]^{1/2}\left[\int_0^{+\infty}g^2(\varsigma)\mathrm{d}\varsigma\right]^{1/2}, \tag{2}$$

where the constants π and $\frac{\pi}{\sin\frac{\pi}{p}}$ are the best possible.

Over the past decade, a great number of dynamic Hilbert-type inequalities on time scales have been established by many researchers who were motivated by some applications; see the papers [2–13]. For more details on time scales calculus, see [14].

In this paper, we extend some generalizations of the integral Hardy-Hilbert inequality to a general time scale. As special cases of our results, we will recover some dynamic integral and discrete inequalities known in the literature.

A time scale $\mathbb{T}$ is an arbitrary non-empty closed subset of the real number. We define forward and backward jump operators $\sigma : \mathbb{T} \to \mathbb{T}$ and $\rho : \mathbb{T} :\to \mathbb{T}$ respectively by

$$\sigma(\iota) := \inf\{s \in \mathbb{T} : s > \iota\}, \qquad \iota \in \mathbb{T},$$

$$\rho(\iota) := \sup\{s \in \mathbb{T} : s < \iota\}, \qquad \iota \in \mathbb{T}.$$

We will need the following important relations between calculus on time scales $\mathbb{T}$ and either continuous calculus on $\mathbb{R}$ or discrete calculus on $\mathbb{Z}$. Note that:

(*i*) If $\mathbb{T} = \mathbb{R}$, then

$$\sigma(\iota) = \iota, \quad \mu(\iota) = 0, \quad F^{\Delta}(\iota) = F'(\iota), \quad \int_a^b F(\iota)\Delta\iota = \int_a^b F(\iota)d\iota. \tag{3}$$

(*ii*) If $\mathbb{T} = \mathbb{Z}$, then

$$\sigma(\iota) = \iota + 1, \quad \mu(\iota) = 1, \quad F^{\Delta}(\iota) = F(\iota+1) - F(\iota), \quad \int_a^b F(\iota)\Delta\iota = \sum_{\iota=a}^{b-1} F(\iota). \tag{4}$$

Next, we write Hölder's inequality and Jensen's inequality on time scales.

Lemma 1 ([3])**.** *Let $u, v \in \mathbb{T}$ with $u < v$. Assume $F^*, g^* \in CC^1_{rd}([u,v]_{\mathbb{T}} \times [u,v]_{\mathbb{T}}, \mathbb{R})$ be integrable functions and $\frac{1}{p} + \frac{1}{q} = 1$ with $p > 1$. Then*

$$\begin{aligned} \int_u^v \int_u^v |F^*(r^*,\iota^*)g^*(r^*,\iota^*)|\Delta r^* \Delta\iota^* \quad &\leq \quad \left[\int_u^v \int_u^v |F^*(r^*,\iota^*)|^p \Delta r^* \Delta\iota^*\right]^{\frac{1}{p}} \\ &\times \left[\int_u^v \int_u^v |g^*(r^*,\iota^*)|^q \Delta r^* \Delta\iota^*\right]^{\frac{1}{q}}. \end{aligned} \tag{5}$$

This inequality is reversed if $0 < p < 1$ and if $p < 0$ or $q < 0$.

Lemma 2 ([15])**.** *Let $r^*, \iota^* \in \mathbb{R}$ and $-+\infty \leqslant m^*, n^* \leqslant +\infty$. If $F^* \in CC^1_{rd}(\mathbb{R}, (m^*, n^*))$, and $\phi : (m^*, n^*) \longrightarrow \mathbb{R}$ is convex, then*

$$\phi\left(\frac{\int_u^v \int_\omega^s F^*(r^*,\iota^*)\Delta_1 r^* \Delta_2 \iota^*}{\int_u^v \int_\omega^s \Delta_1 r^* \Delta_2 \iota^*}\right) \leqslant \frac{\int_u^v \int_\omega^s \phi(F^*(r^*,\iota^*))\Delta_1 r^* \Delta_2 \iota^*}{\int_u^v \int_\omega^s \Delta_1 r^* \Delta_2 \iota^*}. \tag{6}$$

This inequality is reversed if $\phi \in C_{rd}((c,d),\mathbb{R})$ is concave.

Theorem 1 (Chain rule on time scales [14])**.** *Assume $g : \mathbb{R} \to \mathbb{R}$ is continuous, $g : \mathbb{T} \to \mathbb{R}$ is Δ-differentiable on $\mathbb{T}^\kappa$, and $F : \mathbb{R} \to \mathbb{R}$ is continuously differentiable. Then there exists $c \in [\iota, \sigma(\iota)]_{\mathbb{R}}$ with*

$$(F \circ g)^{\Delta}(\iota) = F'(g(c))(g)^{\Delta}(\iota). \tag{7}$$

Definition 1. *Φ is called a supermultiplicative function on $[0, +\infty)$ if*

$$\Phi(\vartheta\varsigma) \geq \Phi(\vartheta)\Phi(\varsigma), \ \textit{for all} \ \vartheta, \varsigma \geqslant 0. \tag{8}$$

Next, we write Fubini's theorem on time scales.

Lemma 3 (Fubini's Thoerem, see [16])**.** *Assume that $(\vartheta, \Sigma_1, \mu_\Delta)$ and $(\varsigma, \Sigma_2, \nu_\Delta)$ are two finite-dimensional time scales measure spaces. Moreover, suppose that $F : \vartheta \times \varsigma \to \mathbb{R}$ is a delta integrable function and define the functions*

$$\phi(\varsigma) = \int_\vartheta F(\vartheta,\varsigma)d\mu_\Delta(\vartheta), \quad \varsigma \in \varsigma,$$

and

$$\psi(\vartheta) = \int_\varsigma F(\vartheta,\varsigma)d\nu_\Delta(\varsigma), \quad \vartheta \in \vartheta.$$

Then ϕ is delta integrable on ς, and ψ is delta integrable on ϑ and

$$\int_\vartheta d\mu_\Delta(\vartheta) \int_\varsigma F(\vartheta,\varsigma)d\nu_\Delta(\varsigma) = \int_\varsigma d\nu_\Delta(\varsigma) \int_\vartheta F(\vartheta,\varsigma)d\mu_\Delta(\vartheta).$$

Now we are ready to state and prove our main results.

2. Main Results

First, we enlist the following assumptions for the proofs of our main results:

(S_1) $\mathbb{T}$ be time scales with $\iota_0, v_\ell, \varsigma_\ell, s_\ell, \iota_\ell \in \mathbb{T}$, $(\ell = 1, \ldots, n)$.
(S_2) $\lambda_\ell(s_\ell, \iota_\ell)$ are nonnegative, delta integrable functions defined on $[\iota_0, v_\ell)_\mathbb{T} \times [\iota_0, \varsigma_\ell)_\mathbb{T}$ $(\ell = 1, \ldots, n)$.
(S_3) $\lambda_\ell(s_\ell, \iota_\ell)$ have a partial Δ- derivatives $\lambda_\ell^{\Delta_1}(s_\ell, \iota_\ell)$ and $\lambda_\ell^{\Delta_2}(s_\ell, \iota_\ell)$ with respect s_ℓ and ι_ℓ respectively.
(S_4) All functions used in this section are integrable according to Δ sense.
(S_5) $\lambda_\ell(s_\ell, \iota_\ell) \in C^2_{rd}\big([\iota_0, v_\ell)_\mathbb{T} \times [\iota_0, \varsigma_\ell)_\mathbb{T}, [0, \infty)\big)$ $(\ell = 1, \ldots, n)$.
(S_6) $p_\ell(\xi_\ell, \tau_\ell)$ are n positive delta integrable functions defined for $\xi_\ell \in (\iota_0, s_\ell)_\mathbb{T}$, $\tau_\ell \in (\iota_0, \iota_\ell)_\mathbb{T}$.
(S_7) $p_\ell(\xi_\ell)$ and $q_\ell(\tau_\ell)$ are positive delta integrable functions defined for $\xi_\ell \in (\iota_0, s_\ell)_\mathbb{T}$, $\tau_\ell \in (\iota_0, \iota_\ell)_\mathbb{T}$.
(S_8) Φ_ℓ $(\ell = 1, \ldots, n)$ are n real-valued nonnegative concave and supermultiplicative functions defined on $(0, \infty)$.
(S_9) v_ℓ and ς_ℓ are positive real numbers.
(S_{10}) $s_\ell \in [\iota_0, v_\ell)_\mathbb{T}$ and $\iota_\ell \in [\iota_0, \varsigma_\ell)_\mathbb{T}$.
(S_{11}) $\lambda_\ell(\iota_0, \iota_\ell) = \lambda_\ell(s_\ell, \iota_0) = 0$, $(\ell = 1, \ldots, n)$.
(S_{12}) $\lambda_\ell^{\Delta_1\Delta_2}(s_\ell, \iota_\ell) = \lambda_\ell^{\Delta_2\Delta_1}(s_\ell, \iota_\ell)$.
(S_{13}) $P_\ell(s_\ell, \iota_\ell) = \int_{\iota_0}^{\iota_\ell}\int_{\iota_0}^{s_\ell} p_\ell(\xi_\ell) q_\ell(\tau_\ell) \Delta\xi_\ell \Delta\tau_\ell$.
(S_{14}) $\Lambda_\ell(s_\ell, \iota_\ell) = \int_{\iota_0}^{s_\ell}\int_{\iota_0}^{\iota_\ell} \lambda_\ell(\xi_\ell, \tau_\ell) \Delta\xi_\ell \Delta\tau_\ell$.
(S_{15}) $P_\ell(s_\ell, \iota_\ell) = \int_{\iota_0}^{s_\ell}\int_{\iota_0}^{\iota_\ell} p_\ell(\xi_\ell, \tau_\ell) \Delta\xi_\ell \Delta\tau_\ell$.
(S_{16}) $\Lambda_\ell(s_\ell, \iota_\ell) = \frac{1}{P_\ell(\xi_\ell, \tau_\ell)} \int_{\iota_0}^{s_\ell}\int_{\iota_0}^{\iota_\ell} p_\ell(\xi_\ell, \tau_\ell) \lambda_\ell(\xi_\ell, \tau_\ell) \Delta\xi_\ell \Delta\tau_\ell$.
(S_{17}) $\gamma_\ell \in (1, \infty)$, $\gamma'_\ell = 1 - \gamma_\ell$, $\gamma = \sum_{\ell=1}^n \gamma_\ell$, and $\gamma' = \sum_{\ell=1}^n \gamma'_\ell = n - \gamma$, $(\ell = 1, \ldots, n)$.
(S_{18}) $0 < \beta_\ell < 1$.
(S_{19}) $h_\ell \geqslant 2$.
(S_{20}) $\sum_{\ell=1}^n \frac{1}{\gamma_\ell} = \frac{1}{\gamma}$.
(S_{21}) $h_\ell \geqslant 1$.
(S_{22}) $\lambda_\ell(\xi_\ell) \in C^1_{rd}[\iota_0, v_\ell]_\mathbb{T}$, $(\ell = 1, \ldots, n)$.
(S_{23}) v_ℓ is positive real number.
(S_{24}) $\Lambda_\ell(s_\ell) = \int_{\iota_0}^{s_\ell} \lambda_\ell(\xi_\ell) \Delta\xi_\ell$.
(S_{25}) $s_\ell \in [\iota_0, v_\ell)_\mathbb{T}$.
(S_{26}) $p_\ell(\xi_\ell)$ are n positive functions.
(S_{27}) $P_\ell(s_\ell) = \int_{\iota_0}^{s_\ell} p_\ell(\xi_\ell) \Delta\xi_\ell$.
(S_{28}) $\Lambda_\ell(s_\ell) = \frac{1}{P_\ell(s_\ell)} \int_{\iota_0}^{s_\ell} p(\xi_\ell) \lambda(\xi_\ell) \Delta\xi_\ell$.
(S_{29}) $\lambda_\ell(\iota_0) = 0$.

