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Research Article

The Extended Laguerre Polynomials $\{A_{q,n}^{(\alpha)}(x)\}$ Involving ${}_qF_q$, $q > 2$

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In this paper, for the proposed extended Laguerre polynomials $\left\{A_{q,n}^{(\alpha)}(x)\right\}$, the generalized hypergeometric function of the type ${}_qF_q$, $q > 2$ and extension of the Laguerre polynomial are introduced. Similar to those related to the Laguerre polynomials, the generating function, recurrence relations, and Rodrigue's formula are determined. Some corollaries are also discussed at the end.

1. Introduction and Applications

Due to its wide applications, the study of orthogonal polynomials has been a popular research topic for many years. Many of these polynomials are generated by hypergeometric functions. Indeed, the orthogonal polynomials have numerous properties of interest, e.g., recurrence relations and differential equations. Based on their Rodrigues formulae, generating functions and solutions of integral equations with orthogonal polynomials as kernels have been extensively investigated.

Generalizations and extensions of orthogonal polynomials are in the another familiar direction of research. One of the polynomial set which has been extended is a set of Laguerre polynomials. Laguerre polynomials are well-known to form an orthogonal set with respect to the weight function $z^{\alpha}e^{-z}$ on the interval $(0, \infty)$.

A set of Laguerre polynomials is generated by well-known confluent hypergeometric function ${}_1F_1$. It can be also generated by hypergeometric function ${}_0F_1$. Another direction is the study of Laguerre polynomials based on more than one variable which are often used in physical and statistical model. One, too, combinatorial polynomial images, moments, orthogonality relation, and a combinatorial understanding Ikyrana coefficients Al-Salam and Chihara q Laguerre polynomial, can study various aspects. Orthogonal polynomials, namely, Hermite polynomials and Legendre polynomials can also be studied through the finite series involving Laguerre polynomials.

Laguerre polynomials are used to solve noncentral Chi-square distribution. Laguerre polynomials are the orthogonal polynomial satisfied the recurrence relations. Various specializations are studied with application to classical orthogonal polynomials. Kinetic theory of particles based on Laguerre polynomial macroscopic hydrodynamic quantities and kinetic coefficients of different medium is used to set.

There are a large number of generalizations and extensions of Laguerre polynomials, e.g., Shively's polynomials. Many of these generalizations are based on its Rodrigues formulae in addition to hypergeometric functions. Recently, an interesting integral representation of generalized hypergeometric functions has been determined. It is now natural to point to a generalization of Laguerre polynomials based on such a discovery. This idea has motivated the current work. Also, it will explore deeper investigation and extensions of results which we proved in our early studies and research.

In this work, we discuss the features of Extending Laguerre polynomial involving ${}_qF_q$, $q > 2$. Extending Laguerre polynomial set has been a popular research issue well considered for years. There have a number of directions to do so. One direction is to follow the definition of Laguerre polynomials based on the confluent hypergeometric function, explicitly

$$L_n^{(\alpha)}(x) = \frac{(1+\alpha)_n}{n!} {}_1F_1(-n\,; 1+\alpha\,; x). \tag{1}$$

Shively [1] extended the Laguerre polynomials as

$$R_n(a,x) = \frac{(a)_{2n}}{n!(a)_n} {}_1F_1(-n\,; a+n\,; x). \tag{2}$$

He used a factor $a+n$ instead of $1+\alpha$ in Laguerre polynomials. In his study, he found a large number of its properties including the result that a finite sum of Laguerre polynomials is Shively's polynomials. Habibullah [2] proved the Rodrigues formula for Shively's polynomials in the following form

$$R_n(a+1,x) = \frac{e^x x^{-\alpha-n}}{n!} D^n\left(x^{\alpha+2n} e^{-x}\right), \tag{3}$$

similar to the Rodrigues formula

$$L_n^{(\alpha)}(x) = \frac{e^x x^{-\alpha}}{n!} D^n\left(x^{\alpha+n} e^{-x}\right), \tag{4}$$

for the Laguerre polynomials.

Researchers have also often based their generalization on extension of Rodrigues formula and subsequently determined properties of extended polynomials. Chatterjea [3] developed an extension of the Laguerre polynomial by strengthening the Rodrigues formula. Chatterjea and Das [4] restructured their definition and the resultant study by considering another version of the Laguerre polynomials.

Chen and Srivastava [5] found a stronger Rodrigues formula to develop a generalization of the Laguerre polynomial.

The forms generalized Rodrigues formulae by Chak [6] show that robust following of this method of defining extensions of the Laguerre polynomial. Since comprehensive literature is available on special functions, we follow Shively's tradition to introduce the definition of the extended Laguerre polynomials set based on special functions similar to that contained the original definition.

Dattoli et al. [7] used an exponential generating functions approach involving Hermite polynomials and Bessel functions introduced new families. He, too, studied their respective recurrence relations and showed that they fulfill different differential equations. Trickovic and Stankovic [8] of the Jacobi and Laguerre polynomial orthogonality of rational functions that have proved equally. Trickovic and Stankovic [8] have proved the orthogonality of the Jacobi and the Laguerre polynomials.

Khan and Shukla [9] have introduced a novel method to give operator representations of certain polynomials. They gave binomial and trinomial operators representations of certain polynomials. Grinshpan [10] has shown that all solutions to the equations of a family of integral equations fulfill modulus inequality. Duenas et al. [11] a derivative of a Dirac delta by adding a perturbation of a Laguerre-Hahn functional gain catalog.

Kim et al. [12] have studied some interesting identities and also studied Bernoulli and Euler's numbers in connection with the properties of Laguerre polynomials. They derived identities by using the orthogonality of Laguerre polynomials w.r.t the relevant inner product. Marinkovic et al. [13] have demonstrated the theory of deformed Laguerre derivative defined by iterated deformed Laguerre operator. Nowak et al. [14] convolution type Laguerre function expansions in order to prove the standard estimates has developed a technique. Khan and Habibullah [15] have introduced $A_{2,n}(x) = {}_2F_2(-n/2, (-n+1/2)\,; 1/2, 1\,; x^2)$.

Khan and Kalim [16] have introduced

$$A_{3,m}^{(\alpha)}(y) = \frac{(1+\alpha)_m}{m!} {}_3F_3\left(\frac{-m}{3}, \frac{-m+1}{3}, \frac{-m+2}{3}\,; \frac{1+\alpha}{3}, \frac{2+\alpha}{3}, \frac{3+\alpha}{3}\,; y^3\right). \tag{5}$$

Doha et al. [17] modified generalized Laguerre expansion coefficients of the derivatives of a function in terms of its original expansion coefficients, and an explicit expression for the derivatives of modified generalized Laguerre polynomials of any degree and for any order as a linear combination of modified generalized Laguerre polynomials themselves is also deduced.

Dattoli et al. [18] applied operational techniques to introduce suitable families of special functions. Andrews et al. [19], Trickovic and Stankovic [20], Radulescu [21], and Doha and Youssri [22] have done a lot of work for properties of Laguerre polynomials. Akbary et al. [23] can be referred for other applications of Laguerre polynomials. Li [24], Aksoy et al. [25], Wang [26], and Krasikov and Zarkh [27] have studied problems of permutation of polynomials, bijections that can induce polynomials with integer coefficients is modulo m.

We organize our manuscript as: we present the properties and applications of extended polynomials in Section 2. We give the extended Laguerre polynomials in Section 3. We discuss the generating functions in Section 4. We present the recurrence relations in Section 5. We give the differential equations in Section 6. We discuss the Rodrigues formula in Section 7. We give the special properties in Section 8. We present some other generating functions in Section 9. We give the expansion of the polynomials in Section 10. We present the conclusion in the last section.

2. Extended Polynomial Properties and Application Elementary Results

Das [28] has modified the work of Al-Salam [29]. Carlitz [30] has given a generating function and an explicit polynomial expression for the polynomial $Y_n^c(x\,; k)$, a variant of Laguerre polynomials. Srivastava [31] has derived the several bilinear generating functions by using generalized hypergeometric functions. Explicitly, we can mention [Erdélyi p, 190] [32].

$$D^m\left[x^{\alpha+m} L_n^{(\alpha+m)}(x)\right] = \frac{\Gamma(\alpha+m+n+1)}{\Gamma(\alpha+n+1)} x^\alpha L_n^{(\alpha)}(x), D = \frac{d}{dx}. \tag{6}$$

One generalization of Laguerre polynomials is $R_n(a,x)$ as Shively defined it by [Rainville p, 298] [33].

$$R_n(a,x)=\frac{(a)_{2n}}{n!(a)_n}\,{}_1F_1(-n\,;a+n\,;x), \tag{7}$$

he used these results as an integral equation involving Shively's polynomials. Karande and Thakare [34] have derived the generating functions, bilinear generating functions, and recurrence relations by using the biorthogonal set of Konhausar. Panda [35] has studied a new generalization based on several known polynomials systems belonging to the families of the classical Jacobi, Hermite, and Laguerre polynomials. Parashar [36] has introduced a new set of Laguerre polynomials $L_n^{(\alpha,h)}(x)$ related to the Laguerre polynomials $L_n^{(\alpha)}(x)$. Sharma and Chongdar [37] have proved an extension of bilateral generating functions of the modified Laguerre polynomials.

