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On some fractional operators generated from Abel's formula

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Abstract: This work aims to share some fractional integrals and derivatives containing three real parameters. The main tool to introduce such operators is the corresponding Abel's equation. Solvability conditions for the Abel's equations are shared. Semigroup properties for fractional integrals are introduced. Integration by parts rule is given. Moreover, mean value theorems and related results are shared. At the end of the paper, some directions for some fractional operators are given.

Key words: Fractional integrals, fractional derivatives, mean value theorems

1. Introduction

The integral containing a kernel of the form $(x-t)^{\beta-1}$ with $a \leq t \leq x \leq b$, $[a,b] \subset (-\infty,\infty)$ and $0<\beta<1$, was studied firstly by Abel while he was studying on tautochrone problem [3], [4]. Abel showed that the equation

$$\frac{1}{\Gamma(\beta)} \int_a^x \frac{f(t)}{(x-t)^{1-\beta}} dt = g(x), \tag{1.1}$$

where $a < x \leq b$, Γ is the Gamma function and $0<\beta<1$, is solvable in $L^1(a,b)$ (L^p denotes the Lebesgue space) if and only if the integral

$$\frac{1}{\Gamma(1-\beta)} \int_a^x \frac{g(t)}{(x-t)^{\beta}} dt$$

is absolutely continuous on $[a,b]$ and

$$\lim_{x\to a^+} \int_a^x \frac{g(t)}{(x-t)^{\beta}} dt = 0.$$

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Although Abel's work only dealt with solving an integral equation containing a parameter β with $0<\beta<1$ the left-hand side of (1.1) was used by Riemann [12] to introduce an expression for the fractional integral as

$$^{\beta}I_{a^+}f(x) := \frac{1}{\Gamma(\beta)}\int_a^x \frac{f(t)}{(x-t)^{1-\beta}}dt,\ a<x\le b,\ 0<\beta<1. \tag{1.2}$$

As was shared in Abel's papers such operators have many (real-world) applications. Among these applications one of the areas is the mean value theorem. First mean value theorem related with the fractional integrals $^{\beta}I_{a^+}f$ of a function f was shared by Riesz [13]. In [13], Riesz proved that for a function f belonging to $L^1(a,b+\in),\in>0$, such that $^{\beta}I_{a^+}f(x)$ is continuous on $[a,b]$ and $^{\beta}I_{a^+}f(a^+)=0$ there exists a $\tau\in[a,b)$ such that

$$^{\beta}I_{a^+}f(x) -^{\beta} I_{b^+}f(x) =^{\beta} I_{a^+}f(\tau),\ 0<\beta<1. \tag{1.3}$$

Eq. (1.3) gives rise to the following inequalities [14]

$$|^{\gamma}I_{a^+}f(x)| \le const.p(x)^{\left(1-\frac{\gamma}{\beta}\right)}q(x)^{\frac{\gamma}{\beta}},\ 0<\gamma<\beta, \tag{1.4}$$

where p and q are nondecreasing functions such that

$$|f(x)|\le p(x),\ \ \left|^{\beta}I_{a^+}f(x)\right|\le q(x),\ x>a,$$

and

$$\left|\int_a^b (x-t)^{\beta-1}f(t)dt\right| \le esssup_{\zeta\in[a,b]}\left|\int_a^{\zeta}(\zeta-t)^{\beta-1}f(t)dt\right|,\ x>b. \tag{1.5}$$

In 1953, Isaacs [8] proved that (1.5) holds if the lower bound is negative infinity, i.e.

$$\left|\int_{-\infty}^b (x-t)^{\beta-1}f(t)dt\right| \le esssup_{\zeta<b}\left|\int_{-\infty}^{\zeta}(\zeta-t)^{\beta-1}f(t)dt\right|,\ x>b. \tag{1.6}$$

where $0<\beta<1$, provided that left-hand integral converges, or upper bound is infinity, i.e.

$$\left|\int_a^{\infty} (t-x)^{\beta-1}f(t)dt\right| \le esssup_{\zeta>a}\left|\int_{\zeta}^{\infty}(t-\zeta)^{\beta-1}f(t)dt\right|,\ x<a, \tag{1.7}$$

where $0<\beta<1$, provided that left-hand integral converges.

Choudhary and Kumar [5] introduced an abstract inequality related with (1.7) (or (1.6)) as

$$\left|\int_a^{\infty} G(t-x)f(t)dt\right| \le esssup_{\zeta>a}\left|\int_{\zeta}^{\infty}G(t-\zeta)f(t)dt\right|,\ x<a, \tag{1.8}$$

provided that left-hand integral converges absolutely, where $G(t), H(t)$ are decreasing and positive functions for $t > 0$, $G(t), H(t)$ and $H'(t)$ are continuous such that

$$\int_0^x G(x,t)H(t)dt = \begin{cases} 1, & x > 0, \\ 0, & x = 0. \end{cases} \tag{1.9}$$

Clearly for $G(t) = t^{\beta-1}/\Gamma(\beta)$ and $H(t) = t^{-\beta}/\Gamma(1-\beta)$, (1.8) and (1.9) are contained in (1.7).

Erdélyi and Osler introduced a fractional integral by just changing the kernel $(x-t)^{\beta-1}$ with $(\varphi(x) - \varphi(t))^{\beta-1}\varphi'(t)$, where φ is an infinitely differentiable function and $\varphi'(x) > 0$ on a given interval $[a,b]$ [6], [7], [11]. Samko et al. [14] shared some properties of this fractional integral for the monotone functions φ having continuous derivatives. Moreover the readers may see the papers [1], [2], [10].

In this paper, we will consider the following fractional integral operators

$${}^{\beta,\alpha}I^{\rho}_{a+}f(x) := \frac{\alpha}{\Gamma(\beta)} \int_a^x \frac{\left[(x-\rho)^{\alpha} - (t-\rho)^{\alpha}\right]^{\beta-1}}{(t-\rho)^{1-\alpha}} f(t)dt,$$

where $\alpha > 0$, $0 < \beta < 1$, $-\infty < \rho < \infty$ such that $x \neq \rho$ and $-\infty < a < x \leq b < \infty$, and

$${}^{\beta,\alpha}I^{\delta}_{b-}f(x) := \frac{\alpha}{\Gamma(\beta)} \int_x^b \frac{\left[(\delta-x)^{\alpha} - (\delta-t)^{\alpha}\right]^{\beta-1}}{(\delta-t)^{1-\alpha}} f(t)dt,$$

where $\alpha > 0$, $0 < \beta < 1$, $-\infty < \delta < \infty$ such that $x \neq \delta$ and $-\infty < a \leq x < b < \infty$.

At this stage we should note that these operators can be obtained from the kernel introduced by Erdélyi and Osler without sharing any reason. But we will share a way to get these operators using Abel's equations. Beside this, fractional integrals $\alpha^{-\beta}\left\{{}^{\beta,\alpha}I^{a}_{a+}f\right\}$ and $\alpha^{-\beta}\left\{{}^{\beta,\alpha}I^{b}_{b-}f\right\}$ have been introduced in [9] using iterated $n-$fold integrals and conformable derivatives. However, as can be seen that the operators $\alpha^{-\beta}\left\{{}^{\beta,\alpha}I^{\rho}_{a+}\right\}$ and $\alpha^{-\beta}\left\{{}^{\beta,\alpha}I^{\delta}_{b-}\right\}$ can not be obtained from iterated integrals unless $\rho = a$ and $\delta = b$. On the other side, iterated integrals method is one of the methods to construct the corresponding fractional integral and derivative operators. In this paper, we construct the corresponding fractional integral (and derivative) using only Abel's equation for $0 < \beta < 1$ and this differs from Erdélyi and Oslers' ideas. Moreover, we will share mean value theorems for ${}^{\beta,\alpha}I^{\rho}_{a+}f$ and ${}^{\beta,\alpha}I^{\delta}_{b-}f$ and some inequalities generalizing (1.3) and (1.5). Letting one of the bounds of the fractional integrals infinity we will generalize the results given in (1.6) and (1.7).

In this paper we will use the notation $L^p((a,b);w)$ to denote the Lebesgue space consisting of all functions f such that $|w|\,|f|^p$ is integrable on (a,b). For $w = 1$ we will use the notation $L^p(a,b)$.

2. On Abel's equation

In this section, firstly, we will introduce some results on solvability of Abel's equations and then we will share the related results.

Theorem 2.1. *Let $\alpha > 0,\ 0 < \beta < 1$ and $-\infty < \rho < \infty$. For $(x-\rho)^{\alpha-1}\psi(x) \in L^1(a,b)$ the equation*

$$\frac{\alpha}{\Gamma(\beta)}\int_a^x \frac{[(x-\rho)^\alpha-(t-\rho)^\alpha]^{\beta-1}}{(t-\rho)^{1-\alpha}}f(t)dt=\psi(x) \tag{2.1}$$

has a solution f with $(x-\rho)^{\alpha-1}f(x)\in L^1(a,b)$, where $x\in(a,b]\subset(-\infty,\infty)$ and $x\neq\rho$ for $\rho\in(a,b]$, if and only if

$$^{\beta,\alpha}(\eta\psi)^{\rho}_{a+}(x) := \frac{1}{\Gamma(1-\beta)}\int_a^x \frac{[(x-\rho)^\alpha-(t-\rho)^\alpha]^{-\beta}}{(t-\rho)^{1-\alpha}}\psi(t)dt$$

is absolutely continuous on $[a,b]$ and $^{\beta,\alpha}(\eta\psi)^{\rho}_{a+}(a^+)=0$. In this case we have the following

$$f(x)=(x-\rho)^{1-\alpha}\frac{d}{dx}\,^{\beta,\alpha}(\eta\psi)^{\rho}_{a+}(x).$$

Proof With a direct calculation we get that

$$\int_a^b \frac{^{\beta,\alpha}(\eta\psi)^{\rho}_{a+}(x)}{(x-\rho)^{1-\alpha}}dx=\frac{\alpha^{-1}}{\Gamma(2-\beta)}\int_a^b\frac{[(b-\rho)^\alpha-(x-\rho)^\alpha]^{1-\beta}}{(x-\rho)^{1-\alpha}}\psi(x)dx.$$

Therefore we obtain that

$$\left|\int_a^b \frac{^{\beta,\alpha}(\eta\psi)^{\rho}_{a+}(x)}{(x-\rho)^{1-\alpha}}dx\right|\leq\frac{\alpha^{-1}}{\Gamma(2-\beta)}\int_a^b\left|[(b-\rho)^\alpha-(x-\rho)^\alpha]^{1-\beta}\right|\left|\frac{\psi(x)}{(x-\rho)^{1-\alpha}}\right|dx.$$

Hence $(x-\rho)^{\alpha-1}\psi(x)\in L^1(a,b)$ implies that $(x-\rho)^{\alpha-1}\left\{^{\beta,\alpha}(\eta\psi)^{\rho}_{a+}(x)\right\}\in L^1(a,b)$. Now we shall multiply both sides of (2.1) by $[(x-\rho)^\alpha-(t-\rho)^\alpha]^{-\beta}(t-\rho)^{\alpha-1},\ x\neq\rho$, and integrate them with respect to t on $[a,x]$, $x\leq b$, to get

$$\frac{\alpha}{\Gamma(\beta)}\int_a^x\frac{f(s)}{(s-\rho)^{1-\alpha}}\int_s^x\frac{[(t-\rho)^\alpha-(s-\rho)^\alpha]^{\beta-1}}{[(x-\rho)^\alpha-(t-\rho)^\alpha]^\beta}(t-\rho)^{\alpha-1}dtds=\int_a^x\frac{[(x-\rho)^\alpha-(t-\rho)^\alpha]^{-\beta}}{(t-\rho)^{1-\alpha}}\psi(t)dt.$$

