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Research article

Oscillation result for half-linear delay difference equations of second-order

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Abstract: In this paper, we obtain the new single-condition criteria for the oscillation of second-order half-linear delay difference equation. Even in the linear case, the sharp result is new and, to our knowledge, improves all previous results. Furthermore, our method has the advantage of being simple to prove, as it relies just on sequentially improved monotonicities of a positive solution. Examples are provided to illustrate our results.

Keywords: oscillation; non-oscillation; second-order; delay; half-linear; difference equations

1. Introduction

In this paper, we study the oscillation for the a second-order half-linear delay difference equation of the type

$$\Delta(\phi(\psi)(\Delta x(\psi))^{\nu}) + \rho(\psi)x^{\nu}(\psi - \eta) = 0; \quad n \geq \psi_0, \tag{1.1}$$

where the forward difference operator Δ is defined by $\Delta x(\psi) = x(\psi + 1) - x(\psi)$.

The following conditions are assumed throughout the paper:

(A_1) η is a non-negative integer;
(A_2) $\{\phi(\psi)\}_{\psi=\psi_0}^{\infty}$ is a positive real sequence;
(A_3) $\{\rho(\psi)\}_{\psi=\psi_0}^{\infty}$ is a sequence of non-negative real numbers and $\rho(\psi) \equiv 0$ for infinitely many values of ψ;
(A_4) $\nu \in \{\frac{a}{b} : a$ and b are odd integers$\}$;
(A_5) the equation (1.1) is called non-canonical form as

$$\theta(\psi) := \sum_{s=\psi}^{\infty} \frac{1}{\phi^{\frac{1}{\nu}}(s)} < \infty. \tag{1.2}$$

A solution of (1.1) is a real sequence $\{x(\psi)\}$ which is defined for $\psi \geq -\eta$ and satisfies (1.1) for $\psi \geq \psi_0$. A solution $\{x(\psi)\}$ is said to be oscillatory, if the terms $\{x(\psi)\}$ of the solution are not eventually positive or eventually negative. Otherwise the solution is called non-oscillatory.

The oscillation theory of delay differential equations has been significantly developed in recent decades. In recent years, the oscillation theory of discrete analogues of delay differential equations has received much interest. For the second-order difference equations, oscillation and non-oscillation problems have recently received considerable attention. This is likely due to the similarity of such phenomena to equivalent differential equations. Furthermore, these equations have a wide range of applications in physics and other domains. In [1] and [2], the authors have discussed the oscillation theorems for nonlinear fractional difference equations. The oscillation results for nonlinear second-order difference equations gives in [3–5] and difference equations with mixed neutral terms are discussed in [6–8].

Agarwal et al. [9–12] investigate discrete oscillatory theory, advanced topics in difference equations and oscillation theory for difference equations. In [13, 14], the authors gives the theory of difference equations and oscillation theory of delay difference equations. The stability and periodic solutions of neutral difference equations are discussed in [15, 16]. Park et al. [17–19] gives the results of stability analysis, neutral dynamic systems with delay in control input and design of dynamic controller for neutral differential systems. Also, stability criteria for uncertain neutral systems are discussed in [20]. In 2019, Thandapani and Selvarangam [21] gives oscillation results for third-order half-linear neutral difference equations. In [22–25], S. S. Santra et al. and in [26, 27] M. Ruggieri et al. investigate various oscillation results of linear and non-linear differential systems. Oscillatory properties of even-order ordinary differential equations with variable coefficients is discussed in O. Bazighifan [28].

In [29], Murugesan et al. have established the result that the second-order non-canonical advanced difference equation

$$\Delta(\phi(\psi)(\Delta x(\psi))^{\nu}) + \rho(\psi)x^{\nu}(\psi + \eta) = 0; \quad \psi \geq \psi_0 \tag{1.3}$$

is oscillatory if

$$\sum_{\psi=\psi_0}^{\infty}\left(\frac{1}{\phi(\psi)}\sum_{s=\psi_0}^{\psi-1}\theta^{\nu}(s+\eta)\rho(s)\right)^{\frac{1}{\nu}}=\infty.$$

In [30], authors have studied the linear equations

$$\Delta(\phi(\psi)\Delta x(\psi))+\rho(\psi)x(\psi-\eta)=0;\quad \psi\geq\psi_0 \tag{1.4}$$

and established the oscillation criteria for (1.4).

