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Research article

Qualitative analysis of a fuzzy Volterra-Fredholm integrodifferential equation with an Atangana-Baleanu fractional derivative

Mohammed A. Almalahi[1,2], Satish K. Panchal[1], Fahd Jarad[3,4,*], Mohammed S. Abdo [5], Kamal Shah[6,7] and Thabet Abdeljawad[6]

[1] Department of Mathematics, Dr. Babasaheb Ambedkar Marathwada University, Aurangabad, India

[2] Department of Mathematics, Hajjah University, Hajjah, Yemen

[3] Department of Mathematics, Çankaya University, Etimesgut, Ankara, Turkey

[4] Department of Medical Research, China Medical University Hospital, China Medical University, Taichung, Taiwan

[5] Department of Mathematics, Hodeidah University, Al-Hodeidah, Yemen

[6] Department of Mathematics and Sciences, Prince Sultan University, Riyadh, Saudi Arabia

[7] Department of Mathematics, University of Malakand, Chakdara Dir (Lower), Khyber Pakhtunkhawa, Pakistan

* **Correspondence:** E-mail: fahd@cankaya.edu.tr.

Abstract: The point of this work was to analyze and investigate the sufficient conditions of the existence and uniqueness of solutions for the nonlinear fuzzy fractional Volterra Fredholm integro-differential equation in the frame of the Atangana-Baleanu-Caputo fractional derivative methodology. To begin with, we give the parametric interval form of the Atangana-Baleanu-Caputo fractional derivative on fuzzy set-valued functions. Then, by employing Schauder's and Banach's fixed point procedures, we examine the existence and uniqueness of solutions for fuzzy fractional Volterra Fredholm integro-differential equation with the Atangana-Baleanu-Caputo fractional operator. It turns out that the last interval model is a combined arrangement of nonlinear equations. In addition, we consider results by applying the Adams Bashforth fractional technique and present two examples that have been numerically solved using graphs.

Keywords: Atangana Baleanu fractional derivative; fractional differential equations; fuzzy fractional derivatives; fuzzy valued functions; generalized Hukuhara differentiability; fixed point theorem
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1. Introduction

Fractional calculus (FC) was introduced around the completion of the seventeenth century as a branch of mathematical analysis that manages the investigations of different prospects of characterizing real number (or complex number) orders of differentiation (and integration) operators. FC is a generalization of classical calculus that is concerned with operations of differentiation (and integration) of non-integer order. This calculus which was first proposed by Leibniz and L'Hôspital in 1695; since then, it has been refined by various mathematicians including Euler, Laplace, Liouville, Riemann and many others. FC has acquired considerable significance due to the fact that it has become a useful tool for modeling various complex phenomena in various fields of science and engineering. The utilization of FC to explain natural phenomena and mathematical models has gotten progressively popular lately. Unlike the classical differential equations, the fractional differential equations (FDEs) have the advantage that they can better describe some natural physics and dynamic system processes [1–4] due to the fractional operators being nonlocal operators. FC for fuzzy-valued mappings was highly developed in the work of Dubois and Prade [5], wherein real line fuzzy-set-valued mappings were examined and seen as fuzzy relations. The integral of such fuzzy mappings over a crisp interval was established using Zadeh's extension concept. Also the authors of [6–8] studied the relationships and the differences between the various points of views regarding several definitions that were proposed. Ahmad et al. [9] used the fixed point theory to prove the existence and uniqueness of the fuzzy fractional Volterra-Fredholm integro-differential equation in the Caputo sense.

There has been quite a bit of investigations into and studies on approaches to fuzzy fractional derivatives (FDs) like the Riemann-Liouville (RL), Caputo, Hadamard, Caputo-Hadamard and Caputo-Katugampola. In particular, theoretical portions like the existence and uniqueness of fuzzy fractional differential equations (FFDEs) were discussed in the frame of the fuzzy RL operator (see [10–12]). The authors of [13, 14] discussed the fundamental theories and the numerical solutions of FFDEs by using a fuzzy Caputo FD. In [15], Lupulescu gave a general theory for interval fractional analysis which is an essential tool in the survey of FFDEs. Additionally, diverse approaches to determine the existence and stability of the solution of FFDEs involving a fuzzy Caputo operator have been given in papers [16–18]. In addition, sundry types of optimal control problems of fuzzy fractional evolution equations have been studied by Agarwal et al. [19]. In another context, to solve the FFDEs, some modified setups of the Euler, Laplace transform and Adams-Bashforth-Moulton within the fuzzy framework was suggested in papers [20–23]. Recently, Ahmadian et al. [24, 25] proposed a method based on an operational matrix of shifted Chebyshev polynomials and the spectral tau to solve FFDEs. Vinothkumar et al. [26] introduced the finite-difference technique to solve approximate solutions of fuzzy wave equations involving the Caputo operator. There are new analytical studies related to FFDEs that have beeen presented by various researchers [27–29]. On the other hand, Atangana and Baleanu [30] proposed a novel operator and used it in many applied science field. Recent works of various problems and epidemic models involving this operator can be seen in [30–33]. Ndolane Sene [34] introduced a new four-dimensional hyperchaotic financial model using the quadratic function to accurately model the financial market; Sene also studied the existence and uniqueness of its solutions to justify the physical adequacy of the model and the numerical scheme proposed in the resolution. Ndolane Sene [35] presented a modified chaotic system that incorporates the fractional operator with singularity. He studied the influence of the new model's parameters and its fractional order using the bifurcation diagrams and the Lyapunov exponents. Owolabi and colleagues [36–40] studied some applications of the Atangana-Baleanu-Caputo fractional operator to model some symbiotic systems such as commensalism and predator-prey processes, the parasitic predator-prey model, the commensalism system and the mutualism case. He et al. [41] presented a new fractional-order discrete-time susceptible-infected-recovered epidemic model with vaccination. Jin et al. [42] developed the dynamical behavior of the drinking population through the fractional drinking model by applying the Caputo-Fabrizio arbitrary order operator along with the special non-singular kernel. The authors of [43] investigated a fractional model in the Caputo sense to explore the dynamics of the Zika virus.

Motivated by the preceding works mentioned above, in the current paper, we investigate some existence, uniqueness and numerical results by applying the fractional Adams Bashforth method (FABM) of the following nonlinear fuzzy fractional Volterra Fredholm integro-differential equation (FFVFIE)

$$\begin{cases} {}_0^{ABC}D_\eta^{\mathfrak{q}}\widehat{\omega}(\eta,\alpha) = \mathfrak{g}(\eta) + \mathfrak{a}(\eta)\widehat{\omega}(\eta,\alpha) + \int_0^\eta \mathcal{K}_1(\eta,s)\mathcal{N}_1(\widehat{\omega}(s,\alpha))ds \\ \qquad\qquad + \int_0^1 \mathcal{K}_2(\eta,s)\mathcal{N}_2(\widehat{\omega}(s,\alpha))ds, \\ \widehat{\omega}(0,\alpha) = \widehat{\omega}_0, \end{cases} \tag{1.1}$$

where ${}_0^{ABC}D_\eta^{\mathfrak{q}}$ represents the ABC-type FDs of order $\mathfrak{q}$ and $\widehat{\omega}(\eta,\alpha)$ is a fuzzy number such that $\widehat{\omega}(\eta,\alpha) = \left[\underline{\omega}(\eta,\alpha), \overline{\omega}(\eta,\alpha)\right]$, $\mathcal{K}_1, \mathcal{K}_2 : \mathcal{J}\times\mathcal{J} \to \mathbb{R}$, $\mathfrak{g} : \mathcal{J} \to \mathbb{R}$ are continuous functions, $\mathcal{N}_1, \mathcal{N}_2 : \mathcal{J} \to \mathbb{R}$ are Lipschitz continuous functions $\mathcal{J} := [0,1]$ and $\widehat{\omega}(\eta,\alpha) \in C^F(\mathcal{J},\mathbb{R}) \cap L^F(\mathcal{J},\mathbb{R})$ where $C^F(\mathcal{J},\mathbb{R})$ and $L^F(\mathcal{J},\mathbb{R})$ are the space of all continuous fuzzy valued functions and the space of all Lebesgue integrable fuzzy valued functions on $\mathcal{J}$ respectively.