Lemma 1. *If $j\in\mathbb{Z}^+$ and n is any nonnegative integer, then*

$$\left(\frac{-n}{q}\right)_j\left(\frac{-n+1}{q}\right)_j\cdots\left(\frac{-n+q-1}{q}\right)_j=(-1)^{qj}\frac{n!}{q^{qj}(n-qj)!}. \tag{8}$$

Proof.

$$\begin{aligned}
&\left(\frac{-n}{q}\right)_j\left(\frac{-n+1}{q}\right)_j\cdots\left(\frac{-n+q-1}{q}\right)_j=\left(\frac{-n}{q}\right)\left(\frac{-n}{q}+1\right)\left(\frac{-n}{q}+2\right)\cdots\left(\frac{-n}{q}+j-1\right),\\
&\left(\frac{-n+1}{q}\right)\left(\frac{-n+1}{q}+1\right)\left(\frac{-n+1}{q}+2\right)\cdots\left(\frac{-n+1}{q}+j-1\right),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\left(\frac{-n+q-1}{q}\right)\left(\frac{-n+q-1}{q}+1\right)\left(\frac{-n+q-1}{q}+2\right)\cdots\left(\frac{-n+q-1}{q}+j-1\right)=\left(\frac{-n}{q}\right)\left(\frac{-n+q}{q}\right)\left(\frac{-n+2q}{q}\right)\cdots\left(\frac{-n+qj-q}{q}\right),\\
&\left(\frac{-n+1}{q}\right)\left(\frac{-n+q+1}{q}\right)\left(\frac{-n+2q+1}{q}\right)\cdots\left(\frac{-n+qj-q+1}{q}\right),\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\
&\left(\frac{-n+q-1}{q}\right)\left(\frac{-n+2q-1}{q}\right)\left(\frac{-n+3q-1}{q}\right)\cdots\left(\frac{-n+qj-1}{q}\right)=(-1)^{qj}\frac{n!}{qj^{qj}(n-qj)!}.
\end{aligned} \tag{9}$$

Lemma 2. *If $k\in\mathbb{Z}^+$ and n is any nonnegative integer, thus*

$$(\alpha)_{kn}=k^{kn}\left(\frac{\alpha}{k}\right)_n\left(\frac{\alpha+1}{k}\right)_n\cdots\left(\frac{\alpha+k-1}{k}\right)_n, \tag{10}$$

Rainville [33] (p 22).

Lemma 3. *If $k\in\mathbb{Z}^+$ and n is any nonnegative integer, thus*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n}B(k,n)=\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}B(k,n+k), \tag{11}$$

Rainville [33] (p 57).

Lemma 4. *If $k\in\mathbb{Z}^+$ and n is any nonnegative integer, then*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}B(k,n)=\sum_{n=0}^{\infty}\sum_{k=0}^{n}B(k,n-k), \tag{12}$$

Rainville [33] (p 56).

3. The Extended Laguerre Polynomials $A_{q,n}^{(\alpha)}(x)$

We define the extended Laguerre polynomial set $\left\{A_{q,n}^{(\alpha)}(x)\right\}$ by

$$A_{q,n}^{(\alpha)}(x)=\frac{e^x(q+\alpha)_n}{n!}\,{}_qF_q\left(\begin{matrix}\dfrac{-n}{q},\dfrac{-n+1}{q},\cdots,\dfrac{-n+q-1}{q}\,;\\ \\ \dfrac{q+\alpha}{q},\dfrac{q+1+\alpha}{q},\cdots,\dfrac{2q+\alpha-1}{q}\end{matrix}\;;x^q\right), \tag{13}$$

where $\alpha\in\mathbb{R}$, $n,q\in\mathbb{Z}^+$.

Theorem 5. *If $\left\{A_{q,n}^{(\alpha)}(x)\right\}$ are the extended Laguerre polynomials, then*

$$A_{q,n}^{(\alpha)}(x)=e^x(q+\alpha)_n\sum_{j=0}^{[n/q]}\frac{(-1)^{qj}}{(n-qj)!(q+\alpha)_{qj}}\frac{(x)^{qj}}{(qj)!},\ \alpha\in\mathbb{R},n\in\mathbb{Z}^+. \tag{14}$$

Proof. Consider

$$A_{q,n}^{(\alpha)}(x) = \frac{e^x (q+\alpha)_n}{n!} {}_qF_q\left(\begin{matrix} \frac{-n}{q}, \frac{-n+1}{q}, \cdots, \frac{-n+q-1}{q}; \\ \\ \frac{q+\alpha}{q}, \frac{q+1+\alpha}{q}, \cdots, \frac{2q+\alpha-1}{q} \end{matrix} ; x^q \right)$$
$$= \frac{e^x (q+\alpha)_n}{n!} \times \sum_{j=0}^{[n/q]} \left\{ \frac{(-n/q)_j (-n+1/q)_j \cdots (-n+q-1/q)_j}{(q+\alpha/q)_j (q+1+\alpha/q)_j \cdots (2q+\alpha-1/q)_j} \right\} \frac{(x)^{qj}}{(qj)!}. \tag{15}$$

By using Lemma 1

$$A_{q,n}^{(\alpha)}(x) = \frac{e^x (q+\alpha)_n}{n!} \times \sum_{j=0}^{[n/q]} \left[\frac{(-1)^{qj} n!}{q^{qj}(n-qj)!(q+\alpha/q)_j (q+1+\alpha/q)_j \cdots (2q+\alpha-1/q)_j} \right] \frac{(x)^{qj}}{(qj)!}. \tag{16}$$

Then from Lemma 2, we have

$$A_{q,n}^{(\alpha)}(x) = e^x (q+\alpha)_n \sum_{j=0}^{[n/q]} \frac{(-1)^{qj}}{(n-qj)!(q+\alpha)_{qj}} \frac{(x)^{qj}}{(qj)!}. \tag{17}$$

□

4. Generating Functions

The following theorem formulates a generating function for the extended Laguerre polynomials $A_{q,n}^{(\alpha)}(x)$.

Theorem 6. *If $n, j \in \mathbb{Z}^+$, then*

$$\sum_{n=0}^{\infty} \sum_{j=0}^{[n/q]} \frac{(-1)^{qj} e^x t^n}{(n-qj)!(q+\alpha)_{qj}} \frac{(x)^{qj}}{(qj)!} = e^{x+t} {}_0F_q\left(--; \frac{q+\alpha}{q}, \frac{q+1+\alpha}{q}, \cdots, \frac{2q+\alpha-1}{q}; \left(\frac{-xt}{q}\right)^q\right). \tag{18}$$

Proof. By using Lemma 3, we acquire

$$\sum_{n=0}^{\infty} \sum_{j=0}^{[n/q]} \frac{(-1)^{qj} e^x t^n}{(n-qj)!(q+\alpha)_{qj}} \frac{(x)^{qj}}{(qj)!} = \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^{qj} e^x t^{n+qj}}{n!(q+\alpha)_{qj}} \frac{(x)^{qj}}{(qj)!}$$
$$= e^x \left[\sum_{n=0}^{\infty} \frac{t^n}{n!}\right] \left[\sum_{j=0}^{\infty} \frac{(-1)^{qk} t^{qj}}{(q+\alpha)_{qj}} \frac{(x)^{qj}}{(qj)!}\right]$$
$$= e^{x+t} \sum_{j=0}^{\infty} \frac{(-xt)^{qj}}{(q+\alpha)_{qj}(qj)!}. \tag{19}$$

By using Lemma 2, we acquire

$$\sum_{n=0}^{\infty} \sum_{j=0}^{[n/q]} \frac{(-1)^{qj} e^x t^n}{(n-qj)!(q+\alpha)_{qj}} \frac{(x)^{qj}}{(qj)!}$$
$$= e^{x+t} \sum_{j=0}^{\infty} \frac{(-xt)^{qj}}{q^{qj}(q+\alpha/q)_j (q+1+\alpha/q)_j \cdots (2q+\alpha-1/q)_j (qj)!}$$
$$= e^{x+t} {}_0F_q\left(--; \frac{q+\alpha}{q}, \frac{q+1+\alpha}{q}, \cdots, \frac{2q+\alpha-1}{q}; \left(\frac{-xt}{q}\right)^q\right). \tag{20}$$