Using the substitution

$$v=\frac{(t-\rho)^\alpha-(s-\rho)^\alpha}{(x-\rho)^\alpha-(s-\rho)^\alpha}$$

we obtain that

$$\int_a^x\frac{f(t)}{(t-\rho)^{1-\alpha}}dt=\frac{1}{\Gamma(1-\beta)}\int_a^x\frac{[(x-\rho)^\alpha-(t-\rho)^\alpha]^{-\beta}}{(t-\rho)^{1-\alpha}}\psi(t)dt.$$

From the assumption we get that $^{\beta,\alpha}(\eta\psi)^{\rho}_{a+}(x)$ is absolutely continuous on $[a,b]$ and $^{\beta,\alpha}(\eta\psi)^{\rho}_{a+}(a^+)=0$. Now suppose that $^{\beta,\alpha}(\eta\psi)^{\rho}_{a+}(x)$ is absolutely continuous on $[a,b]$ and $^{\beta,\alpha}(\eta\psi)^{\rho}_{a+}(a^+)=0$. This implies, in particular, that ${}^{\beta,\alpha}\left(\eta\psi\right)_{a+}^{\rho}\in L^{1}(a,b)$ and

$$\frac{d}{dx}\left\{{}^{\beta,\alpha}\left(\eta\psi\right)_{a+}^{\rho}(x)\right\}=(x-\rho)^{\alpha-1}f(x)$$

exist a.e. belonging to $L^{1}(a,b)$. Hence the equation

$$\frac{\alpha}{\Gamma(\beta)}\int\limits_{a}^{x}\frac{\left[(x-\rho)^{\alpha}-(t-\rho)^{\alpha}\right]^{\beta-1}}{(t-\rho)^{1-\alpha}}\frac{d}{dt}\left\{{}^{\beta,\alpha}\left(\eta\psi\right)_{a+}^{\rho}(t)\right\}dt=\varphi(x) \tag{2.2}$$

has a solution

$$\frac{d}{dx}\left\{{}^{\beta,\alpha}\left(\eta\psi\right)_{a+}^{\rho}(x)\right\}=\frac{1}{\Gamma(1-\beta)}(x-\rho)^{1-\alpha}\frac{d}{dx}\int\limits_{a}^{x}\tfrac{\left[(x-\rho)^{\alpha}-(t-\rho)^{\alpha}\right]^{-\beta}}{(t-\rho)^{1-\alpha}}\varphi(t)dt.$$

This implies that

$$\frac{d}{dx}\left\{{}^{\beta,\alpha}\left(\eta\psi\right)_{a+}^{\rho}(x)\right\}=\frac{d}{dx}\left\{{}^{\beta,\alpha}\left(\eta\varphi\right)_{a+}^{\rho}(x)\right\}. \tag{2.3}$$

Note that both sides of (2.3) are absolutely continuous on $[a,b]$. Hence ${}^{\beta,\alpha}\left(\eta\psi\right)_{a+}^{\rho}(x)\ -{}^{\beta,\alpha}\left(\eta\varphi\right)_{a+}^{\rho}(x)=c$, where c is a constant. From the assumption and (2.2) we have ${}^{\beta,\alpha}\left(\eta\psi\right)_{a+}^{\rho}(a^{+})=0\ \ {}^{\beta,\alpha}\left(\eta\varphi\right)_{a+}^{\rho}(a^{+})=0$. Therefore $c=0$ and this completes the proof. □

A similar result can also be introduced as follows.

Theorem 2.2. *Let $\alpha>0,\ 0<\beta<1$ and $-\infty<\delta<\infty$. For $(\delta-x)^{\alpha-1}\psi(x)\in L^{1}(a,b)$ the equation*

$$\frac{\alpha}{\Gamma(\beta)}\int\limits_{x}^{b}\frac{\left[(\delta-x)^{\alpha}-(\delta-t)^{\alpha}\right]^{\beta-1}}{(\delta-t)^{1-\alpha}}f(t)dt=\psi(x) \tag{2.4}$$

has a solution f with $(\delta-x)^{\alpha-1}f(x)\in L^{1}(a,b)$, where $x\in[a,b)\subset(-\infty,\infty)$ and $x\neq\delta$ for $\delta\in[a,b)$, if and only if

$${}^{\beta,\alpha}\left(\eta\psi\right)_{b-}^{\delta}(x):=\frac{1}{\Gamma(1-\beta)}\int\limits_{x}^{b}\frac{\left[(\delta-x)^{\alpha}-(\delta-t)^{\alpha}\right]^{-\beta}}{(\delta-t)^{1-\alpha}}\psi(t)dt$$

is absolutely continuous on $[a,b]$ and ${}^{\beta,\alpha}\left(\eta\psi\right)_{b-}^{\delta}(b^{-})=0$. In this case we have the following

$$f(x)=-(\delta-x)^{1-\alpha}\frac{d}{dx}\ {}^{\beta,\alpha}\left(\eta\psi\right)_{b-}^{\delta}(x).$$

Remark 2.3. *To justify the results in Theorem 2.2 it is enough to use the substitution*

$$v=\frac{(\delta-t)^{\alpha}-(\delta-s)^{\alpha}}{(\delta-x)^{\alpha}-(\delta-s)^{\alpha}}$$

as in the proof of Theorem 2.1.

Theorem 2.1 and Theorem 2.2 introduce some criteria for solvability of (2.1) and (2.4). However, we may share the following simple criteria.

Theorem 2.4. *Let* $\alpha > 0$, $0 < \beta < 1$ *and* $-\infty < \rho < \infty$. *If* f *is absolutely continuous on* $[a,b]$ *then* ${}^{\beta,\alpha}\left(\eta f\right)_{a+}^{\rho}(x)$ *is absolutely continuous and*

$$ {}^{\beta,\alpha}\left(\eta f\right)_{a+}^{\rho}(x) = \frac{\alpha^{-1}}{\Gamma(2-\beta)}\left\{\frac{f(a)}{\left[(x-\rho)^{\alpha}-(a-\rho)^{\alpha}\right]^{\beta-1}} + \int_a^x \frac{f'(t)}{\left[(x-\rho)^{\alpha}-(t-\rho)^{\alpha}\right]^{\beta-1}}dt\right\}, $$

where $a < x \leq b$ *and* $x \neq \rho$.

Proof Observe firstly that

$$ \left[(x-\rho)^{\alpha}-(a-\rho)^{\alpha}\right]^{1-\beta} = \alpha\left(1-\beta\right)\int_a^x \frac{\left[(t-\rho)^{\alpha}-(a-\rho)^{\alpha}\right]^{-\beta}}{(t-\rho)^{1-\alpha}}dt \tag{2.5} $$

and

$$ \int_a^x \frac{\left[(x-\rho)^{\alpha}-(t-\rho)^{\alpha}\right]^{-\beta}}{(t-\rho)^{1-\alpha}}\int_a^t f'(s)dsdt = \int_a^x\left(\int_a^t \frac{\left[(t-\rho)^{\alpha}-(s-\rho)^{\alpha}\right]^{-\beta}}{(t-\rho)^{1-\alpha}}f'(s)ds\right)dt. \tag{2.6} $$

Note that the left and right hand-sides of (2.6) represent the same integral. Since f is absolutely continuous on $[a,b]$ we have the following

$$ \begin{aligned} {}^{\beta,\alpha}\left(\eta f\right)_{a+}^{\rho}(x) &= \tfrac{1}{\Gamma(1-\beta)}\int_a^x \tfrac{\left[(x-\rho)^{\alpha}-(t-\rho)^{\alpha}\right]^{-\beta}}{(t-\rho)^{1-\alpha}}\left(f(a)+\int_a^t f'(s)ds\right)dt \\ &= f(a)\tfrac{\alpha^{-1}}{\Gamma(2-\beta)}\left[(x-\rho)^{\alpha}-(a-\rho)^{\alpha}\right]^{1-\beta} + \tfrac{\alpha^{-1}}{\Gamma(2-\beta)}\int_a^x \tfrac{f'(t)}{\left[(x-\rho)^{\alpha}-(t-\rho)^{\alpha}\right]^{\beta-1}}dt. \end{aligned} \tag{2.7} $$

From (2.5) one may see that $\left[(x-\rho)^{\alpha}-(a-\rho)^{\alpha}\right]^{1-\beta}$ is absolutely continuous in x and from (2.6) we may infer that second term at right hand side in (2.7) is a primitive of an integrable function and hence it is absolutely continuous. This completes the proof. □

Corollary 2.5. *Let* $\alpha > 0$, $0 < \beta < 1$ *and* $-\infty < \rho < \infty$. *If* ψ *is absolutely continuous on* $[a,b]$ *then* *(2.1) has a solution* f *with* $(x-\rho)^{\alpha-1}f(x) \in L^1(a,b)$, $x \neq \rho$, *and*

$$ f(x) = \frac{1}{\Gamma(1-\beta)}\left\{\frac{\psi(a)}{\left[(x-\rho)^{\alpha}-(a-\rho)^{\alpha}\right]^{\beta}} + \int_a^x \frac{\psi'(t)}{\left[(x-\rho)^{\alpha}-(t-\rho)^{\alpha}\right]^{\beta}}dt\right\}. $$

Proof From Theorem 2.4 we get that

$$ \begin{aligned} &\tfrac{d}{dx}\left\{{}^{\beta,\alpha}\left(\eta\psi\right)_{a+}^{\rho}(x)\right\} = \frac{1}{\Gamma(1-\beta)}\left[\frac{\psi(a)(x-\rho)^{\alpha-1}}{\left[(x-\rho)^{\alpha}-(a-\rho)^{\alpha}\right]^{\beta}}\right\} \\ &+(x-\rho)^{\alpha-1}\int_a^x \frac{\psi'(t)}{\left[(x-\rho)^{\alpha}-(t-\rho)^{\alpha}\right]^{\beta}}dt\Bigg]. \end{aligned} $$

Since $(x-\rho)^{\alpha-1}f(x)=\frac{d}{dx}\left\{ {}^{\beta,\alpha}\left(\eta\psi\right)^{\rho}_{a+}(x)\right\}$ we complete the proof. □

Now we may introduce the following results.

Theorem 2.6. *Let $\alpha>0,\ 0<\beta<1$ and $-\infty<\delta<\infty$. If f is absolutely continuous on $[a,b]$ then ${}^{\beta,\alpha}\left(\eta f\right)^{\delta}_{b-}(x)$ is absolutely continuous and*

$$ {}^{\beta,\alpha}\left(\eta f\right)^{\delta}_{b-}(x)=\frac{\alpha^{-1}}{\Gamma(2-\beta)}\left\{\frac{f(b)}{\left[(\delta-x)^{\alpha}-(\delta-b)^{\alpha}\right]^{\beta-1}}-\int_{x}^{b}\frac{f'(t)}{\left[(\delta-x)^{\alpha}-(\delta-t)^{\alpha}\right]^{\beta-1}}dt\right\}, $$

where $a\le x<b$ and $x\neq\delta$.