We note that the fuzzy calculus and the fuzzy differential equations have attracted a lot of attention from scholars in recent years. This is because fuzzy calculus and fuzzy differential equations have a wide range of applications in many mathematical and computer models of deterministic real-world processes with uncertainty. Because of the applications of FC and FDEs in real-world systems, as well as the existence of uncertainties and disturbances in dynamic systems, fuzzy FC and FDEs have recently emerged as significant topics and the topic of FC and fractional dynamic systems in the fuzzy setting can be used as an important mathematical tool for modeling real-world systems. As a result, considering and analyzing fractional-order uncertain dynamical systems is critical in terms of both study and application and this topic has piqued scholars' interest in recent years.

In this work, we develop some theorems of FC related to the nonlinear FFVFIE and learn more properties of the proposed ABC problem which makes use of non-singular kernel derivatives with fractional order. The task was to determine whether or not the proposed ABC problem has position solutions. Volterra Fredholm integro-differential equations are used to investigate a more reliable and appropriate ABC problem for some novel real-world problems.

The major contribution of the current paper is to generalize the idea proposed by Ahmed et al. [9] by using fuzzy concepts and studying the problem FFVFIE (1.1) in the Caputo sense. In the beginning, we define the parametric interval form of the ABC-FD on fuzzy set-valued functions. Then, by utilizing the fixed-point techniques of Schauder and Banach, we investigate the existence and uniqueness of the solution for FFVFIE (1.1). In general, it turns out that the last interval model is a coupled system of nonlinear equations. The numerical results are discussed by applying the FABM. For more clarification, sundry examples are solved numerically and analyzed by using graphs. To the best of our knowledge, this is the first work in the literature that deals with FFVFIEs involving the interval

ABC-fractional derivative. The results of this work will therefore make a useful contribution to the existing literature on this subject.

This paper is organized as follows: In Section 2, we render the rudimentary definitions and prove some lemmas that are applied throughout this paper and present the concepts of some fixed point theorems. In Section 3, we prove the existence and uniqueness of solutions for the proposed problem defined by Eq (3.1). Section 5 gives pertinent examples to illustrate our results. We provide the conclusion in the last section.

2. Background material and auxiliary results

In this section, we introduce the concept of the ABC fractional derivative in the fuzzy sense and supply some fundamental theories. Furthermore, we establish some necessary results employed throughout this paper.

Definition 2.1. *[30] Let* $(0, b)$ *be an open interval and let* $\omega(\eta)$ *be a function differentiable in* $(0, b)$*. Then, the left-sided ABC fractional derivative with the lower limit zero of order* $\mathfrak{q} \in (0, 1]$ *of a function* $\omega(\eta)$ *is given as*

$$^{ABC}D_0^{\mathfrak{q}}\omega(\eta) = \frac{\Psi(\mathfrak{q})}{1-\mathfrak{q}}\int_0^{\eta} E_{\mathfrak{q}}\left(\frac{-\mathfrak{q}}{\mathfrak{q}-1}(\eta - s)^{\mathfrak{q}}\right)ds,$$

where $\Psi(\mathfrak{q}) = 1 - \mathfrak{q} + \frac{\mathfrak{q}}{\Gamma(\mathfrak{q})}$ *and* $E_{\mathfrak{q}}$ *is the Mittag-Leffler function defined by*

$$E_{\mathfrak{q}}(\eta) = \sum_{i=0}^{\infty}\frac{\eta^i}{\Gamma(i\mathfrak{q}+1)},\ Re(\mathfrak{q}) > 0, \eta \in \mathbb{C}.$$

Definition 2.2. *[30] If* $\omega(\eta)$ *is a continuous function in the interval* $(0, b)$*, then, the left-sided AB fractional integral with the lower limit zero of order* $\mathfrak{q} \in (0, 1]$ *for a function* ω *is defined as:*

$$^{AB}I_0^{\mathfrak{q}}\omega(\eta) = \frac{1-\mathfrak{q}}{\Psi(\mathfrak{q})}\omega(\eta) + \frac{\mathfrak{q}}{\Psi(\mathfrak{q})}\frac{1}{\Gamma(\mathfrak{q})}\int_0^{\eta}(\eta - s)^{\mathfrak{q}-1}\omega(s)ds.$$

Definition 2.3. *[44] A fuzzy number* ϑ *is a map* $\vartheta : \mathbb{R} \to \mathcal{J}$ *such that* ϑ *satisfies the following properties*

- ϑ *is upper semi-continuous, fuzzy convex and normal.*
- *The closure of supp*(ϑ) *is compact.*

Definition 2.4. *[44] Let* $\widehat{\vartheta}$ *be a fuzzy number. Then the parametric interval form of* $\widehat{\vartheta}$ *is given by*

$$\widehat{\vartheta} = \left[\underline{\vartheta}(\alpha), \overline{\vartheta}(\alpha)\right], \alpha \in [0, 1],$$

where

- $\underline{\vartheta}(\alpha)$ *is a left continuous and nondecreasing function with respect to* α*.*
- $\overline{\vartheta}(\alpha)$ *is a right-continuous and nondecreasing function with respect to* α*.*
- *For each* $\alpha \in \mathcal{J}$*, we have* $\overline{\vartheta}(\alpha) \geq \underline{\vartheta}(\alpha)$*.*

Definition 2.5. *[44] Let* ϑ *and* v *be two fuzzy numbers. Then, the arithmetic operations are given as*

$$\left(\widehat{\vartheta} \oplus \widehat{v}\right) = \left[\underline{\vartheta}(\alpha) + \overline{\vartheta}(\alpha), \underline{v}(\alpha) + \overline{v}(\alpha)\right],$$

$$\left(\lambda \odot \widehat{\vartheta}\right) = \begin{cases} \left[\lambda\underline{\vartheta}(\alpha), \lambda\overline{\vartheta}(\alpha)\right], \lambda \geq 0, \\ \left[\lambda\overline{\vartheta}(\alpha), \lambda\underline{\vartheta}(\alpha)\right], \lambda < 0, \end{cases}$$

where $\alpha \in \mathcal{J}$.

Assume that the fuzzy valued function $\widehat{\omega}(\eta, \alpha) \in C^F(\mathcal{J}, \mathbb{R}) \cap L^F(\mathcal{J}, \mathbb{R})$. *Then, the parametric interval form of* $\omega(\eta, \alpha)$ *is*

$$\widehat{\omega}(\eta, \alpha) = \left[\underline{\omega}(\eta; \alpha), \overline{\omega}(\eta; \alpha)\right], \alpha \in [0, 1].$$

Definition 2.6. *[45] Let* ϑ, v *be two fuzzy numbers belonging to* $\mathbb{R}^F$ *where* $\mathbb{R}^F$ *is the set of all fuzzy numbers on real numbers. If there exists a fuzzy number* $w \in \mathbb{R}^F$ *such that* $\vartheta = v + w$, *then* w *is called the Hukuhara difference of* ϑ *and* v *and it is denoted by* $\vartheta \ominus v$.

Definition 2.7. *[45] (Generalized Hukuhara derivative) Let* $\omega(\eta)$ *be a solution of FFVFIE (1.1). Then the gH-derivative of the function* $\omega(\eta)$ *can be defined as*

$$_{gH}\omega'(\eta) = \lim_{h\to 0^+} \frac{\omega(\tau + h) \ominus_{gH} \omega(\tau)}{h} = \lim_{h\to 0^+} \frac{\omega(\tau) \ominus_{gH} \omega(\tau + h)}{h}, \tag{2.1}$$

where

$$_{gH}\omega'(\eta) \in C^F(I) \cap L^F(I),$$

and

$$\omega(\tau + h) \ominus_{gH} \omega(\tau) = g(\tau) \text{ iff } \begin{cases} i)\ \omega(\tau + h) = \omega(\tau) \oplus g(\tau), \\ ii)\ \omega(\tau) = \omega(\tau + h) \oplus (-1)\, g(\tau). \end{cases} \tag{2.2}$$

The case (i) $\Longrightarrow$ *definition of Hukuhara difference of* $\omega(\tau + h) \ominus \omega(\tau)$. *Taking an* α*-cut of both sides of Eq (2.1) and using the definition of the gH-difference given by Eq (2.2), we get the following two cases:*

Case 2.1. *(i-Differentiability)*

$$_{i-gH}\left[\widehat{\omega}'(\eta; \alpha)\right] = \left[\underline{\omega}'(\eta; \alpha), \overline{\omega}'(\eta; \alpha)\right].$$

Case 2.2. *(ii-Differentiability)*

$$_{ii-gH}\left[\widehat{\omega}'(\eta; \alpha)\right] = \left[\overline{\omega}'(\eta; \alpha), \underline{\omega}'(\eta; \alpha)\right].$$

In the forthcoming definitions, we will introduce the concept of the fuzzy fractional derivative and fractional integral in the frame of the ABC fractional operator.

