PRABHAKAR FRACTIONAL DERIVATIVE AND ITS APPLICATIONS IN THE TRANSPORT PHENOMENA CONTAINING NANOPARTICLES

by

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In this paper, a new approach of analytical solutions is carried out on the thermal transport phenomena of Brinkman fluid based on Prabhakar's fractional derivative with generalized Fourier's law. The governing equations are obtained through constitutive relations and analytical solutions obtained via Laplace transform technique. Solutions for temperature and velocity field were analyzed through graphical description by MathCad software. The fluid properties revealed various aspects for different flow parameters as well as fractional parameter values and found important results. As a result, it is found that fluid properties can be enhanced by increasing the values of fractional parameters and can be useful in some experimental data where suitable.

Key words: *Prabhakar's fractional derivative, Mittag-Leffler kernel, heat transfer, Brinkman fluid, nanoparticles*

Introduction

It is found that the mathematical models of integer-order derivatives and, counting non-linear models does not work on streamline properly in different aspects. Fractional calculus has many applications in the field of viscoelasticity, signals processing, electromagnetics, fluid mechanics and optics. Though in the current scenario the models of fractional calculus has been widely used in the field of engineering, where these are described by fractional differential equation. The main crux of fractional derivative is to sort out the modeling in an appropriate way. Some of the most prominent area of research regarding to the above applications can be found in [1-8].

The non-Newtonian fluids are the wider class of fluids having many practical applications like cell isolation, drug transport, positron emission tomography, thermal management, cooling, and dynamic sealing. Because of the extensive applications of ferro-nanofluids, scholars' primary goal is to devote their full attention to revealing even more qualities. The scientist

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in the beginning which introduced nanofluid in 1995 was Choi [9], he also sums up those nanoscaled particles which have a diameter less than (100 nm) diameter. He also investigated the stretching sheet having properties of thermal transport and wide applications of nanofluid in a science field such as cancer therapy and many more. Some related applications of different kinds of fractional derivatives with nanofluids can be seen in [10-14].

In the mentioned literature problems are modeled and solved with C, CF, and ABC, with artificial replacement approach. Present problem deals with governing equations for the fluid-flow are obtained by means of generalized Fourier law with Prabhakar fractional derivatives with kernel Mittage-Leffler function of three parameters. Recently, the same approach worked out by Elnaqeeb [15] for viscous fluid. Our intention is to extend to non-Newtonian fluid namely Brinkman fluid under different thermal and geometric conditions and water based silver nanoparticles. In the existing literature there is no result exist for considered assumptions.

Mathematical formulation

Let us consider a vertical plate on a rectangular co-ordinate system consisting thermal transportation of an incompressible fractional nanofluid placed at the plane $y = 0$, at temperature T_∞. At time $t = 0$, both the plate and fluid are in rest state. After passing some time the temperature of the wall raised to $T_\infty + (T_\mathrm{w} - T_\infty) f(t)$ and plate moves with a constant velocity in its own plan. All the flow properties are of function of y and t only. Further assumes that momentum equation contains no pressure gradient and thermal equation without the effect of viscous dissipation term. Then the governing equations are:

– The PDE of velocity:

$$\rho_\mathrm{nf} u_t(y,t) + \rho_\mathrm{nf} \beta_p u(y,t) = [T(y,t) - T_\infty] \mathrm{g}(\rho\beta_T)_\mathrm{nf} - \sigma_\mathrm{nf} B_0^2 u(y,t) + \mu_\mathrm{nf} u_{yy}(y,t) \tag{1}$$

– The heat equation:

$$T_t(y,t) + \frac{q_y(y,t)}{(\rho C_p)_\mathrm{nf}} = 0 \tag{2}$$

– Classical Fourier's law:

$$q(y,t) = -k_\mathrm{nf} T_t(y,t) \tag{3}$$

Boundary conditions are written:

$$T(y,0) = T_\infty, \quad y \ge 0, \quad T(0,t) = T_\infty + (T_\mathrm{w} - T_\infty)t, \quad t \ge 0, \quad T(y,t) \to T_\infty \quad \text{as} \quad y \to \infty \tag{4}$$

$$u(y,0) = 0, \quad y \ge 0, \quad u(0,t) = 0, \quad t \ge 0, \quad u(y,t) \to 0, \quad \text{as} \quad y \to \infty \tag{5}$$

The thermophysical parameters of nanofluid are [16].