Motivated by the above results, we derive new sufficient condition for the oscillation of all solutions to (1.1). Even in the linear situation, this sharp conclusion is unique. Our results are improved all previous results in the literature. Moreover, in the linear case, we can express comparable results for canonical equations.

We divided the paper in the following structure: We proved some auxiliary lemmas in section 2. Section 3 deals with the main results of the paper. Finally, two examples are offered in section 4 to demonstrate our results.

2. Auxiliary lemmas

Let us define

$$\delta_*=\liminf_{\psi\to\infty}\frac{1}{\nu}\phi^{\frac{1}{\nu}}(\psi)\theta^{\nu+1}(\psi+1)\rho(\psi) \tag{2.1}$$

and

$$\mu_*=\liminf_{\psi\to\infty}\frac{\theta(\psi-\eta)}{\theta(\psi)}<\infty. \tag{2.2}$$

The proofs rely on the existence of positivity δ_*, which is also required for Theorems 3.1 and 3.4 to be valid. Then there is a $\psi_1\geq\psi_0$ for every arbitrary fixed $\delta\in(0,\delta_*)$ and $\mu\in[1,\mu_*)$ such that

$$\frac{1}{\nu}\rho(\psi)\phi^{\frac{1}{\nu}}(\psi)\theta^{\nu+1}(\psi+1)\geq\delta$$

and

$$\frac{\theta(\psi-\eta)}{\theta(\psi)}\geq\mu,\quad \psi\geq\psi_0. \tag{2.3}$$

In the following section, we presume that all functional inequalities are satisfied; eventually, that is, for all ψ large enough.

Using the procedure used in [8, Theorem 2], one can prove the following result.

Lemma 2.1. *Suppose that*

$$\sum_{\psi=\psi_0}^{\infty}\frac{1}{r^{\frac{1}{\nu}}(\psi)}\left(\sum_{s=\psi_0}^{\psi-1}\rho(s)\right)^{\frac{1}{\nu}}=\infty. \tag{2.4}$$

If $\{x(\psi)\}$ *is eventually positive solution of* (1.1)*, then* $\Delta x(\psi)<0$ *and* $\lim_{\psi\to\infty}x(\psi)=0$.

Lemma 2.2. *Let* $\delta_*>0$. *If* (1.1) *has an eventually positive solution* $\{x(\psi)\}$*, then*

(*i*) $\{x(\psi)\}$ *is eventually decreasing with* $\lim_{\psi\to\infty}x(\psi)=0$*;*

(ii) $\{\frac{x(\psi)}{\theta(\psi)}\}$ *is eventually non-decreasing.*

Proof. (i) By using (1.2), (2.3) and the decreasing nature of $\{\theta(\psi)\}$, we have

$$\begin{aligned}\sum_{u=\psi_1}^{\psi-1}\frac{1}{r^{\frac{1}{\nu}}(u)}\left(\sum_{s=\psi_1}^{u-1}\rho(s)\right)^{\frac{1}{\nu}} &\geq \sqrt[\nu]{\delta}\sum_{u=\psi_1}^{\psi-1}\frac{1}{r^{\frac{1}{\nu}}(u)}\left(\sum_{s=\psi_1}^{u-1}\frac{\nu}{r^{\frac{1}{\nu}}(s)\theta^{\nu+1}(s+1)}\right)^{\frac{1}{\nu}}\\ &\geq \sqrt[\nu]{\delta}\sum_{u=\psi_1}^{\psi-1}\frac{1}{r^{\frac{1}{\nu}}(u)}\left(-\nu\sum_{s=\psi_1}^{u-1}\frac{\theta(s)}{\theta^{\nu+1}(s+1)}\right)^{\frac{1}{\nu}}\\ &\geq \sqrt[\nu]{\delta}\sum_{u=\psi_1}^{\psi-1}\frac{1}{r^{\frac{1}{\nu}}(u)}\left(\frac{1}{\theta^{\nu}(u)}-\frac{1}{\theta^{\nu}(n_1)}\right)^{\frac{1}{\nu}}.\end{aligned}$$

Since $\theta^{-\nu}(\psi)\to\infty$ as $\psi\to\infty$, for any $l\in(0,1)$ and ψ large enough, we have $\theta^{-\nu}(\psi)-\theta^{-\nu}(\psi_1)\geq l^{\nu}\theta^{-\nu}(\psi)$ and hence

$$\begin{aligned}\sum_{u=\psi_1}^{\psi-1}\frac{1}{r^{\frac{1}{\nu}}(u)}\left(\sum_{s=\psi_1}^{u-1}\rho(s)\right)^{\frac{1}{\nu}} &\geq l\sqrt[\nu]{\delta}\sum_{u=\psi_1}^{\psi-1}\frac{1}{r^{\frac{1}{\nu}}(u)\theta(u)}\\ &\geq l\sqrt[\nu]{\delta}\ln\frac{\theta(\psi_1)}{\theta(\psi)}\\ &\geq 0.\end{aligned}$$

By Lemma 2.1, the conclusion follows.