Now, we are ready to state and prove the main results that extend several results in the literature.

Theorem 2. *Let S_1, S_2, S_9, S_{11}, S_7, S_{13}, S_3, S_{12}, S_8 and S_{17} be satisfied. Then for S_{10} we have*

$$\int_{\iota_0}^{v_1}\int_{\iota_0}^{\varsigma_1} \cdots \int_{\Im_0}^{v_n}\int_{\Im_0}^{\varsigma_n} \prod_{\ell=1}^n \frac{\Phi_\ell(\lambda_\ell(s_\ell, \Im_\ell))}{\left(\frac{1}{\gamma'}\sum_{\ell=1}^n \gamma'_\ell (s_\ell - \Im_0)(\Im_\ell - \Im_0)\right)^{\gamma'}} \Delta s_n \Delta\Im_n \ldots \Delta s_1 \Delta\Im_1 \tag{9}$$

$$\geqslant G(v_1\varsigma_1, \ldots, v_n\varsigma_n)$$

$$\times \prod_{\ell=1}^n \left(\int_{\iota_0}^{v_\ell}\int_{\iota_0}^{\varsigma_\ell} (\rho(v_\ell) - s_\ell)(\rho(\varsigma_\ell) - \iota_\ell) \left(p_\ell(s_\ell) q_\ell(\iota_\ell) \Phi_\ell\left(\frac{\lambda_\ell^{\Delta_2\Delta_1}(s_\ell, \iota_\ell)}{p_\ell(s_\ell) q_\ell(\iota_\ell)}\right)\right)^{\frac{1}{\gamma_\ell}} \Delta s_\ell \Delta\iota_\ell \right)^{\gamma_\ell}$$

where

$$G(\upsilon_1\varsigma_1,\ldots,\upsilon_n\varsigma_n)=\prod_{\ell=1}^{n}\left(\int_{\iota_0}^{\upsilon_\ell}\int_{\iota_0}^{\varsigma_\ell}\left(\frac{\Phi_\ell(P_\ell(s_\ell,\iota_\ell))}{P_\ell(s_\ell,\iota_\ell)}\right)^{\frac{1}{\gamma'_\ell}}\Delta s_\ell\Delta\iota_\ell\right)^{\gamma'_\ell}.$$

Proof. From the hypotheses of Theorem 2, we obtain

$$\lambda_\ell(s_\ell,\iota_\ell)=\int_{\iota_0}^{s_\ell}\int_{\iota_0}^{\Im_\ell}\lambda_\ell^{\Delta_2\Delta_1}(\xi_\ell,\tau_\ell)\Delta\xi_\ell\Delta\tau_\ell. \tag{10}$$

From (10) and S_8, it is easy to observe that

$$\begin{aligned}\Phi_\ell(\lambda_\ell(s_\ell,\iota_\ell)) &= \Phi_\ell\left(\frac{P_\ell(s_\ell,\iota_\ell)\int_{\iota_0}^{s_\ell}\int_{\Im_0}^{\Im_\ell}p_\ell(\xi_\ell)q_\ell(\tau_\ell)\left(\frac{\lambda_\ell^{\Delta_2\Delta_1}(\xi_\ell,\tau_\ell)}{p_\ell(\xi_\ell)q_\ell(\tau_\ell)}\right)\Delta\xi_\ell\Delta\tau_\ell}{\int_{\iota_0}^{s_\ell}\int_{\iota_0}^{\iota_\ell}p_\ell(\xi_\ell)q_\ell(\tau_\ell)\Delta\xi_\ell\Delta\tau_\ell}\right)\\ &\geqslant \Phi_\ell(P_\ell(s_\ell,\iota_\ell))\Phi_\ell\left(\frac{\int_{\iota_0}^{s_\ell}\int_{\iota_0}^{\iota_\ell}p_\ell(\xi_\ell)q_\ell(\tau_\ell)\left(\frac{\lambda_\ell^{\Delta_2\Delta_1}(\xi_\ell,\tau_\ell)}{p_\ell(\xi_\ell)q_\ell(\tau_\ell)}\right)\Delta\xi_\ell\Delta\tau_\ell}{\int_{\iota_0}^{s_\ell}\int_{\iota_0}^{\iota_\ell}p_\ell(\xi_\ell)q_\ell(\tau_\ell)\Delta\xi_\ell\Delta\tau_\ell}\right). \end{aligned} \tag{11}$$

By using inverse Jensen's dynamic inequality, we get

$$\Phi_\ell(\lambda_\ell(s_\ell,\iota_\ell)) \geqslant \frac{\Phi_\ell(P_\ell(s_\ell,\iota_\ell))}{P_\ell(s_\ell,\iota_\ell)}\int_{\iota_0}^{s_\ell}\int_{\iota_0}^{\iota_\ell}p_\ell(\xi_\ell)q_\ell(\tau_\ell)\Phi_\ell\left(\frac{\lambda_\ell^{\Delta_2\Delta_1}(\xi_\ell,\tau_\ell)}{p_\ell(\xi_\ell)q_\ell(\tau_\ell)}\right)\Delta\xi_\ell\Delta\tau_\ell. \tag{12}$$

Applying inverse Hölder's inequality on the right hand side of (12) with indices $1/\gamma_\ell$ and $1/\gamma'_\ell$, we obtain

$$\begin{aligned}\Phi_\ell(\lambda_\ell(s_\ell,\iota_\ell)) &\geqslant \frac{\Phi_\ell(P_\ell(s_\ell,\iota_\ell))}{P_\ell(s_\ell,\iota_\ell)}\left[(s_\ell-\iota_0)(\iota_\ell-\iota_0)\right]^{\gamma'_\ell}\\ &\times\left(\int_{\iota_0}^{s_\ell}\int_{\iota_0}^{\iota_\ell}\left(p_\ell(\xi_\ell)q_\ell(\tau_\ell)\Phi_\ell\left(\frac{\lambda_\ell^{\Delta_2\Delta_1}(\xi_\ell,\tau_\ell)}{p_\ell(\xi_\ell)q_\ell(\tau_\ell)}\right)\right)^{\frac{1}{\gamma_\ell}}\Delta\xi_\ell\Delta\tau_\ell\right)^{\gamma_\ell}. \end{aligned} \tag{13}$$

Using the following inequality on the term $\left[(s_\ell-\iota_0)(\Im_\ell-\Im_0)\right]^{\gamma'_\ell}$, where $\gamma'_\ell<0$ and $\varrho_\ell>0$

$$\prod_{\ell=1}^{n}\varrho_\ell^{\gamma'_\ell}\geqslant\left(\frac{1}{\gamma'}\Big(\sum_{\ell=1}^{n}\gamma'_\ell\varrho_\ell\Big)\right)^{\gamma'}, \tag{14}$$

we obtain that

$$\begin{aligned}\prod_{\ell=1}^{n}\Phi_\ell(\lambda_\ell(s_\ell,\iota_\ell)) &\geqslant \prod_{\ell=1}^{n}\frac{\Phi_\ell(P_\ell(s_\ell,\iota_\ell))}{P_\ell(s_\ell,\iota_\ell)}\left(\frac{1}{\gamma'}\sum_{\ell=1}^{n}\gamma'_\ell(s_\ell-\iota_0)(\iota_\ell-\iota_0)\right)^{\gamma'}\\ &\times\left(\int_{\iota_0}^{s_\ell}\int_{\iota_0}^{\iota_\ell}\left(p_\ell(\xi_\ell)q_\ell(\tau_\ell)\Phi_\ell\left(\frac{\lambda_\ell^{\Delta_2\Delta_1}(\xi_\ell,\tau_\ell)}{p_\ell(\xi_\ell)q_\ell(\tau_\ell)}\right)\right)^{\frac{1}{\gamma_\ell}}\Delta\xi_\ell\Delta\tau_\ell\right)^{\gamma_\ell}. \end{aligned} \tag{15}$$

From (15), we obtain that

$$\begin{aligned}&\prod_{\ell=1}^{n}\frac{\Phi_\ell(\lambda_\ell(s_\ell,\iota_\ell))}{\left(\frac{1}{\gamma'}\sum_{\ell=1}^{n}\gamma'_\ell(s_\ell-\iota_0)(\iota_\ell-\iota_0)\right)^{\gamma'}}\\ &\geqslant\prod_{\ell=1}^{n}\frac{\Phi_\ell(P_\ell(s_\ell,\iota_\ell))}{P_\ell(s_\ell,\iota_\ell)}\left(\int_{\iota_0}^{s_\ell}\int_{\iota_0}^{\iota_\ell}\left(p_\ell(\xi_\ell)q_\ell(\tau_\ell)\Phi_\ell\left(\frac{\lambda_\ell^{\Delta_2\Delta_1}(\xi_\ell,\tau_\ell)}{p_\ell(\xi_\ell)q_\ell(\tau_\ell)}\right)\right)^{\frac{1}{\gamma_\ell}}\Delta\xi_\ell\Delta\tau_\ell\right)^{\gamma_\ell}. \end{aligned} \tag{16}$$