Introducing dimensionless quantities to make a problem-free flow regime:

$$\tilde{y} = \frac{v_0 y}{\upsilon_f}, \quad \tilde{t} = \frac{v_0^2 t}{\upsilon_f}, \upsilon_f = \frac{\mu_f}{\rho_f}, \quad \tilde{u} = \frac{u}{v_0}, \quad \tilde{T} = \frac{T - T_\infty}{T_w - T_\infty}, \quad \tilde{q} = \frac{q}{q_o}, \quad q_0 = \frac{k_f (T_w - T_\infty) v_0}{\upsilon_f}$$

$$\mathrm{Pr} = \frac{(\mu c_p)_f}{k_f}, \quad \mathrm{Gr} = \frac{(T_w - T_\infty)\mathrm{g}(\upsilon\beta_T)_f}{v_0^3}, \quad M = \frac{\sigma_f B_0^2 \upsilon_f^2}{v_0^2 \mu_f}, \quad \beta_p^* = \frac{\beta\rho\upsilon_f}{v_0^2} \tag{6}$$

into eqs. (1)-(5), after neglecting the star notations we have:

$$\phi_3 u_t(y,t) - u_{yy}(y,t) + \phi_3 \beta_p^* u(y,t) = \phi_5 \mathrm{Gr} T(y,t) - \phi_4 u(y,t) \tag{7}$$

$$\phi_6 \Pr T_t(y,t) = -q_y(y,t) \tag{8}$$

$$q(y,t) = -\phi_7 T_y(y,t) \tag{9}$$

With dimensionless boundary conditions:

$$T(y,0)=0, \quad y \ge 0, \quad T(0,t)=t, \quad t \ge 0, \quad T(y,t) \to 0, \quad \text{as} \quad y \to \infty \tag{10}$$

$$u(y,0)=0, \quad y \ge 0, \quad u(0,t)=0, \quad t \ge 0, \quad u(y,t) \to 0, \quad \text{as} \quad y \to \infty \tag{11}$$

In our problem, we considered a new model with Prabhakar's fractional derivative based upon generalized Fourier's law [15]:

$$q(y,t) = -{}^{C}\mathrm{D}_{\alpha,\beta,\alpha}^{\gamma} \phi_7 T_y(y,t) \tag{12}$$

Solution of fractional model

In the recent paper, we suppose $\beta \varepsilon [0,1)$ in eq. (12), m is zero. By utilizing the Laplace transform method and using it to eqs. (8), (10), and (12) we attain transformed form of the temperature field:

$$\phi_6 \Pr s\bar{T}(y,s) = -\bar{q}_y(y,s) \tag{15}$$

$$\bar{q}(y,s) = -\phi_7 s^{\beta}(1-as^{-\alpha})^{\gamma} \bar{T}_y(y,s) \tag{16}$$

$$\bar{T}(0,s) = \frac{1}{s^2}, \quad \bar{T}(y,s)=0 \quad \text{as} \quad y \to \infty \tag{17}$$

By solving eq. (16) in eq. (15) and rearranging, we attain the following differential equation:

$$\bar{T}_{yy}(y,s) = \frac{\phi_8 \Pr s}{s^{\beta}(1-as^{\alpha})^{\gamma}} \bar{T}(y,s) \tag{18}$$

Result of eq. (18), with the help of eq. (17) is:

$$\bar{T}(y,s) = s^{-2} \exp\left[\frac{-y(\phi_8 \Pr s)^{1/2}}{s^{\beta/2}(1-as^{-\alpha})^{\gamma/2}} \right] \tag{19}$$

The series form of eq. (19):

$$\bar{T}(y,s) = s^{-2} + s^{-2} \sum_{z=1}^{\infty} \frac{(-y\sqrt{\phi_8 \Pr})^z}{z!} s^{-(\beta-1)/2} (1-as^{-\alpha})^{-z\gamma/2} \tag{20}$$

The Laplace inverse of eq. (20) is:

$$T(y,t)=t+t*\sum_{z=1}^{\infty}\frac{(-y\sqrt{\phi_8 \Pr})^z}{z!}t^{\frac{(\beta-1)z}{2}-1}E_{\alpha,\frac{(\beta-1)z}{2}}^{\frac{\gamma z}{2}}(a;t) \tag{21}$$

Solution of fluid velocity

The Laplace transform of eqs. (7) and (11) we attain the velocity field:

$$\bar{u}(y,s)=\frac{s^{-2}\mathrm{Gr}}{\left[1-\dfrac{\phi_8 \Pr}{s^\beta(1-as^{-\alpha})^\gamma(s\phi_3+\phi_4+\phi_3\beta_p^*)}\right]}\left\{\frac{\mathrm{e}^{-y\left[\sqrt{\frac{\phi_8 \Pr s}{s^\beta(1-as^{-\alpha})^\gamma}}\right]}}{(s\phi_3+\phi_4+\phi_3\beta_p^*)}-\frac{\mathrm{e}^{-y\left[\sqrt{(s\phi_3+\phi_4+\phi_3\beta_p^*)}\right]}}{(s\phi_3+\phi_4+\phi_3\beta_p^*)}\right\} \tag{22}$$

where

$$\phi_0=(1-\phi)+\frac{\phi\rho_s}{\rho_f},\quad \phi_1=\frac{1}{(1-\phi)^{2.5}},\quad \phi_2=(1-\phi)+\frac{\phi(\rho\beta T)_s}{(\rho\beta T)_f},\quad \phi_3=\frac{\phi_0}{\phi_1},\quad \phi_4=\frac{M}{\phi_1},\quad \phi_5=\frac{\phi_2}{\phi_1}$$

$$\phi_6=(1-\phi)+\frac{\phi(\rho cp)_s}{(\rho cp)_f},\quad \phi_7=\frac{k_{nf}}{k_f},\quad \phi_8=\frac{\phi_6}{\phi_7}$$