□

Corollary 7. *If $\alpha \in \mathbb{R}$ and $n, q, j \in \mathbb{Z}^+$, then*

$$\sum_{n=0}^{\infty} \frac{A_{q,n}^{(\alpha)}(x) t^n}{(q+\alpha)_n} = e^{x+t} {}_0F_q\left(\begin{matrix} --; \\ \frac{q+\alpha}{q}, \frac{q+1+\alpha}{q}, \cdots, \frac{2q+\alpha-1}{q}; \end{matrix} \left(\frac{-xt}{q}\right)^q\right). \tag{21}$$

Proof. From Equation (14), we acquire

$$\sum_{n=0}^{\infty} \left[\frac{A_{q,n}^{(\alpha)}(x)}{(q+\alpha)_n}\right] t^n = \sum_{n=0}^{\infty} \left[\sum_{j=0}^{[n/q]} \left[\frac{(-1)^{qj}}{(n-qj)!(q+\alpha)_{qj}}\right] \frac{(x)^{qj}}{(qj)!}\right] t^n. \tag{22}$$

A use of Theorem (18), therefore, shows that the extended Laguerre polynomials have the generating function given by

$$\sum_{n=0}^{\infty} \frac{A_{q,n}^{(\alpha)}(x) t^n}{(q+\alpha)_n} = e^{x+t} {}_0F_q\left(\begin{matrix} --; \\ \frac{q+\alpha}{q}, \frac{q+1+\alpha}{q}, \cdots, \frac{2q+\alpha-1}{q}; \end{matrix} \left(\frac{-xt}{q}\right)^q\right). \tag{23}$$

□

Theorem 8. *If $c \in \mathbb{Z}^+$, then*

$$\sum_{n=0}^{\infty} \frac{(c)_n A_{q,n}^{(\alpha)}(x) t^n}{(q+\alpha)_n} = \frac{e^x}{(1-t)^c} {}_qF_q\left(\begin{matrix} \frac{c}{q}, \frac{c+1}{q}, \cdots, \frac{c+q-1}{q}; \\ \frac{q+\alpha}{q}, \frac{q+1+\alpha}{q}, \cdots, \frac{2q+\alpha-1}{q}; \end{matrix} \left(\frac{-xt}{1-t}\right)^q\right). \tag{24}$$

Proof. From Equation (22), we note that

$$\sum_{n=0}^{\infty} (c)_n \left[\frac{A_{q,n}^{(\alpha)}(x)}{(q+\alpha)_n}\right] t^n = \sum_{n=0}^{\infty} (c)_n e^x \left[\sum_{j=0}^{\left[\frac{n}{q}\right]} \left[\frac{(-1)^{qj}}{(n-qj)!(q+\alpha)_{qj}}\right] \frac{(x)^{qj}}{(qj)!}\right] t^n. \tag{25}$$

By using Lemma 3, we acquire

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(c)_n A_{q,n}^{(\alpha)}(x) t^n}{(q+\alpha)_n} &= \sum_{n=0}^{\infty}\sum_{j=0}^{\infty} \frac{(c)_{n+qj} e^x t^{n+qj}}{n!} \frac{(-1)^{qj}(x)^{qj}}{(q+\alpha)_{qj}(qj)!} \\ &= \sum_{j=0}^{\infty}\left[\sum_{n=0}^{\infty} \frac{(c+qj)_n t^n}{n!}\right]\left[\frac{(c)_{qj}}{(q+\alpha)_{qj}}\right] \frac{e^x(-xt)^{qj}}{(qj)!}. \end{aligned} \tag{26}$$

Since $(c)_{n+qj} = (c+qj)_n (c)_{qj}$ and $(1-t)^{-m} = \sum_{n=0}^{\infty} (m)_n t^n / n!$, it thus implies that

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{(c)_n A_{q,n}^{(\alpha)}(x) t^n}{(q+\alpha)_n} &= \sum_{j=0}^{\infty}\left[\frac{(c)_{qj}}{\left[(1-t)^{c+qj}\right](q+\alpha)_{qj}}\right] \frac{e^x(-xt)^{qj}}{(qj)!} \\ &= \frac{e^x}{(1-t)^c} \sum_{k=0}^{\infty}\left[\frac{(c)_{qj}}{(q+\alpha)_{qj}}\right] \frac{1}{(qj)!}\left(\frac{-xt}{1-t}\right)^{qj}. \end{aligned} \tag{27}$$

By using Lemma 2, we consequently obtain the required result

$$\sum_{n=0}^{\infty} \frac{(c)_n A_{q,n}^{(\alpha)}(x) t^n}{(q+\alpha)_n} = \frac{e^x}{(1-t)^c} {}_qF_q \cdot \left(\begin{matrix} \frac{c}{q}, \frac{c+1}{q}, \cdots, \frac{c+q-1}{q}; \\ \frac{q+\alpha}{q}, \frac{q+1+\alpha}{q}, \cdots, \frac{2q+\alpha-1}{q}; \end{matrix} \left(\frac{-xt}{1-t}\right)^q\right). \tag{28}$$

□

Corollary 9. *If $\alpha \in \mathbb{R}$ and $n, m, j \in \mathbb{Z}^+$, then*

$$\sum_{n=0}^{\infty} A_{q,n}^{(\alpha)}(x) t^n = \frac{1}{(1-t)^{q+\alpha}} \exp\left(\frac{x-2xt}{1-t}\right). \tag{29}$$

Proof. Put $c = q+\alpha$ in Equation (24), we obtain our desired result. □

5. Recurrence Relations

We describe the recurrence relations for the extended Laguerre polynomials $A_{q,n}^{(\alpha)}(x)$.

Theorem 10. *If $\alpha \in \mathbb{R}$ and $n, j \in \mathbb{Z}^+$, then*

$$\begin{aligned} xDA_{q,n}^{(\alpha)}(x) &= (n+x)A_{q,n}^{(\alpha)}(x) \\ &\quad - (q+\alpha+n-1)A_{q,n-1}^{(\alpha)}(x), D = \frac{d}{dx}. \end{aligned} \tag{30}$$

Proof. From Equation (18)

$$\sum_{n=0}^{\infty}\frac{A_{q,n}^{(\alpha)}(x)t^n}{(q+\alpha)_n}=e^{x+t}{}_0F_q\left(\begin{matrix} --; \\ \frac{q+\alpha}{q},\frac{q+1+\alpha}{q},\cdots,\frac{2q+\alpha-1}{q}; \end{matrix}\ \left(\frac{-xt}{q}\right)^q\right). \tag{31}$$

Let $\sigma_{q,n}(x)=A_{q,n}^{(\alpha)}(x)/(q+\alpha)_n$.

Suppose that

$${}_0F_q\left(\begin{matrix} --; \\ \frac{q+\alpha}{q},\frac{q+1+\alpha}{q},\cdots,\frac{2q+\alpha-1}{q}; \end{matrix}\ \left(\frac{-xt}{q}\right)^q\right)=\psi\left(\frac{x^qt^q}{q}\right). \tag{32}$$

$$\text{Then } F=e^{x+t}\psi\left(\frac{x^qt^q}{q}\right)=\sum_{n=0}^{\infty}\sigma_{q,n}(x)t^n, \tag{33}$$

provide that the series is uniformly convergent. By taking partial derivatives,

$$\frac{\partial F}{\partial x}=e^{x+t}\psi+x^{q-1}t^qe^{x+t}\psi', \tag{34}$$

$$\frac{\partial F}{\partial t}=e^{x+t}\psi+x^qt^{q-1}e^{x+t}\psi', \tag{35}$$

$$x\frac{\partial F}{\partial x}-t\frac{\partial F}{\partial t}=xF-tF. \tag{36}$$

Now, since $F=\sum_{n=0}^{\infty}\sigma_{q,n}(x)t^n$, therefore,

$$\frac{\partial F}{\partial x}=\sum_{n=0}^{\infty}\sigma'_{q,n}(x)t^n \quad \text{and} \quad t\frac{\partial F}{\partial t}=\sum_{n=0}^{\infty}n\sigma_{q,n}(x)t^n. \tag{37}$$