Integrating both sides of (16) over s_ℓ, ι_ℓ from $\Im_0$ to υ_ℓ, ς_ℓ $(\ell=1,\ldots,n)$, we get

$$\int_{\iota_0}^{\upsilon_1}\int_{\iota_0}^{\varsigma_1}\cdots\int_{\Im_0}^{\upsilon_n}\int_{\Im_0}^{\varsigma_n}\prod_{\ell=1}^{n}\frac{\Phi_\ell(\lambda_\ell(s_\ell,\Im_\ell))}{\left(\frac{1}{\gamma'}\sum_{\ell=1}^{n}\gamma'_\ell(s_\ell-\Im_0)(\Im_\ell-\Im_0)\right)^{\gamma'}}\Delta s_n\Delta\Im_n\ldots\Delta s_1\Delta\Im_1$$
$$\geqslant\prod_{\ell=1}^{n}\int_{\iota_0}^{\upsilon_\ell}\int_{\iota_0}^{\varsigma_\ell}\frac{\Phi_\ell(P_\ell(s_\ell,\iota_\ell))}{P_\ell(s_\ell,\iota_\ell)}\left(\int_{\iota_0}^{s_\ell}\int_{\iota_0}^{\iota_\ell}\left(p_\ell(\xi_\ell)q_\ell(\tau_\ell)\Phi_\ell\left(\frac{\lambda_\ell^{\Delta_2\Delta_1}(\xi_\ell,\tau_\ell)}{p_\ell(\xi_\ell)q_\ell(\tau_\ell)}\right)\right)^{\frac{1}{\gamma_\ell}}\Delta\xi_\ell\Delta\tau_\ell\right)^{\gamma_\ell}\Delta s_\ell\Delta\iota_\ell. \tag{17}$$

Applying inverse Hölder's inequality on the right hand side of (17) with indices $1/\gamma_\ell$ and $1/\gamma'_\ell$, we obtain

$$\int_{\iota_0}^{\upsilon_1}\int_{\iota_0}^{\varsigma_1}\cdots\int_{\Im_0}^{\upsilon_n}\int_{\Im_0}^{\varsigma_n}\prod_{\ell=1}^{n}\frac{\Phi_\ell(\lambda_\ell(s_\ell,\Im_\ell))}{\left(\frac{1}{\gamma'}\sum_{\ell=1}^{n}\gamma'_\ell(s_\ell-\Im_0)(\Im_\ell-\Im_0)\right)^{\gamma'}}\Delta s_1\Delta\Im_1\ldots\Delta s_n\Delta\Im_n$$
$$\geqslant\prod_{\ell=1}^{n}\left(\int_{\iota_0}^{\upsilon_\ell}\int_{\iota_0}^{\varsigma_\ell}\left(\frac{\Phi_\ell(P_\ell(s_\ell,\iota_\ell))}{P_\ell(s_\ell,\iota_\ell)}\right)^{\frac{1}{\gamma'_\ell}}\Delta s_\ell\Delta\iota_\ell\right)^{\gamma'_\ell} \tag{18}$$
$$\times\prod_{\ell=1}^{n}\left(\int_{\iota_0}^{\upsilon_\ell}\int_{\iota_0}^{\varsigma_\ell}\left(\int_{\iota_0}^{s_\ell}\int_{\iota_0}^{\iota_\ell}\left(p_\ell(\xi_\ell)q_\ell(\tau_\ell)\Phi_\ell\left(\frac{\lambda_\ell^{\Delta_2\Delta_1}(\xi_\ell,\tau_\ell)}{p_\ell(\xi_\ell)q_\ell(\tau_\ell)}\right)\right)^{\frac{1}{\gamma_\ell}}\Delta\xi_\ell\Delta\tau_\ell\right)\Delta s_\ell\Delta\iota_\ell\right)^{\gamma_\ell}.$$

By using Fubini's theorem, we observe that

$$\int_{\iota_0}^{\upsilon_1}\int_{\iota_0}^{\varsigma_1}\cdots\int_{\Im_0}^{\upsilon_n}\int_{\Im_0}^{\varsigma_n}\prod_{\ell=1}^{n}\frac{\Phi_\ell(\lambda_\ell(s_\ell,\Im_\ell))}{\left(\frac{1}{\gamma'}\sum_{\ell=1}^{n}\gamma'_\ell(s_\ell-\Im_0)(\Im_\ell-\Im_0)\right)^{\gamma'}}\Delta s_n\Delta\Im_n\ldots\Delta s_1\Delta\Im_1 \tag{19}$$
$$\geqslant G(\upsilon_1\varsigma_1,\ldots,\upsilon_n\varsigma_n)$$
$$\times\prod_{\ell=1}^{n}\left(\int_{\iota_0}^{\upsilon_\ell}\int_{\iota_0}^{\varsigma_\ell}(\upsilon_\ell-s_\ell)(\varsigma_\ell-\iota_\ell)\left(p_\ell(s_\ell)q_\ell(\iota_\ell)\Phi_\ell\left(\frac{\lambda_\ell^{\Delta_2\Delta_1}(s_\ell,\iota_\ell)}{p_\ell(s_\ell)q_\ell(\iota_\ell)}\right)\right)^{\frac{1}{\gamma_\ell}}\Delta s_\ell\Delta\iota_\ell\right)^{\gamma_\ell}.$$

By using the facts $\upsilon_\ell\geqslant\rho(\upsilon_\ell)$ and $\varsigma_\ell\geqslant\rho(\varsigma_\ell)$, we get

$$\int_{\iota_0}^{\upsilon_1}\int_{\iota_0}^{\varsigma_1}\cdots\int_{\Im_0}^{\upsilon_n}\int_{\Im_0}^{\varsigma_n}\prod_{\ell=1}^{n}\frac{\Phi_\ell(\lambda_\ell(s_\ell,\Im_\ell))}{\left(\frac{1}{\gamma'}\sum_{\ell=1}^{n}\gamma'_\ell(s_\ell-\Im_0)(\Im_\ell-\Im_0)\right)^{\gamma'}}\Delta s_n\Delta\Im_n\ldots\Delta s_1\Delta\Im_1$$
$$\geqslant G(\upsilon_1\varsigma_1,\ldots,\upsilon_n\varsigma_n)$$
$$\times\prod_{\ell=1}^{n}\left(\int_{\iota_0}^{\upsilon_\ell}\int_{\iota_0}^{\varsigma_\ell}(\rho(\upsilon_\ell)-s_\ell)(\rho(\varsigma_\ell)-\iota_\ell)\left(p_\ell(s_\ell)q_\ell(\iota_\ell)\Phi_\ell\left(\frac{\lambda_\ell^{\Delta_2\Delta_1}(s_\ell,\iota_\ell)}{p_\ell(s_\ell)q_\ell(\iota_\ell)}\right)\right)^{\frac{1}{\gamma_\ell}}\Delta s_\ell\Delta\iota_\ell\right)^{\gamma_\ell}.$$

This completes the proof. □

Remark 1. *In Theorem 2, if $\mathbb{T}=\mathbb{Z}$, we get the result due to Zhao et al. ([17], Theorem 1.5).*

Remark 2. *In Theorem 2, if we take $\mathbb{T}=\mathbb{R}$, we get inequality due to Zhao et al. [17].*

Remark 3. *Let S_1, S_2, S_9, S_{11}, S_7, S_{13}, S_3 and S_{12} be satisfied and let Φ_ℓ, γ_ℓ, γ'_ℓ, γ, and γ' be as in Theorem 2. Similar to proof of Theorem 2, we have*

$$\int_{\iota_0}^{\upsilon_1}\int_{\iota_0}^{\varsigma_1}\cdots\int_{\Im_0}^{\upsilon_n}\int_{\Im_0}^{\varsigma_n}\prod_{\ell=1}^{n}\frac{\Phi_\ell(\lambda_\ell(s_\ell,\Im_\ell))}{\left(\frac{1}{\gamma'}\sum_{\ell=1}^{n}\gamma'_\ell(s_\ell-\Im_0)(\Im_\ell-\Im_0)\right)^{\gamma'}}\Delta s_n\Delta\Im_n\ldots\Delta s_1\Delta\Im_1$$
$$\leqslant G^*(\upsilon_1\varsigma_1,\ldots,\upsilon_n\varsigma_n)$$
$$\times\prod_{\ell=1}^{n}\left(\int_{\iota_0}^{\upsilon_\ell}\int_{\iota_0}^{\varsigma_\ell}(\sigma(\upsilon_\ell)-s_\ell)(\sigma(\varsigma_\ell)-\iota_\ell)\left(p_\ell(s_\ell)q_\ell(\iota_\ell)\Phi_\ell\left(\frac{\lambda_\ell^{\Delta_2\Delta_1}(s_\ell,\iota_\ell)}{p_\ell(s_\ell)q_\ell(\iota_\ell)}\right)\right)^{\frac{1}{\gamma_\ell}}\Delta s_\ell\Delta\iota_\ell\right)^{\gamma_\ell}.$$

where

$$G^*(\upsilon_1\varsigma_1,\ldots,\upsilon_n\varsigma_n)=\frac{1}{(\gamma')^{\gamma'}}\prod_{\ell=1}^{n}\left(\int_{\iota_0}^{\upsilon_\ell}\int_{\iota_0}^{\varsigma_\ell}\left(\frac{\Phi_\ell(P_\ell(s_\ell,\iota_\ell))}{P_\ell(s_\ell,\iota_\ell)}\right)^{\frac{1}{\gamma'_\ell}}\Delta s_\ell\Delta\iota_\ell\right)^{\gamma'_\ell}.$$

This is an inverse form of the inequality (9).