Definition 2.8. *[45] The ABC fractional derivative is given in two cases, as follows:*

$$\begin{aligned} {}^{ABC}_{\ 0}D^{i,\mathfrak{q}}_{\eta}\widehat{\omega}(\eta; \alpha) &= \left[{}^{ABC}_{\ 0}D^{i,\mathfrak{q}}_{\eta}\underline{\omega}(\eta; \alpha), {}^{ABC}_{\ 0}D^{i,\mathfrak{q}}_{\eta}\overline{\omega}(\eta; \alpha)\right], \quad \textit{Case (2.1)}, \\ {}^{ABC}_{\ 0}D^{ii,\mathfrak{q}}_{\eta}\widehat{\omega}(\eta; \alpha) &= \left[{}^{ABC}_{\ 0}D^{ii,\mathfrak{q}}_{\eta}\overline{\omega}(\eta; \alpha), {}^{ABC}_{\ 0}D^{i,\mathfrak{q}}_{\eta}\underline{\omega}(\eta; \alpha)\right], \quad \textit{Case (2.2)}, \end{aligned}$$

where

$${}^{ABC}_{\ 0}D^{i,\mathfrak{q}}_{\eta}\widehat{\omega}(\eta; \alpha) = \begin{cases} \frac{\Psi(\mathfrak{q})}{1-\mathfrak{q}} \int_0^{\eta} \left({}_{i-gH}\widehat{\omega}'(\tau; \alpha)\right) E_{\mathfrak{q}}\left(-\frac{\mathfrak{q}}{1-\mathfrak{q}}(\eta - \tau)^{\mathfrak{q}}\right) |_{E_{\mathfrak{q}} \geq 0}\, d\tau \\ + \frac{\Psi(\mathfrak{q})}{1-\mathfrak{q}} \int_0^{\eta} \left({}_{ii-gH}\widehat{\omega}'(\tau; \alpha)\right) E_{\mathfrak{q}}\left(-\frac{\mathfrak{q}}{1-\mathfrak{q}}(\eta - \tau)^{\mathfrak{q}}\right) |_{E_{\mathfrak{q}} < 0}\, d\tau, \end{cases}$$

and

$$ {}_{0}^{ABC}D_{\eta}^{ii,\mathrm{q}}\widehat{\omega}(\eta;\alpha) = \begin{cases} \frac{\Psi(\mathrm{q})}{1-\mathrm{q}} \int_0^\eta \left({}_{ii-gH}\widehat{\omega}'(\tau;\alpha) \right) E_{\mathrm{q}} \left(-\frac{\mathrm{q}}{1-\mathrm{q}} (\eta-\tau)^{\mathrm{q}} \right) |_{E_{\mathrm{q}} \geq 0} \, d\tau \\ + \frac{\Psi(\mathrm{q})}{1-\mathrm{q}} \int_0^\eta \left({}_{i-gH}\widehat{\omega}'(\tau;\alpha) \right) E_{\mathrm{q}} \left(-\frac{\mathrm{q}}{1-\mathrm{q}} (\eta-\tau)^{\mathrm{q}} \right) |_{E_{\mathrm{q}} < 0} \, d\tau. \end{cases} $$

Then, the end points of the ABC fractional derivative are

$$ {}_{0}^{ABC}D_{\eta}^{*,\mathrm{q}}\underline{\omega}(\eta;\alpha) = \begin{cases} \frac{\Psi(\mathrm{q})}{1-\mathrm{q}} \int_0^\eta \underline{\omega}'(\tau;\alpha) E_{\mathrm{q}} \left(-\frac{\mathrm{q}}{1-\mathrm{q}} (\eta-\tau)^{\mathrm{q}} \right) |_{E_{\mathrm{q}} \geq 0} \, d\tau \\ + \frac{\Psi(\mathrm{q})}{1-\mathrm{q}} \int_0^\eta \overline{\omega}'(\tau;\alpha) E_{\mathrm{q}} \left(-\frac{\mathrm{q}}{1-\mathrm{q}} (\eta-\tau)^{\mathrm{q}} \right) |_{E_{\mathrm{q}} < 0} \, d\tau, \end{cases} $$

and

$$ {}_{0}^{ABC}D_{\eta}^{*,\mathrm{q}}\overline{\omega}(\eta;\alpha) = \begin{cases} \frac{\Psi(\mathrm{q})}{1-\mathrm{q}} \int_0^\eta \overline{\omega}'(\tau;\alpha) E_{\mathrm{q}} \left(-\frac{\mathrm{q}}{1-\mathrm{q}} (\eta-\tau)^{\mathrm{q}} \right) |_{E_{\mathrm{q}} \geq 0} \, d\tau \\ + \frac{\Psi(\mathrm{q})}{1-\mathrm{q}} \int_0^\eta \underline{\omega}'(\tau;\alpha) E_{\mathrm{q}} \left(-\frac{\mathrm{q}}{1-\mathrm{q}} (\eta-\tau)^{\mathrm{q}} \right) |_{E_{\mathrm{q}} < 0} \, d\tau, \end{cases} $$

where $* \in \{i, ii\}$.

Definition 2.9. *[45] The AB fractional integral associated with the ABC fractional derivative on fuzzy-valued functions of the interval parametric type is defined by*

$$ \begin{aligned} & {}_{0}^{AB}I_{\eta}^{\mathrm{q}}\widehat{\omega}(\eta;\alpha) \\ = & \left[{}_{0}^{ABC}I_{\eta}^{\mathrm{q}}\underline{\omega}(\eta;\alpha), {}_{0}^{ABC}I_{\eta}^{\mathrm{q}}\overline{\omega}(\eta;\alpha) \right] \\ = & \begin{bmatrix} \frac{1-\mathrm{q}}{\Psi(\mathrm{q})}\underline{\omega}(\eta;\alpha) + \frac{\mathrm{q}}{\Psi(\mathrm{q})\Gamma(\mathrm{q})} \int_0^\eta (\eta-\tau)^{\mathrm{q}-1} \underline{\omega}(\tau;\alpha) d\tau, \\ \frac{1-\mathrm{q}}{\Psi(\mathrm{q})}\overline{\omega}(\eta;\alpha) + \frac{\mathrm{q}}{\Psi(\mathrm{q})\Gamma(\mathrm{q})} \int_0^\eta (\eta-\tau)^{\mathrm{q}-1} \overline{\omega}(\tau;\alpha) d\tau \end{bmatrix}. \end{aligned} $$