(ii) Using the fact that $\{r^{\frac{1}{\nu}}(n)\Delta x(n)\}$ is non-increasing, we obtain

$$\begin{aligned}x(\psi) &\geq -\sum_{s=\psi}^{\infty}\frac{1}{r^{\frac{1}{\nu}}(s)}r^{\frac{1}{\nu}}(s)\Delta x(\psi)\\ &\geq -r^{\frac{1}{\nu}}(\psi)\Delta x(\psi)\sum_{s=\psi}^{\infty}\frac{1}{r^{\frac{1}{\nu}}(s)}\\ &= -r^{\frac{1}{\nu}}(\psi)\Delta x(\psi)\theta(\psi),\end{aligned}$$

i.e.,

$$\Delta\left(\frac{x(\psi)}{\theta(\psi)}\right)=\frac{r^{\frac{1}{\nu}}(\psi)\Delta x(\psi)\theta(\psi)+x(\psi)}{r^{\frac{1}{\nu}}(\psi)\theta(\psi)\theta(\psi+1)}\geq 0.$$

The proof is complete. □

To develop the (i) - part of Lemma 2.2, let us define a sequence $\{\delta_k\}$ by

$$\begin{aligned}\delta_0 &= \sqrt[\nu]{\delta_*},\quad k=0\\ \delta_k &= \frac{\delta_0\mu_*^{\delta_{k-1}}}{\sqrt[\nu]{1-\delta_{k-1}}},\quad k\in\mathbb{N}.\end{aligned}\tag{2.5}$$

We can easily show by induction that if for any $k \in \mathbb{N}$, $\delta_i < 1$, $i = 0, 1, 2, \ldots, k,$. then δ_{k+1} exists and

$$\delta_{k+1} = \xi_k \delta_k > \delta_k, \tag{2.6}$$

where ξ_k is defined by

$$\xi_0 = \frac{\mu_*^{\delta_0}}{\sqrt[\nu]{1-\delta_0}}, \quad k = 0 \tag{2.7}$$

$$\xi_{k+1} = \mu_*^{\delta_0(\xi_k - 1)} \sqrt[\nu]{\frac{1-\delta_k}{1-\xi_k\delta_k}}, \quad k \in \mathbb{N}_0. \tag{2.8}$$

Lemma 2.3. *Let $\delta_* > 0$ and $\mu_* < \infty$. If* (1.1) *has an eventually positive solution $\{x(\psi)\}$, then for any $k \in \mathbb{N}$, $\{\frac{x(\psi)}{\theta^{\delta_k}(\psi)}\}$ is eventually decreasing.*

Proof. Let $\{x(\psi)\}$ be an eventually positive solution of (1.1). Then there exists a $\psi_1 \geq \psi_0$ such that $x(\psi - \eta) > 0$ for $\psi \geq \psi_1$. Summing (1.1) from ψ_1 to $\psi - 1$, we have

$$-\phi(\psi)(\Delta x(\psi))^\nu = -\phi(\psi_1)(\Delta x(\psi_1))^\nu + \sum_{s=\psi_1}^{\psi-1} \rho(s) x^\nu(s-\eta). \tag{2.9}$$

By (i) of Lemma 2.2, $\{x(\psi)\}$ is decreasing and $x(\psi - \eta) \geq x(\psi)$ for $\psi \geq \psi_1$. Therefore,

$$\begin{aligned}
-\phi(\psi)(\Delta x(\psi))^\nu &\geq -\phi(\psi_1)(\Delta x(\psi_1))^\nu + \sum_{s=\psi_1}^{\psi-1} \rho(s) x^\nu(s-\eta) \\
&\geq -\phi(\psi_1)(\Delta x(\psi_1))^\nu + x^\nu(\psi) \sum_{s=\psi_1}^{\psi-1} \rho(s).
\end{aligned}$$