Corollary 1. *Let S_{22}, S_{23}, S_{25}, S_{26}, S_{27}, S_{29}, S_{17}, and S_8 be satisfied. Then we have*

$$\int_{\iota_0}^{\upsilon_1}\cdots\int_{\iota_0}^{\upsilon_n}\prod_{\ell=1}^{n}\frac{\Phi_\ell(\lambda_\ell(s_\ell))}{\left(\frac{1}{\gamma'}\sum_{\ell=1}^{n}\gamma'_\ell(s_\ell-\iota_0)\right)^{\gamma'}}\Delta s_n\ldots\Delta s_1 \tag{20}$$
$$\geqslant G^{**}(\upsilon_1,\ldots,\upsilon_n)$$
$$\times\prod_{\ell=1}^{n}\left(\int_{\iota_0}^{\upsilon_\ell}(\rho(\upsilon_\ell)-s_\ell)\left(p_\ell(s_\ell)\Phi_\ell\left(\frac{\lambda_\ell^{\Delta}(s_\ell)}{p_\ell(s_\ell)}\right)\right)^{\frac{1}{\gamma_\ell}}\Delta s_\ell\right)^{\gamma_\ell},$$

where

$$G^{**}(\upsilon_1,\ldots,\upsilon_n)=\prod_{\ell=1}^{n}\left(\int_{\iota_0}^{\upsilon_\ell}\left(\frac{\Phi_\ell(P_\ell(s_\ell))}{P_\ell(s_\ell)}\right)^{\frac{1}{\gamma'_\ell}}\Delta s_\ell\right)^{\gamma'_\ell}.$$

Remark 4. *In Corollary 1, if we take $\mathbb{T}=\mathbb{Z}$, we get an inverse form of inequality due to Handley [18].*

Remark 5. *In Corollary 1, if we take $\mathbb{T}=\mathbb{R}$, we get an inverse form of inequality due to Handley [18].*

Remark 6. *In inequality (20) taking $n=2$, $\gamma_1=\gamma_2=2$, then $\gamma'_1=\gamma'_2=-1$, we have*

$$\int_{\iota_0}^{\upsilon_1}\int_{\iota_0}^{\upsilon_2}\prod_{\ell=1}^{n}\frac{\Phi_1(\lambda_1(s_1))\Phi_1(\lambda_2(s_2))}{\left((s_1-\iota_0)+(s_2-\iota_0)\right)^{-2}}\Delta s_1\Delta s_2 \tag{21}$$
$$\geqslant D(\upsilon_1,\upsilon_2)\left(\int_{\iota_0}^{\upsilon_1}(\rho(\upsilon_1)-s_1)\left(p_1(s_1)\Phi_1\left(\frac{\lambda_1^{\Delta}(s_1)}{p_1(s_1)}\right)\right)^{\frac{1}{2}}\Delta s_1\right)^2$$
$$\times\left(\int_{\iota_0}^{\upsilon_2}(\rho(\upsilon_2)-s_2)\left(p_2(s_2)\Phi_2\left(\frac{\lambda_2^{\Delta}(s_2)}{p_2(s_2)}\right)\right)^{\frac{1}{2}}\Delta s_2\right)^2.$$

where

$$D(\upsilon_1,\upsilon_2)=4\left(\int_{\iota_0}^{\upsilon_1}\left(\frac{\Phi_1(P_1(s_1))}{P_2(s_1)}\right)^{-1}\Delta s_1\right)^{-1}\left(\int_{\iota_0}^{\upsilon_2}\left(\frac{\Phi_2(P_2(s_2))}{P_2(s_2)}\right)^{-1}\Delta s_2\right)^{-1}.$$

Remark 7. *If we take $\mathbb{T}=\mathbb{Z}$, the inequality (21) is an inverse of inequality due to Pachpatte [19].*

Remark 8. *If we take $\mathbb{T} = \mathbb{R}$, the inequality (21) is an inverse of inequality due to Pachpatte [19].*

Theorem 3. *Let S_1, S_5, S_{14}, S_6, S_{15}, and S_8 be satisfied. Then for S_{10}, S_{18} and S_{20}, we have*

$$\int_{\iota_0}^{\upsilon_1}\int_{\iota_0}^{\varsigma_1}\cdots\int_{\Im_0}^{\upsilon_n}\int_{\Im_0}^{\varsigma_n}\frac{\prod_{\ell=1}^{n}\Phi_\ell(\Lambda_\ell(s_\ell,\Im_\ell))}{\left(\gamma\sum_{\ell=1}^{n}\frac{1}{\gamma_\ell}(s_\ell-\Im_0)(\Im_\ell-\Im_0)\right)^{\frac{1}{\gamma}}}\Delta s_n\Delta\Im_n\ldots\Delta s_1\Delta\Im_1 \tag{22}$$

$$\geqslant L(\upsilon_1\varsigma_1,\ldots,\upsilon_n\varsigma_n)$$

$$\times\prod_{\ell=1}^{n}\left(\int_{\iota_0}^{\upsilon_\ell}\int_{\iota_0}^{\varsigma_\ell}(\rho(\upsilon_\ell)-s_\ell)(\rho(\varsigma_\ell)-\iota_\ell)\left(p_\ell(s_\ell,\iota_\ell)\Phi_\ell\left(\frac{\lambda_\ell(s_\ell,\iota_\ell)}{p_\ell(s_\ell,\iota_\ell)}\right)\right)^{\beta_\ell}\Delta s_\ell\Delta\iota_\ell\right)^{\frac{1}{\beta_\ell}}$$

where

$$L(\upsilon_1\varsigma_1,\ldots,\upsilon_n\varsigma_n)=\prod_{\ell=1}^{n}\left(\int_{\iota_0}^{\upsilon_\ell}\int_{\iota_0}^{\varsigma_\ell}\left(\frac{\Phi_\ell(P_\ell(s_\ell,\iota_\ell))}{P_\ell(s_\ell,\iota_\ell)}\right)^{\gamma_\ell}\Delta s_\ell\Delta\iota_\ell\right)^{\frac{1}{\gamma_\ell}}.$$

Proof. From the hypotheses of Theorem 3, S_{14}, S_{15}, and S_8, it is easy to observe that

$$\begin{aligned}\Phi_\ell(\Lambda_\ell(s_\ell,\iota_\ell)) &= \Phi_\ell\left(\frac{P_\ell(s_\ell,\iota_\ell)\int_{\iota_0}^{s_\ell}\int_{\iota_0}^{\iota_\ell}p_\ell(\xi_\ell,\tau_\ell)\left(\frac{\lambda_\ell(\xi_\ell,\tau_\ell)}{p_\ell(\xi_\ell,\tau_\ell)}\right)\Delta\xi_\ell\Delta\tau_\ell}{\int_{\iota_0}^{s_\ell}\int_{\iota_0}^{\Im_\ell}p_\ell(\xi_\ell,\tau_\ell)\Delta\xi_\ell\Delta\tau_\ell}\right)\\ &\geqslant \Phi_\ell(P_\ell(s_\ell,\iota_\ell))\Phi_\ell\left(\frac{\int_{\iota_0}^{s_\ell}\int_{\iota_0}^{\iota_\ell}p_\ell(\xi_\ell,\tau_\ell)\left(\frac{\lambda_\ell(\xi_\ell,\tau_\ell)}{p_\ell(\xi_\ell,\tau_\ell)}\right)\Delta\xi_\ell\Delta\tau_\ell}{\int_{\iota_0}^{s_\ell}\int_{\iota_0}^{\Im_\ell}p_\ell(\xi_\ell,\tau_\ell)\Delta\xi_\ell\Delta\tau_\ell}\right).\end{aligned} \tag{23}$$

By using inverse Jensen dynamic inequality, we obtain that

$$\Phi_\ell(\Lambda_\ell(s_\ell,\iota_\ell)) \geqslant \frac{\Phi_\ell(P_\ell(s_\ell,\iota_\ell))}{P_\ell(s_\ell,\iota_\ell)}\int_{\iota_0}^{s_\ell}\int_{\Im_0}^{\Im_\ell}p_\ell(\xi_\ell,\tau_\ell)\Phi_\ell\left(\frac{\lambda_\ell(\xi_\ell,\tau_\ell)}{p_\ell(\xi_\ell,\tau_\ell)}\right)\Delta\xi_\ell\Delta\tau_\ell. \tag{24}$$

Applying inverse Hölder's inequality on the right hand side of (24) with indices γ_ℓ and β_ℓ, it is easy to observe that

$$\Phi_\ell(\Lambda_\ell(s_\ell,\iota_\ell)) \geqslant \frac{\Phi_\ell(P_\ell(s_\ell,\iota_\ell))}{P_\ell(s_\ell,\iota_\ell)}[(s_\ell-\iota_0)(\iota_\ell-\iota_0)]^{\frac{1}{\gamma_\ell}}\left(\int_{\iota_0}^{s_\ell}\int_{\iota_0}^{\iota_\ell}\left(p_\ell(\xi_\ell,\tau_\ell)\Phi_\ell\left(\frac{\lambda_\ell(\xi_\ell,\tau_\ell)}{p_\ell(\xi_\ell,\tau_\ell)}\right)\right)^{\beta_\ell}\Delta\xi_\ell\Delta\tau_\ell\right)^{\frac{1}{\beta_\ell}}. \tag{25}$$

By using inequality the following inequality on the term $\left[(s_\ell-\iota_0)(\iota_\ell-\iota_0)\right]^{\frac{1}{\gamma_\ell}}$, we get

$$\prod_{\ell=1}^{n}m_\ell^{\frac{1}{\alpha_\ell}}\geqslant\left(\alpha\sum_{\ell=1}^{n}\frac{1}{\alpha_\ell}m_\ell\right)^{\frac{1}{\alpha}}, \tag{26}$$

$$\frac{\prod_{\ell=1}^{n}\Phi_\ell(\Lambda_\ell(s_\ell,\iota_\ell))}{\left(\gamma\sum_{\ell=1}^{n}\frac{1}{\gamma_\ell}(s_\ell-\iota_0)(\iota_\ell-\iota_0)\right)^{\frac{1}{\gamma}}}$$

$$\geqslant\prod_{\ell=1}^{n}\frac{\Phi_\ell(P_\ell(s_\ell,\iota_\ell))}{P_\ell(s_\ell,\iota_\ell)}\left(\int_{\iota_0}^{s_\ell}\int_{\iota_0}^{\iota_\ell}\left(p_\ell(\xi_\ell,\tau_\ell)\Phi_\ell\left(\frac{\lambda_\ell(\xi_\ell,\tau_\ell)}{p_\ell(\xi_\ell,\tau_\ell)}\right)\right)^{\frac{1}{\beta_\ell}}\Delta\xi_\ell\Delta\tau_\ell\right)^{\frac{1}{\beta_\ell}}. \tag{27}$$

Integrating both sides of (27) over s_ℓ, ι_ℓ from $\Im_0$ to υ_ℓ, ς_ℓ $(\ell = 1,\ldots,n)$, we obtain that

$$\int_{\iota_0}^{\upsilon_1}\int_{\iota_0}^{\varsigma_1}\cdots\int_{\Im_0}^{\upsilon_n}\int_{\Im_0}^{\varsigma_n}\frac{\prod_{\ell=1}^{n}\Phi_\ell(\Lambda_\ell(s_\ell,\Im_\ell))}{\left(\gamma\sum_{\ell=1}^{n}\frac{1}{\gamma_\ell}(s_\ell-\Im_0)(\Im_\ell-\Im_0)\right)^{\frac{1}{\gamma}}}\Delta s_n\Delta\Im_n\ldots\Delta s_1\Delta\Im_1 \tag{28}$$
$$\geqslant\prod_{\ell=1}^{n}\int_{\iota_0}^{\upsilon_\ell}\int_{\iota_0}^{\varsigma_\ell}\frac{\Phi_\ell(P_\ell(s_\ell,\iota_\ell))}{P_\ell(s_\ell,\iota_\ell)}\left(\int_{\iota_0}^{s_\ell}\int_{\iota_0}^{\iota_\ell}\left(p_\ell(\xi_\ell,\tau_\ell)\Phi_\ell\left(\frac{\lambda_\ell(\xi_\ell,\tau_\ell)}{p_\ell(\xi_\ell,\tau_\ell)}\right)\right)^{\beta_\ell}\Delta\xi_\ell\Delta\tau_\ell\right)^{\frac{1}{\beta_\ell}}\Delta s_\ell\Delta\iota_\ell.$$