Definition 2.10. *[45] Let* $\mathrm{q} \in (0,1)$ *. Then*

$$ \begin{aligned} {}_{0}^{AB}I_{\eta}^{\mathrm{q}}\left({}_{0}^{AB}D_{\eta}^{i,\mathrm{q}}\widehat{\omega}(\eta;\alpha) \right) &= \left[{}_{0}^{AB}I_{\eta}^{\mathrm{q}}\left({}_{0}^{ABC}D_{\eta}^{i,\mathrm{q}}\underline{\omega}(\eta;\alpha) \right), {}_{0}^{AB}I_{\eta}^{\mathrm{q}}\left({}_{0}^{ABC}D_{\eta}^{i,\mathrm{q}}\overline{\omega}(\eta;\alpha) \right) \right], \quad \textit{Case (2.1)}, \\ {}_{0}^{AB}I_{\eta}^{\mathrm{q}}\left({}_{0}^{ABC}D_{\eta}^{ii,\mathrm{q}}\widehat{\omega}(\eta;\alpha) \right) &= \left[{}_{0}^{AB}I_{\eta}^{\mathrm{q}}\left({}_{0}^{ABC}D_{\eta}^{ii,\mathrm{q}}\overline{\omega}(\eta;\alpha) \right), {}_{0}^{AB}I_{\eta}^{\mathrm{q}}\left({}_{0}^{ABC}D_{\eta}^{ii,\mathrm{q}}\underline{\omega}(\eta;\alpha) \right) \right], \quad \textit{Case (2.2)}, \end{aligned} $$

where

$$ {}_{0}^{AB}I_{\eta}^{\mathrm{q}}\left[{}_{0}^{ABC}D_{\eta}^{*,\mathrm{q}}\underline{\omega}(\eta;\alpha) \right] = \begin{cases} {}_{0}^{AB}I_{\eta}^{\mathrm{q}}\left(\frac{\Psi(\mathrm{q})}{1-\mathrm{q}} \int_0^\eta \underline{\omega}'(\tau;\alpha) E_{\mathrm{q}} \left(-\frac{\mathrm{q}}{1-\mathrm{q}} (\eta-\tau)^{\mathrm{q}} \right) |_{E_{\mathrm{q}} \geq 0} \, d\tau \right) \\ + {}_{0}^{AB}I_{\eta}^{\mathrm{q}}\left(\frac{\Psi(\mathrm{q})}{1-\mathrm{q}} \int_0^\eta \overline{\omega}'(\tau;\alpha) E_{\mathrm{q}} \left(-\frac{\mathrm{q}}{1-\mathrm{q}} (\eta-\tau)^{\mathrm{q}} \right) |_{E_{\mathrm{q}} < 0} \, d\tau \right), \end{cases} $$

and

$$ {}_{0}^{AB}I_{\eta}^{\mathrm{q}}\left[{}_{0}^{ABC}D_{\eta}^{*,\mathrm{q}}\overline{\omega}(\eta;\alpha) \right] = \begin{cases} {}_{0}^{AB}I_{\eta}^{\mathrm{q}}\left(\frac{\Psi(\mathrm{q})}{1-\mathrm{q}} \int_0^\eta \overline{\omega}'(\tau;\alpha) E_{\mathrm{q}} \left(-\frac{\mathrm{q}}{1-\mathrm{q}} (\eta-\tau)^{\mathrm{q}} \right) |_{E_{\mathrm{q}} \geq 0} \, d\tau \right) \\ + {}_{0}^{AB}I_{\eta}^{\mathrm{q}}\left(\frac{\Psi(\mathrm{q})}{1-\mathrm{q}} \int_0^\eta \underline{\omega}'(\tau;\alpha) E_{\mathrm{q}} \left(-\frac{\mathrm{q}}{1-\mathrm{q}} (\eta-\tau)^{\mathrm{q}} \right) |_{E_{\mathrm{q}} < 0} \, d\tau \right), \end{cases} $$

where $* \in \{i, ii\}$.

Lemma 2.1. *[46] (Leray-Schauder alternative) Let* $G : \mathcal{X} \to \mathcal{X}$ *be a completely continuous operator and* $\xi(G) = \{y \in \mathcal{X} : y = \delta G(y), \delta \in [0,1]\}$ *. Then either the set* $\xi(G)$ *is unbounded or G has at least one fixed point.*

Theorem 2.1. *[47] (Banach fixed point theorem) Let* $\mathcal{X}$ *be a Banach space,* $K \subset \mathcal{X}$ *be closed, and* $G : K \to K$ *be a strict contraction, i.e.,* $\|G(x) - G(y)\| \leq L\|x - y\|$ *for some* $0 < L < 1$ *for all* $x, y \in K$*. Then G has a fixed point in K.*

Lemma 2.2. *If we suppose that,*

$$ {}_{0}^{ABC}D_{\eta}^{\mathfrak{q}}\widehat{\omega}(\eta;\alpha)=\widehat{\vartheta}(\eta;\alpha), $$

then, the solution is found in each case as

$$ \omega(\eta;\alpha)=\begin{cases} \frac{1-\mathfrak{q}}{\Psi(\mathfrak{q})}\odot\vartheta(\eta;\alpha)\oplus\frac{\mathfrak{q}}{\Psi(\mathfrak{q})\Gamma(\mathfrak{q})}\odot\int_0^{\eta}\vartheta(\tau;\alpha)\odot(\eta-\tau)^{\mathfrak{q}-1}\,d\tau & \text{Case (2.1)},\\ \ominus(-1)\frac{1-\mathfrak{q}}{\Psi(\mathfrak{q})}\odot\vartheta(\eta;\alpha)\ominus(-1)\frac{\mathfrak{q}}{\Psi(\mathfrak{q})\Gamma(\mathfrak{q})}\odot\int_0^{\eta}\vartheta(\tau;\alpha)\odot(\eta-\tau)^{\mathfrak{q}-1}\,d\tau, & \text{Case (2.2)}. \end{cases} $$

In the interval parametric form, they are

In Case (2.1)

$$ \begin{aligned} \left[\underline{\omega}(\eta;\alpha),\overline{\omega}(\eta;\alpha)\right] &= \left[\frac{1-\mathfrak{q}}{\Psi(\mathfrak{q})}\underline{\vartheta}(\eta;\alpha)+\frac{\mathfrak{q}}{\Psi(\mathfrak{q})\Gamma(\mathfrak{q})}\int_0^{\eta}\underline{\vartheta}(\eta;\alpha)\,(\eta-\tau)^{\mathfrak{q}-1}\,d\tau\right.\\ &\quad\left.,\frac{1-\mathfrak{q}}{\Psi(\mathfrak{q})}\overline{\vartheta}(\eta;\alpha)+\frac{\mathfrak{q}}{\Psi(\mathfrak{q})\Gamma(\mathfrak{q})}\int_0^{\eta}\overline{\vartheta}(\eta;\alpha)\,(\eta-\tau)^{\mathfrak{q}-1}\,d\tau\right]. \end{aligned} $$

In Case (2.2)

$$ \begin{aligned} \left[\underline{\omega}(\eta;\alpha),\overline{\omega}(\eta;\alpha)\right] &= \left[\frac{1-\mathfrak{q}}{\Psi(\mathfrak{q})}\overline{\vartheta}(\eta;\alpha)+\frac{\mathfrak{q}}{\Psi(\mathfrak{q})\Gamma(\mathfrak{q})}\int_0^{\eta}\overline{\vartheta}(\eta;\alpha)\,(\eta-\tau)^{\mathfrak{q}-1}\,d\tau\right.\\ &\quad\left.,\frac{1-\mathfrak{q}}{\Psi(\mathfrak{q})}\underline{\vartheta}(\eta;\alpha)+\frac{\mathfrak{q}}{\Psi(\mathfrak{q})\Gamma(\mathfrak{q})}\int_0^{\eta}\underline{\vartheta}(\eta;\alpha)\,(\eta-\tau)^{\mathfrak{q}-1}\,d\tau\right]. \end{aligned} $$

Proof. See [9]. □

Remark 2.1. *In this research paper, we shall prove the existence and uniqueness results for Case (2.1) only.*

3. Existence and uniqueness of solution

In this part, we discuss the existence and uniqueness of the solution to the problem of FFVFIE (1.1). Let $\mathcal{J}=[0,1]\subset\mathbb{R}$ and $C^F(\mathcal{J},\mathbb{R})$ be the space of all continuous functions $\omega:\mathcal{J}\to\mathbb{R}$ with the norm $\|\omega\|_\infty=\max\{|\omega(\sigma)|:\sigma\in\mathcal{J}\}$. Then $\left(C^F(\mathcal{J},\mathbb{R}),\|\cdot\|_\infty\right)$ is a Banach space. For our analysis, the following hypotheses must be satisfied