Corollary 2.7. *Let $\alpha>0,\ 0<\beta<1$ and $-\infty<\delta<\infty$. If ψ is absolutely continuous on $[a,b]$ then (2.4) has a solution f with $(\delta-x)^{\alpha-1}f(x)\in L^1(a,b)$, $x\neq\delta$, and*

$$ f(x)=\frac{1}{\Gamma(1-\beta)}\left\{\frac{\psi(b)}{\left[(\delta-x)^{\alpha}-(\delta-b)^{\alpha}\right]^{\beta}}-\int_{x}^{b}\frac{\psi'(t)}{\left[(\delta-x)^{\alpha}-(\delta-t)^{\alpha}\right]^{\beta}}dt\right\}. $$

Lemma 2.8. *Let $\alpha>0,\ 0<\beta<1$, $-\infty<\rho<\infty$ and $1<p<\beta^{-1}$. Then for $q=p/(1-\beta p)$*

i) ${}^{\beta,\alpha}I^{\rho}_{a+}f(x)\in L^q(a,b)$ for $f(x)\in L^p((a,b);(x-\rho)^{p\alpha-p})$, where $\rho<a$ and $\rho>b$ with $\alpha>0$,

ii) ${}^{\beta,\alpha}I^{\rho}_{a+}f(x)\in L^q(a,b)$ for $f(x)\in L^p((a,b);(x-\rho)^{p\alpha-p})$, where $\rho\in[a,b]$ with $1\ge\alpha>0$,

iii) $(x-\rho)^{\frac{\alpha-1}{r}}\left\{ {}^{\beta,\alpha}I^{\rho}_{a+}f(x)\right\}\in L^q(a,b)$ for $f(x)\in L^p((a,b);(x-\rho)^{p\alpha-p})$, where $\rho<a$ and $\rho>b$ with $\alpha>0$, and $p<r<\frac{1}{1-\beta p}$,

iv) $(x-\rho)^{\frac{\alpha-1}{r}}\left\{ {}^{\beta,\alpha}I^{\rho}_{a+}f(x)\right\}\in L^q(a,b)$ for $f(x)\in L^p((a,b);(x-\rho)^{p\alpha-p})$, where $\rho\in[a,b]$ with $1\ge\alpha>0$, and $p<r<\frac{1}{1-\beta p}$.

Proof Let $1<p<\beta^{-1}$ and $1\le r<q=p/(1-\beta p)$. For $r>p$ and $\varkappa=(1/r-1/q)/2$ and $1/p+1/p'=1$ we get that

$$ \begin{aligned} &\frac{\Gamma(\beta)}{\alpha}\left|{}^{\beta,\alpha}I^{\rho}_{a+}f(x)\right| \\ &\le\int_{a}^{x}\left|\tfrac{f(t)}{(t-\rho)^{1-\alpha}}\right|^{\frac{p}{r}}\left|(x-\rho)^{\alpha}-(t-\rho)^{\alpha}\right|^{\varkappa-\frac{1}{r}}\left|\tfrac{f(t)}{(t-\rho)^{1-\alpha}}\right|^{1-\frac{p}{r}}\left|(x-\rho)^{\alpha}-(t-\rho)^{\alpha}\right|^{\varkappa-\frac{1}{p'}}dt \\ &\le\left(\int_{a}^{x}\left|\tfrac{f(t)}{(t-\rho)^{1-\alpha}}\right|^{p}\left|(x-\rho)^{\alpha}-(t-\rho)^{\alpha}\right|^{r\varkappa-1}dt\right)^{\frac{1}{r}}\left(\int_{a}^{x}\left|\tfrac{f(t)}{(t-\rho)^{1-\alpha}}\right|^{p}dt\right)^{\frac{1}{p}-\frac{1}{r}} \\ &\left(\int_{a}^{x}\left|(x-\rho)^{\alpha}-(t-\rho)^{\alpha}\right|^{\varkappa p'-1}dt\right)^{\frac{1}{p'}}\le\left\|(x-\rho)^{\alpha-1}f(x)\right\|^{1-\frac{p}{r}}_{L^p}\times \\ &\left(\int_{a}^{x}\left|\tfrac{f(t)}{(t-\rho)^{1-\alpha}}\right|^{p}\left|(x-\rho)^{\alpha}-(t-\rho)^{\alpha}\right|^{r\varkappa-1}\right)^{\frac{1}{r}}\left(\int_{a}^{x}\left|(x-\rho)^{\alpha}-(t-\rho)^{\alpha}\right|^{\varkappa p'-1}dt\right)^{\frac{1}{p'}}. \end{aligned} $$

Note that for $\rho < a$ and $\rho > b$ with $\alpha > 0$ or $\rho \in [a,b]$ with $1 \geq \alpha > 0$ the integral

$$\int_a^x |(x-\rho)^\alpha - (t-\rho)^\alpha|^{\varkappa p'-1} dt = \int_a^x \frac{|(x-\rho)^\alpha - (t-\rho)^\alpha|^{\varkappa p'-1}}{|t-\rho|^{1-\alpha}} |t-\rho|^{1-\alpha} dt$$

converges. Hence we have

$$\begin{aligned} \frac{\Gamma(\beta)}{\alpha} \left| {}^{\beta,\alpha}I^\rho_{a+} f(x) \right| &\leq const. \left\| (x-\rho)^{\alpha-1} f(x) \right\|^{1-\frac{p}{r}}_{L^p} \\ &\times \left(\int_a^x \left| \frac{f(t)}{(t-\rho)^{1-\alpha}} \right|^p |(x-\rho)^\alpha - (t-\rho)^\alpha|^{r\varkappa-1} dt \right)^{\frac{1}{r}} \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} \frac{\Gamma(\beta)}{\alpha} \left\| {}^{\beta,\alpha}I^\rho_{a+} f(x) \right\|_{L^r} &\leq const. \left\| (x-\rho)^{\alpha-1} f(x) \right\|^{1-\frac{p}{r}}_{L^p} \\ &\times \left(\int_a^b \left| \frac{f(t)}{(t-\rho)^{1-\alpha}} \right|^p dt \int_a^b \frac{|(x-\rho)^\alpha - (t-\rho)^\alpha|^{r\varkappa-1}}{|x-\rho|^{1-\alpha}} |x-\rho|^{1-\alpha} dx \right)^{\frac{1}{r}} \\ &\leq const. \left\| (x-\rho)^{\alpha-1} f(x) \right\|^{1-\frac{p}{r}}_{L^p} \left\| (x-\rho)^{\alpha-1} f(x) \right\|^{\frac{p}{r}}_{L^p}. \end{aligned}$$

This implies that ${}^{\beta,\alpha}I^\rho_{a+}$ is bounded from $L^p((a,b);(x-\rho)^{p\alpha-p})$ into $L^q(a,b)$ for $\rho < a$ and $\rho > b$ with $\alpha > 0$ or $\rho \in [a,b]$ with $1 \geq \alpha > 0$. From (2.8) one may also have that

$$\begin{aligned} &\frac{\Gamma(\beta)}{\alpha} \left\| |x-\rho|^{\frac{\alpha-1}{r}} \left\{ {}^{\beta,\alpha}I^\rho_{a+} f(x) \right\} \right\|_{L^r} \leq const. \left\| (x-\rho)^{\alpha-1} f(x) \right\|^{1-\frac{p}{r}}_{L^p} \\ &\times \left(\int_a^b \left| \frac{f(t)}{(t-\rho)^{1-\alpha}} \right|^p dt \int_a^b \frac{|(x-\rho)^\alpha - (t-\rho)^\alpha|^{r\varkappa-1}}{|x-\rho|^{1-\alpha}} dx \right)^{\frac{1}{r}} \leq const. \left\| (x-\rho)^{\alpha-1} f(x) \right\|^{1-\frac{p}{r}}_{L^p} \\ &\times \left\| (x-\rho)^{\alpha-1} f(x) \right\|^{\frac{p}{r}}_{L^p}. \end{aligned}$$

Therefore the proof is completed. □

Now we may introduce the following integration by parts formula.

Theorem 2.9. *Let $\rho < a$ and $\rho > b$ with $\alpha > 0$, or $\rho \in [a,b]$ with $0 < \alpha \leq 1$, and $0 < \beta < 1$. For $f \in L^p((a,b);\ (x-\rho)^{p\alpha-p}) \cap L^p(a,b)$, $g \in L^q((a,b);(x-\rho)^{q\alpha-q}) \cap L^q(a,b)$, $1/p + 1/q \leq 1+\beta$ we have the following*

$$\int_a^b \frac{f(x)}{(x-\rho)^{1-\alpha}} \left\{ {}^{\beta,\alpha}I^\rho_{a+} g(x) \right\} dx = (-1)^{\alpha\beta-1} \int_a^b \frac{g(x)}{(x-\rho)^{1-\alpha}} \left\{ {}^{\beta,\alpha}I^\rho_{b-} f(x) \right\} dx.$$

Proof The proof follows from Lemma 2.8 and the definitions of the operators ${}^{\beta,\alpha}I^\rho_{a+}$ and ${}^{\beta,\alpha}I^\rho_{b-}$. □

Following Theorem is about the semigroup properties of ${}^{\beta,\alpha}I^\rho_{a+}$ and ${}^{\beta,\alpha}I^\delta_{b-}$.

Theorem 2.10. *Let $\alpha > 0$, $0 < \beta, \theta < 1$ and $-\infty < \rho, \delta < \infty$. For $(x-\rho)^{\alpha-1} f(x) \in L^1(a,b)$* we have the following equations

$$
\begin{aligned}
{}^{\beta,\alpha}I^{\rho}_{a+} \circ {}^{\theta,\alpha}I^{\rho}_{a+} f(x) &= {}^{(\beta+\theta),\alpha}I^{\rho}_{a+} f(x),\ x \in (a,b],\ \rho \neq x,\\
{}^{\beta,\alpha}I^{\delta}_{b-} \circ {}^{\theta,\alpha}I^{\delta}_{b-} f(x) &= {}^{(\beta+\theta),\alpha}I^{\delta}_{b-} f(x),\ x \in [a,b),\ \delta \neq x.
\end{aligned}
$$

Proof Let $(x-\rho)^{\alpha-1} f(x) \in L^1(a,b)$. Then with a direct calculation we get that

$$
\begin{aligned}
&{}^{\beta,\alpha}I^{\rho}_{a+}\left({}^{\theta,\alpha}I^{\rho}_{a+} f(t)\right)(x) = \tfrac{\alpha}{\Gamma(\beta)} \int_a^x \tfrac{[(x-\rho)^\alpha-(t-\rho)^\alpha]^{\beta-1}}{(t-\rho)^{1-\alpha}} \tfrac{\alpha}{\Gamma(\theta)} \int_a^t \tfrac{[(t-\rho)^\alpha-(s-\rho)^\alpha]^{\beta-1}}{(s-\rho)^{1-\alpha}} f(s)dsdt\\
&= \tfrac{\alpha^2}{\Gamma(\beta)\Gamma(\theta)} \int_a^x \tfrac{f(s)}{(s-\rho)^{1-\alpha}} \int_s^x \tfrac{[(x-\rho)^\alpha-(t-\rho)^\alpha]^{\beta-1}[(t-\rho)^\alpha-(s-\rho)^\alpha]^{\beta-1}}{(t-\rho)^{1-\alpha}} dtds.
\end{aligned}
$$

Using the substitution

$$
w = \frac{(t-\rho)^\alpha-(s-\rho)^\alpha}{(x-\rho)^\alpha-(s-\rho)^\alpha}
$$

we get that

$$
{}^{\beta,\alpha}I^{\rho}_{a+}\left({}^{\theta,\alpha}I^{\rho}_{a+} f(t)\right)(x) = \frac{\alpha}{\Gamma(\beta+\theta)} \int_a^x \tfrac{[(x-\rho)^\alpha-(s-\rho)^\alpha]^{\beta+\theta-1}}{(s-\rho)^{1-\alpha}} f(s)ds = {}^{(\beta+\theta),\alpha}I^{\rho}_{a+} f(x).
$$

This completes the proof of the first assertion. The proof of the second assertion can be proved similarly and this completes the proof. □

3. Fractional derivatives

In this section we will introduce suitable fractional derivatives related with Abel's equations.