Equation (36) then yields

$$\begin{aligned} x\sum_{n=0}^{\infty}\sigma'_{q,n}(x)t^n-\sum_{n=0}^{\infty}n\sigma_{q,n}(x)t^n &= x\sum_{n=0}^{\infty}\sigma_{q,n}(x)t^n-\sum_{n=0}^{\infty}\sigma_{q,n}(x)t^{n+1} \\ &= x\sum_{n=0}^{\infty}\sigma_{q,n}(x)t^n-\sum_{n=1}^{\infty}\sigma_{q,n-1}(x)t^n. \end{aligned} \tag{38}$$

We get $\sigma'_{2,0}(x)=0$, and for $n>1$,

$$x\sigma'_{q,n}(x)-n\sigma_{q,n}(x)=x\sigma_{q,n}(x)-\sigma_{q,n-1}(x). \tag{39}$$

This implies that

$$xDA_{q,n}^{(\alpha)}(x)=(n+x)A_{q,n}^{(\alpha)}(x)-(q+\alpha+n-1)A_{q,n-1}^{(\alpha)}(x). \tag{40}$$

□

Theorem 11. *If $\alpha\in\mathbb{R}$ and $n\geq 2$ then*

$$DA_{q,n}^{(\alpha)}(x)=DA_{q,n-1}^{(\alpha)}(x)+A_{q,n}^{(\alpha)}(x)-2A_{q,n-1}^{(\alpha)}(x). \tag{41}$$

Proof. From Equation (29), we get the following

$$(1-t)^{-q-\alpha}\exp\left[x\left(\frac{1-2t}{1-t}\right)\right]=\sum_{n=0}^{\infty}A_{q,n}^{(\alpha)}(x)t^n. \tag{42}$$

$$\text{Let } F=A(t)\exp\left[x\left(\frac{1-2t}{1-t}\right)\right]=\sum_{n=0}^{\infty}y_{q,n}(x)t^n, \tag{43}$$

$$\frac{\partial F}{\partial x}=\left(\frac{1-2t}{1-t}\right)A(t)\exp\left[x\left(\frac{1-2t}{1-t}\right)\right], \tag{44}$$

$$(1-t)\frac{\partial F}{\partial x}=(1-2t)A(t)\exp\left[x\left(\frac{1-2t}{1-t}\right)\right]. \tag{45}$$

By using Equation (42), we obtain

$$(1-t)\frac{\partial F}{\partial x}=(1-2t)F. \tag{46}$$

$$\text{Since } F=\sum_{n=0}^{\infty}y_{q,n}(x)t^n, \text{ therefore we have } \frac{\partial F}{\partial x}=\sum_{n=0}^{\infty}y'_{q,n}(x)t^n. \tag{47}$$

Equation (46) can be expressed as

$$\sum_{n=0}^{\infty}y'_{q,n}(x)t^n-\sum_{n=0}^{\infty}y'_{q,n}(x)t^{n+1}=\sum_{n=0}^{\infty}y_{q,n}(x)t^n-2\sum_{n=0}^{\infty}y_{q,n}(x)t^{n+1}, \tag{48}$$

$$\sum_{n=0}^{\infty}y'_{q,n}(x)t^n-\sum_{n=1}^{\infty}y'_{q,n-1}(x)t^n=\sum_{n=0}^{\infty}y_{q,n}(x)t^n-2\sum_{n=1}^{\infty}y_{q,n-1}(x)t^n. \tag{49}$$

We reach $y'_{q,0}(x)=0$, $y'_{q,1}(x)=0$ and for $n>2$,

$$DA_{q,n}^{(\alpha)}(x)=DA_{q,n-1}^{(\alpha)}(x)+A_{q,n}^{(\alpha)}(x)-2A_{q,n-1}^{(\alpha)}(x). \tag{50}$$

□

Theorem 12. *If $\alpha \in \mathbb{R}$ and $n \geq q$, then*

$$DA_{q,n}^{(\alpha)}(x) = A_{q,n}^{(\alpha)}(x) - \sum_{j=0}^{n-1} A_{q,j}^{(\alpha)}(x). \tag{51}$$

Proof. Equation (46) can be written as

$$\frac{\partial F}{\partial x} = \left[1 - \frac{t}{1-t}\right] F. \tag{52}$$

By using Equation (42), we obtain

$$\frac{\partial F}{\partial x} = \left[1 - \frac{t}{1-t}\right] \sum_{n=0}^{\infty} y_{q,n}(x) t^n. \tag{53}$$

By using Equation (47), we obtain.

$$\begin{aligned} \sum_{n=0}^{\infty} y'_{q,n}(x) t^n &= \sum_{n=0}^{\infty} y_{q,n}(x) t^n - \left[\sum_{n=0}^{\infty} t^{n+1}\right] \left[\sum_{n=0}^{\infty} y_{q,n}(x) t^n\right] \\ &= \sum_{n=0}^{\infty} y_{q,n}(x) t^n - \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} y_{q,j}(x) t^j t^{n+1}. \end{aligned} \tag{54}$$

Since $\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} B(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} B(k, n-k)$, (Rainville [33], (p 56)).

$$\begin{aligned} \sum_{n=0}^{\infty} y'_{q,n}(x) t^n &= \sum_{n=0}^{\infty} y_{q,n}(x) t^n - \sum_{n=0}^{\infty} \sum_{j=0}^{n} y_{q,j}(x) t^{n+1} \\ &= \sum_{n=0}^{\infty} y_{q,n}(x) t^n - \sum_{n=1}^{\infty} \sum_{j=0}^{n-1} y_{q,j}(x) t^n. \end{aligned} \tag{55}$$

It follows that $y'_{q,0}(x) = 0$, $y'_{q,1}(x) = 0$ and for $n > q$, $y'_{q,n}(x) = y_{q,n}(x) - \sum_{j=0}^{n-1} y_{q,j}(x)$, and $DA_{q,n}^{(\alpha)}(x) = A_{q,n}^{(\alpha)}(x) - \sum_{j=0}^{n-1} A_{q,j}^{(\alpha)}(x)$. □

Theorem 13. *If $\alpha \in \mathbb{R}$ and $n \geq q + 1$, then*

$$nA_{q,n}^{(\alpha)}(x) = (3x - q - \alpha) A_{q,n-1}^{(\alpha)}(x) - (q + \alpha + n - 2) A_{q,n-2}^{(\alpha)}(x). \tag{56}$$

□

Proof. We can have the following equation after eliminating the derivatives from Equations (30) and (41).

$$\begin{aligned} 0 &= nA_{q,n}^{(\alpha)}(x) - xDA_{q,n-1}^{(\alpha)}(x) \\ &\quad + (2x - q - \alpha - n + 1) A_{q,n-1}^{(\alpha)}(x) nA_{q,n}^{(\alpha)}(x) \\ &= xDA_{q,n-1}^{(\alpha)}(x) - (2x - q - \alpha - n + 1) A_{q,n-1}^{(\alpha)}(x). \end{aligned} \tag{57}$$

Now, by using Equation (30), we finally have

$$\begin{aligned} nA_{q,n}^{(\alpha)}(x) &= (n - 1 + x) A_{q,n-1}^{(\alpha)}(x) \\ &\quad - (q + \alpha + n - 2) A_{q,n-2}^{(\alpha)}(x) \\ &\quad + (2x - q - \alpha - n + 1) A_{q,n-1}^{(\alpha)}(x), \end{aligned} \tag{58}$$

$$nA_{q,n}^{(\alpha)}(x) = (3x - q - \alpha) A_{q,n-1}^{(\alpha)}(x) - (q + \alpha + n - 2) A_{q,n-2}^{(\alpha)}(x). \tag{59}$$

□

Theorem 14. *If $\alpha \in \mathbb{R}$ and $n, q, j \in \mathbb{Z}^+$, then*

$$A_{q,n-1}^{(1+\alpha)}(x) + A_{q,n}^{(\alpha)}(x) = A_{q,n}^{(1+\alpha)}(x).. \tag{60}$$

Proof. From Equation (14), we obtain

$$A_{q,n-1}^{(1+\alpha)}(x) = e^x (q + 1 + \alpha)_{n-1} \sum_{j=0}^{\left[\frac{n-1}{q}\right]} \frac{(-1)^{qj}}{(n - 1 - qj)!(q + 1 + \alpha)_{qj}} \frac{x^{qj}}{(qj)!}, \tag{61}$$

so that $A_{q,n}^{(\alpha)}(x) = e^x (q + \alpha)_n \sum_{j=0}^{[n/q]} ((-1)^{qj}/(n - qj)!(q + \alpha)_{qj}) (x^{qj}/(qj)!)$.