Numerical results and discussion

To see the physical insight of fractional parameters α, β, and γ on-field variable fig. 1 is depicted. The Ag is considered as nanoparticle and water is taken as base fluid. By fixing other parameters constant and vary the values of fractional parameters α, β, and γ, respectively. It is observed that for $t = 1$, field variable temperature can be enhanced for larger values of α, β, and γ, respectively. In fig. 2 by fixing other parameters constant and varies the value of fraction parameter ϕ it can be observed that field variable temperature can be enhanced for a large value of volume fraction ϕ. Physically, increases in volume fraction ϕ causes increase in thermal conductivity and fluid becomes more heated and whenever the profile of temperature increased,

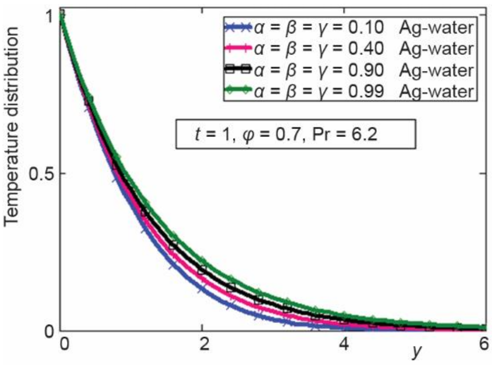

Figure 1. Temperature profile against *y* due to *α*, *β*, and *γ*

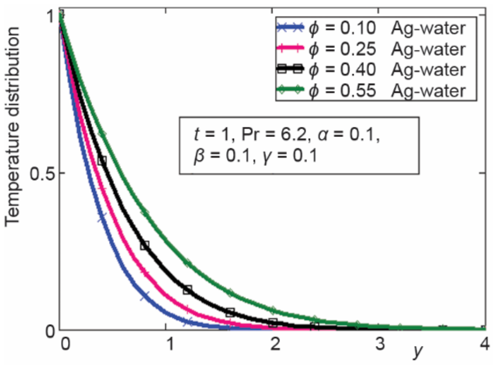

Figure 2. Temperature profile against *y* due to *φ*

this means that the transfer of heat is also increased. Figure 3 shows how fractional parameters α, β, and γ effects velocity profile. If we increase the value of fractional parameters the velocity of fluid increases. Since with the increase in fractional parameter's momentum layers increase which causes the increase in the velocity profile. Via fig. 4 it can be seen that how the volume fraction ϕ influences the fluid velocity. It can be seen that variable fluid velocity cannot be enhanced because for large values of ϕ fluid becomes more viscous which causes decrement in fluid velocity.

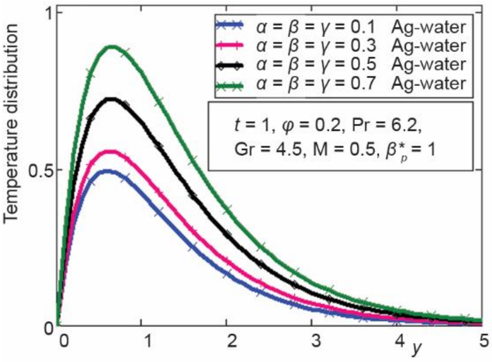

Figure 3. Velocity profile against *y* due to *α*, *β*, and *γ*

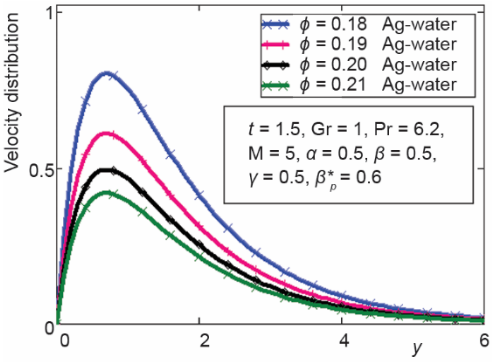

Figure 4. Velocity profile against *y* due to *φ*

Conclusions

The key resolution of this article is to introduce the unsteady thermal transport flow of Brinkman nanofluids with generalized Mittag-Leffler. In this model, we introduced fractionalized heat equation by applying Prabhakar fractional derivative with generalized Fourier's law and analytical solutions are obtained with Laplace transform. Some useful results of the present work:

- Temperature and velocity of water based silver nanoparticles can be enhanced away from the plate in the main stream region by increasing values of fractional parameters α, β, and γ.
- Further, noticed that thermal as well as momentum boundary layer thickness increases which is responsible for increasing behavior of the fluid properties by larger values of α, β, and γ.
- By increasing volume concentration of nanoparticles temperature and velocity show opposite trend and momentum boundary layer decreases.

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