Firstly, for $\alpha > 0$ and $0 < \beta < 1$ we shall define the following operators

$$
{}^{\beta,\alpha}D^{\rho}_{a+} f(x) := \frac{1}{\Gamma(1-\beta)} (x-\rho)^{1-\alpha} \frac{d}{dx} \int_a^x \frac{[(x-\rho)^\alpha-(t-\rho)^\alpha]^{-\beta}}{(t-\rho)^{1-\alpha}} f(t)dt,
$$

where $x \in (a,b]$ and $-\infty < \rho < \infty$ such that $\rho \neq x$ when $\rho \in (a,b]$, and

$$
{}^{\beta,\alpha}D^{\delta}_{b-} f(x) := \frac{-1}{\Gamma(1-\beta)} (\delta-x)^{1-\alpha} \frac{d}{dx} \int_x^b \frac{[(\delta-x)^\alpha-(\delta-t)^\alpha]^{-\beta}}{(\delta-t)^{1-\alpha}} f(t)dt,
$$

where $x \in [a,b)$ and $-\infty < \delta < \infty$ such that $\delta \neq x$ when $\delta \in [a,b)$.

Using Theorem 2.4 and Theorem 2.6 we may introduce the following.

Theorem 3.1. *Let $\alpha > 0$, $0 < \beta < 1$ and $-\infty < \rho, \delta < \infty$. If f is absolutely continuous on $[a,b]$ then ${}^{\beta,\alpha}D^{\rho}_{a+} f(x)$ and ${}^{\beta,\alpha}D^{\delta}_{b-} f(x)$ exist a.e. with the rules*

$$
{}^{\beta,\alpha}D^{\rho}_{a+} f(x) = \tfrac{\alpha^{-1}}{\Gamma(1-\beta)} \left\{ \frac{f(a)}{[(x-\rho)^\alpha-(a-\rho)^\alpha]^\beta} + \int_a^x \frac{f'(t)}{[(x-\rho)^\alpha-(t-\rho)^\alpha]^\beta} dt \right\},
$$

where $x \neq \rho$ when $\rho \in (a,b]$, and

$$ {}^{\beta,\alpha}D^{\delta}_{b-}f(x) = \tfrac{\alpha^{-1}}{\Gamma(1-\beta)}\left\{\frac{f(b)}{[(\delta-x)^{\alpha}-(\delta-b)^{\alpha}]^{\beta}} - \int_x^b \frac{f'(t)}{[(\delta-x)^{\alpha}-(\delta-t)^{\alpha}]^{\beta}}dt\right\}, $$

where $x \neq \delta$ *when* $\delta \in [a,b)$.

We will use the notations

$$ {}^{\beta,\alpha}D^{\rho}_{a+}f(x) = {}^{-\beta,\alpha}I^{\rho}_{a+}f(x),\ {}^{\beta,\alpha}D^{\delta}_{b-}f(x) = {}^{-\beta,\alpha}I^{\delta}_{b-}f(x). $$

We denote by ${}^{\beta,\alpha}I^{\rho}_{a+}(L^p(a,b))$, $\alpha > 0$, $0 < \beta < 1$, $-\infty < \rho < \infty$, the space of functions f such that $f = {}^{\beta,\alpha}I^{\rho}_{a+}\widetilde{f}$ with $\widetilde{f} \in L^p(a,b)$, $1 \leq p < \infty$. ${}^{\beta,\alpha}I^{\delta}_{b-}(L^p(a,b))$ can be defined similarly.

Theorem 3.2. *Let* $\alpha > 0$, $0 < \beta < 1$ *and* $-\infty < \rho < \infty$. *Then*

(i) If $(x-\rho)^{\alpha-1}f(x) \in L^1(a,b)$ *then* ${}^{\beta,\alpha}D^{\rho}_{a+}\circ {}^{\beta,\alpha}I^{\rho}_{a+}f(x) = f(x)$,

(ii) If $(x-\rho)^{\alpha-1}f(x) \in {}^{\beta,\alpha}I^{\rho}_{a+}(L^1(a,b))$ *then* ${}^{\beta,\alpha}I^{\rho}_{a+}\circ {}^{\beta,\alpha}D^{\rho}_{a+}f(x) = f(x)$,

(iii) If $(x-a)^{\alpha-1}f(x) \in L^1(a,b)$ *such that* ${}^{\beta,\alpha}I^{a}_{a+}\circ {}^{\beta,\alpha}D^{a}_{a+}f(x)$ *exists* $[a,b]$ *and* $[(x-a)^{\alpha}]^{1-\beta}f(x)$ *is continuous at* a *then*

$$ {}^{\beta,\alpha}I^{a}_{a+} \circ {}^{\beta,\alpha}D^{a}_{a+}f(x) = f(x) - \left\{[(x-a)^{\alpha}]^{1-\beta}f(x)\right\}(a^{+})\,[(x-a)^{\alpha}]^{\beta-1}. $$

Proof For $(x-\rho)^{\alpha-1}f(x) \in L^1(a,b)$ a direct calculation gives that

$$ \begin{aligned} &{}^{\beta,\alpha}D^{\rho}_{a+} \circ {}^{\beta,\alpha}I^{\rho}_{a+}f(x) \\ &= \tfrac{\alpha}{\Gamma(\beta)\Gamma(1-\beta)}(x-\rho)^{1-\alpha}\tfrac{d}{dx}\int_a^x \tfrac{f(s)}{(s-\rho)^{1-\alpha}}\int_s^x \left[\tfrac{(t-\rho)^{\alpha}-(s-\rho)^{\alpha}}{(x-\rho)^{\alpha}-(t-\rho)^{\alpha}}\right]^{\beta}\tfrac{(t-\rho)^{\alpha-1}}{(t-\rho)^{\alpha}-(s-\rho)^{\alpha}}dtds. \end{aligned} $$

Using the substitution

$$ v = \frac{(t-\rho)^{\alpha}-(s-\rho)^{\alpha}}{(x-\rho)^{\alpha}-(s-\rho)^{\alpha}} $$

we obtain that

$$ {}^{\beta,\alpha}D^{\rho}_{a+} \circ {}^{\beta,\alpha}I^{\rho}_{a+}f(x) = (x-\rho)^{1-\alpha}\frac{d}{dx}\int_a^x \frac{f(s)}{(s-\rho)^{1-\alpha}}ds = f(x) $$

and this proves (i). (ii) follows from (i). Let ${}^{\beta,\alpha}I^{a}_{a+}\circ {}^{\beta,\alpha}D^{a}_{a+}f(x) = f(x) + \chi(x)$ [15]. Then by (i) ${}^{\beta,\alpha}D^{a}_{a+}\chi(x) = 0$ and hence $\chi(x) = c\frac{[(x-a)^{\alpha}]^{\beta-1}}{\alpha\Gamma(\beta)}$. Therefore

$$ \begin{aligned} c &= -\alpha\Gamma(\beta)\lim_{x\to a^{+}}[(x-a)^{\alpha}]^{1-\beta}\left\{f(x) - {}^{\beta,\alpha}I^{\rho}_{a+} \circ {}^{\beta,\alpha}D^{\rho}_{a+}f(x)\right\} \\ &= -\alpha\Gamma(\beta)\lim_{x\to a^{+}}[(x-a)^{\alpha}]^{1-\beta}f(x). \end{aligned} $$

This proves (iii). Therefore we complete the proof. □

Using Theorem 2.9 we may introduce another version of integration by parts as follows.

Theorem 3.3. *For* $g \in {}^{\beta,\alpha}I^{\rho}_{a+}\left(L^q(a,b);(x-\rho)^{q-\alpha q}\right) \cap {}^{\beta,\alpha}I^{\rho}_{a+}\left(L^q(a,b)\right)$ *and* $f \in {}^{\beta,\alpha}I^{\rho}_{b-}(L^p(a,b);$ $(x-\rho)^{p-\alpha p}) \cap {}^{\beta,\alpha}I^{\rho}_{a+}\left(L^p(a,b)\right)$ *we have the following*

$$\int_a^b \frac{f(x)}{(x-\rho)^{1-\alpha}}\left\{{}^{\beta,\alpha}D^{\rho}_{a+}g(x)\right\}dx = (-1)^{\alpha\beta-1}\int_a^b \frac{g(x)}{(x-\rho)^{1-\alpha}}\left\{{}^{\beta,\alpha}D^{\rho}_{b-}f(x)\right\}dx,$$

where $1/p+1/q \le 1+\beta$.

Now we may introduce the following extended semigroup property.

Theorem 3.4. *Let* $\alpha>0$ *and* $-\infty<\rho<\infty$. *Then for each of the following conditions*

(i) $0<\theta<1,\ 0<\beta+\theta<1,\ (x-\rho)^{\alpha-1}f(x)\in L^1(a,b)$,

(ii) $-1<\theta<0,\ 0<\beta<1,\ (x-\rho)^{\alpha-1}f(x)\in {}^{-\theta,\alpha}I^{\rho}_{a+}(L^1(a,b))$,

(iii) $-1<\beta<0,\ -1<\beta+\theta<0,\ (x-\rho)^{\alpha-1}f(x)\in {}^{-(\theta-\beta),\alpha}I^{\rho}_{a+}(L^1(a,b))$,

we have

$${}^{\beta,\alpha}I^{\rho}_{a+}\circ {}^{\theta,\alpha}I^{\rho}_{a+}f(x) = {}^{(\beta+\theta),\alpha}I^{\rho}_{a+}f(x).$$

Proof The relation has been proved for $0<\beta<1$ and $0<\theta<1$ in Theorem 2.10. Let $\beta=0$ and $0<\theta<1$. Then we have

$${}^{0,\alpha}I^{\rho}_{a+}\circ {}^{\theta,\alpha}I^{\rho}_{a+}f(x) = \frac{\alpha^2}{\Gamma(\theta)}\int_a^x \frac{f(s)}{(s-\rho)^{1-\alpha}}\int_s^x \frac{\left[(t-\rho)^{\alpha}-(s-\rho)^{\alpha}\right]^{\theta-1}}{\left[(x-\rho)^{\alpha}-(t-\rho)^{\alpha}\right](t-\rho)^{1-\alpha}}dtds.$$

Letting $v=(x-\rho)^{\alpha}-(t-\rho)^{\alpha}$ we get that

$${}^{0,\alpha}I^{\rho}_{a+}\circ {}^{\theta,\alpha}I^{\rho}_{a+}f(x) = \frac{\alpha}{\Gamma(\theta)}\int_a^x \frac{\left[(x-\rho)^{\alpha}-(t-\rho)^{\alpha}\right]^{\theta-1}}{(t-\rho)^{1-\alpha}}f(t)dt = {}^{\theta,\alpha}I^{\rho}_{a+}f(x).$$

For the case $-1<\beta<\theta,\ 0<\theta<1,\ 0<\beta+\theta<1$ we may introduce the following

$${}^{\beta,\alpha}I^{\rho}_{a+}\circ {}^{\theta,\alpha}I^{\rho}_{a+}f(x) = {}^{-\beta,\alpha}D^{\rho}_{a+}\circ {}^{-\beta,\alpha}I^{\rho}_{a+}\circ {}^{(\theta+\beta),\alpha}I^{\rho}_{a+}f(x).$$