By adding the above equations, we get

Theorem 15. *If $\alpha \in \mathbb{R}$ and $n \geq q$, then*

$$\begin{aligned} &xD^2 A_{q,n}^{(\alpha)}(x) + (q + \alpha - 3x) DA_{q,n}^{(\alpha)}(x) \\ &\quad + (2x + n - q - \alpha) A_{q,n}^{(\alpha)}(x) = 0. \end{aligned} \tag{63}$$

6. Differential Equation

Since the Extended Laguerre polynomial is a constant multiple of hypergeometric functions ${}_qF_q$, we may obtain the differential equation.

By using Equation (30), we have

$$
\begin{aligned}
A_{q,n-1}^{(1+a)}(x)+A_{q,n}^{(a)}(x)&=e^{x}(q+1+\alpha)_{n-1}\sum_{j=0}^{\left[\frac{n-1}{q}\right]}\frac{(-1)^{qj}}{(n-1-qj)!(q+1+\alpha)_{qj}}\frac{x^{qj}}{(qj)!}+e^{x}(q+\alpha)_{n}\sum_{j=0}^{\left[\frac{n}{q}\right]}\frac{(-1)^{qj}}{(n-qj)!(q+\alpha)_{qj}}\frac{x^{qj}}{(qj)!}\\
&=e^{x}\left[\sum_{j=0}^{\left[\frac{n-1}{q}\right]}\frac{(q+\alpha+n-1)!(-1)^{qj}}{(n-1-qj)!(q+\alpha+qj)!}\frac{x^{qj}}{(qj)!}+\sum_{k=0}^{\left[\frac{n}{q}\right]}\frac{(q+\alpha+n-1)!(-1)^{qj}}{(n-qj)!(q+\alpha+qj-1)!}\frac{x^{qj}}{(qj)!}\right]\\
&=e^{x}\left[\sum_{k=0}^{\left[\frac{n-1}{q}\right]}\frac{(q+\alpha+n-1)!(-1)^{qj}}{(n-1-qj)!(q+\alpha+qj)!}\frac{x^{qj}}{(qj)!}+\sum_{j=0}^{\left[\frac{n-1}{q}\right]}\frac{(q+\alpha+n-1)!(-1)^{qj}}{(n-qj)!(q+\alpha+qj-1)!}\frac{x^{qj}}{(qj)!}+\frac{x^{qn}}{(qn)!}\right]\\
&=e^{x}\left[\sum_{k=0}^{\left[\frac{n-1}{q}\right]}\frac{(q+\alpha+n-1)!x^{qj}(-1)^{qj}}{(qj)!}\left\{\frac{1}{(n-1-qj)!(q+\alpha+qj)!}+\frac{1}{(n-qj)!(q+\alpha+qj-1)!}\right\}+\frac{x^{qn}}{(qn)!}\right]\\
&=e^{x}\left[\sum_{k=0}^{\left[\frac{n-1}{q}\right]}\frac{(q+\alpha+n-1)!(-1)^{qj}}{(n-qj)!(q+\alpha+qj)!}\{q+\alpha+n\}\frac{x^{qj}}{(qj)!}+\frac{x^{qn}}{(qn)!}\right]\\
&=e^{x}\left[\sum_{j=0}^{\left[\frac{n-1}{q}\right]}\frac{(q+\alpha+n)!(-1)^{qj}}{(n-qj)!(q+\alpha+qj)!}\frac{x^{qj}}{(qj)!}+\frac{x^{qn}}{(qn)!}\right]=e^{x}(q+1+\alpha)_{n}\sum_{j=0}^{\left[\frac{n}{q}\right]}\frac{(-1)^{qj}}{(n-qj)!(q+1+\alpha)_{qj}}\frac{x^{qj}}{(qj)!}=A_{q,n}^{(1+\alpha)}(x).
\end{aligned}
\tag{62}
$$

Proof. By taking partial derivatives of Equation (30), we have

$$
\begin{aligned}
xD^{2}A_{q2,n}^{(\alpha)}(x)+DA_{q,n}^{(\alpha)}(x)&=(n+x)DA_{q,n}^{(\alpha)}(x)+A_{q,n}^{(\alpha)}(x)\\
&\quad-(q+\alpha+n-1)DA_{q,n-1}^{(\alpha)}(x).
\end{aligned}
\tag{64}
$$

By using Equation (41), we have

$$
\begin{aligned}
xD^{2}A_{q,n}^{(\alpha)}(x)+DA_{q,n}^{(\alpha)}(x)&=(n+x)DA_{q,n}^{(\alpha)}(x)+A_{q,n}^{(\alpha)}(x)\\
&\quad-(q+\alpha+n-1)\left[DA_{q,n}^{(\alpha)}(x)-A_{q,n}^{(\alpha)}(x)+2A_{q,n-1}^{(\alpha)}(x)\right],
\end{aligned}
\tag{65}
$$

or

$$
\begin{aligned}
xD^{2}A_{q,n}^{(\alpha)}(x)+(q+\alpha-x)DA_{q,n}^{(\alpha)}(x)&=(q+\alpha+n)A_{q,n}^{(\alpha)}(x)\\
&\quad-2(q+\alpha+n-1)A_{q,n-1}^{(\alpha)}(x).
\end{aligned}
\tag{66}
$$

By using Equation (30), we have

$$
\begin{aligned}
xD^{2}A_{q,n}^{(\alpha)}(x)+(q+\alpha-x)DA_{q,n}^{(\alpha)}(x)&=(q+\alpha+n)A_{q,n}^{(\alpha)}(x)\\
&\quad+2xDA_{q,n}^{(\alpha)}(x)-2(n+x)DA_{q,n}^{(\alpha)}(x),
\end{aligned}
\tag{67}
$$

or

$$
xD^{2}A_{q,n}^{(\alpha)}(x)+(q+\alpha-3x)DA_{q,n}^{(\alpha)}(x)+(2x+n-q-\alpha)A_{q,n}^{(\alpha)}(x)=0.
\tag{68}
$$

□

7. Rodrigues Formula

The Rodrigues formula for the Laguerre polynomials is presented as

$$
L_{n}^{(\alpha)}(x)=\frac{x^{-\alpha}e^{x}}{n!}D^{n}\left(x^{\alpha+n}e^{-x}\right),
\tag{69}
$$

but we intend to extend this Rodrigues formula.

Theorem 16. *If $\alpha \in \mathbb{R}$ and $n, j \in \mathbb{Z}^+$, then*

$$A_{q,n}^{(\alpha)}(x) = \frac{x^{-(q-1)-\alpha} e^{2x}}{n!} D^n \left(x^{(q-1)+\alpha+n} e^{-x} \right). \tag{70}$$

Proof. Consider the extended Laguerre polynomials involving ${}_qF_q, q > 2$

$$A_{q,n}^{(\alpha)}(x) = \frac{e^x (q+\alpha)_n}{n!} {}_qF_q \left(\begin{matrix} \frac{-n}{q}, \frac{-n+1}{q}, \cdots, \frac{-n+q-1}{q}; \\ \frac{q+\alpha}{q}, \frac{q+1+\alpha}{q}, \cdots, \frac{2q+\alpha-1}{q}; \end{matrix} x^q \right). \tag{71}$$

By Theorem (14), we have

$$\begin{aligned} A_{q,n}^{(\alpha)}(x) &= \frac{e^x}{n!} \sum_{j=0}^{\left[\frac{n}{q}\right]} \left[\frac{n!}{(n-qj)!(qj)!} \right] \frac{(q+\alpha)_n x^{qj}}{(q+\alpha)_{qj}} \\ &= \frac{e^x x^{-(q-1)-\alpha}}{n!} \sum_{j=0}^{\left[\frac{n}{q}\right]} \left[\frac{(-1)^{qj} n!}{(n-qj)!(qj)!} \right] \frac{(q+m)_n x^{qj+\alpha+(q-1)}}{(q+m)_{qj}}. \end{aligned} \tag{72}$$

Since $D^{n-qj}(x^{n+\alpha+(q-1)}) = (q+\alpha)_n x^{qj+\alpha+(q-1)} / (q+\alpha)_{qj}$, therefore, we write it as

$$\begin{aligned} A_{q,n}^{(\alpha)}(x) &= \frac{x^{-(q-1)-\alpha} e^{2x}}{n!} \sum_{j=0}^{\left[\frac{n}{q}\right]} \left[\frac{n!}{(n-qj)!(qj)!} \right] \left[(-1)^{qj} e^{-x} \right] \\ &\cdot \left[D^{n-qj} \left(x^{n+\alpha+(q-1)} \right) \right] = \frac{x^{-(q-1)-\alpha} e^{2x}}{n!} \sum_{j=0}^{\left[\frac{n}{q}\right]^n} C_{qj} D^{n-qj} \\ &\cdot \left(x^{n+\alpha+(q-1)} \right) D^{qj}(e^{-x}). \end{aligned} \tag{73}$$

Lastly, we use the Leibnitz formula for the nth derivative to obtain the following

$$A_{q,n}^{(\alpha)}(x) = \frac{x^{-(q-1)-\alpha} e^{2x}}{n!} D^n \left(x^{(q-1)+\alpha+n} e^{-x} \right). \tag{74}$$

□

8. Special Properties

In this section, we determine the special features of the extended Laguerre polynomials $A_{q,n}^{(\alpha)}(x)$.