Using (2.3) in the above inequality, we get

$$\begin{aligned}
-\phi(\psi)(\Delta x(\psi))^\nu &\geq -\phi(\psi_1)(\Delta x(\psi_1))^\nu + \delta x^\nu(\psi) \sum_{s=\psi_1}^{\psi-1} \frac{c}{\phi^{\frac{1}{\nu}}(s)\theta^{\nu+1}(s+1)} \\
&\geq -\phi(\psi_1)(\Delta x(\psi_1))^\nu + \delta \frac{x^\nu(\psi)}{\theta^\nu(\psi)} - \delta \frac{x^\nu(\psi)}{\theta^\nu(\psi_1)}.
\end{aligned} \tag{2.10}$$

From (i)-part of Lemma 2.2, we have that $\lim_{\psi\to\infty} x(\psi) = 0$. Hence, there is a $\psi_2 \geq \psi_1$ such that

$$-\phi(\psi_1)(\Delta x(\psi_1))^\nu - \delta \frac{x^\nu(\psi)}{\theta^\nu(\psi_1)} > 0, \quad \psi \geq \psi_2.$$

Thus,

$$-\phi(\psi)(\Delta x(\psi))^\nu > \delta \frac{x^\nu(\psi)}{\theta^\nu(\psi)} \tag{2.11}$$

or

$$-\phi^{\frac{1}{\nu}}(\psi)\Delta x(\psi)\theta(\psi) > \sqrt[\nu]{\delta} x(\psi) = \epsilon_0 \delta_0 x(\psi),$$

where $\epsilon_0 = \frac{\sqrt[\nu]{\delta}}{\delta_0}$ is an arbitrary constant from $(0, 1)$. Therefore,

$$\Delta\left(\frac{x(\psi)}{\theta^{\sqrt[\nu]{\delta}}(\psi)}\right) = \frac{\phi^{\frac{1}{\nu}}(\psi)\Delta x(\psi)\theta^{\sqrt[\nu]{\delta}}(\psi) + \sqrt[\nu]{\delta}\theta^{\sqrt[\nu]{\delta}-1}(\psi)x(\psi)}{\phi^{\frac{1}{\nu}}(\psi)\theta^{\sqrt[\nu]{\delta}}(\psi)\theta^{\sqrt[\nu]{\delta}}(\psi+1)}$$
$$= \frac{\theta^{\sqrt[\nu]{\delta}-1}(\psi)(\sqrt[\nu]{\delta}x(\psi) + \theta(\psi)\phi^{\frac{1}{\nu}}(\psi)\Delta x(\psi))}{\phi^{\frac{1}{\nu}}(\psi)\theta^{\sqrt[\nu]{\delta}}(\psi)\theta^{\sqrt[\nu]{\delta}}(\psi+1)} \leq 0, \quad \psi \geq \psi_2. \tag{2.12}$$

Summing (1.1) from ψ_2 to $\psi - 1$ and using that $\{\frac{x(\psi)}{\theta^{\sqrt[\nu]{\delta}}(\psi)}\}$ is decreasing, we have

$$-\phi(\psi)(\Delta x(\psi))^\nu \geq -\phi(\psi_2)(\Delta x(\psi_2))^\nu + \left(\frac{x(\psi-\eta)}{\theta^{\sqrt[\nu]{\delta}}(\psi-\eta)}\right)^\nu \sum_{s=\psi_2}^{\psi-1}\rho(s)\theta^{\sqrt[\nu]{\delta}}(s-\eta)$$
$$\geq -\phi(\psi_2)(\Delta x(\psi_2))^\nu + \left(\frac{x(\psi)}{\theta^{\sqrt[\nu]{\delta}}(\psi)}\right)^\nu \sum_{s=\psi_2}^{\psi-1}\rho(s)\left(\frac{\theta(s-\eta)}{\theta(s)}\right)^{\sqrt[\nu]{\delta}}\theta^{\sqrt[\nu]{\delta}}(s).$$

By virtue of (2.3), we see that

$$-\phi(\psi)(\Delta x(\psi))^\nu \geq -\phi(\psi_2)(\Delta x(\psi_2))^\nu + \delta\left(\frac{x(\psi)}{\theta^{\sqrt[\nu]{\delta}}(\psi)}\right)^\nu \sum_{s=\psi_2}^{\psi-1}\frac{\nu\left(\frac{\theta(s-\eta)}{\theta(s)}\right)^{\sqrt[\nu]{\delta}}}{\phi^{\frac{1}{\nu}}(s)\theta^{\nu+1-\nu\sqrt[\nu]{\delta}}(s+1)}$$