Applying inverse Hölder's inequality on the right hand side of (28) with indices γ_ℓ and β_ℓ, it is easy to observe that

$$\int_{\iota_0}^{\upsilon_1}\int_{\iota_0}^{\varsigma_1}\cdots\int_{\Im_0}^{\upsilon_n}\int_{\Im_0}^{\varsigma_n}\frac{\prod_{\ell=1}^{n}\Phi_\ell(\Lambda_\ell(s_\ell,\Im_\ell))}{\left(\gamma\sum_{\ell=1}^{n}\frac{1}{\gamma_\ell}(s_\ell-\Im_0)(\Im_\ell-\Im_0)\right)^{\frac{1}{\gamma}}}\Delta s_n\Delta\Im_n\ldots\Delta s_1\Delta\Im_1$$
$$\geqslant\prod_{\ell=1}^{n}\left(\int_{\iota_0}^{\upsilon_\ell}\int_{\iota_0}^{\varsigma_\ell}\left(\frac{\Phi_\ell(P_\ell(s_\ell,\iota_\ell))}{P_\ell(s_\ell,\iota_\ell)}\right)^{\gamma_\ell}\Delta s_\ell\Delta\iota_\ell\right)^{\frac{1}{\gamma_\ell}} \tag{29}$$
$$\times\prod_{\ell=1}^{n}\left(\int_{\iota_0}^{\upsilon_\ell}\int_{\iota_0}^{\varsigma_\ell}\left(\int_{\iota_0}^{s_\ell}\int_{\iota_0}^{\iota_\ell}\left(p_\ell(\xi_\ell,\tau_\ell)\Phi_\ell\left(\frac{\lambda_\ell(\xi_\ell,\tau_\ell)}{p_\ell(\xi_\ell,\tau_\ell)}\right)\right)^{\beta_\ell}\Delta\xi_\ell\Delta\tau_\ell\right)\Delta s_\ell\Delta\iota_\ell\right)^{\frac{1}{\beta_\ell}}.$$

Using Fubini's theorem, we observe that

$$\int_{\iota_0}^{\upsilon_1}\int_{\iota_0}^{\varsigma_1}\cdots\int_{\Im_0}^{\upsilon_n}\int_{\Im_0}^{\varsigma_n}\frac{\prod_{\ell=1}^{n}\Phi_\ell(\Lambda_\ell(s_\ell,\Im_\ell))}{\left(\gamma\sum_{\ell=1}^{n}\frac{1}{\gamma_\ell}(s_\ell-\Im_0)(\Im_\ell-\Im_0)\right)^{\frac{1}{\gamma}}}\Delta s_n\Delta\Im_n\ldots\Delta s_1\Delta\Im_1$$
$$\geqslant L(\upsilon_1\varsigma_1,\ldots,\upsilon_n\varsigma_n)$$
$$\times\prod_{\ell=1}^{n}\left(\int_{\iota_0}^{\upsilon_\ell}\int_{\iota_0}^{\varsigma_\ell}(\upsilon_\ell-s_\ell)(\varsigma_\ell-\iota_\ell)\left(p_\ell(s_\ell,\iota_\ell)\Phi_\ell\left(\frac{\lambda_\ell(s_\ell,\iota_\ell)}{p_\ell(s_\ell,\iota_\ell)}\right)\right)^{\beta_\ell}\Delta s_\ell\Delta\iota_\ell\right)^{\frac{1}{\beta_\ell}}.$$

By using the facts $\upsilon_\ell\geqslant\rho(\upsilon_\ell)$ and $\varsigma_\ell\geqslant\rho(\varsigma_\ell)$, we get

$$\int_{\iota_0}^{\upsilon_1}\int_{\iota_0}^{\varsigma_1}\cdots\int_{\Im_0}^{\upsilon_n}\int_{\Im_0}^{\varsigma_n}\frac{\prod_{\ell=1}^{n}\Phi_\ell(\Lambda_\ell(s_\ell,\Im_\ell))}{\left(\gamma\sum_{\ell=1}^{n}\frac{1}{\gamma_\ell}(s_\ell-\Im_0)(\Im_\ell-\Im_0)\right)^{\frac{1}{\gamma}}}\Delta s_n\Delta\Im_n\ldots\Delta s_1\Delta\Im_1$$
$$\geqslant L(\upsilon_1\varsigma_1,\ldots,\upsilon_n\varsigma_n)$$
$$\times\prod_{\ell=1}^{n}\left(\int_{\iota_0}^{\upsilon_\ell}\int_{\iota_0}^{\varsigma_\ell}(\rho(\upsilon_\ell)-s_\ell)(\rho(\varsigma_\ell)-\iota_\ell)\left(p_\ell(s_\ell,\iota_\ell)\Phi_\ell\left(\frac{\lambda_\ell(s_\ell,\iota_\ell)}{p_\ell(s_\ell,\iota_\ell)}\right)\right)^{\beta_\ell}\Delta s_\ell\Delta\iota_\ell\right)^{\frac{1}{\beta_\ell}}.$$

This completes the proof. □

Remark 9. *In Theorem 3, if $\mathbb{T}=\mathbb{R}$, we get the result due to Zhao et al. [20] (Theorem 2).*

As a special case of Theorem 3, when $\mathbb{T}=\mathbb{Z}$, we have $\rho(n)=n-1$, and we get the following result.

Corollary 2. *Let* $\{a_{s_\ell,\iota_\ell,m_{s_\ell},m_{\iota_\ell}}\}$ *and* $\{p_{s_\ell,\iota_\ell,m_{s_\ell},m_{\iota_\ell}}\}$, $(\ell = 1,\ldots,n)$ *be n sequences of non-negative numbers defined for* $m_{s_\ell} = 1,\ldots,k_{s_\ell}$, *and* $m_{\iota_\ell} = 1,\ldots,k_{\Im_\ell}$, *and define*

$$A_{s_\ell,\iota_\ell,m_{s_\ell},m_{\iota_\ell}} = \sum_{m_{\xi_\ell}}^{m_{s_\ell}} \sum_{m_{\eta_\ell}}^{m_{\iota_\ell}} a_{s_\ell,\iota_\ell,m_{\xi_\ell},m_{\eta_\ell}}$$

$$P_{s_\ell,\iota_\ell,m_{s_\ell},m_{\iota_\ell}} = \sum_{m_{\xi_\ell}}^{m_{s_\ell}} \sum_{m_{\eta_\ell}}^{m_{\iota_\ell}} p_{s_\ell,\iota_\ell,m_{\xi_\ell},m_{\eta_\ell}}. \tag{30}$$

Then

$$\sum_{m_{s_1}}^{k_{s_1}} \sum_{m_{\iota_1}}^{k_{\iota_1}} \cdots \sum_{m_{s_n}}^{k_{s_n}} \sum_{m_{\iota_n}}^{k_{\iota_n}} \frac{\prod_{\ell=1}^n \Phi_\ell(A_{s_\ell,\iota_\ell,m_{s_\ell},m_{\iota_\ell}})}{\left(\gamma \sum_{\ell=1}^n \frac{1}{\gamma_\ell}(m_{s_\ell} m_{\iota_\ell})\right)^{\frac{1}{\gamma}}}$$
$$\geqslant C(k_{s_1}k_{\iota_1},\ldots,k_{s_n}k_{\iota_n})$$
$$\times \prod_{\ell=1}^n \left(\sum_{m_{s_\ell}}^{k_{s_\ell}} \sum_{m_{\iota_\ell}}^{k_{\iota_\ell}} (k_{s_\ell} - (m_{s_\ell} - 1))(k_{\iota_\ell} - (m_{\iota_\ell} - 1)) \left(P_{s_\ell,\iota_\ell,m_{s_\ell},m_{\iota_\ell}} \Phi_\ell\left(\frac{a_{s_\ell,\iota_\ell,m_{s_\ell},m_{\iota_\ell}}}{P_{s_\ell,\iota_\ell,m_{s_\ell},m_{\iota_\ell}}} \right) \right)^{\beta_\ell} \right)^{\frac{1}{\beta_\ell}}$$

where

$$C(k_{s_1}k_{\iota_1},\ldots,k_{s_n}k_{\iota_n}) = \prod_{\ell=1}^n \left(\sum_{m_{s_\ell}}^{k_{s_\ell}} \sum_{m_{\iota_\ell}}^{k_{\iota_\ell}} \left(\frac{\Phi_\ell(P_{s_\ell,\iota_\ell,m_{s_\ell},m_{\iota_\ell}})}{P_{s_\ell,\iota_\ell,m_{s_\ell},m_{\iota_\ell}}} \right)^{\beta_\ell} \right)^{\frac{1}{\beta_\ell}}.$$

Remark 10. *Let* $\lambda_\ell(\xi_\ell,\tau_\ell)$, $p_\ell(\xi_\ell,\tau_\ell)$, $P_\ell(\xi_\ell,\tau_\ell)$, *and* $\lambda_\ell(\xi_\ell,\tau_\ell)$ *change to* $\lambda_\ell(\xi_\ell)$, $p_\ell(\xi_\ell)$, $P_\ell(s_\ell)$, *and* $\lambda_\ell(s_\ell)$, *respectively, and with suitable changes, and we have the following result:*

Corollary 3. *Let* S_{22}, S_{23}, S_{24}, S_{26}, S_{27} *and* S_8 *be satisfied. Then for* S_{18}, S_{20} *and* S_{25} *we have that*

$$\int_{\iota_0}^{v_1} \cdots \int_{\iota_0}^{v_n} \frac{\prod_{\ell=1}^n \Phi_\ell(\Lambda_\ell(s_\ell))}{\left(\gamma \sum_{\ell=1}^n \frac{1}{\gamma_\ell}(s_\ell - \iota_0)\right)^{\frac{1}{\gamma}}} \Delta s_n \ldots \Delta s_1 \tag{31}$$
$$\geqslant L^*(v_1,\ldots,v_n) \prod_{\ell=1}^n \left(\int_{\iota_0}^{v_\ell} (\rho(v_\ell) - s_\ell) \left(p_\ell(s_\ell) \Phi_\ell\left(\frac{\lambda_\ell(s_\ell)}{p_\ell(s_\ell)} \right) \right)^{\beta_\ell} \Delta s_\ell \right)^{\frac{1}{\beta_\ell}}$$

where

$$L^*(v_1,\ldots,v_n) = \prod_{\ell=1}^n \left(\int_{\iota_0}^{v_\ell} \left(\frac{\Phi_\ell(P_\ell(s_\ell))}{P_\ell(s_\ell)} \right)^{\gamma_\ell} \Delta s_\ell \right)^{\frac{1}{\gamma_\ell}}.$$