Theorem 17. *If $\alpha, \beta \in \mathbb{R}$ and $n, j, q \in \mathbb{Z}^+$, then*

$$A_{q,n}^{(\alpha)}(x) = \sum_{j=0}^{\left[\frac{n}{q}\right]} \frac{(\alpha-\beta)_{qj} A_{q,n-qj}^{(\beta)}(x)}{(qj)!}. \tag{75}$$

Proof. From Equation (29)

$$\sum_{n=0}^{\infty} A_{q,n}^{(\alpha)}(x) t^n = \frac{1}{(1-t)^{q+\alpha}} \exp\left(x \left(\frac{1-2t}{1-t} \right) \right). \tag{76}$$

Also, consider

$$\begin{aligned} \frac{1}{(1-t)^{q+\alpha}} \exp\left(x \left(\frac{1-2t}{1-t} \right) \right) &= (1-t)^{-(\alpha-\beta)} (1-t)^{-q-\beta} \exp \\ &\cdot \left(x \left(\frac{1-2t}{1-t} \right) \right), \end{aligned} \tag{77}$$

$$\begin{aligned} \sum_{n=0}^{\infty} A_{q,n}^{(\alpha)}(x) t^n &= (1-t)^{-(\alpha-\beta)} \sum_{n=0}^{\infty} A_{q,n}^{(\beta)}(x) t^n \\ &= \sum_{n=0}^{\infty} \frac{(\alpha-\beta)_{qn} t^{qn}}{(qn)!} \sum_{n=0}^{\infty} A_{q,n}^{(\beta)}(x) t^n \\ &= \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \frac{(\alpha-\beta)_{qj} t^{qj} A_{q,n}^{(\beta)}(x) t^n}{(qj)!}. \end{aligned} \tag{78}$$

By utilizing Lemma 4, we acquire

$$\begin{aligned} \sum_{n=0}^{\infty} A_{q,n}^{(\alpha)}(x) t^n &= \sum_{n=0}^{\infty} \sum_{j=0}^{\left[\frac{n}{q}\right]} \frac{(\alpha-\beta)_{qj} t^{qj} A_{q,n-qj}^{(\beta)}(x) t^{n-qj}}{(qj)!} \\ &= \sum_{n=0}^{\infty} \sum_{j=0}^{\left[\frac{n}{q}\right]} \frac{(\alpha-\beta)_{qj} A_{q,n-qj}^{(\beta)}(x) t^n}{(qj)!}. \end{aligned} \tag{79}$$

On comparing the coefficients of t^n, we acquire

$$A_{q,n}^{(\alpha)}(x) = \sum_{j=0}^{\left[\frac{n}{q}\right]} \frac{(\alpha-\beta)_{qj} A_{q,n-qj}^{(\beta)}(x)}{(qj)!}. \tag{80}$$

□

Theorem 18. *If $\alpha \in \mathbb{R}$ and $n, j \in \mathbb{Z}^+$, then*

$$A_{q,n}^{(\alpha+\beta+q)}(x+y) = \sum_{j=0}^{\left[\frac{n}{q}\right]} A_{q,n-qj}^{(\beta)}(y) A_{q,qj}^{(\alpha)}(x). \tag{81}$$

Proof. Consider

$$(1-t)^{-q-\alpha}\exp\left(x\left(\frac{1-2t}{1-t}\right)\right)(1-t)^{-q-\beta}\exp\left(y\left(\frac{1-2t}{1-t}\right)\right) = (1-t)^{-q-(\alpha+\beta+q)}\exp\left\{(x+y)\left(\frac{1-2t}{1-t}\right)\right\}. \tag{82}$$

By using Equation (75), we acquire

$$\sum_{n=0}^{\infty} A_{q,n}^{(\alpha)}(x)t^n \sum_{n=0}^{\infty} A_{q,n}^{(\beta)}(y)t^n = \sum_{n=0}^{\infty} A_{q,n}^{(\alpha+\beta+q)}(x+y)t^n, \tag{83}$$

$$\sum_{n=0}^{\infty} A_{q,n}^{(\alpha+\beta+q)}(x+y)t^n = \sum_{n=0}^{\infty}\sum_{j=0}^{\infty} A_{q,n}^{(\beta)}(y)t^n A_{q,qj}^{(\alpha)}(x)t^{qj}. \tag{84}$$

By using Lemma 4, we acquire

$$\sum_{n=0}^{\infty} A_{q,n}^{(\alpha+\beta+q)}(x+y)t^n = \sum_{n=0}^{\infty}\sum_{j=0}^{\left[\frac{n}{q}\right]} A_{q,n-qj}^{(\beta)}(y) A_{q,qj}^{(\alpha)}(x)t^n. \tag{85}$$

On comparing the coefficients of t^n, we acquire

$$A_{q,n}^{(\alpha+\beta+q)}(x+y) = \sum_{j=0}^{\left[\frac{n}{q}\right]} A_{q,n-qj}^{(\beta)}(y) A_{q,qj}^{(\alpha)}(x). \tag{86}$$

□

Theorem 19. *If $\alpha\in\mathbb{R}$ and $n, j\in\mathbb{Z}^+$, then*

$$A_{q,n}^{(\alpha)}(xy) = \sum_{j=0}^{\left[\frac{n}{q}\right]} \frac{(q+\alpha)_n A_{q,qj}^{(\alpha)}(x^q)y^{qj}}{(q+\alpha)_{qj}}\frac{(1-y)^{n-qj}}{(n-qj)!}. \tag{87}$$

Proof. Consider

$$e^{x+t}\,{}_0F_q\left(\begin{matrix} --; \\ \frac{q+\alpha}{q}, \frac{q+1+\alpha}{q}, \dots, \frac{2q+\alpha-1}{q}; \end{matrix}\ \left(\frac{-xyt}{q}\right)^q\right) = e^{(1-y)t}e^{x+yt}\,{}_0F_q\left(\begin{matrix} --; \\ \frac{q+\alpha}{q}, \frac{q+1+\alpha}{q}, \dots, \frac{2q+\alpha-1}{q}; \end{matrix}\ \left(\frac{-xyt}{q}\right)^q\right). \tag{88}$$

By using Equation (21), we get

$$\sum_{n=0}^{\infty}\frac{A_{q,n}^{(\alpha)}(xy)t^n}{(q+\alpha)_n} = \sum_{n=0}^{\infty}\frac{(1-y)^n t^n}{n!}\sum_{n=0}^{\infty}\frac{A_{q,n}^{(\alpha)}(x)y^n t^n}{(q+\alpha)_n} = \sum_{n=0}^{\infty}\sum_{j=0}^{\infty}\frac{A_{q,qj}^{(\alpha)}(x)y^{qj}t^{qj}}{(q+\alpha)_{qj}}\frac{(1-y)^n t^n}{n!}. \tag{89}$$

By using Lemma 4, we acquire

$$\sum_{n=0}^{\infty}\frac{A_{q,n}^{(\alpha)}(xy)t^n}{(q+\alpha)_n} = \sum_{n=0}^{\infty}\sum_{j=0}^{\left[\frac{n}{q}\right]}\frac{A_{q,qj}^{(\alpha)}(x)y^{qj}t^{qj}}{(q+\alpha)_{qj}}\frac{(1-y)^{n-qj}t^{n-qj}}{(n-qj)!} = \sum_{n=0}^{\infty}\sum_{j=0}^{\left[\frac{n}{q}\right]}\frac{A_{q,qj}^{(\alpha)}(x)y^{qj}}{(q+\alpha)_{qj}}\frac{(1-y)^{n-qj}t^n}{(n-qj)!}. \tag{90}$$

On comparing the coefficients of t^n, we get

$$A_{q,n}^{(\alpha)}(xy) = \sum_{j=0}^{\left[\frac{n}{q}\right]}\frac{(q+\alpha)_n A_{q,qj}^{(\alpha)}(x^q)y^{qj}}{(q+\alpha)_{qj}}\frac{(1-y)^{n-qj}}{(n-qj)!}. \tag{91}$$