$$-\phi(\psi)(\Delta x(\psi))^\nu \geq -\phi(\psi_2)(\Delta x(\psi_2))^\nu + \frac{\delta}{1-\sqrt[\nu]{\delta}}\mu^{\nu\sqrt[\nu]{\delta}}\left(\frac{x(\psi)}{\theta^{\sqrt[\nu]{\delta}}(\psi)}\right)^\nu \sum_{s=\psi_2}^{\psi-1}\frac{\nu(1-\sqrt[\nu]{\delta})}{\phi^{\frac{1}{\nu}}(s)\theta^{\nu+1-\nu\sqrt[\nu]{\delta}}(s+1)} \tag{2.13}$$

$$-\phi(\psi)(\Delta x(\psi))^\nu \geq -\phi(\psi_2)(\Delta x(\psi_2))^\nu + \frac{\delta}{1-\sqrt[\nu]{\delta}}\mu^{\nu\sqrt[\nu]{\delta}}\left(\frac{x(\psi)}{\theta^{\nu\sqrt[\nu]{\delta}}(\psi)}\right)^\nu\left(\frac{1}{\theta^{\nu(1-\sqrt[\nu]{\delta})}(\psi)} - \frac{1}{\theta^{\nu(1-\sqrt[\nu]{\delta})}(\psi_2)}\right). \tag{2.14}$$

Now, we claim that $\lim_{\psi\to\infty}\frac{x(\psi)}{\theta^{\sqrt[\nu]{\delta}}(\psi)} = 0$. It sufficies to show that there is $\epsilon > 0$ such that $\left\{\frac{x(\psi)}{\theta^{\sqrt[\nu]{\delta}+\epsilon}(\psi)}\right\}$ is eventually decreasing sequence. Since $\{\theta(\psi)\}$ tends to zero, there exists is a constant.

$$\xi \in \left(\frac{\sqrt[\nu]{1-\sqrt[\nu]{\delta}}}{\mu^{\sqrt[\nu]{\delta}}}, 1\right)$$

and a $\psi_3 \geq \psi_2$ such that

$$\frac{1}{\theta^{\nu(1-\sqrt[\nu]{\delta})}(\psi)} - \frac{1}{\theta^{\nu(1-\sqrt[\nu]{\delta})}(\psi_2)} > \xi^\nu\frac{1}{\theta^{\nu(1-\sqrt[\nu]{\delta})}(\psi)}, \quad \psi \geq \psi_3.$$

Using the above inequality in (2.14) yields

$$-\phi(\psi)(\Delta x(\psi))^{\nu} \geq \frac{\xi^{\nu}\delta}{1-\sqrt[\nu]{\delta})}\mu^{\nu\sqrt[\nu]{\delta}}\left(\frac{x(\psi)}{\theta(\psi)}\right)^{\nu},$$

i.e.,

$$-\phi^{\frac{1}{\nu}}(\psi)\Delta x(\psi) \geq \left(\sqrt[\nu]{\delta}+\epsilon\right)\frac{x(\psi)}{\theta(\psi)}, \tag{2.15}$$

where

$$\epsilon = \sqrt[\nu]{\delta}\left(\frac{\xi\mu^{\sqrt[\nu]{\delta}}}{\sqrt[\nu]{1-\sqrt[\nu]{\delta}}}-1\right) > 0.$$

Then, from (2.15),

$$\Delta\left(\frac{x(\psi)}{\theta^{\sqrt[\nu]{\delta}+\epsilon}(\psi)}\right) \leq 0, \quad \psi \geq \psi_3,$$

and hence the claim holds. Therefore, for $\psi_4 \geq \psi_3$,

$$-\phi(\psi_2)(\Delta x(\psi_2))^{\nu} - \frac{\delta}{1-\sqrt[\nu]{\delta}}\mu^{\nu\sqrt[\nu]{\delta}}\left(\frac{x(\psi)}{\theta^{\sqrt[\nu]{\delta}}(\psi)}\right)^{\nu}\frac{1}{\theta^{\nu-\nu\sqrt[\nu]{\delta}}(\psi_2)} > 0, \quad \psi \geq \psi_4.$$