(H_1) For each $\widehat{\omega}_1,\widehat{\omega}_2\in C^F(\mathcal{J})$, there exist two constants $c_1,c_2>0$ such that

$$ \begin{aligned} \left|\mathcal{N}_1(\widehat{\omega}_1(\eta;\alpha))-\mathcal{N}_1(\widehat{\omega}_2(\eta;\alpha))\right| &\le c_1\left|\widehat{\omega}_1(\eta;\alpha)-\widehat{\omega}_2(\eta;\alpha)\right|,\\ \left|\mathcal{N}_2(\widehat{\omega}_1(\eta;\alpha))-\mathcal{N}_2(\widehat{\omega}_2(\eta;\alpha))\right| &\le c_2\left|\widehat{\omega}_1(\eta;\alpha)-\widehat{\omega}_2(\eta;\alpha)\right|. \end{aligned} $$

(H_2) For the set

$$ F=\left\{(\eta,s)\in\mathbb{R}^2:0\le s\le\eta\le1\right\}, $$

there exist the functions $\mathcal{K}_1^*$ and $\mathcal{K}_2^*$ such that

$$ \begin{aligned} \mathcal{K}_1^* &= \sup_{\eta\in\mathcal{J}}\int_0^{\eta}|\mathcal{K}_1(\eta,s)|\,ds<\infty,\\ \mathcal{K}_2^* &= \sup_{\eta\in\mathcal{J}}\int_0^{\eta}|\mathcal{K}_2(\eta,s)|\,ds<\infty. \end{aligned} $$

(H_3) The functions $\mathfrak{a}$ and $\mathfrak{g}$ are continuous

Lemma 3.1. *Let* $\mathfrak{q} \in (0, 1]$ *. Then, the solution of the problem*

$$\begin{aligned} {}_{0}^{ABC}D_{\eta}^{\mathfrak{q}}\widehat{\omega}(\eta;\alpha) &= \widehat{g}(\eta;\alpha) \\ \widehat{\omega}(0;\alpha) &= \widehat{\omega}_0, \end{aligned} \tag{3.1}$$

is given by

$$\widehat{\omega}(\eta;\alpha) = \widehat{\omega}_0 + \frac{1-\mathfrak{q}}{\Psi(\mathfrak{q})}\widehat{g}(\eta;\alpha) + \frac{\mathfrak{q}}{\Psi(\mathfrak{q})\Gamma(\mathfrak{q})}\int_0^{\eta}(\eta-s)^{\mathfrak{q}-1}\widehat{g}(s;\alpha)ds.$$

Proof. Assume that $\widehat{\omega}(\eta;\alpha)$ is a solution of the first equation of Eq (3.1). Applying the operator ${}_{0}^{AB}I_{\eta}^{\mathfrak{q}}$ to both sides of the first equation of Eq (3.1), we get

$${}_{0}^{AB}I_{\eta}^{\mathfrak{q}}\,{}_{0}^{ABC}D_{\eta}^{\mathfrak{q}}\widehat{\omega}(\eta;\alpha) = {}_{0}^{AB}I_{\eta}^{\mathfrak{q}}\widehat{g}(\eta;\alpha).$$

Then, we have

$$\begin{aligned} \widehat{\omega}(\eta;\alpha) &= \widehat{\omega}(0;\alpha) + \frac{1-\mathfrak{q}}{\Psi(\mathfrak{q})}\widehat{g}(\eta;\alpha) + \frac{\mathfrak{q}}{\Psi(\mathfrak{q})\Gamma(\mathfrak{q})}\int_0^{\eta}(\eta-s)^{\mathfrak{q}-1}\widehat{g}(s;\alpha)ds \\ &= \widehat{\omega}_0 + \frac{1-\mathfrak{q}}{\Psi(\mathfrak{q})}\widehat{g}(\eta;\alpha) + \frac{\mathfrak{q}}{\Psi(\mathfrak{q})\Gamma(\mathfrak{q})}\int_0^{\eta}(\eta-s)^{\mathfrak{q}-1}\widehat{g}(s;\alpha)ds. \end{aligned}$$

□

Theorem 3.1. *Assume that* (H_1)*–*(H_3) *are satisfied. If*

$$\Omega_1 = \left(\frac{1-\mathfrak{q}}{\Psi(\mathfrak{q})} + \frac{\mathfrak{q}}{\Psi(\mathfrak{q})\Gamma(\mathfrak{q}+1)}\right)\|\mathfrak{a}\|_{\infty} < 1, \tag{3.2}$$

then the problem given by FFVFIE (1.1) has at least one solution $\widehat{\omega}(\eta,\alpha)$*.*

Proof. In light of Lemma 3.1, the solution of the FFVFIE (1.1) is given as

$$\begin{aligned} \widehat{\omega}(\eta;\alpha) &= \widehat{\omega}_0 + \frac{1-\mathfrak{q}}{\Psi(\mathfrak{q})} \\ &\left[\mathfrak{g}(\eta) + \mathfrak{a}(\eta)\widehat{\omega}(\eta;\alpha) + \int_0^{\eta}\mathcal{K}_1(\eta,s)\mathcal{N}_1(\widehat{\omega}(s,\alpha))ds + \int_0^1\mathcal{K}_2(\eta,s)\mathcal{N}_2(\widehat{\omega}(s,\alpha))ds\right] \\ &+\frac{\mathfrak{q}}{\Psi(\mathfrak{q})\Gamma(\mathfrak{q})}\int_0^{\eta}(\eta-s)^{\mathfrak{q}-1} \\ &\left[\mathfrak{g}(s) + \mathfrak{a}(s)\widehat{\omega}(s,\alpha) + \int_0^{s}\mathcal{K}_1(s,\tau)\mathcal{N}_1(\widehat{\omega}(\tau,\alpha))d\tau + \int_0^1\mathcal{K}_2(s,\tau)\mathcal{N}_2(\widehat{\omega}(\tau,\alpha))d\tau\right]ds. \end{aligned}$$

Define the operator $\Xi : C^F(\mathcal{J},\mathbb{R}) \cap L^F(\mathcal{J},\mathbb{R}) \to C^F(\mathcal{J},\mathbb{R}) \cap L^F(\mathcal{J},\mathbb{R})$ as

$$\begin{aligned} \Xi\widehat{\omega}(\eta;\alpha) &= \widehat{\omega}_0 + \frac{1-\mathfrak{q}}{\Psi(\mathfrak{q})} \\ &\left[\mathfrak{g}(\eta) + \mathfrak{a}(\eta)\widehat{\omega}(\eta;\alpha) + \int_0^{\eta}\mathcal{K}_1(\eta,s)\mathcal{N}_1(\widehat{\omega}(s;\alpha))ds + \int_0^1\mathcal{K}_2(\eta,s)\mathcal{N}_2(\widehat{\omega}(s;\alpha))ds\right] \end{aligned}$$

$$+\frac{\mathfrak{q}}{\Psi(\mathfrak{q})\Gamma(\mathfrak{q})}\int_0^{\eta}(\eta-s)^{\mathfrak{q}-1}$$
$$\left[\mathfrak{g}(s)+\mathfrak{a}(s)\widehat{\omega}(s;\alpha)+\int_0^{s}\mathcal{K}_1(s,\tau)\mathcal{N}_1(\widehat{\omega}(\tau;\alpha))d\tau+\int_0^{1}\mathcal{K}_2(s,\tau)\mathcal{N}_2(\widehat{\omega}(\tau;\alpha))d\tau\right]ds.$$

Now, we will prove that the operator Ξ has a fixed point by using Theorem 2.1. For that, we divide the proof into the following steps.

Step 1. Ξ is continuous.