These prove (i). Now let $-1<\theta<0,\ 0<\beta<1$. Then for $(x-\rho)^{\alpha-1}g(x)\in L^1(a,b)$ we may write $f={}^{-\theta,\alpha}I^{\rho}_{a+}g(x)$ and

$${}^{(\beta+\theta),\alpha}I^{\rho}_{a+}f(x) = {}^{(\beta+\theta),\alpha}I^{\rho}_{a+}\circ {}^{-\theta,\alpha}I^{\rho}_{a+}g(x).$$

Since $0<\beta+\theta+(-\theta)<1$ we get from (i) that

$$\begin{aligned}&{}^{(\beta+\theta),\alpha}I^{\rho}_{a+}\circ {}^{-\theta,\alpha}I^{\rho}_{a+}g(x) = {}^{\beta,\alpha}I^{\rho}_{a+}g(x) = {}^{\beta,\alpha}I^{\rho}_{a+}\circ {}^{-\theta,\alpha}D^{\rho}_{a+}f(x)\\ &= {}^{\beta,\alpha}I^{\rho}_{a+}\circ {}^{\theta,\alpha}I^{\rho}_{a+}f(x)\end{aligned}$$

and this proves (ii). Finally we shall let $f(x) = {}^{(-\beta-\theta),\alpha}I^{\rho}_{a+}g(x),\ (x-\rho)^{\alpha-1}g(x) \in L^1(a,b)$, where $-1 < \beta < 0$ and $-1 < \beta+\theta < 0$. Then we get that

$$ {}^{\beta,\alpha}I^{\rho}_{a+} \circ {}^{\theta,\alpha}I^{\rho}_{a+}f(x) = {}^{-\beta,\alpha}D^{\rho}_{a+} \circ {}^{-\beta,\alpha}I^{\rho}_{a+}g(x) = g(x) = {}^{(\beta+\theta),\alpha}I^{\rho}_{a+}f(x). $$

This completes the proof of (iii) and Theorem 3.4. □

4. Mean value theorems

In this section we will firstly introduce mean value theorems and then we will share some inequalities on finite intervals. Finally we will show that the inequality is true when the lower (upper) bound is negative (positive) infinity.

4.1. Finite interval case

Firstly we shall introduce the following mean value theorem.

Theorem 4.1.1. *Let $\alpha > 0,\ 0 < \beta < 1$ and $-\infty < \rho < a$. If $(x-\rho)^{\alpha-1}f(x) \in L^1(a,b)$ and ${}^{\beta,\alpha}I^{\rho}_{a+}f(x)$ is continuous on $[a,b]$ such that ${}^{\beta,\alpha}I^{\rho}_{a+}f(a^+) = 0$ then for $x > b$ there exists $\xi \in [a,b)$ such that*

$$ \int_a^b \frac{[(x-\rho)^{\alpha}-(t-\rho)^{\alpha}]^{\beta-1}}{(t-\rho)^{1-\alpha}}f(t)dt = \int_a^{\xi} \frac{[(\xi-\rho)^{\alpha}-(t-\rho)^{\alpha}]^{\beta-1}}{(t-\rho)^{1-\alpha}}f(t)dt. $$

Proof For $x > b,\ a \le t < b$ and $-\infty < \rho < a$ we shall consider the integral

$$ \int_t^b \frac{[(u-\rho)^{\alpha}-(t-\rho)^{\alpha}]^{\beta-1}\,[(b-\rho)^{\alpha}-(u-\rho)^{\alpha}]^{-\beta}}{[(x-\rho)^{\alpha}-(u-\rho)^{\alpha}]\,(u-\rho)^{1-\alpha}}du \tag{4.1} $$

where $\alpha > 0,\ 0 < \beta < 1$. The substitution

$$ s = \frac{[(u-\rho)^{\alpha}-(t-\rho)^{\alpha}]\,[(b-\rho)^{\alpha}-(x-\rho)^{\alpha}]}{[(b-\rho)^{\alpha}-(t-\rho)^{\alpha}]\,[(u-\rho)^{\alpha}-(x-\rho)^{\alpha}]} $$

implies that (4.1) is equivalent to

$$ \frac{1}{\alpha}\left[\frac{[(x-\rho)^{\alpha}-(t-\rho)^{\alpha}]}{[(x-\rho)^{\alpha}-(b-\rho)^{\alpha}]}\right]^{\beta-1}\frac{\pi}{[(x-\rho)^{\alpha}-(b-\rho)^{\alpha}]\sin(\beta\pi)}. \tag{4.2} $$

On the other side we have the following

$$ \begin{aligned} &\frac{[(x-\rho)^{\alpha}-(b-\rho)^{\alpha}]^{\beta}}{\Gamma(1-\beta)}\int_a^b \frac{(u-\rho)^{\alpha-1}\left\{{}^{\beta,\alpha}I^{\rho}_{a+}f(u)\right\}}{[(b-\rho)^{\alpha}-(u-\rho)^{\alpha}]^{\beta}[(x-\rho)^{\alpha}-(u-\rho)^{\alpha}]}du \\ &= \frac{[(x-\rho)^{\alpha}-(b-\rho)^{\alpha}]^{\beta}}{\Gamma(\beta)\Gamma(1-\beta)\alpha^{-1}}\int_a^b \frac{f(t)}{(t-\rho)^{1-\alpha}}\int_t^b \frac{[(u-\rho)^{\alpha}-(t-\rho)^{\alpha}]^{\beta-1}(u-\rho)^{\alpha-1}}{[(b-\rho)^{\alpha}-(u-\rho)^{\alpha}]^{\beta}[(x-\rho)^{\alpha}-(u-\rho)^{\alpha}]}dudt. \end{aligned} \tag{4.3} $$

Using (4.1) and (4.2) one may see that (4.3) can be written as follows

$$\int_a^b \frac{[(x-\rho)^\alpha-(t-\rho)^\alpha]^{\beta-1}}{(t-\rho)^{1-\alpha}} f(t)dt = \frac{[(x-\rho)^\alpha-(b-\rho)^\alpha]^\beta}{\Gamma(1-\beta)} \int_a^b \frac{(u-\rho)^{\alpha-1}\left\{{}^{\beta,\alpha}I^\rho_{a+}f(u)\right\}}{[(b-\rho)^\alpha-(u-\rho)^\alpha]^\beta[(x-\rho)^\alpha-(u-\rho)^\alpha]} du. \tag{4.4}$$

One may see that the sign of

$$[(b-\rho)^\alpha-(u-\rho)^\alpha]^{-\beta}\,[(x-\rho)^\alpha-(u-\rho)^\alpha]^{-1}\,(u-\rho)^{\alpha-1},\ \rho<a,\ x>b$$

does not change for $u\in[a,b]$. Therefore using mean value theorem for integrals we obtain from (4.4) that

$$\frac{1}{\Gamma(\beta)}\int_a^b \frac{[(x-\rho)^\alpha-(t-\rho)^\alpha]^{\beta-1}}{(t-\rho)^{1-\alpha}} f(t)dt = \left\{{}^{\beta,\alpha}I^\rho_{a+}f(\xi_1)\right\}E(x), \tag{4.5}$$

where $a\le\xi_1\le b$ and

$$E(x)=\frac{[(x-\rho)^\alpha-(b-\rho)^\alpha]^\beta\sin(\beta\pi)}{\pi}\int_a^b \frac{(u-\rho)^{\alpha-1}}{[(b-\rho)^\alpha-(u-\rho)^\alpha]^\beta[(x-\rho)^\alpha-(u-\rho)^\alpha]}du. \tag{4.6}$$

The substitution

$$(b-\rho)^\alpha-(u-\rho)^\alpha=[(x-\rho)^\alpha-(b-\rho)^\alpha]\,w(1-w)^{-1}$$

or equivalently

$$w=\frac{(b-\rho)^\alpha-(u-\rho)^\alpha}{(x-\rho)^\alpha-(u-\rho)^\alpha}$$

in (4.6) gives that

$$E(x)=\frac{\sin(\beta\pi)}{\alpha\pi}\int_0^{\frac{(b-\rho)^\alpha-(a-\rho)^\alpha}{(x-\rho)^\alpha-(a-\rho)^\alpha}}\frac{dw}{w^\beta(1-w)^\beta},\ x>b,\ \rho<a.$$

Therefore $0<\alpha E(x)<1$ for all $x>b$ and $\rho<a$. From the assumption ${}^{\beta,\alpha}I^\rho_{a+}f(x)$ is continuous on $[a,\xi_1]$, $\xi_1\le b$. Hence there exists a $\xi\in[a,\xi_1]$ such that

$$E(x)\left\{{}^{\beta,\alpha}I^\rho_{a+}f(\xi_1)\right\}=\frac{1}{\Gamma(\beta)}\int_a^\xi \frac{[(\xi-\rho)^\alpha-(t-\rho)^\alpha]^{\beta-1}}{(t-\rho)^{1-\alpha}}$$

or equivalently

$$\frac{1}{\Gamma(\beta)}\int_a^b \frac{[(x-\rho)^\alpha-(t-\rho)^\alpha]^{\beta-1}}{(t-\rho)^{1-\alpha}} f(t)dt=\frac{1}{\Gamma(\beta)}\int_a^\xi \frac{[(\xi-\rho)^\alpha-(t-\rho)^\alpha]^{\beta-1}}{(t-\rho)^{1-\alpha}} f(t)dt \tag{4.7}$$

and this completes the proof. □

Remark 4.1.2. *Theorem 4.1.1 is true when* $\rho \leq a$ *if* α *is restricted only on* $(0,1]$.

Corollary 4.1.3. *Let* f *be a function on* $[a,\infty)$ *such that the conditions of Theorem 4.1.1 hold. Then*

$$^{\beta,\alpha}I^{\rho}_{a+}f(x) -^{\beta,\alpha} I^{\rho}_{b+}f(x) = \ ^{\beta,\alpha}I^{\rho}_{a+}f(\xi)$$

provided that f *is integrable beyond [a, b].*

Using (4.7) we may introduce the following.

Corollary 4.1.4. *Let the conditions of Theorem 4.1.1 hold. Then*

$$\left|\int\limits_a^b \frac{[(x-\rho)^\alpha-(t-\rho)^\alpha]^{\beta-1}}{(t-\rho)^{1-\alpha}}f(t)dt\right| \leq \max_{\xi\in[a,b]}\left|\int\limits_a^\xi \frac{[(\xi-\rho)^\alpha-(t-\rho)^\alpha]^{\beta-1}}{(t-\rho)^{1-\alpha}}f(t)dt\right|.$$

Corollary 4.1.5. *Let* $\alpha>0,\ 0<\beta<1$ *and* $-\infty<\rho<a$. *If* $(x-\rho)^{\alpha-1}f(x)\in L^1(a,b)$ *then for* $x>b$

$$\left|\int\limits_a^b \frac{[(x-\rho)^\alpha-(t-\rho)^\alpha]^{\beta-1}}{(t-\rho)^{1-\alpha}}f(t)dt\right| \leq esssup_{\xi\in[a,b]}\left|\int\limits_a^\xi \frac{[(\xi-\rho)^\alpha-(t-\rho)^\alpha]^{\beta-1}}{(t-\rho)^{1-\alpha}}f(t)dt\right|.$$

Proof The proof follows from the property

$$\alpha\int\limits_a^b |E'(t)|\,dt = -\alpha\int\limits_a^b E'(t)dt = \alpha E(a) < 1$$

and (4.5). □

Theorem 4.1.6. *Let* $\alpha>0,\ 0<\beta<1$ *and* $-\infty<\rho<a$. *If for* $(x-\rho)^{\alpha-1}f(x)\in L^1(a,\infty)$ *there exist nondecreasing functions* φ *and* ψ *such that for* $0<\gamma<1$ *the inequalities hold*

$$|f(x)|\leq\varphi(x),\ \ \left|^{\beta,\alpha}I^{\rho}_{a+}f(x)\right|\leq\psi(x),\ x>a, \tag{4.8}$$

then for $0<\beta<\gamma$ *we have*

$$\left|^{\beta,\alpha}I^{\rho}_{a+}f(x)\right|\leq const.\,[\varphi(x)]^{\left(1-\frac{\beta}{\gamma}\right)}[\psi(x)]^{\frac{\beta}{\gamma}},$$

where the constant does not depend on x,β *and* α.