Returning to (2.14) and applying the above inequality,

$$\begin{aligned} -\phi(\psi)(\Delta x(\psi))^{\nu} \geq &-\phi(\psi_2)(\Delta x(\psi_2))^{\nu} \\ &+ \frac{\delta}{1-\sqrt[\nu]{\delta}}\mu^{\nu\sqrt[\nu]{\delta}}\left(\frac{x(\psi)}{\theta(\psi)}\right)^{\nu} \\ &- \frac{\delta}{1-\sqrt[\nu]{\delta}}\mu^{\nu\sqrt[\nu]{\delta}}\left(\frac{x(\psi)}{\theta^{\sqrt[\nu]{\delta}}(\psi)}\right)^{\nu}\frac{1}{\theta^{\nu-\nu\sqrt[\nu]{\delta}}(\psi_2)} \\ > &\frac{\delta}{1-\sqrt[\nu]{\delta}}\mu^{\nu\sqrt[\nu]{\delta}}\left(\frac{x(\psi)}{\theta(\psi)}\right)^{\nu}, \end{aligned}$$

or

$$-\phi^{\frac{1}{\nu}}(\psi)\Delta x(\psi)\theta(\psi) > \frac{\sqrt[\nu]{\delta}}{\sqrt[\nu]{1-\sqrt[\nu]{\delta}}}\mu^{\nu\sqrt[\nu]{\delta}}x(\psi) = \epsilon_1\delta_1 x(\psi), \quad \psi \geq \psi_4,$$

where

$$\epsilon_1 = \sqrt[\nu]{\frac{\delta(1-\sqrt[\nu]{\delta_*})}{\delta_*(1-\sqrt[\nu]{\delta})}}\frac{\mu^{\sqrt[\nu]{\delta}}}{\mu_*^{\sqrt[\nu]{\delta_*}}}$$

is an arbitrary constant from $(0,1)$ tends to 1 if $\delta \to \delta_*$ and $\mu \to \mu_*$. Hence,

$$\Delta\left(\frac{x(\psi)}{\theta^{\epsilon_1\delta_1}(\psi)}\right) < 0, \quad \psi \geq \psi_4.$$

By induction, one can show that for any $k \in \mathbb{N}_0$ and ψ large enough,

$$\Delta\left(\frac{x(\psi)}{\theta^{\epsilon_k\delta_k}(\psi)}\right) < 0,$$

where ϵ_k given by $\epsilon_0 = \sqrt[\nu]{\frac{\delta}{\delta_*}}$

$$\epsilon_{k+1} = \epsilon_0 \sqrt[\nu]{\frac{1-\delta_k}{1-\epsilon_k\delta_k}} \frac{\mu^{\epsilon_k\delta_k}}{\mu_*^{\delta_k}}, \quad k \in \mathbb{N}_0$$

is an arbitrary constant from $(0, 1)$ tends to 1 if $\delta \to \delta_*$ and $\mu \to \mu_*$. Now, we assert that from any $k \in \mathbb{N}_0$, $\left\{\frac{x(\psi)}{\theta^{\epsilon_{k+1}\delta_{k+1}}(\psi)}\right\}$ decreasing implies that $\{\frac{x(\psi)}{\theta^{\delta_k}}\}$ is a decreasing sequence as well. Using (2.6) and the fact that ϵ_{k+1} is arbitrarly closed to 1, we see that

$$\epsilon_{k+1}\delta_{k+1} > \delta_k.$$

Then, for ψ sufficiently large enough,

$$-\phi^{\frac{1}{\nu}}(\psi)\Delta x(\psi)\theta(\psi) > \epsilon_{k+1}\delta_{k+1}x(\psi) > \delta_k x(\psi)$$

and so for any $\psi \in \mathbb{N}_0$ and ψ large enough,

$$\Delta\left(\frac{x(\psi)}{\theta^{\delta_k}(\psi)}\right) < 0.$$

The proof is complete. □

3. Main results

Theorem 3.1. *Let*

$$\mu_* := \liminf_{\psi\to\infty} \frac{\theta(\psi-\eta)}{\theta(\psi)} < \infty. \tag{3.1}$$

If

$$\liminf_{\psi\to\infty} \phi^{\frac{1}{\nu}}(\psi)\theta^{\nu+1}(\psi+1)\rho(\psi) > \max\{c(\omega) : \nu\omega^{\nu}(1-\omega)\mu_*^{-\nu\omega} : 0 < \omega < 1\}, \tag{3.2}$$

then (1.1) *is oscillatory.*

Proof. Assume that $\{x(\psi)\}$ is an eventually positive solution of (1.1). Lemma 2.2 and 2.3 ensure that $\Delta\{\frac{x(\psi)}{\theta(\psi)}\} \geq 0$ and $\Delta\{\frac{x(\psi)}{\theta^{\delta_k}(\psi)}\} < 0$ for any $k \in \mathbb{N}_0$ and ψ sufficiently large enough. This case occurs when $\delta_k < 1$ for any $k \in \mathbb{N}_0$.