Let $\widehat{\omega}_n$ be a sequence such that $\widehat{\omega}_n \to \widehat{\omega}$ in $C\left(\mathcal{J},\mathbb{R}^F\right)$. Then, for $\eta\in\mathcal{J}$, we have

$$\begin{aligned}
&\left|\Xi\widehat{\omega}_n(\eta,\alpha)-\Xi\widehat{\omega}(\eta,\alpha)\right|\\
\leq\ &\frac{1-\mathfrak{q}}{\Psi(\mathfrak{q})}\left[|\mathfrak{a}(\eta)|\left|\widehat{\omega}_n(\eta,\alpha)-\widehat{\omega}(\eta,\alpha)\right|+\int_0^{\eta}|\mathcal{K}_1(\eta,s)|\left|\mathcal{N}_1(\widehat{\omega}_n(s,\alpha))-\mathcal{N}_1(\widehat{\omega}(s,\alpha))\right|ds\right.\\
&\left.+\int_0^{1}|\mathcal{K}_2(\eta,s)|\left|\mathcal{N}_2(\widehat{\omega}_n(s,\alpha))-\mathcal{N}_2(\widehat{\omega}(s,\alpha))\right|ds\right]+\frac{\mathfrak{q}}{\Psi(\mathfrak{q})\Gamma(\mathfrak{q})}\int_0^{\eta}(\eta-s)^{\mathfrak{q}-1}\\
&\left[|\mathfrak{a}(s)|\left|\widehat{\omega}_n(s,\alpha)-\widehat{\omega}(s,\alpha)\right|+\int_0^{s}|\mathcal{K}_1(s,\tau)|\left|\mathcal{N}_1(\widehat{\omega}_n(\tau,\alpha))-\mathcal{N}_1(\widehat{\omega}(\tau,\alpha))\right|d\tau\right.\\
&\left.+\int_0^{1}|\mathcal{K}_2(s,\tau)|\left|\mathcal{N}_2(\widehat{\omega}_n(\tau,\alpha))-\mathcal{N}_2(\widehat{\omega}(\tau,\alpha))\right|d\tau\right]ds.
\end{aligned}$$

Taking the supremum on both sides, we get

$$\begin{aligned}
&\left\|\Xi\widehat{\omega}_n(\eta,\alpha)-\Xi\widehat{\omega}(\eta,\alpha)\right\|_\infty\\
\leq\ &\frac{1-\mathfrak{q}}{\Psi(\mathfrak{q})}\\
&\left[\|\mathfrak{a}\|_\infty\left\|\widehat{\omega}_n-\widehat{\omega}\right\|_\infty+\mathcal{K}_1^*\left\|\mathcal{N}_1(\widehat{\omega}_n)-\mathcal{N}_1(\widehat{\omega})\right\|_\infty+\mathcal{K}_2^*\left\|\mathcal{N}_2(\widehat{\omega}_n)-\mathcal{N}_2(\widehat{\omega})\right\|_\infty\right]\\
&+\frac{\mathfrak{q}\eta^{\mathfrak{q}}}{\Psi(\mathfrak{q})\Gamma(\mathfrak{q}+1)}\\
&\left[\|\mathfrak{a}\|_\infty\left\|\widehat{\omega}_n-\widehat{\omega}\right\|_\infty+\mathcal{K}_1^*\left\|\mathcal{N}_1(\widehat{\omega}_n)-\mathcal{N}_1(\widehat{\omega})\right\|_\infty+\mathcal{K}_2^*\left\|\mathcal{N}_2(\widehat{\omega}_n)-\mathcal{N}_2(\widehat{\omega})\right\|_\infty\right]\\
\leq\ &\left(\frac{1-\mathfrak{q}}{\Psi(\mathfrak{q})}+\frac{\mathfrak{q}\eta^{\mathfrak{q}}}{\Psi(\mathfrak{q})\Gamma(\mathfrak{q}+1)}\right)\\
&\left[\|\mathfrak{a}\|_\infty\left\|\widehat{\omega}_n-\widehat{\omega}\right\|_\infty+\mathcal{K}_1^*\left\|\mathcal{N}_1(\widehat{\omega}_n)-\mathcal{N}_1(\widehat{\omega})\right\|_\infty+\mathcal{K}_2^*\left\|\mathcal{N}_2(\widehat{\omega}_n)-\mathcal{N}_2(\widehat{\omega})\right\|_\infty\right].
\end{aligned}$$

$\mathcal{N}_1$ and $\mathcal{N}_2$ are continuous. Then

$$\left\|\Xi\widehat{\omega}_n(\eta,\alpha)-\Xi\widehat{\omega}(\eta,\alpha)\right\|_\infty\to 0 \text{ as } \widehat{\omega}_n\to\widehat{\omega}.$$

Hence Ξ is continuous.

Step 2. Ξ is compact.

Define a bounded, closed and convex set $\mathbb{B}_R=\left\{\widehat{\omega}\in C\left(\mathcal{J},\mathbb{R}^F\right):\left\|\widehat{\omega}\right\|_\infty\leq R\right\}$ with

$$R\geq\frac{\Omega_2}{1-\Omega_1}\quad\text{where}\quad\Omega_2:=\left|\widehat{\omega}_0\right|+\left(\frac{1-\mathfrak{q}}{\Psi(\mathfrak{q})}+\frac{\mathfrak{q}}{\Psi(\mathfrak{q})\Gamma(\mathfrak{q}+1)}\right)\left[\|\mathfrak{g}\|_\infty+\mathcal{K}_1^*\zeta_1+\mathcal{K}_2^*\zeta_2\right],\tag{3.3}$$

where $\zeta_i := \sup_{\widehat{\omega}\in\mathcal{J}\times[0,R]} \mathcal{N}_i(\widehat{\omega}(s,\alpha) + 1,\ i = 1,2.$

First, we show that Ξ is uniformly bounded on $\mathbb{B}_R$. For each $\widehat{\omega} \in \mathbb{B}_R$, we have

$$\begin{aligned}
\left|\Xi\widehat{\omega}(\eta,\alpha)\right| \leq\ & \left|\widehat{\omega}_0\right| + \frac{1-\mathfrak{q}}{\Psi(\mathfrak{q})} \\
& \left[|\mathfrak{g}(\eta)| + |\mathfrak{a}(\eta)|\left|\widehat{\omega}(\eta,\alpha)\right| + \int_0^\eta |\mathcal{K}_1(\eta,s)|\left|\mathcal{N}_1(\widehat{\omega}(s,\alpha))\right| ds + \int_0^1 |\mathcal{K}_2(\eta,s)|\left|\mathcal{N}_2(\widehat{\omega}(s,\alpha))\right| ds\right] \\
& + \frac{\mathfrak{q}}{\Psi(\mathfrak{q})\Gamma(\mathfrak{q})}\int_0^\eta (\eta-s)^{\mathfrak{q}-1} \\
& \left[|\mathfrak{g}(s)| + |\mathfrak{a}(s)|\left|\widehat{\omega}(s,\alpha)\right| + \int_0^s |\mathcal{K}_1(s,\tau)|\left|\mathcal{N}_1(\widehat{\omega}(\tau,\alpha))\right| d\tau + \int_0^1 |\mathcal{K}_2(s,\tau)|\left|\mathcal{N}_2(\widehat{\omega}(\tau,\alpha))\right| d\tau\right] ds \\
\leq\ & \left|\widehat{\omega}_0\right| + \frac{1-\mathfrak{q}}{\Psi(\mathfrak{q})}\left[|\mathfrak{g}(\eta)| + |\mathfrak{a}(\eta)|\left|\widehat{\omega}(\eta,\alpha)\right| + \mathcal{K}_1^*\zeta_1 + \mathcal{K}_2^*\zeta_2\right] \\
& + \frac{\mathfrak{q}\eta^{\mathfrak{q}}}{\Psi(\mathfrak{q})\Gamma(\mathfrak{q}+1)}\left[|\mathfrak{g}(\eta)| + |\mathfrak{a}(\eta)|\left|\widehat{\omega}(\eta,\alpha)\right| + \mathcal{K}_1^*\zeta_1 + \mathcal{K}_2^*\zeta_2\right].
\end{aligned}$$