Proof For $\alpha>0,\ 0<\beta<\gamma<1$ and $-\infty<\rho<a$, we shall consider the following

$$\Gamma(\beta)^{\beta,\alpha}I^{\rho}_{a+}f(x) = I_1+I_2, \tag{4.9}$$

where

$$I_1 = \alpha\int\limits_a^\zeta \frac{[(x-\rho)^\alpha-(t-\rho)^\alpha]^{\beta-1}}{(t-\rho)^{1-\alpha}}f(t)dt,$$
$$I_2 = \alpha\int\limits_\zeta^x \frac{[(x-\rho)^\alpha-(t-\rho)^\alpha]^{\beta-1}}{(t-\rho)^{1-\alpha}}f(t)dt,$$

and

$$\zeta=\begin{cases}\rho+\left\{(x-\rho)^{\alpha}-\left[\frac{\psi(x)}{\varphi(x)}\right]^{\frac{1}{\gamma}}\right\}^{\frac{1}{\alpha}}, & x>\rho+\left[\frac{\psi(x)}{\varphi(x)}\right]^{\frac{1}{\alpha\gamma}},\\ a,\ x<\rho+\left[\frac{\psi(x)}{\varphi(x)}\right]^{\frac{1}{\alpha\gamma}}.\end{cases}$$

Note that

$$I_1=\alpha\int_a^{\zeta}\frac{[(x-\rho)^{\alpha}-(t-\rho)^{\alpha}]^{\gamma-1}[(x-\rho)^{\alpha}-(t-\rho)^{\alpha}]^{\beta-\gamma}}{(t-\rho)^{1-\alpha}}f(t)dt.$$

For fixed x, the function $[(x-\rho)^{\alpha}-(t-\rho)^{\alpha}]^{\beta-\gamma}$ increases in t on $[a,\zeta]$. Using the mean value theorem for integrals we get that

$$I_1=\frac{[(x-\rho)^{\alpha}-(\zeta-\rho)^{\alpha}]^{\beta-\gamma}}{\alpha^{-1}}\int_u^{\zeta}\frac{[(x-\rho)^{\alpha}-(t-\rho)^{\alpha}]^{\gamma-1}}{(t-\rho)^{1-\alpha}}f(t)dt, \tag{4.10}$$

where $a\le u\le\zeta$. Using Corollary 4.1.5, (4.8) and (4.10) we obtain that

$$\begin{aligned}|I_1| &\le\Gamma(\gamma)[(x-\rho)^{\alpha}-(\zeta-\rho)^{\alpha}]^{\beta-\gamma}\,esssup_{\nu\in[u,\zeta]}\left|^{\beta,\alpha}I^{\rho}_{a+}f(\nu)\right|\\ &\le\Gamma(\gamma)[(x-\rho)^{\alpha}-(\zeta-\rho)^{\alpha}]^{\beta-\gamma}\psi(\zeta)\\ &\le\Gamma(\gamma)\left[\frac{\psi(x)}{\varphi(x)}\right]^{\frac{\beta-\gamma}{\gamma}}\psi(x)=\Gamma(\gamma)[\varphi(x)]^{\left(1-\frac{\beta}{\gamma}\right)}[\psi(x)]^{\frac{\beta}{\gamma}}.\end{aligned}\tag{4.11}$$

Moreover we get that

$$\begin{aligned}|I_2| &\le\alpha\int_{\zeta}^{x}\frac{[(x-\rho)^{\alpha}-(t-\rho)^{\alpha}]^{\beta-1}}{(t-\rho)^{1-\alpha}}|f(t)|\,dt\le\varphi(x)\frac{[(x-\rho)^{\alpha}-(\zeta-\rho)^{\alpha}]^{\beta}}{\beta}\\ &\le\frac{\varphi(x)}{\beta}\left[\frac{\psi(x)}{\varphi(x)}\right]^{\frac{\beta}{\gamma}}\le\frac{1}{\beta}[\varphi(x)]^{\left(1-\frac{\beta}{\gamma}\right)}[\psi(x)]^{\frac{\beta}{\gamma}}.\end{aligned}\tag{4.12}$$

From (4.9), (4.11) and (4.12) we obtain that

$$\left|^{\beta,\alpha}I^{\rho}_{a+}f(x)\right|\le\left(\frac{1}{\beta\Gamma(\beta)}+\frac{\Gamma(\gamma)}{\Gamma(\beta)}\right)[\varphi(x)]^{\left(1-\frac{\beta}{\gamma}\right)}[\psi(x)]^{\frac{\beta}{\gamma}}.$$

The constant $1/[\beta\Gamma(\beta)]+\Gamma(\gamma)/\Gamma(\beta)$ is dominated by a constant not depending on α,β,γ and this completes the proof. □

Taking $^{\beta,\alpha}I^{\rho}_{a+}f(x)=g(x),\ \beta-\gamma=\theta$ we obtain the following.

Theorem 4.1.7. *Let* $\alpha>0,\ 0<\beta<1$ *and* $-\infty<\rho<a$. *If* $g\in{}^{\gamma,\alpha}I^{\rho}_{a+}\left[L^1(a,\infty)\right],\ 0<\gamma<1$ *and there exist nondecreasing functions* φ *and* ψ *such that*

$$\left|^{\beta,\alpha}D^{\rho}_{a+}g(x)\right|\le\varphi(x),\ \ |g(x)|\le\psi(x),\ x>a,$$

then for $0<\theta<\gamma<1$ *we have*

$$\left|^{\theta,\alpha}D^{\rho}_{a+}g(x)\right|\le const.\,[\psi(x)]^{\left(1-\frac{\theta}{\gamma}\right)}[\varphi(x)]^{\frac{\theta}{\gamma}}.$$

Corollary 4.1.8. *Let* $\alpha > 0,\ 0 < \beta < 1$ *and* $-\infty < \rho < a$*. If* $g \in {}^{\gamma,\alpha}I^{\rho}_{a+}\left[L^1(a,\infty)\right],\ 0 < \gamma < 1,$ *such that* $g(x)$ *and* ${}^{\gamma,\alpha}D^{\rho}_{a+}g(x)$ *are bounded on* (a,∞) *then*

$$\max_{x\in(a,\infty)}\left|{}^{\theta,\alpha}D^{\rho}_{a+}g(x)\right| \leq const. \max_{x\in(a,\infty)}|g(x)|^{\left(1-\frac{\theta}{\gamma}\right)} esssup_{x\in(a,\infty)}\left|{}^{\gamma,\alpha}D^{\rho}_{a+}g(x)\right|^{\frac{\theta}{\gamma}},$$

where $0 \leq \theta \leq \gamma \leq 1$.

4.2. Infinite interval case

In this subsection we will consider the infinite interval case.

Firstly we shall introduce the equivalent form of Theorem 4.1.1, Corollary 4.1.3 and Corollary 4.1.4 as follows.

Theorem 4.2.1. *Let* $\alpha > 0,\ 0 < \beta < 1$ *and* $b < \delta < \infty$*. If* $(\delta - x)^{\alpha-1}f(x) \in L^1(a,b)$ *and* ${}^{\beta,\alpha}I^{\delta}_{b-}f(x)$ *is continuous on* $[a,b]$ *such that* ${}^{\beta,\alpha}I^{\delta}_{b-}f(b^-) = 0$ *then for* $x < a$ *there exists* $\xi \in (a,b]$ *such that*

$$(-1)^{\beta-1}\int\limits_a^b \frac{\left[(\delta-x)^\alpha - (\delta-t)^\alpha\right]^{\beta-1}}{(\delta-t)^{1-\alpha}}f(t)dt = \int\limits_\xi^b \frac{\left[(\delta-\xi)^\alpha - (\delta-t)^\alpha\right]^{\beta-1}}{(\delta-t)^{1-\alpha}}f(t)dt$$

and

$$\left|\int\limits_a^b \frac{\left[(\delta-x)^\alpha - (\delta-t)^\alpha\right]^{\beta-1}}{(\delta-t)^{1-\alpha}}f(t)dt\right| \leq \max_{\xi\in[a,b]}\left|\int\limits_\xi^b \frac{\left[(\delta-\xi)^\alpha - (\delta-t)^\alpha\right]^{\beta-1}}{(\delta-t)^{1-\alpha}}f(t)dt\right|.$$

If $(\delta - x)^{\alpha-1}f(x) \in L^1(a,b)$ *then for* $x < a$

$$\left|\int\limits_a^b \frac{\left[(\delta-x)^\alpha - (\delta-t)^\alpha\right]^{\beta-1}}{(\delta-t)^{1-\alpha}}f(t)dt\right| \leq esssup_{\xi\in[a,b]}\left|\int\limits_\xi^b \frac{\left[(\delta-\xi)^\alpha - (\delta-t)^\alpha\right]^{\beta-1}}{(\delta-t)^{1-\alpha}}f(t)dt\right|. \tag{4.13}$$

In this subsection we will show that (4.13) is true when the upper bound is infinity.

Firstly we shall adopt the notation

$$e(x,u) := \frac{\alpha}{\Gamma(\beta)\Gamma(1-\beta)}\int\limits_w^x \frac{\left[(\delta-u)^\alpha - (\delta-v)^\alpha\right]^{\beta-1}}{\left[(\delta-v)^\alpha - (\delta-x)^\alpha\right]^\beta}\frac{1}{(\delta-v)^{1-\alpha}}dv,$$

where $\alpha = 2n+1, n = 0,1,2,...,\ 0 < \beta < 1,\ b < \delta < \infty,\ u < w$ and w is fixed.

Lemma 4.2.2. $\dfrac{\partial}{\partial x}e(x,u) \geq 0$ for all $x \geq \delta \geq w > u$.