Thus, the sequence $\{\delta_k\}$ given by (2.5) is increasing and bounded sequence from above which implies that there exists a finite limit $\liminf_{k\to\infty} \delta_k = \omega$, where ω is the smallest positive root of the equation

$$c(\omega) = \liminf_{\psi\to\infty} \phi^{\frac{1}{\nu}}(\psi)\theta^{\nu+1}(\psi+1)\rho(\psi). \tag{3.3}$$

Because of (3.2), equation (3.3) cannot have a positive solution.

This contradiction completes the proof. □

Corollary 3.1. *By simple computations, we obtain*

$$max\{c(\omega) : 0 < \omega < 1\} = c(max),$$

where

$$\omega_{max} = \begin{cases} \frac{\nu}{\nu+1}, & \text{for } \mu_* = 1 \\ \frac{-\sqrt{(\nu\phi+\nu+1)^2-4\nu^2\phi}+\nu\phi+\nu+1}{2\nu\phi}, & \text{for } \mu_* \neq 1 \text{ and } \phi = \ln \mu_*, \end{cases}$$

and $c(\omega)$ is defined by (3.2).

We get the following result when (3.1) *is failed.*

Theorem 3.2. *Let*

$$\lim_{\psi\to\infty} \frac{\theta(\psi-\eta)}{\theta(\psi)} = \infty. \tag{3.4}$$

If

$$\liminf_{\psi\to\infty} \phi^{\frac{1}{\nu}}(\psi)\theta^{\nu+1}(\psi+1)\rho(\psi) > 0 \tag{3.5}$$

then (1.1) *is oscillatory.*

Proof. Let $\{x(\psi)\}$ be an eventually positive solution of (1.1). Then there exists a $\psi_1 \geq \psi_0$ such that $x(\psi-\eta) > 0$ for $\psi \geq \psi_1$. By virtue of (3.4), we see that for any $M > 0$ there exists a ψ sufficiently large enough such that

$$\frac{\theta(\psi-\eta)}{\theta(\psi)} \geq \left(\frac{M}{\sqrt[\nu]{\delta}}\right)^{\frac{1}{\sqrt[\nu]{\delta}}}. \tag{3.6}$$

As in the proof of Lemma 2.3, we can show that $\left\{\frac{x(\psi)}{\theta^{\sqrt[\nu]{\delta}}(\psi)}\right\}$ is decreasing eventually, say for $\psi \geq \psi_2 \geq \psi_1$. Using this monotonicity in (2.9), we have

$$\begin{aligned} -\phi(\psi)(\Delta x(\psi))^\nu &= -\phi(\psi_2)(\Delta x(\psi_2))^\nu + \sum_{s=\psi_2}^{\psi-1} \rho(s)x^\nu(s-\eta) \\ &\geq -\phi(\psi_2)(\Delta x(\psi_2))^\nu + M^\nu x^\nu(\psi) \sum_{s=\psi_2}^{\psi-1} \frac{\nu}{\phi^{\frac{1}{\nu}}(s)\theta^{\nu+1}(s+1)} \\ &> M^\nu \left(\frac{x(\psi)}{\theta(\psi)}\right)^\nu, \end{aligned}$$

from which we derive that $\{\frac{x(\psi)}{\theta^M(\psi)}\}$ is decreasing sequence. From the fact that M is a arbitrary, we have $\{\frac{x(\psi)}{\theta(\psi)}\}$ is non-decreasing sequence.This is a contradiction with (*ii*)-part of Lemma 2.2 and this completes the proof. □

Below, we study the oscillation behaviour of (1.1) to the canonical equations in linear case when $\nu = 1$, that is, we consider the equation

$$\Delta(\widetilde{\phi}(\psi)\Delta u(\psi)) + \widetilde{\rho}(\psi)u(\psi-\eta) = 0, \quad \psi \geq \psi_0, \tag{3.7}$$

where $\{\widetilde{\phi}(\psi)\}$ is a positive real sequence and $\{\widetilde{\rho}(\psi)\}$ is a nonnegative real sequence with $\rho(\psi) \equiv 0$ for infinitely many values of ψ, and

$$R(\psi) = \sum_{s=\psi_0}^{\psi-1} \frac{1}{\widetilde{\phi}(s)} \to \infty \quad as \quad \psi \to \infty.$$