Corollary 4. *In Corollary 3, if we take* $n = 2$, $\beta_\ell = \frac{1}{2}$, *then the inequality (31) changes to*

$$\int_{\iota_0}^{v_1} \int_{\iota_0}^{v_2} \frac{\Phi_1(\Lambda_1(s_1))\Phi_2(\Lambda_2(s_2))}{((s_1 - \iota_0) + (s_2 - \iota_0))^{-2}} \Delta s_1 \Delta s_2 \geqslant L^{**}(v_1,v_2) \left(\int_{\iota_0}^{v_1} (\rho(v_1) - s_1) \left(p_1(s_1) \Phi\left(\frac{\lambda_1(s_1)}{p_1(s_1)} \right) \right)^2 \Delta s_1 \right)^{\frac{1}{2}}$$
$$\times \left(\int_{\iota_0}^{v_2} (\rho(v_2) - s_2) \left(p_2(s_2) \Psi\left(\frac{\lambda_2(s_2)}{p_2(s_2)} \right) \right)^2 \Delta s_2 \right)^{\frac{1}{2}}. \tag{32}$$

where

$$L^{**}(v_1,v_2) = 4\left(\int_{\iota_0}^{v_1} \left(\frac{\Phi_1(P_1(s_1))}{P_1(s_1)} \right)^{-1} \Delta s_1 \right)^{-1} \left(\int_{\iota_0}^{v_2} \left(\frac{\Phi_2(P_2(s_2))}{P_2(s_2)} \right)^{-1} \Delta s_2 \right)^{-1}.$$

Remark 11. *In Corollary 4, if we take* $\mathbb{T} = \mathbb{R}$, *then the inequality (32) changes to*

$$\int_0^{v_1}\int_0^{v_1}\frac{\Phi_1(\Lambda_1(s_1))\Phi_2(\Lambda_2(s_2))}{(s_1+s_2)^{-2}}ds_1ds_2 \geqslant L^{**}(v_1,v_2)\left(\int_0^{v_1}(v_1-s_1)\left(p_1(s_1)\Phi\left(\frac{\lambda_1(s_1)}{p_1(s_1)}\right)\right)^2 ds_1\right)^{\frac{1}{2}}$$
$$\times\left(\int_0^{v_2}(v_2-s_2)\left(p_2(s_2)\Psi\left(\frac{\lambda_2(s_2)}{p_2(s_2)}\right)\right)^2 ds_2\right)^{\frac{1}{2}}, \tag{33}$$

where

$$L^{**}(v_1,v_2)=4\left(\int_0^{v_1}\left(\frac{\Phi_1(P_1(s_1))}{P_1(s_1)}\right)^{-1}ds_1\right)^{-1}\left(\int_0^{v_2}\left(\frac{\Phi_2(P_2(s_2))}{P_2(s_2)}\right)^{-1}ds_2\right)^{-1}.$$

This is an inverse of the inequality due to Pachpatte [21].

Corollary 5. *In Corollary 3, if we take $\beta_\ell=\frac{n-1}{n}$, the inequality (31) becomes*

$$\int_{\iota_0}^{v_1}\cdots\int_{\iota_0}^{v_n}\frac{\prod_{\ell=1}^n\Phi_\ell(\lambda_\ell(s_\ell)}{\left(\sum_{\ell=1}^n(s_\ell-\iota_0)\right)^{\frac{-n}{n-1}}}\Delta s_1\ldots\Delta s_n$$
$$\geqslant L^*(v_1,\ldots,v_n)\prod_{\ell=1}^n\left(\int_{\iota_0}^{v_\ell}(\rho(v_\ell)-s_\ell)\left(p_\ell(s_\ell)\Phi_\ell\left(\frac{\lambda_\ell(s_\ell)}{p_\ell(s_\ell)}\right)\right)^{\frac{n-1}{n}}\Delta s_\ell\right)^{\frac{n}{n-1}}$$

where

$$L^*(v_1,\ldots,v_n)=n^{\frac{n}{n-1}}\prod_{\ell=1}^n\left(\int_{\iota_0}^{v_\ell}\left(\frac{\Phi_\ell(P_\ell(s_\ell))}{P_\ell(s_\ell)}\right)^{-(n-1)}\Delta s_\ell\right)^{\frac{-1}{n-1}}.$$

Theorem 4. *Let S_1, S_5, S_6, S_9, S_{15}, and S_{16} be satisfied. Then for S_{10}, S_{18}, and S_{20}, we have*

$$\int_{\iota_0}^{v_1}\int_{\iota_0}^{\varsigma_1}\cdots\int_{\Im_0}^{v_n}\int_{\Im_0}^{\varsigma_n}\frac{\prod_{\ell=1}^n P_\ell(s_\ell,\Im_\ell)\Phi_\ell(\Lambda_\ell(s_\ell,\Im_\ell))}{\left(\gamma\sum_{\ell=1}^n\frac{1}{\gamma_\ell}(s_\ell-\Im_0)(\Im_\ell-\Im_0)\right)^{\frac{1}{\gamma}}}\Delta s_n\Delta\Im_n\ldots\Delta s_1\Delta\Im_1 \tag{34}$$
$$\geqslant\prod_{\ell=1}^n\left[(v_\ell-\iota_0)(\varsigma_\ell-\iota_0)\right]^{\frac{1}{\gamma_\ell}}\left(\int_{\iota_0}^{v_\ell}\int_{\iota_0}^{\varsigma_\ell}(\rho(v_\ell)-s_\ell)(\rho(\varsigma_\ell)-\iota_\ell)\left(p_\ell(s_\ell,\iota_\ell)\Phi_\ell(\lambda_\ell(s_\ell,\iota_\ell))\right)^{\beta_\ell}\Delta s_\ell\Delta\iota_\ell\right)^{\frac{1}{\beta_\ell}}.$$

Proof. From the hypotheses of Theorem 4, and by using inverse Jensen dynamic inequality, we have

$$\begin{aligned}\Phi_\ell(\Lambda_\ell(s_\ell,\iota_\ell)) &= \Phi_\ell\left(\frac{1}{P_\ell(s_\ell,\Im_\ell)}\int_{\Im_0}^{s_\ell}\int_{\Im_0}^{\Im_\ell}p_\ell(\xi_\ell,\tau_\ell)\lambda_\ell(\xi_\ell,\tau_\ell)\Delta\xi_\ell\Delta\tau_\ell\right)\\ &\geqslant\frac{1}{P_\ell(s_\ell,\iota_\ell)}\int_{\iota_0}^{s_\ell}\int_{\iota_0}^{\iota_\ell}p_\ell(\sigma_\ell,\tau_\ell)\Phi_\ell(\lambda_\ell(\xi_\ell,\tau_\ell))\Delta\xi_\ell\Delta\tau_\ell.\end{aligned} \tag{35}$$

Applying inverse Hölder's inequality on the right hand side of (35) with indices γ_ℓ and β_ℓ, it is easy to observe that

$$\Phi_\ell(\Lambda_\ell(s_\ell,\iota_\ell))\geqslant\frac{1}{P_\ell(s_\ell,\Im_\ell)}\left[(s_\ell-\iota_0)(\iota_\ell-\iota_0)\right]^{\frac{1}{\gamma_\ell}}\left(\int_{\Im_0}^{s_\ell}\int_{\Im_0}^{\Im_\ell}\left(p_\ell(\xi_\ell,\tau_\ell)\Phi_\ell(\lambda_\ell(\xi_\ell,\tau_\ell))\right)^{\beta_\ell}\Delta\xi_\ell\Delta\tau_\ell\right)^{\frac{1}{\beta_\ell}}.$$

By using the inequality (26) on the term $\left[(s_\ell-\iota_0)(\iota_\ell-\iota_0)\right]^{\frac{1}{\gamma_\ell}}$, we get

$$\frac{P_\ell(s_\ell,\iota_\ell)\Phi_\ell(\Lambda_\ell(s_\ell,\iota_\ell))}{\left(\gamma\sum_{\ell=1}^n\frac{1}{\gamma_\ell}(s_\ell-\iota_0)(\iota_\ell-\iota_0)\right)^{\frac{1}{\gamma}}} \geqslant \left(\int_{\iota_0}^{s_\ell}\int_{\iota_0}^{\iota_\ell}\left(p_\ell(\xi_\ell,\tau_\ell)\Phi_\ell\left(\lambda_\ell(\xi_\ell,\tau_\ell)\right)\right)^{\beta_\ell}\Delta\xi_\ell\Delta\tau_\ell\right)^{\frac{1}{\beta_\ell}}. \tag{36}$$

Integrating both sides of (36) over s_ℓ, ι_ℓ from ι_0 to v_ℓ, ς_ℓ ($\ell = 1,\ldots,n$), we get

$$\int_{\iota_0}^{v_1}\int_{\iota_0}^{\varsigma_1}\cdots\int_{\Im_0}^{v_n}\int_{\Im_0}^{\varsigma_n}\frac{\prod_{\ell=1}^n P_\ell(s_\ell,\Im_\ell)\Phi_\ell(\Lambda_\ell(s_\ell,\Im_\ell))}{\left(\gamma\sum_{\ell=1}^n\frac{1}{\gamma_\ell}(s_\ell-\Im_0)(\Im_\ell-\Im_0)\right)^{\frac{1}{\gamma}}}\Delta s_n\Delta\Im_n\ldots\Delta s_1\Delta\Im_1$$
$$\geqslant\prod_{\ell=1}^n\int_{\iota_0}^{v_\ell}\int_{\Im_0}^{\varsigma_\ell}\left(\int_{\Im_0}^{s_\ell}\int_{\Im_0}^{\Im_\ell}\left(p_\ell(\xi_\ell,\tau_\ell)\Phi_\ell\left(\lambda_\ell(\xi_\ell,\tau_\ell)\right)\right)^{\beta_\ell}\Delta\sigma_\ell\Delta\tau_\ell\right)^{\frac{1}{\beta_\ell}}.$$