□

Theorem 20. *If $\alpha\in\mathbb{R}$ and $n, j, q\in\mathbb{Z}^+$, then*

$$\sum_{n=0}^{\infty}\frac{(n+qj)!A_{q,n+qj}^{(\alpha)}(x)t^n}{(qj)!n!} = (1-t)^{-q-\alpha-qj}\exp\left(\frac{-xt}{1-t}\right)A_{q,j}^{(\alpha)}\left(\frac{x}{1-t}\right). \tag{92}$$

Proof. Consider the series

$$\begin{aligned}\sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\frac{(n+qj)!A_{q,n+qj}^{(\alpha)}(x)t^n y^{qj}}{(qj)!n!} &= \sum_{n=0}^{\infty}\sum_{j=0}^{\left[\frac{n}{q}\right]}\frac{n!A_{q,n}^{(\alpha)}(x)t^{n-qj}y^{qj}}{(qj)!(n-qj)!} \\ &= \sum_{n=0}^{\infty}A_{q,n}^{(\alpha)}(x)\sum_{j=0}^{\left[\frac{n}{q}\right]}{}^{n}C_{qj}t^{n-qj}y^{qj} \\ &= \sum_{n=0}^{\infty}A_{q,n}^{(\alpha)}(x)(t+y)^n \\ &= (1-t-y)^{-q-\alpha}\exp\left(\frac{x(1-2y-2t)}{1-y-t}\right).\end{aligned} \tag{93}$$

Since $(1-t-y)^{-q-\alpha} = (1-t)^{-q-\alpha}(1-y/1-t)^{-q-\alpha}$

$$\begin{aligned}\exp\left(\frac{x(1-2y-2t)}{1-t}\right) &= \exp(x)\exp\left(\frac{-x(y+t)}{(1-t-y)}\right)\\ &= \exp(x)\exp\left(\frac{-xt}{1-t}\right)\exp\\ &\cdot\left(\frac{(-x/1-t)(y/1-t)}{(1-y/1-t)}\right).\end{aligned} \tag{94}$$

Therefore, Equation (93) becomes

$$\begin{aligned}\sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\frac{(n+qj)!A_{q,n+qj}^{(\alpha)}(x^q)t^n y^{qj}}{(qj)!n!} &= (1-t)^{-q-\alpha}\\ &\cdot\left(1-\frac{y}{1-t}\right)^{-q-\alpha}\exp(x)\exp\left(\frac{-xt}{1-t}\right)\exp\\ &\cdot\left(\frac{(-x/1-t)(y/1-t)}{(1-y/1-t)}\right).\end{aligned} \tag{95}$$

By using Equation (29), we get

$$\begin{aligned}\sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\frac{(n+qj)!A_{q,n+qj}^{(\alpha)}(x^q)t^n y^{qj}}{(qj)!n!} &= (1-t)^{-q-\alpha}\exp\\ &\cdot\left(\frac{-xt}{1-t}\right)\sum_{j=0}^{\infty}A_{q,j}^{(\alpha)}\left(\frac{x}{1-t}\right)\left(\frac{y}{1-t}\right)^{qj}.\end{aligned} \tag{96}$$

On comparing the coefficients of y^{qj}, we get

$$\begin{aligned}\sum_{n=0}^{\infty}\frac{(n+qj)!A_{q,n+qj}^{(\alpha)}(x^q)t^n}{(qj)!n!} &= (1-t)^{-q-\alpha-qj}\exp\\ &\cdot\left(\frac{-xt}{1-t}\right)A_{q,j}^{(\alpha)}\left(\frac{x}{1-t}\right).\end{aligned} \tag{97}$$

□

9. Other Generating Functions

In this section, we study some other generating functions.

Theorem 21. *If $\alpha \in \mathbb{R}$ and $n, j, q \in \mathbb{Z}^+$, then*

$$\begin{aligned}\sum_{j=0}^{\infty}\frac{n!A_{q,n}^{(\alpha)}(x)A_{q,n}^{(\alpha)}(y)t^n}{(q+\alpha)_n} &= (1-t)^{-q-\alpha}\exp\left(\frac{-xt}{1-t}\right)\exp\left(\frac{x-yt}{1-t}\right){}_0F_q\\ &\cdot\left(--;\frac{q+\alpha}{q},\frac{q+1+\alpha}{q},\cdots,\frac{2q+\alpha-1}{q};\right.\\ &\cdot\left.\left(\frac{xyt}{q(1-t)}\right)^q\right).\end{aligned} \tag{98}$$

Proof. Consider the series

$$\sum_{n=0}^{\infty}\frac{n!A_{q,n}^{(\alpha)}(x)A_{q,n}^{(\alpha)}(y)t^n}{(q+\alpha)_n} = \sum_{n=0}^{\infty}\sum_{j=0}^{\left[\frac{n}{q}\right]}\frac{n!y^{qj}A_{q,n}^{(\alpha)}(x)(-1)^{qj}t^n}{(qj)!(n-qj)!(q+\alpha)_{qj}}. \tag{99}$$

By using Lemma 3, we get

$$\sum_{n=0}^{\infty}\frac{n!A_{q,n}^{(\alpha)}(x)A_{q,n}^{(\alpha)}(y)t^n}{(q+\alpha)_n} = \sum_{n=0}^{\infty}\sum_{j=0}^{\infty}\frac{(n+qj)!y^{qj}A_{q,n+qj}^{(\alpha)}(x)(-1)^{qj}t^{n+qj}}{(qj)!n!(q+\alpha)_{qj}}, \tag{100}$$

$$= \sum_{j=0}^{\infty}\sum_{n=0}^{\infty}\frac{(n+qj)!A_{q,n+qj}^{(\alpha)}(x)t^n}{(qj)!n!}\frac{(-yt)^{qj}}{(q+\alpha)_{qj}}. \tag{101}$$

By using Theorem (92), we get

$$\begin{aligned}\sum_{j=0}^{\infty}\frac{n!A_{q,n}^{(\alpha)}(x)A_{q,n}^{(\alpha)}(y)t^n}{(q+\alpha)_n} &= \sum_{j=0}^{\infty}(1-t)^{-q-\alpha-qj}\exp\left(\frac{-xt}{1-t}\right)A_{q,j}^{(\alpha)}\\ &\cdot\left(\frac{x}{1-t}\right)\frac{(-yt)^{qj}}{(q+\alpha)_{qj}} = (1-t)^{-q-\alpha}\exp\\ &\cdot\left(\frac{-xt}{1-t}\right)\sum_{j=0}^{\infty}(1-t)^{-qj}A_{q,j}^{(\alpha)}\\ &\cdot\left(\frac{x}{1-t}\right)\frac{(-yt)^{qj}}{(q+\alpha)_{qj}} = (1-t)^{-q-\alpha}\exp\\ &\cdot\left(\frac{-xt}{1-t}\right)\times\sum_{j=0}^{\infty}A_{q,j}^{(\alpha)}\\ &\cdot\left(\frac{x}{1-t}\right)\frac{(-yt/1-t)^{qj}}{q^{qj}(q+\alpha/q)_j(q+1+\alpha/q)_j\cdots(2q+\alpha-1/q)_j}\\ &= (1-t)^{-q-\alpha}\exp\left(\frac{-xt}{1-t}\right)\sum_{j=0}^{\infty}A_{q,j}^{(\alpha)}\\ &\cdot\left(\frac{x}{1-t}\right)\frac{(-yt/q(1-t))^{qj}}{(q+\alpha/q)_j(q+1+\alpha/q)_j\cdots(2q+\alpha-1/q)_j}.\end{aligned} \tag{102}$$

By using Equation (21), we get

$$\sum_{j=0}^{\infty}\frac{n!A_{q,n}^{(\alpha)}(x)A_{q,n}^{(\alpha)}(y)t^n}{(q+\alpha)_n}=(1-t)^{-q-\alpha}\exp\left(\frac{-xt}{1-t}\right)\exp\left(\frac{x-yt}{1-t}\right){}_0F_q\cdot\left(--;\frac{q+\alpha}{q},\frac{q+1+\alpha}{q},\cdots,\frac{2q+\alpha-1}{q};\cdot\left(\frac{xyt}{q(1-t)}\right)^q\right). \tag{103}$$