Taking the supremum on both sides, we get

$$\begin{aligned}
\left\|\Xi\widehat{\omega}\right\|_\infty \leq\ & \left|\widehat{\omega}_0\right| + \left(\frac{1-\mathfrak{q}}{\Psi(\mathfrak{q})} + \frac{\mathfrak{q}}{\Psi(\mathfrak{q})\Gamma(\mathfrak{q}+1)}\right)\left[\|\mathfrak{g}\|_\infty + \|\mathfrak{a}\|_\infty R + \mathcal{K}_1^*\zeta_1 + \mathcal{K}_2^*\zeta_2\right] \\
\leq\ & \left|\widehat{\omega}_0\right| + \left(\frac{1-\mathfrak{q}}{\Psi(\mathfrak{q})} + \frac{\mathfrak{q}}{\Psi(\mathfrak{q})\Gamma(\mathfrak{q}+1)}\right)\left[\|\mathfrak{g}\|_\infty + \mathcal{K}_1^*\zeta_1 + \mathcal{K}_2^*\zeta_2\right] \\
& + \left(\frac{1-\mathfrak{q}}{\Psi(\mathfrak{q})} + \frac{\mathfrak{q}}{\Psi(\mathfrak{q})\Gamma(\mathfrak{q}+1)}\right)\|\mathfrak{a}\|_\infty R \\
\leq\ & \Omega_2 + \Omega_1 R \leq R.
\end{aligned}$$

Hence Ξ is uniformly bounded. Now, we show that Ξ is equicontinuous. Let $\eta_1, \eta_2 \in \mathcal{J}$ such that $\eta_1 < \eta_2$. Then, we have

$$\begin{aligned}
& \left|\Xi\widehat{\omega}(\eta_2,\alpha) - \Xi\widehat{\omega}(\eta_1,\alpha)\right| \\
=\ & \left|\frac{1-\mathfrak{q}}{\Psi(\mathfrak{q})}\left(\mathfrak{g}(\eta_2) + \mathfrak{a}(\eta_2)\widehat{\omega}(\eta_2,\alpha) + \int_0^{\eta_2}\mathcal{K}_1(\eta_2,s)\mathcal{N}_1(\widehat{\omega}(s,\alpha))ds + \int_0^1\mathcal{K}_2(\eta_2,s)\mathcal{N}_2(\widehat{\omega}(s,\alpha))ds\right)\right. \\
& + \frac{\mathfrak{q}}{\Psi(\mathfrak{q})\Gamma(\mathfrak{q})}\int_0^{\eta_2}(\eta_2-s)^{\mathfrak{q}-1} \\
& \left(\mathfrak{g}(s) + \mathfrak{a}(s)\widehat{\omega}(s,\alpha) + \int_0^s\mathcal{K}_1(s,\tau)\mathcal{N}_1(\widehat{\omega}(\tau,\alpha))d\tau + \int_0^1\mathcal{K}_2(s,\tau)\mathcal{N}_2(\widehat{\omega}(\tau,\alpha))d\tau\right)ds \\
& - \left[\frac{1-\mathfrak{q}}{\Psi(\mathfrak{q})}\left(\mathfrak{g}(\eta_1) + \mathfrak{a}(\eta_1)\widehat{\omega}(\eta_1,\alpha) + \int_0^{\eta_1}\mathcal{K}_1(\eta_1,s)\mathcal{N}_1(\widehat{\omega}(s,\alpha))ds + \int_0^1\mathcal{K}_2(\eta_1,s)\mathcal{N}_2(\widehat{\omega}(s,\alpha))ds\right)\right. \\
& + \frac{\mathfrak{q}}{\Psi(\mathfrak{q})\Gamma(\mathfrak{q})}\int_0^{\eta_1}(\eta_1-s)^{\mathfrak{q}-1} \\
& \left.\left.\left(\mathfrak{g}(s) + \mathfrak{a}(s)\widehat{\omega}(s,\alpha) + \int_0^s\mathcal{K}_1(s,\tau)\mathcal{N}_1(\widehat{\omega}(\tau,\alpha))d\tau + \int_0^1\mathcal{K}_2(s,\tau)\mathcal{N}_2(\widehat{\omega}(\tau,\alpha))d\tau\right)ds\right]\right|
\end{aligned}$$

$$
\begin{aligned}
\leq\ & \frac{1-\mathfrak{q}}{\Psi(\mathfrak{q})}\Big(|\mathfrak{g}(\eta_2)-\mathfrak{g}(\eta_1)|+\big|\mathfrak{a}(\eta_2)\widehat{\omega}(\eta_2,\alpha)-\mathfrak{a}(\eta_1)\widehat{\omega}(\eta_1,\alpha)\big| \\
& +\int_0^{\eta_1}\left(\mathcal{K}_1(\eta_2,s)-\mathcal{K}_1(\eta_1,s)\right))\,\mathcal{N}_1(\widehat{\omega}(s,\alpha)ds+\int_{\eta_1}^{\eta_2}\mathcal{K}_1(\eta_2,s)\mathcal{N}_1(\widehat{\omega}(s,\alpha))ds \\
& +\int_0^1\left(\mathcal{K}_2(\eta_2,s)-\mathcal{K}_2(\eta_2,s)\right)\mathcal{N}_2(\widehat{\omega}(s,\alpha))ds\Big) \\
& +\frac{\mathfrak{q}}{\Psi(\mathfrak{q})\Gamma(\mathfrak{q})}\int_0^{\eta_1}\left((\eta_2-s)^{\mathfrak{q}-1}-(\eta_1-s)^{\mathfrak{q}-1}\right) \\
& \left(\mathfrak{a}(s)\widehat{\omega}(s,\alpha)+\int_0^s\mathcal{K}_1(s,\tau)\mathcal{N}_1(\widehat{\omega}(\tau,\alpha))d\tau+\int_0^1\mathcal{K}_2(s,\tau)\mathcal{N}_2(\widehat{\omega}(\tau,\alpha))d\tau\right)ds \\
& +\frac{\mathfrak{q}}{\Psi(\mathfrak{q})\Gamma(\mathfrak{q})}\int_{\eta_1}^{\eta_2}(\eta_2-s)^{\mathfrak{q}-1} \\
& \left(\mathfrak{a}(s)\widehat{\omega}(s,\alpha)+\int_0^s\mathcal{K}_1(s,\tau)\mathcal{N}_1(\widehat{\omega}(\tau,\alpha))d\tau+\int_0^1\mathcal{K}_2(s,\tau)\mathcal{N}_2(\widehat{\omega}(\tau,\alpha))d\tau\right)ds \\
=\ & \mathcal{A}_1+\mathcal{A}_2+\mathcal{A}_3,
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{A}_1 &= \frac{1-\mathfrak{q}}{\Psi(\mathfrak{q})}\Big(|\mathfrak{g}(\eta_2)-\mathfrak{g}(\eta_1)|+\big|\mathfrak{a}(\eta_2)\widehat{\omega}(\eta_2,\alpha)-\mathfrak{a}(\eta_1)\widehat{\omega}(\eta_1,\alpha)\big| \\
&\quad +\int_0^{\eta_1}\left(\mathcal{K}_1(\eta_2,s)-\mathcal{K}_1(\eta_1,s)\right))\,\mathcal{N}_1(\widehat{\omega}(s,\alpha)ds+\int_{\eta_1}^{\eta_2}\mathcal{K}_1(\eta_2,s)\mathcal{N}_1(\widehat{\omega}(s,\alpha))ds \\
&\quad +\int_0^1\left(\mathcal{K}_2(\eta_2,s)-\mathcal{K}_2(\eta_2,s)\right)\mathcal{N}_2(\widehat{\omega}(s,\alpha))ds\Big) \\
&\to 0,\ \text{as } \eta_2\to\eta_1,
\end{aligned}
\tag{3.4}
$$