Applying inverse Hölder's inequality on the right hand side of (37) with indices γ_ℓ and β_ℓ, it is easy to observe that

$$\int_{\iota_0}^{v_1}\int_{\iota_0}^{\varsigma_1}\cdots\int_{\Im_0}^{v_n}\int_{\Im_0}^{\varsigma_n}\frac{\prod_{\ell=1}^n P_\ell(s_\ell,\Im_\ell)\Phi_\ell(\Lambda_\ell(s_\ell,\Im_\ell))}{\left(\gamma\sum_{\ell=1}^n\frac{1}{\gamma_\ell}(s_\ell-\Im_0)(\Im_\ell-\Im_0)\right)^{\frac{1}{\gamma}}}\Delta s_1\Delta\Im_1\ldots\Delta s_n\Delta\Im_n \tag{37}$$
$$\geqslant\prod_{\ell=1}^n\left[(v_\ell-\iota_0)(\varsigma_\ell-\iota_0)\right]^{\frac{1}{\gamma_\ell}}\left(\int_{\iota_0}^{v_\ell}\int_{\Im_0}^{\varsigma_\ell}\int_{\Im_0}^{s_\ell}\int_{\Im_0}^{\Im_\ell}\left(p_\ell(\xi_\ell,\tau_\ell)\Phi_\ell\left(\lambda_\ell(\xi_\ell,\tau_\ell)\right)\right)^{\beta_\ell}\Delta\xi_\ell\Delta\tau_\ell\Delta s_\ell\Delta\Im_\ell\right)^{\frac{1}{\beta_\ell}}.$$

By using Fubini's theorem, we observe that

$$\int_{\iota_0}^{v_1}\int_{\iota_0}^{\varsigma_1}\cdots\int_{\Im_0}^{v_n}\int_{\Im_0}^{\varsigma_n}\frac{\prod_{\ell=1}^n P_\ell(s_\ell,\Im_\ell)\Phi_\ell(\Lambda_\ell(s_\ell,\Im_\ell))}{\left(\gamma\sum_{\ell=1}^n\frac{1}{\gamma_\ell}(s_\ell-\Im_0)(\Im_\ell-\Im_0)\right)^{\frac{1}{\gamma}}}\Delta s_n\Delta\Im_n\ldots\Delta s_1\Delta\Im_1$$
$$\geqslant\prod_{\ell=1}^n\left[(v_\ell-\iota_0)(\varsigma_\ell-\iota_0)\right]^{\frac{1}{\gamma_\ell}}\left(\int_{\iota_0}^{v_\ell}\int_{\iota_0}^{\varsigma_\ell}(v_\ell-s_\ell)(\varsigma_\ell-\Im_\ell)\left(p_\ell(s_\ell,\Im_\ell)\Phi_\ell\left(\lambda_\ell(s_\ell,\Im_\ell)\right)\right)^{\beta_\ell}\Delta s_\ell s_\ell\Delta\Im_\ell\right)^{\frac{1}{\beta_\ell}}.$$

By using the facts $v_\ell \geqslant \rho(v_\ell)$ and $\varsigma_\ell \geqslant \rho(\varsigma_\ell)$, we get

$$\int_{\iota_0}^{v_1}\int_{\iota_0}^{\varsigma_1}\cdots\int_{\Im_0}^{v_n}\int_{\Im_0}^{\varsigma_n}\frac{\prod_{\ell=1}^n P_\ell(s_\ell,\Im_\ell)\Phi_\ell(\Lambda_\ell(s_\ell,\Im_\ell))}{\left(\gamma\sum_{\ell=1}^n\frac{1}{\gamma_\ell}(s_\ell-\Im_0)(\Im_\ell-\Im_0)\right)^{\frac{1}{\gamma}}}\Delta s_n\Delta\Im_n\ldots\Delta s_1\Delta\Im_1$$
$$\geqslant\prod_{\ell=1}^n\left[(v_\ell-\iota_0)(\varsigma_\ell-\iota_0)\right]^{\frac{1}{\gamma_\ell}}\left(\int_{\iota_0}^{v_\ell}\int_{\iota_0}^{\varsigma_\ell}(\rho(v_\ell)-s_\ell)(\rho(\varsigma_\ell)-\iota_\ell)\left(p_\ell(s_\ell,\iota_\ell)\Phi_\ell\left(\lambda_\ell(s_\ell,\iota_\ell)\right)\right)^{\beta_\ell}\Delta s_\ell\Delta\iota_\ell\right)^{\frac{1}{\beta_\ell}}.$$

This completes the proof. □

Remark 12. *In Theorem 4, if $\mathbb{T} = \mathbb{R}$, we get the result due to Zhao et al. [20] (Theorem 3).*

As a special case of Theorem 4, when $\mathbb{T} = \mathbb{Z}$, we have $\rho(n) = n - 1$, and we get the following result.

Corollary 6. *Let* $\{a_{s_\ell,\iota_\ell,m_{s_\ell},m_{\iota_\ell}}\}$ *and* $\{p_{s_\ell,\iota_\ell,m_{s_\ell},m_{\iota_\ell}}\}$, $(\ell = 1,\ldots,n)$ *be* n *sequences of non-negative numbers defined for* $m_{s_\ell} = 1,\ldots,k_{s_\ell}$ *and* $m_{\iota_\ell} = 1,\ldots,k_{\iota_\ell}$, *and define*

$$\begin{aligned} A_{s_\ell,\iota_\ell,m_{s_\ell},m_{\iota_\ell}} &= \frac{1}{P_{s_\ell,\iota_\ell,m_{s_\ell},m_{\iota_\ell}}}\sum_{m_{\xi_\ell}}^{m_{s_\ell}}\sum_{m_{\eta_\ell}}^{m_{\iota_\ell}} a_{s_\ell,\iota_\ell,m_{\xi_\ell},m_{\eta_\ell}} p_{s_\ell,\iota_\ell,m_{\xi_\ell},m_{\eta_\ell}}, \\ P_{s_\ell,\iota_\ell,m_{s_\ell},m_{\iota_\ell}} &= \sum_{m_{\xi_\ell}}^{m_{s_\ell}}\sum_{m_{\eta_\ell}}^{m_{\iota_\ell}} p_{s_\ell,\iota_\ell,m_{\xi_\ell},m_{\eta_\ell}}. \end{aligned} \tag{38}$$

Then

$$\begin{aligned} &\sum_{m_{s_1}}^{k_{s_1}}\sum_{m_{\iota_1}}^{k_{\iota_1}}\cdots\sum_{m_{s_n}}^{k_{s_n}}\sum_{m_{\iota_n}}^{k_{\iota_n}} \frac{\prod_{\ell=1}^{n} P_{s_\ell,\iota_\ell,m_{s_\ell},m_{\iota_\ell}}\Phi_\ell(A_{s_\ell,\iota_\ell,m_{s_\ell},m_{\iota_\ell}})}{\left(\gamma\sum_{\ell=1}^{n}\frac{1}{\gamma_\ell}(m_{s_\ell}m_{\iota_\ell})\right)^{\frac{1}{\gamma}}} \\ &\geqslant \prod_{\ell=1}^{n}(k_{s_\ell}k_{\iota_\ell})^{\frac{1}{\gamma_\ell}}\left(\sum_{m_{s_\ell}}^{k_{s_\ell}}\sum_{m_{\iota_\ell}}^{k_{\iota_\ell}}(k_{s_\ell}-(m_{s_\ell}-1))(k_{\iota_\ell}-(m_{\iota_\ell}-1))\left(p_{s_\ell,\iota_\ell,m_{s_\ell},m_{\iota_\ell}}\Phi_\ell\left(a_{s_\ell,\iota_\ell,m_{s_\ell},m_{\iota_\ell}}\right)\right)^{\beta_\ell}\right)^{\frac{1}{\beta_\ell}}. \end{aligned}$$

Remark 13. *Let* $\lambda_\ell(\xi_\ell,\tau_\ell)$, $p_\ell(\xi_\ell,\tau_\ell)$, $P_\ell(\xi_\ell,\tau_\ell)$ *and*

$$\lambda_\ell(s_\ell,\iota_\ell) = \frac{1}{P_\ell(s_\ell,\iota_\ell)}\int_{\iota_0}^{s_\ell}\int_0^{\iota_\ell} p_\ell(\xi_\ell,\tau_\ell)\Lambda_\ell(\xi_\ell,\tau_\ell)\Delta\xi_\ell\Delta\tau_\ell$$

changes to $\lambda_\ell(\xi_\ell)$, $p_\ell(\xi_\ell)$, $P_\ell(s_\ell)$ *and*

$$\lambda_\ell(s_\ell) = \frac{1}{P_\ell(s_\ell)}\int_{\iota_0}^{s_\ell} p_\ell(\xi_\ell)\lambda_\ell(\xi_\ell)\Delta\xi_\ell.$$

respectively, and with suitable changes, we have the following result:

Corollary 7. *Let* S_{22}, S_{23}, S_{26}, S_{27}, *and* S_{28} *be satisfied. Then for* S_{18}, S_{20}, *and* S_{25}, *we have*

$$\begin{aligned} &\int_{\iota_0}^{\upsilon_1}\cdots\int_{\iota_0}^{\upsilon_n}\frac{\prod_{\ell=1}^{n}P_\ell(s_\ell)\Phi_\ell(\Lambda_\ell(s_\ell))}{\left(\gamma\sum_{\ell=1}^{n}\frac{1}{\gamma_\ell}(s_\ell-\iota_0)\right)^{\frac{1}{\gamma}}}\Delta s_n\ldots\Delta s_1 \\ &\geqslant \prod_{\ell=1}^{n}(\upsilon_\ell-\iota_0)^{\frac{1}{\gamma_\ell}}\left(\int_{\iota_0}^{\upsilon_\ell}(\rho(\upsilon_\ell)-s_\ell)\left(p_\ell(s_\ell)\Phi_\ell(\lambda_\ell(s_\ell))\right)^{\beta_\ell}\Delta s_\ell\right)^{\frac{1}{\beta_\ell}}. \end{aligned} \tag{39}$$

Corollary 8. *In Corollary 7, if we take* $n = 2$, $\beta_\ell = \frac{1}{2}$, *then the inequality (39) changes to*

$$\begin{aligned} &\int_{\iota_0}^{\upsilon_1}\int_{\iota_0}^{\upsilon_2}\frac{P_1(s_1)P_2(s_2)\Phi_1(\Lambda_1(s_1))\Phi_2(\Lambda_2(s_2))}{((s_1-\iota_0)+(s_2-\iota_0))^{-2}}\Delta s_1\Delta s_2 \geqslant 4\left[(\upsilon_1-\iota_0)(\upsilon_2-\iota_0)\right]^{-1} \\ &\times\left(\int_{\iota_0}^{\upsilon_1}(\rho(\upsilon_1)-s_1)\left(p_1(s_1)\Phi_1(\lambda_1(s_1))\right)^2\Delta s_1\right)^{\frac{1}{2}}\left(\int_{\iota_0}^{\upsilon_2}(\rho(\upsilon_2)-s_2)\left(p_2(s_2)\Phi_2\left(\lambda_2(s_2)\right)\right)^2\Delta s_2\right)^{\frac{1}{2}}. \end{aligned} \tag{40}$$

Remark 14. *In Corollary 8, if we take* $\mathbb{T} = \mathbb{R}$, *then the inequality (40) changes to*

$$\begin{aligned} &\int_0^{\upsilon_1}\int_0^{\upsilon_1}\frac{P_1(s_1)P_2(s_2)\Phi_1(\Lambda_1(s_1))\Phi_2(\Lambda_2(s_2))}{(s_1+s_2)^{-2}}ds_1ds_2 \geqslant 4\left[\upsilon_1\upsilon_2\right]^{-1} \\ &\times\left(\int_0^{\upsilon_1}(\upsilon_1-s_1)\left(p_1(s_1)\Phi_1(\lambda_1(s_1))\right)^2ds_1\right)^{\frac{1}{2}}\left(\int_0^{\upsilon_2}(\upsilon_2-s_2)\left(p_2(s_2)\Phi_2\left(\lambda_2(s_2)\right)\right)^2ds_2\right)^{\frac{1}{2}}. \end{aligned} \tag{41}$$

This is an inverse of the inequality due to Pachpatte [21].