□

Theorem 22. *If $|t|<1$, $\alpha\in\mathbb{R}$ and $c,n\in\mathbb{Z}^+$, then*

$$(1-t)^{-1-\alpha}\exp\left(\frac{x}{1-t}\right)\left(1-\frac{yt}{1-t}\right)^{-c}{}_qF_q\cdot\left(\begin{matrix}\frac{c}{q},\frac{c+1}{q},\cdots,\frac{c+q-1}{q};\\ \frac{q+\alpha}{q},\frac{q+1+\alpha}{q},\cdots,\frac{2q+\alpha-1}{q};\end{matrix}\left(\frac{(-x/(1-t))(yt/1-t)}{1-yt/1-t}\right)^q\right)=\sum_{n=0}^{\infty}{}_{2q}F_q\left(\begin{matrix}\frac{-n}{q},\frac{-n+1}{q},\cdots,\frac{-n+q-1}{q},\frac{c}{q},\frac{c+1}{q},\cdots,\frac{c+q-1}{q};\\ \frac{q+\alpha}{q},\frac{q+1+\alpha}{q},\cdots,\frac{2q+\alpha-1}{q};(qy)^q\end{matrix}\right)A_{q,n}^{(\alpha)}(x)t^n. \tag{104}$$

Proof. Consider the series

$$\sum_{j=0}^{\infty}\frac{n!A_{q,n}^{(\alpha)}(x)A_{q,n}^{(\alpha)}(y)t^n}{(q+\alpha)_n}=(1-t)^{-q-\alpha}\exp\left(\frac{-(x+y)t}{1-t}\right)\times{}_0F_q\cdot\left(--;\frac{q+\alpha}{q},\frac{q+1+\alpha}{q},\cdots\frac{2q+\alpha-1}{q};\cdot\left(\frac{xyt}{q(1-t)}\right)^q\right). \tag{105}$$

Applying Equation (92), we get

$$\frac{e^x}{(1-t)^c}{}_qF_q\left(\begin{matrix}\frac{c}{q},\frac{c+1}{q},\cdots,\frac{c+q-1}{q};\\ \frac{q+\alpha}{q},\frac{q+1+\alpha}{q},\cdots,\frac{2q+\alpha-1}{q};\end{matrix}\left(\frac{-xt}{1-t}\right)^q\right)=\sum_{j=0}^{\infty}\frac{(c)_{qj}A_{q,qj}^{(\alpha)}(x)t^{qj}}{(q+\alpha)_{qj}}. \tag{106}$$

Replacing x by $x(1-t)^{-1}$ and t by $yt(1-t)^{-1}$ yields

$$\exp\left(\frac{x}{1-t}\right)\left(1-\frac{yt}{1-t}\right)^{-c}{}_qF_q\cdot\left(\begin{matrix}\frac{c}{q},\frac{c+1}{q},\cdots,\frac{c+q-1}{q};\\ \frac{q+\alpha}{q},\frac{q+1+\alpha}{q},\cdots,\frac{2q+\alpha-1}{q};\end{matrix}\left(\frac{(-x/(1-t))(yt/1-t)}{1-yt/1-t}\right)^q\right)=\sum_{j=0}^{\infty}\frac{(c)_{qj}A_{q,qj}^{(\alpha)}(x/1-t)(yt/1-t)^{qj}}{(q+\alpha)_{qj}}, \tag{107}$$

multiplying both sides by $(1-t)^{-q-1}\exp(-xt/1-t)$

$$\begin{aligned}&(1-t)^{-q-\alpha}\exp\left(\frac{x}{1-t}\right)\exp\left(\frac{-xt}{1-t}\right)\left(1-\frac{yt}{1-t}\right)^{-c}\times{}_qF_q\left(\begin{matrix}\frac{c}{q},\frac{c+1}{q},\cdots,\frac{c+q-1}{q};\\ \frac{q+\alpha}{q},\frac{q+1+\alpha}{q},\cdots,\frac{2q+\alpha-1}{q};\end{matrix}\left(\frac{(-x/(1-t))(yt/1-t)}{1-yt/1-t}\right)^q\right)\\&=(1-t)^{-q-\alpha}\exp\left(\frac{-xt}{1-t}\right)^2\sum_{j=0}^{\infty}\frac{(c)_{qj}A_{q,qj}^{(\alpha)}(x/1-t)(yt/1-t)^{qj}}{(q+\alpha)_{qj}}\\&=\sum_{j=0}^{\infty}\frac{(c)_{qj}A_{q,qj}^{(\alpha)}(x/1-t)(1-t)^{-q-\alpha-qj}\exp(-xt/1-t)^q y^{qj}t^{qj}}{(q+\alpha)_{qj}}.\end{aligned} \tag{108}$$

By using Lemma 4, we acquire

$$\begin{aligned}&(1-t)^{-1-\alpha}\exp\left(\frac{x}{1-t}\right)\left(1-\frac{yt}{1-t}\right)^{-c}{}_qF_q\cdot\left(\begin{matrix}\frac{c}{q},\frac{c+1}{q},\cdots,\frac{c+q-1}{q};\\ \frac{q+\alpha}{q},\frac{q+1+\alpha}{q},\cdots,\frac{2q+\alpha-1}{q};\end{matrix}\left(\frac{(-x/(1-t))(yt/1-t)}{1-yt/1-t}\right)^q\right)\\&=\sum_{n=0}^{\infty}\sum_{j=0}^{[n/q]}\frac{(c)_{qj}n!A_{q,qj}^{(\alpha)}(x)t^{n-qj}y^{qj}t^{qj}}{(qj)!(n-qj)!(q+\alpha)_{qj}}=\sum_{n=0}^{\infty}\sum_{j=0}^{[n/q]}\frac{(c)_{qj}(-n)_{qj}A_{q,qj}^{(\alpha)}(x)t^n y^{qj}}{(qj)!(q+\alpha)_{qj}}.\end{aligned} \tag{109}$$

By using Lemma 1 and 2, we get our required result. □

10. Expansion of Polynomials

Since $A_{q,n}^{(\alpha)}(x)$ forms an orthogonal set, the classical technique for expanding a polynomial. As usual, we prefer to treat the problem by obtaining first the expansion of x^{qn} and then using generating function techniques.

Theorem 23. *If $\alpha \in \mathbb{R}$ and $n, j \in \mathbb{Z}^+$, then*

$$x^{qn} = e^{-x} \sum_{j=0}^{[n/q]} \frac{n!(q+\alpha)_n A_{q,qj}^{(\alpha)}(x)}{(n-qj)!(q+\alpha)_{qj}}. \tag{110}$$

Proof. Equation (21) then yields

$$\begin{aligned} &{}_0F_q\left(--; \frac{q+\alpha}{q}, \frac{q+1+\alpha}{q}, \cdots \frac{2q+\alpha-1}{q}; \left(\frac{-xt}{q}\right)^q\right) \\ &= e^{-x-t} \sum_{n=0}^{\infty} \frac{A_{q,n}^{(\alpha)}(x) t^n}{(q+\alpha)_n}, \end{aligned} \tag{111}$$

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{(-xt/q)^{qn}}{(q+\alpha/q)_n (q+1+\alpha/q)_n \cdots (2q+\alpha-1/q)_n (qn)!} \\ &= e^{-x} \sum_{n=0}^{\infty} \frac{(-1)^n t^n}{n!} \sum_{n=0}^{\infty} \frac{A_{q,n}^{(\alpha)}(x) t^n}{(q+\alpha)_n}, \end{aligned} \tag{112}$$

$$\sum_{n=0}^{\infty} \frac{(-xt)^{qn}}{(q+\alpha)_{qn}(qn)!} = e^{-x} \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^n t^n A_{q,qj}^{(\alpha)}(x) t^{qj}}{n!(q+\alpha)_{qj}}. \tag{113}$$

By using Lemma 4, we get

$$\sum_{n=0}^{\infty} \frac{(-1)^n x^n t^n}{(q+\alpha)_n n!} = e^{-x} \sum_{n=0}^{\infty} \sum_{j=0}^{[n/q]} \frac{(-1)^n A_{q,qj}^{(\alpha)}(x) t^n}{(n-qj)!(q+\alpha)_{qj}}. \tag{114}$$

By equating the coefficient of t^n, we get

$$x^{qn} = e^{-x} \sum_{j=0}^{[n/q]} \frac{n!(q+\alpha)_n A_{q,qj}^{(\alpha)}(x)}{(n-qj)!(q+\alpha)_{qj}}. \tag{115}$$

□

11. Conclusion

Finally, in conclusion, we compromised the extended Laguerre polynomials $\left\{A_{q,n}^{(\alpha)}(x)\right\}$ based on the ${}_qF_q$, $q > 2$. We obtained generating functions, recurrence relations, and Rodrigue's formula for these extended Laguerre polynomials. In future work, we can extend it and can get more results. We will apply Laplace transformation, and Elzaki transformation and the same more transformations can apply on the results of extended Laguerre polynomials.

Data Availability

No data were used to support this work.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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