Proof For $x \geq \delta \geq w > u$ we have

$$\begin{aligned}\Gamma(\beta)\Gamma(1-\beta)e(x,u) = \int_u^x \frac{[(\delta-u)^\alpha-(\delta-v)^\alpha]^{\beta-1}}{[(\delta-v)^\alpha-(\delta-x)^\alpha]^\beta}\frac{\alpha}{(\delta-v)^{1-\alpha}}dv \\ -\int_u^w \frac{[(\delta-u)^\alpha-(\delta-v)^\alpha]^{\beta-1}}{[(\delta-v)^\alpha-(\delta-x)^\alpha]^\beta}\frac{\alpha}{(\delta-v)^{1-\alpha}}dv.\end{aligned} \tag{4.14}$$

For the first integral at right-hand side of (4.14) we shall use the substitution

$$z = \frac{(\delta-u)^\alpha-(\delta-v)^\alpha}{(\delta-u)^\alpha-(\delta-x)^\alpha}$$

to get

$$\Gamma(\beta)\Gamma(1-\beta)e(x,u) = \Gamma(\beta)\Gamma(1-\beta) - \int_u^w \frac{[(\delta-u)^\alpha-(\delta-v)^\alpha]^{\beta-1}}{[(\delta-v)^\alpha-(\delta-x)^\alpha]^\beta}\frac{\alpha}{(\delta-v)^{1-\alpha}}dv. \tag{4.15}$$

(4.15) implies that

$$\Gamma(\beta)\Gamma(1-\beta)\frac{\partial}{\partial x}e(x,u) = \alpha^2\beta\int_u^w \frac{[(\delta-u)^\alpha-(\delta-v)^\alpha]^{\beta-1}}{[(\delta-v)^\alpha-(\delta-x)^\alpha]^{\beta+1}}\frac{(\delta-x)^{\alpha-1}}{(\delta-v)^{1-\alpha}}dv \geq 0$$

since $\alpha - 1$ is an even number. This completes the proof. □

Lemma 4.2.3. $0 \leq e(x,u) \leq 1$ *for all* $x \geq \delta \geq w > u$

Proof The proof follows from Lemma 4.2.2. Indeed it is enough to use the substitution

$$t = \frac{(\delta-u)^\alpha-(\delta-v)^\alpha}{(\delta-u)^\alpha-(\delta-x)^\alpha}$$

in the definition of $e(x,u)$. □

Lemma 4.2.4. For all $x \geq \delta \geq w > u$ we have the following

$$\int_w^x \frac{[(\delta-v)^\alpha-(\delta-x)^\alpha]^{\beta-1}}{(\delta-x)^{1-\alpha}}\frac{\partial}{\partial v}e(v,u)dv = (\delta-x)^{\alpha-1}\left[(\delta-u)^\alpha-(\delta-x)^\alpha\right]^{\beta-1}.$$

Proof

$$\begin{aligned}&\alpha\int_w^x \frac{[(\delta-v)^\alpha-(\delta-x)^\alpha]^{\beta-1}}{(\delta-x)^{1-\alpha}}\frac{\partial}{\partial v}e(v,u)dv = \frac{\partial}{\partial x}\int_w^x \frac{[(\delta-v)^\alpha-(\delta-x)^\alpha]^{\beta}}{\beta}\frac{\partial}{\partial v}e(v,u)dv \\ &= \frac{\partial}{\partial x}\left\{-\frac{[(\delta-w)^\alpha-(\delta-x)^\alpha]^\beta e(w,u)}{\beta} + \alpha\int_w^x \frac{[(\delta-v)^\alpha-(\delta-x)^\alpha]^{\beta-1}}{(\delta-v)^{1-\alpha}}e(v,u)dv\right\} \\ &= \alpha\frac{\partial}{\partial x}\int_w^x \frac{[(\delta-v)^\alpha-(\delta-x)^\alpha]^{\beta-1}}{(\delta-v)^{1-\alpha}}e(v,u)dv.\end{aligned}$$

Therefore we get that

$$\begin{aligned}
&\Gamma(\beta)\Gamma(1-\beta)\alpha\int_w^x \frac{[(\delta-v)^\alpha-(\delta-x)^\alpha]^{\beta-1}}{(\delta-x)^{1-\alpha}}\frac{\partial}{\partial v}e(v,u)dv\\
&=\alpha\tfrac{\partial}{\partial x}\int_w^x \frac{[(\delta-v)^\alpha-(\delta-x)^\alpha]^{\beta-1}}{(\delta-v)^{1-\alpha}}\int_w^v \frac{[(\delta-u)^\alpha-(\delta-t)^\alpha]^{\beta-1}}{[(\delta-t)^\alpha-(\delta-v)^\alpha]^\beta}\frac{\alpha}{(\delta-t)^{1-\alpha}}dtdv\\
&=\Gamma(\beta)\Gamma(1-\beta)\frac{\partial}{\partial x}\int_w^x [(\delta-u)^\alpha-(\delta-t)^\alpha]^{\beta-1}\alpha(\delta-t)^{1-\alpha}dt\\
&=\Gamma(\beta)\Gamma(1-\beta)\frac{\partial}{\partial x}\left\{\frac{[(\delta-u)^\alpha-(\delta-x)^\alpha]^\beta-[(\delta-u)^\alpha-(\delta-w)^\alpha]^\beta}{\beta}\right\}\\
&=\Gamma(\beta)\Gamma(1-\beta)[(\delta-u)^\alpha-(\delta-x)^\alpha]^{\beta-1}\alpha(\delta-x)^{\alpha-1}.
\end{aligned}$$

This completes the proof. □

We shall adopt the notation

$$^{\beta,\alpha}\Psi^\delta(v,x)=\frac{\alpha}{\Gamma(\beta)}\int_x^\infty \frac{[(\delta-v)^\alpha-(\delta-t)^\alpha]^{\beta-1}}{(\delta-t)^{1-\alpha}}f(t)dt,\ x\ge v\ge w,$$

where $\alpha=2n+1,\ n=0,1,2,...,$ and $0<\beta<1$.

Theorem 4.2.5. *If* $x>\delta\ge v\ge w$ *and* $^{\beta,\alpha}\Psi^\delta(v,x)$ *converges at infinity then*

$$^{\beta,\alpha}\Psi^\delta(v,x)=\circ\left\{-[(\delta-x)^\alpha-(\delta-v)^\alpha]^{\beta-1}[(\delta-x)^\alpha]^{\beta-1}\right\}$$

as $x\to\infty$, *uniformly in* v.

Proof For $x>\delta\ge v\ge w$ we have

$$\begin{aligned}
&-^{\beta,\alpha}\Psi^\delta(v,x)=\frac{-\alpha}{\Gamma(\beta)}\int_x^\infty \frac{[-(\delta-t)^\alpha]^{\beta-1}}{(\delta-t)^{1-\alpha}}\left[\frac{(\delta-v)^\alpha-(\delta-t)^\alpha}{-(\delta-t)^\alpha}\right]^{\beta-1}f(t)dt\\
&=\frac{\alpha}{\Gamma(\beta)}\left\{\left[\frac{(\delta-v)^\alpha-(\delta-x)^\alpha}{-(\delta-x)^\alpha}\right]^{\beta-1}\int_x^\infty \frac{[-(\delta-t)^\alpha]^{\beta-1}}{(\delta-t)^{1-\alpha}}f(t)dt\right.\\
&\left.+\int_x^\infty\left[\int_t^\infty \frac{[-(\delta-s)^\alpha]^{\beta-1}}{(\delta-s)^{1-\alpha}}f(s)ds\right]\frac{\partial}{\partial t}\left[\frac{(\delta-v)^\alpha-(\delta-t)^\alpha}{-(\delta-t)^\alpha}\right]^{\beta-1}dt\right\}.
\end{aligned}$$

Since $\left\{[(\delta-v)^\alpha-(\delta-t)^\alpha][-(\delta-t)^\alpha]^{-1}\right\}^{\beta-1}$ decreases as t increases we may infer that for any $\epsilon>0$ there exists a $\varepsilon=\varepsilon(\epsilon)>0$ such that the inequalities hold

$$\begin{aligned}
&\left|^{\beta,\alpha}\Psi^\delta(v,x)\right|<\epsilon\left[\frac{(\delta-v)^\alpha-(\delta-x)^\alpha}{-(\delta-x)^\alpha}\right]^{\beta-1}\\
&-\epsilon\int_x^\infty\frac{\partial}{\partial t}\left[\frac{(\delta-v)^\alpha-(\delta-x)^\alpha}{-(\delta-x)^\alpha}\right]^{\beta-1}dt<2\epsilon\left[\frac{(\delta-v)^\alpha-(\delta-x)^\alpha}{-(\delta-x)^\alpha}\right]^{\beta-1},
\end{aligned}$$

where $x\ge\varepsilon=\varepsilon(\epsilon)$. This completes the proof. □

Theorem 4.2.6. *Let* $\alpha = 2n+1,\ n = 0,1,2,...,\ 0 < \beta < 1,\ \delta \geq w > u$ *and* ${}^{\beta,\alpha}\Psi^{\delta}(u,w)$ *converge at infinity. Then we have*

$$\left|{}^{\beta,\alpha}\Psi^{\delta}(u,w)\right| \leq esssup_{w\leq v\leq\infty}\left|{}^{\beta,\alpha}\Psi^{\delta}(v,v)\right|.$$

Proof Using Lemma 4.2.4 we obtain

$$\begin{aligned}
\Gamma(\beta)^{\beta,\alpha}\Psi^{\delta}(u,w) &= \lim_{x\to\infty}\alpha\int_w^x \frac{\left[(\delta-u)^{\alpha}-(\delta-t)^{\alpha}\right]^{\beta-1}}{(\delta-t)^{1-\alpha}}f(t)dt \\
&= \alpha\lim_{x\to\infty}\int_w^x f(t)\int_w^t \frac{\left[(\delta-v)^{\alpha}-(\delta-t)^{\alpha}\right]^{\beta-1}}{(\delta-t)^{1-\alpha}}\frac{\partial}{\partial v}e(v,u)dvdt \\
&= \alpha\lim_{x\to\infty}\int_w^x \frac{\partial}{\partial v}e(v,u)\int_v^{\infty}\frac{\left[(\delta-v)^{\alpha}-(\delta-t)^{\alpha}\right]^{\beta-1}}{(\delta-t)^{1-\alpha}}f(t)dvdt \\
&-\alpha\lim_{x\to\infty}\int_w^x \frac{\partial}{\partial v}e(v,u)\int_x^{\infty}\frac{\left[(\delta-v)^{\alpha}-(\delta-t)^{\alpha}\right]^{\beta-1}}{(\delta-t)^{1-\alpha}}f(t)dvdt,
\end{aligned}$$

where from the assumption both inner integrals converge at infinity. However, at the lower limit the first integral may nonexist in a null set. Then we get that

$$\begin{aligned}
&\int_w^x \frac{\partial}{\partial v}e(v,u)^{\beta,\alpha}\Psi^{\delta}(v,x)dv \\
&= o\left\{-\left[(\delta-x)^{\alpha}\right]^{\beta-1}\int_w^x\left[(\delta-x)^{\alpha}-(\delta-v)^{\alpha}\right]^{\beta-1}\frac{\partial}{\partial v}e(v,u)dv\right\} \\
&= o\left\{-\left[(\delta-x)^{\alpha}\right]^{\beta-1}\left[(\delta-x)^{\alpha}-(\delta-u)^{\alpha}\right]^{\beta-1}\right\} = o(-1).
\end{aligned}$$

On the other side we have

$$\begin{aligned}
&\alpha\lim_{x\to\infty}\int_w^x \frac{\partial}{\partial v}e(v,u)\int_v^{\infty}\frac{\left[(\delta-v)^{\alpha}-(\delta-t)^{\alpha}\right]^{\beta-1}}{(\delta-t)^{1-\alpha}}f(t)dtdv \\
&= \Gamma(\beta)\lim_{x\to\infty}\int_w^x \frac{\partial}{\partial v}e(v,u)^{\beta,\alpha}\Psi^{\delta}(v,v)dv.
\end{aligned}$$