Corollary 9. *In Corollary 8, let $p_1(s_1) = p_2(s_2) = 1$, then $P_1(s_1) = s_1$, $P_2(s_2) = s_2$. Therefore, the inequality (40) changes to*

$$\int_{\iota_0}^{\upsilon_1}\int_{\Im_0}^{\upsilon_2} \frac{\Phi_1(\Lambda_1(s_1))\Phi_2(\Lambda_2(s_2))}{(s_1s_2)^{-1}\big((s_1-\Im_0)+(s_2-\Im_0)\big)^{-2}}\Delta s_1\Delta s_2 \geqslant 4\big[(\upsilon_1-\Im_0)(\upsilon_2-\Im_0)\big]^{-1} \tag{42}$$
$$\times\left(\int_{\iota_0}^{\upsilon_1}(\rho(\upsilon_1)-s_1)\Big(\Phi_1\big(\lambda_1(s_1)\big)\Big)^2\Delta s_1\right)^{\frac{1}{2}}\left(\int_{\iota_0}^{\upsilon_2}(\rho(\upsilon_2)-s_2)\Big(\Phi_2\Big(\lambda_2(s_2)\Big)\Big)^2\Delta s_2\right)^{\frac{1}{2}}.$$

Remark 15. *In Corollary 9, if we take $\mathbb{T}=\mathbb{R}$, then the inequality (42) changes to*

$$\int_0^{\upsilon_1}\int_0^{\upsilon_1}\frac{\Phi_1(\Lambda_1(s_1))\Phi_2(\Lambda_2(s_2))}{(s_1s_2)^{-1}(s_1+s_2)^{-2}}ds_1ds_2 \geqslant 4\big[\upsilon_1\upsilon_2\big]^{-1}$$
$$\times\left(\int_0^{\upsilon_1}(\upsilon_1-s_1)\Big(\Phi_1\big(\lambda_1(s_1)\big)\Big)^2 ds_1\right)^{\frac{1}{2}}\left(\int_0^{\upsilon_2}(\upsilon_2-s_2)\Big(\Phi_2\Big(\lambda_2(s_2)\Big)\Big)^2 ds_2\right)^{\frac{1}{2}}.$$

This is an inverse inequality of the following inequality, which was proved by Pachpatte [20].

$$\int_0^{\upsilon}\int_0^{\varsigma}\frac{\Phi(\Lambda(s))\Psi(G(\iota))}{(s\iota)^{-1}(s+\iota)}dsd\Im \leqslant \frac{1}{2}\big[\upsilon\varsigma\big]^{\frac{1}{2}}$$
$$\times\left(\int_0^{\upsilon}(\upsilon-s_1)\Big(\Phi\big(\lambda(s)\big)\Big)^2 ds\right)^{\frac{1}{2}}\left(\int_0^{\varsigma}(\varsigma-\iota)\Big(\Psi\Big(g(\Im)\Big)\Big)^2 d\Im\right)^{\frac{1}{2}}.$$

Corollary 10. *In Corollary 7, if we take $\beta_\ell = \frac{n-1}{n}$ $(\ell = 1,\ldots,n)$, the inequality (39)*

$$\int_{\iota_0}^{\upsilon_1}\cdots\int_{\iota_0}^{\upsilon_n}\frac{\prod_{\ell=1}^n P_\ell(s_\ell)\Phi_\ell(\Lambda_\ell(s_\ell))}{\left(\sum_{\ell=1}^n(s_\ell-\iota_0)\right)^{\frac{-n}{n-1}}}\Delta s_1\ldots\Delta s_n$$
$$\geqslant n^{\frac{n}{n-1}}\prod_{\ell=1}^n(\upsilon_\ell-\iota_0)^{\frac{-1}{n-1}}\left(\int_{\iota_0}^{\upsilon_\ell}(\rho(\upsilon_\ell)-s_\ell)\Big(p_\ell(s_\ell)\Phi_\ell(\lambda_\ell(s_\ell))\Big)^{\frac{n-1}{n}}\Delta s_\ell\right)^{\frac{n}{n-1}}.$$

3. Conclusions

In this article, we gave some generalizations of the Hardy-Hilbert inequality on a general time scale, and some dynamic integral and discrete inequalities known in the literature were extended as special cases of our results.

Author Contributions: Conceptualization, H.M.E.-O., A.A.E.-D., S.D.M., D.B. and C.C.; formal analysis, H.M.E.-O., A.A.E.-D., S.D.M., D.B. and C.C.; investigation, H.M.E.-O., A.A.E.-D., S.D.M., D.B. and C.C.; writing—original draft preparation, H.M.E.-O., A.A.E.-D., S.D.M., D.B. and C.C.; writing—review and editing, H.M.E.-O., A.A.E.-D., S.D.M., D.B. and C.C. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Hardy, G.H.; Littlewood, J.E.; Polya, G. *Inequalities*; Cambridge University Press: Cambridge, MA, USA, 1952.
2. Yang, B.; Wu, S.; Wang, A. A new Hilbert-type inequality with positive homogeneous kernel and its equivalent forms. *Symmetry* **2020**, *12*, 342. [CrossRef]
3. Agarwal, R.; O'Regan, D.; Saker, S. *Dynamic Inequalities on Time Scales*; Springer: Cham, Switzerland, 2014.

4. Saker, S.H.; El-Deeb, A.A.; Rezk, H.M.; Agarwal, R.P. On Hilbert's inequality on time scales. *Appl. Anal. Discrete Math.* **2017**, *11*, 399–423. [CrossRef]
5. Todorčević, V. *Harmonic Quasiconformal Mappings and Hyperbolic Type Metrics*; Springer: Berlin/Heidelberg, Germany, 2019.
6. Stojiljkovič, V.; Radojevič, S.; Cetin, E., Cavic, V.S.; Radenovič, S. Sharp Bounds for Trigonometric and Hyperbolic Functions with Application to Fractional Calculus. *Symmetry* **2022**, *14*, 1260. [CrossRef]
7. Yang, B.; Wu, S.; Chen, Q. A new extension of Hardy-Hilbert's inequality containing kernel of double power functions. *Mathematics* **2020**, *8*, 894. [CrossRef]
8. El-Deeb, A.A.; Makharesh, S.D.; Askar, S.S.; Awrejcewicz, J. A variety of Nabla Hardy's type inequality on time scales. *Mathematics* **2022**, *10*, 722. [CrossRef]
9. El-Deeb, A.A.; Baleanu, D. Some new dynamic Gronwall-Bellman-Pachpatte type inequalities with delay on time scales and certain applications. *J. Inequal. Appl.* **2022**, *45*, 1–9. [CrossRef]
10. El-Deeb, A.A.; Moaaz, O.; Baleanu, D.; Askar, S.S. A variety of dynamic α-conformable Steffensen-type inequality on a time scale measure space. *AIMS Math.* **2022**, *7*, 11382–11398. [CrossRef]
11. El-Deeb, A.A.; Akin, E.; Kaymakcalan, B. Generalization of Mitrinović-Pečarić inequalities on time scales. *Rocky Mt. J. Math.* **2021**, *51*, 1909–1918. [CrossRef]
12. El-Deeb, A.A.; Makharesh, S.D.; Nwaeze, E.R.; Iyiola, O.S.; Baleanu, D. On nabla conformable fractional Hardy-type inequalities on arbitrary time scales. *J. Inequal. Appl.* **2021**, *192*, 1–23. [CrossRef]
13. El-Deeb, A.A.; Awrejcewicz, J. Novel Fractional Dynamic Hardy–Hilbert-Type Inequalities on Time Scales with Applications. *Mathematics* **2021**, *9*, 2964. [CrossRef]
14. Bohner, M.; Peterson, A. *Dynamic Equations on Time Scales: An Introduction with Applications*; Birkhauser: Boston, MA, USA, 2001.
15. El-Deeb, A.A.; El-Sennary, H.A.; Agarwal, P. Some opial-type inequalities with higher order delta derivatives on time scales. *Rev. Real Acad. Cienc. Exactas Fisicas Nat. Ser. Mat.* **2020**, *114*, 29. [CrossRef]
16. Bibi, R.; Bohner, M.; Pecaric, J.; Varosanec, S. Minkowski and Beckenbach-Dresher inequalities and functionals on time scales. *J. Math. Inequal.* **2013**, *7*, 299–312. [CrossRef]
17. Zhao, C.-J.; Cheung, W.-S. Inverses of new Hilbert-Pachpatte-type inequalities. *J. Inequalities Appl.* **2006**, *1*, 97860. [CrossRef]
18. Handley, G.D.; Koliha, J.J.; Pecaric, J.A. Hilbert type inequality. *Tamkang J. Math.* **2000**, *31*, 311–316. [CrossRef]
19. Pachpatte, B.G. A note on Hilbert type inequality. *Tamkang J. Math.* **1998**, *29*, 293–298. [CrossRef]
20. Zhao, C.-J.; Cheung, W.-S. On new hilbert-pachpatte type integral inequalities. *Taiwanese J. Math.* **2010**, *257*, 1271–1282. [CrossRef]
21. Pachpatte, B.G. On some new inequalities similar to Hilbert's inequality. *J. Math. Anal. Appl.* **1998**, *226*, 166–179. [CrossRef]