Theorem 3.3. *Let*

$$\delta_* := \liminf_{\psi\to\infty} \frac{R(\psi)}{R(\psi-\eta)} < \infty.$$

If

$$\liminf_{\psi\to\infty} \left(\widetilde{\phi}(\psi)\widetilde{\rho}(\psi)R(\psi)R(\psi-\eta)\right) > \max\{\omega(1-\omega)\delta_*^{-\omega} : 0 < \omega < 1\},$$

then (3.7) *is oscillatory.*

Proof. we can readily check that the canonical equation (3.7) is equivalent to a non-canonical equation (1.1) with $\nu = 1$,

$$\phi(\psi) = \widetilde{\phi}(\psi)R(\psi)R(\psi+1)$$
$$\rho(\psi) = \widetilde{\rho}(\psi)R(\psi+1)R(\psi-\eta)$$

and

$$x(\psi) = \frac{u(\psi)}{R(\psi)} > 0.$$

Now,

$$\theta(\psi) = \sum_{s=\psi}^{\infty} \frac{1}{\phi(s)} = \sum_{s=\psi}^{\infty} \frac{\Delta R(s)}{R(s)R(s+1)} = \frac{1}{R(\psi)}.$$

The result derives from Theorem 3.1 immediately. □

Theorem 3.4. *Let*

$$\lim_{\psi\to\infty} \frac{R(\psi)}{R(\psi-\eta)} = \infty.$$

If

$$\liminf_{\psi\to\infty}\{\widetilde{\phi}(\psi)\widetilde{\rho}(\psi)R(\psi)R(\psi-\eta)\} > 0,$$

then (3.7) *is oscillatory.*

Proof. By applying the equivalent non-canonical representation of (3.7) as in the proof of Theorem 3.3, the claim follows from Theorem 3.2. □

4. Examples

Example 4.1. *Let us consider the second-order difference equation*

$$\Delta((\psi(\psi+1))^{\frac{1}{3}}(\Delta x(\psi))^{\frac{1}{3}}) + \lambda_0 \frac{(\psi+1)^{\frac{1}{3}}}{\psi} x^{\frac{1}{5}}(\psi-1) = 0; \quad \psi = 1, 2, 3, \ldots \tag{4.1}$$

Here, we have $\phi(\psi) = (\psi(\psi+1))^{\frac{1}{3}}$, $\rho(\psi) = \lambda_0 \frac{(\psi+1)^{\frac{1}{3}}}{\psi}$, $\nu = \frac{1}{3}$ *and* $\eta = 1$.

By simple computation, we obtain

$$\theta(\psi) = \frac{1}{\psi},$$

$$\lambda_* = \liminf_{\psi\to\infty} \frac{\theta(\psi-1)}{\theta(\psi)} = 1,$$

$$\liminf_{\psi\to\infty} \phi^{\frac{1}{\nu}}(\psi)\theta^{\nu+1}(\psi+1)\rho(\psi) = \lambda_0,$$

and

$$\max\{c(\omega) : \nu\omega^{\nu}(1-\omega) : 0 < \omega < 1\} = \frac{1}{4\sqrt[3]{4}}.$$

Thus, by Theorem 3.1, every solution of (4.1) *is oscillatory if* $\lambda_0 > \frac{1}{4\sqrt[3]{4}}$

Example 4.2. *Let us investigate the oscillatory behaviour of the second-order linear difference equation*

$$\Delta\left(\frac{1}{\psi}\Delta x(\psi)\right) + \frac{4\lambda_0}{(\psi-2)(\psi-1)^2} x(\psi-1) = 0; \quad \psi = 1, 2, 3, \ldots \tag{4.2}$$

We have $\widetilde{\phi}(\psi) = \frac{1}{\psi}$, $\widetilde{\rho}(\psi) = \frac{4\lambda_0}{(\psi-2)(\psi-1)^2}$ *and* $\eta = 1$. *We can easily show that*

$$R(\psi) = \frac{\psi(\psi-1)}{2}, \quad \delta_* = 1,$$

$$\liminf_{\psi\to\infty} \widetilde{\phi}(\psi)\widetilde{\rho}(\psi)R(\psi)R(\psi-1) = \lambda_0,$$

and

$$\max\{\omega(1-\omega) : 0 < \omega < 1\} = \frac{1}{4}.$$

Hence, by Theorem 3.3, the equation (4.2) *is oscillatory for* $\lambda_0 > \frac{1}{4}$.

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Conflicts of interest

The authors declare no conflict of interest.

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