Hence we get that

$$\begin{aligned}
\Gamma(\beta)\left|{}^{\beta,\alpha}\Psi^{\delta}(u,w)\right| &\leq \Gamma(\beta)\int_w^{\infty}\left|\frac{\partial}{\partial v}e(v,u)^{\beta,\alpha}\Psi^{\delta}(v,v)\right|dv \\
&\leq \Gamma(\beta)esssup_{w\leq v\leq\infty}\left|{}^{\beta,\alpha}\Psi^{\delta}(v,v)\right|
\end{aligned}$$

and this completes the proof. □

5. Conclusion and remarks

In this paper we have introduced new fractional integrals ${}^{\beta,\alpha}I^{\rho}_{a+}$, ${}^{\beta,\alpha}I^{\delta}_{b-}$ and new fractional derivatives ${}^{\beta,\alpha}D^{\rho}_{a+}$, ${}^{\beta,\alpha}D^{\delta}_{b-}$. As we have discussed earlier that these operators do not come from iterated $n-$fold integrals unless $\rho = a$ and $\delta = b$ with $\alpha^{-\beta}\left\{{}^{\beta,\alpha}I^{\rho}_{a+}f(x)\right\}$ and $\alpha^{-\beta}\left\{{}^{\beta,\alpha}I^{\delta}_{b-}f(x)\right\}$. These situations have been studied in [9]. In the definitions of the fractional integrals ${}^{\beta,\alpha}I^{\rho}_{a+}f(x)$ and ${}^{\beta,\alpha}I^{\delta}_{b-}f(x)$ we have excluded the constants $\alpha^{-\beta}$ because these factors have no effect in the calculations rather than the extra coefficients α^{β} in the definitions of ${}^{\beta,\alpha}D^{\rho}_{a+}f(x)$ and ${}^{\beta,\alpha}D^{\delta}_{b-}f(x)$. Moreover we should note that we could define the operators

$$ {}^{\beta,\alpha}_{\xi,\tau}I^{\rho}_{a+}f(x) := \frac{\xi^{-\tau}\alpha}{\Gamma(\beta)}\int\limits_a^x \frac{[(x-\rho)^\alpha-(t-\rho)^\alpha]^{\beta-1}}{(t-\rho)^{1-\alpha}}f(t)dt, $$

where $\alpha > 0,\ 0 < \beta < 1,\ -\infty < \rho < \infty,\ \xi > 0$ and $-\infty < \tau < \infty$ such that $x \neq \rho$ and $-\infty < a < x \leq b < \infty$, and

$$ {}^{\beta,\alpha}_{\xi,\tau}I^{\delta}_{b-}f(x) := \frac{\xi^{-\tau}\alpha}{\Gamma(\beta)}\int\limits_x^b \frac{[(\delta-x)^\alpha-(\delta-t)^\alpha]^{\beta-1}}{(\delta-t)^{1-\alpha}}f(t)dt, $$

where $\alpha > 0,\ 0 < \beta < 1,\ -\infty < \delta < \infty,\ \xi > 0$ and $-\infty < \tau < \infty$ such that $x \neq \delta$ and $-\infty < a \leq x < b < \infty$. Then the corresponding fractional derivatives would be

$$ {}^{\beta,\alpha}_{\xi,\tau}D^{\rho}_{a+}f(x) := \frac{\xi^{\tau}}{\Gamma(1-\beta)}(x-\rho)^{1-\alpha}\frac{d}{dx}\int\limits_a^x \frac{[(x-\rho)^\alpha-(t-\rho)^\alpha]^{-\beta}}{(t-\rho)^{1-\alpha}}f(t)dt, $$

where $x \in (a,b]$ and $-\infty < \rho < \infty$ such that $\rho \neq x$ when $\rho \in (a,b]$, and

$$ {}^{\beta,\alpha}_{\xi,\tau}D^{\delta}_{b-}f(x) := \frac{-\xi^{\tau}}{\Gamma(1-\beta)}(\delta-x)^{1-\alpha}\frac{d}{dx}\int\limits_x^b \frac{[(\delta-x)^\alpha-(\delta-t)^\alpha]^{-\beta}}{(\delta-t)^{1-\alpha}}f(t)dt, $$

where $x \in [a,b)$ and $-\infty < \delta < \infty$ such that $\delta \neq x$ when $\delta \in [a,b)$.

We shall note that it is still possible to introduce additional fractional integrals and derivatives. Indeed, we shall consider the following fractional integral

$$ {}^{\beta,\alpha}I^{\rho}_{+}f(x) := \frac{\alpha}{\Gamma(\beta)}\int\limits_{-\infty}^x \frac{[(x-\rho)^\alpha-(t-\rho)^\alpha]^{\beta-1}}{(t-\rho)^{1-\alpha}}f(t)dt, $$

where $\alpha > 0,\ 0 < \beta < 1,\ -\infty < \rho < \infty$ and $x \leq b \leq \infty$. Using the substitution $(x-\rho)^\alpha - (t-\rho)^\alpha = u$ one gets that

$$ {}^{\beta,\alpha}I^{\rho}_{+}f(x) = \frac{-1}{\Gamma(\beta)}\int\limits_{-i^{2\alpha}\infty}^{0} u^{\beta-1}f\left([(x-\rho)^\alpha-u]^{\frac{1}{\alpha}}+\rho\right)du. $$

For $u = i^{2\alpha}t$ we may write the following

$$ {}^{\beta,\alpha}I^{\rho}_{+}f(x) = \frac{-i^{2\alpha\beta}}{\Gamma(\beta)}\int\limits_{-\infty}^{0} t^{\beta-1}f\left(\left[(x-\rho)^\alpha-i^{2\alpha}t\right]^{\frac{1}{\alpha}}+\rho\right)dt. $$

Now corresponding fractional derivative is defined as

$$ {}^{\beta,\alpha}D^{\rho}_{+}f(x) = (x-\rho)^{1-\alpha}\frac{i^{2\alpha\beta}}{\Gamma(1-\beta)}\frac{d}{dx}\int\limits_{-\infty}^{0} t^{-\beta}f\left(\left[(x-\rho)^\alpha-i^{2\alpha}t\right]^{\frac{1}{\alpha}}+\rho\right)dt. $$

Now more calculations are possible. Indeed we get that

$$
\begin{aligned}
{}^{\beta,\alpha}D_{+}^{\rho}f(x) &= (x-\rho)^{1-\alpha}\frac{i^{2\alpha\beta}}{\Gamma(1-\beta)}\frac{d}{dx}\int_{-\infty}^{0} t^{-\beta} f\left(\left[(x-\rho)^{\alpha}-i^{2\alpha}t\right]^{\frac{1}{\alpha}}+\rho\right)dt \\
&= \frac{i^{2\alpha\beta}}{\Gamma(1-\beta)}\int_{-\infty}^{0} t^{-\beta} f'\left(\left[(x-\rho)^{\alpha}-i^{2\alpha}t\right]^{\frac{1}{\alpha}}+\rho\right)\left[(x-\rho)^{\alpha}-i^{2\alpha}t\right]^{\frac{1}{\alpha}-1}dt \\
&= \frac{i^{2\alpha\beta}}{\Gamma(1-\beta)}\int_{-\infty}^{0} f'\left(\left[(x-\rho)^{\alpha}-i^{2\alpha}t\right]^{\frac{1}{\alpha}}+\rho\right)\left[(x-\rho)^{\alpha}-i^{2\alpha}t\right]^{\frac{1}{\alpha}-1}(-\beta)\int_{-\infty}^{t}\frac{d\xi}{\xi^{\beta+1}}dt \\
&= i^{2\alpha(\beta-1)}\frac{\alpha\beta}{\Gamma(1-\beta)}\int_{-\infty}^{0}\frac{f(x)-f\left(\left[(x-\rho)^{\alpha}-i^{2\alpha}\xi\right]^{\frac{1}{\alpha}}+\rho\right)}{\xi^{\beta+1}}d\xi.
\end{aligned}
$$

Now let $\left[(x-\rho)^{\alpha}-i^{2\alpha}\xi\right]^{\frac{1}{\alpha}}+\rho=t$. Then we have

$$
{}^{\beta,\alpha}D_{+}^{\rho}f(x) = -i^{2\alpha(\beta-1)+2\alpha\beta}\frac{\beta}{\Gamma(1-\beta)}\int_{-\infty}^{x}\frac{f(x)-f(t)}{\left[(x-\rho)^{\alpha}-(t-\rho)^{\alpha}\right]^{\beta+1}}(t-\rho)^{\alpha-1}dt. \tag{5.1}
$$

For $\alpha=1$ in (5.1) we get that

$$
{}^{\beta,1}D_{+}f(x) = \frac{\beta}{\Gamma(1-\beta)}\int_{-\infty}^{x}\frac{f(x)-f(t)}{(x-t)^{\beta+1}}dt
$$

which is the Marchaud fractional derivative. This definition given in (5.1) seems to be new in the literature for $\alpha>0$ and $\alpha\neq 1$.

On the other side from Theorem 3.1 we may write for a differentiable function f that

$$
\begin{aligned}
{}^{\beta,\alpha}D_{a+}^{\rho}f(x) &= \frac{1}{\Gamma(1-\beta)}\left\{\frac{f(a)}{\left[(x-\rho)^{\alpha}-(a-\rho)^{\alpha}\right]^{\beta}}+\int_{a}^{x}\frac{d\left[f(t)-f(x)\right]}{\left[(x-\rho)^{\alpha}-(t-\rho)^{\alpha}\right]^{\beta}}\right\} \\
&= \frac{1}{\Gamma(1-\beta)}\left\{\frac{f(a)}{\left[(x-\rho)^{\alpha}-(a-\rho)^{\alpha}\right]^{\beta}}+\frac{f(t)-f(x)}{\left[(x-\rho)^{\alpha}-(t-\rho)^{\alpha}\right]^{\beta}}\Big|_{t=a}^{x}\right. \\
&\left.+\alpha\beta\int_{a}^{x}\frac{f(x)-f(t)}{\left[(x-\rho)^{\alpha}-(t-\rho)^{\alpha}\right]^{\beta+1}}(t-\rho)^{\alpha-1}dt\right\} \\
&= \frac{1}{\Gamma(1-\beta)}\left\{\frac{f(x)}{\left[(x-\rho)^{\alpha}-(a-\rho)^{\alpha}\right]^{\beta}}+\alpha\beta\int_{a}^{x}\frac{f(x)-f(t)}{\left[(x-\rho)^{\alpha}-(t-\rho)^{\alpha}\right]^{\beta+1}}(t-\rho)^{\alpha-1}dt\right\}
\end{aligned}
$$

which is an analogue of (5.1).

Finally in the definition of ${}^{\beta,\alpha}I_{a+}^{\rho}f(x)$ we may use the substitution

$$
\left(\frac{t-\rho}{x-\rho}\right)^{\alpha}=v
$$

to get

$$\begin{aligned} {}^{\beta,\alpha}I^{\rho}_{a^+}f(x) &= \frac{1}{\Gamma(\beta)}\left[(x-\rho)^{\alpha}\right]^{\beta-1}\int\limits_{\left(\frac{a-\rho}{x-\rho}\right)^{\alpha}}^{1} f\left((x-\rho)\,v^{\frac{1}{\alpha}}+\rho\right)(1-v)^{\beta-1}(x-\rho)^{\alpha}dv \\ &= \frac{1}{\Gamma(\beta)}\,(x-\rho)^{\alpha\beta}\int\limits_{\left(\frac{a-\rho}{x-\rho}\right)^{\alpha}}^{1} f\left((x-\rho)\,v^{\frac{1}{\alpha}}+\rho\right)(1-v)^{\beta-1}dv. \end{aligned} \tag{5.2}$$

For $\alpha = 1$ and $\rho = 0$ in (5.2) we get that

$$ {}^{\beta,1}I^{0}_{a^+}f(x) = \frac{x^{\beta}}{\Gamma(\beta)}\int\limits_{\frac{a}{x}}^{1} f(xv)\,(1-v)^{\beta-1}dv $$

which is the Dzherbashyan's generalized fractional integral.

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