$$
\begin{aligned}
\mathcal{A}_2 &= \frac{\mathfrak{q}}{\Psi(\mathfrak{q})\Gamma(\mathfrak{q})}\int_0^{\eta_1}\left((\eta_2-s)^{\mathfrak{q}-1}-(\eta_1-s)^{\mathfrak{q}-1}\right) \\
&\quad \left(\mathfrak{a}(s)\widehat{\omega}(s,\alpha)+\int_0^s\mathcal{K}_1(s,\tau)\mathcal{N}_1(\widehat{\omega}(\tau,\alpha))d\tau+\int_0^1\mathcal{K}_2(s,\tau)\mathcal{N}_2(\widehat{\omega}(\tau,\alpha))d\tau\right)ds \\
&\leq \left(\|\mathfrak{a}\|_\infty R+\mathcal{K}_1^*\zeta_1+\mathcal{K}_2^*\zeta_2\right)\frac{\mathfrak{q}}{\Psi(\mathfrak{q})\Gamma(\mathfrak{q})}\int_0^{\eta_1}\left((\eta_2-s)^{\mathfrak{q}-1}-(\eta_1-s)^{\mathfrak{q}-1}\right)ds \\
&= \frac{\left(\|\mathfrak{a}\|_\infty R+\mathcal{K}_1^*\zeta_1+\mathcal{K}_2^*\zeta_2\right)\mathfrak{q}}{\Psi(\mathfrak{q})\Gamma(\mathfrak{q}+1)}\left((\eta_2-\eta_1)^{\mathfrak{q}}-\eta_2^{\mathfrak{q}}+\eta_1^{\mathfrak{q}}\right) \\
&\to 0,\ \text{as } \eta_2\to\eta_1,
\end{aligned}
\tag{3.5}
$$

and

$$
\begin{aligned}
\mathcal{A}_3 &= \frac{\mathfrak{q}}{\Psi(\mathfrak{q})\Gamma(\mathfrak{q})}\int_{\eta_1}^{\eta_2}(\eta_2-s)^{\mathfrak{q}-1} \\
&\quad \left(\mathfrak{a}(s)\widehat{\omega}(s,\alpha)+\int_0^s\mathcal{K}_1(s,\tau)\mathcal{N}_1(\widehat{\omega}(\tau,\alpha))d\tau+\int_0^1\mathcal{K}_2(s,\tau)\mathcal{N}_2(\widehat{\omega}(\tau,\alpha))d\tau\right)ds
\end{aligned}
$$

$$\leq \left(\|\mathfrak{a}\|_\infty R + \mathcal{K}_1^*\zeta_1 + \mathcal{K}_2^*\zeta_2\right) \frac{\mathfrak{q}}{\Psi(\mathfrak{q})\Gamma(\mathfrak{q})} \int_{\eta_1}^{\eta_2} (\eta_2 - s)^{\mathfrak{q}-1}\, ds$$
$$\leq \frac{\left(\|\mathfrak{a}\|_\infty R + \mathcal{K}_1^*\zeta_1 + \mathcal{K}_2^*\zeta_2\right)\mathfrak{q}}{\Psi(\mathfrak{q})\Gamma(\mathfrak{q}+1)} (\eta_2 - \eta_1)^{\mathfrak{q}} \to 0 \text{ as } \eta_2 \to \eta_1. \tag{3.6}$$

From Eqs (3.4)–(3.6), we get

$$\left\|\Xi\widehat{\omega}(\eta_2, \alpha) - \Xi\widehat{\omega}(\eta_1, \alpha)\right\|_\infty \to 0 \text{ as } \eta_2 \to \eta_1.$$

Hence Ξ is equicontinuous. By the Arzelá-Ascoli theorem, we infer that Ξ is compact in $C\left(\mathcal{J}, \mathbb{R}^F\right)$. Thus, from the above steps, we infer that Ξ is completely continuous.

Step 3. The set $\delta = \left\{\widehat{\omega}(\eta;\alpha) \in C\left(\mathcal{J}, \mathbb{R}^F\right) : \widehat{\omega}(\eta;\alpha) = \varrho\Xi\widehat{\omega}(\eta;\alpha),\ \varrho \in (0,1)\right\}$ is bounded.

Let $\widehat{\omega}(\eta;\alpha) \in \delta$. Then $\widehat{\omega}(\eta;\alpha) = \varrho\Xi\widehat{\omega}(\eta;\alpha)$. Now, for $\eta \in [0,1]$, we have

$$\begin{aligned}
\left|\widehat{\omega}(\eta;\alpha)\right| &= \left|\varrho\Xi\widehat{\omega}(\eta;\alpha)\right| \\
&\leq \left|\Xi\widehat{\omega}(\eta;\alpha)\right| \\
&\leq \left|\widehat{\omega}_0\right| + \left(\frac{1-\mathfrak{q}}{\Psi(\mathfrak{q})} + \frac{\mathfrak{q}}{\Psi(\mathfrak{q})\Gamma(\mathfrak{q}+1)}\right)\left[\|\mathfrak{g}\|_\infty + \|\mathfrak{a}\|_\infty R + \mathcal{K}_1^*\zeta_1 + \mathcal{K}_2^*\zeta_2\right] \\
&\leq \left|\widehat{\omega}_0\right| + \left(\frac{1-\mathfrak{q}}{\Psi(\mathfrak{q})} + \frac{\mathfrak{q}}{\Psi(\mathfrak{q})\Gamma(\mathfrak{q}+1)}\right)\left[\|\mathfrak{g}\|_\infty + \mathcal{K}_1^*\zeta_1 + \mathcal{K}_2^*\zeta_2\right] \\
&\quad + \left(\frac{1-\mathfrak{q}}{\Psi(\mathfrak{q})} + \frac{\mathfrak{q}}{\Psi(\mathfrak{q})\Gamma(\mathfrak{q}+1)}\right)\|\mathfrak{a}\|_\infty R \\
&\leq \Omega_2 + \Omega_2 R.
\end{aligned}$$

From Eq (3.2), we get $\Omega_2 + \Omega_2 R \leq R$. Hence, the set δ is bounded. According to the above steps, and together with Theorem 2.1, we deduce that Ξ has at least one fixed point. Consequently, the problem (1.1) has at least one solution on $\mathcal{J}$. □

In the following theorem, we prove the uniqueness of solutions to problem defined by FFVFIE (1.1) by using Theorem 2.1.

Theorem 3.2. *Suppose (H_1)–(H_3) hold. Then the problem defined by FFVFIE (1.1) has a unique fuzzy number solution $\widehat{\omega}(\eta;\alpha)$, provided that*

$$\Lambda = \left(\frac{1-\mathfrak{q}}{\Psi(\mathfrak{q})} + \frac{\mathfrak{q}\eta^{\mathfrak{q}}}{\Psi(\mathfrak{q})\Gamma(\mathfrak{q}+1)}\right)\left(\|\mathfrak{a}\|_\infty + \mathcal{K}_1^* c_1 + \mathcal{K}_2^* c_2\right) < 1. \tag{3.7}$$

Proof. Let us consider the operator Ξ defined in Theorem 3.1. By applying Theorem 2.1, we shall show that Ξ has a unique fuzzy number solution. Let $\widehat{\omega}(\eta;\alpha), \widehat{v}(\eta;\alpha) \in C\left(\mathcal{J}, \mathbb{R}^F\right)$. Then

$$\begin{aligned}
&\left|\Xi\widehat{\omega}(\eta;\alpha) - \Xi\widehat{v}(\eta;\alpha)\right| \\
\leq\ & \frac{1-\mathfrak{q}}{\Psi(\mathfrak{q})}\left[\mathfrak{a}(\eta)\left|\widehat{\omega}(\eta;\alpha) - \widehat{v}(\eta;\alpha)\right| + \int_0^\eta |\mathcal{K}_1(\eta,s)|\left|\mathcal{N}_1(\widehat{\omega}(s;\alpha)) - \mathcal{N}_1(\widehat{v}(s;\alpha))\right| ds\right. \\
&\left. + \int_0^1 |\mathcal{K}_2(\eta,s)|\left|\mathcal{N}_2(\widehat{\omega}(s;\alpha)) - \mathcal{N}_2(\widehat{v}(s;\alpha))\right| ds\right]
\end{aligned}$$

$$+\frac{\mathfrak{q}}{\Psi(\mathfrak{q})\Gamma(\mathfrak{q})}\int_0^\eta(\eta-s)^{\mathfrak{q}-1}$$
$$\left[\mathfrak{a}(s)\left|\widehat{\omega}(s;\alpha)-\widehat{v}(s;\alpha)\right|+\int_0^s|\mathcal{K}_1(s,\tau)|\left|\mathcal{N}_1(\widehat{\omega}(\tau;\alpha))-\mathcal{N}_1(\widehat{v}(\tau;\alpha))\right|d\tau\right.$$
$$\left.+\int_0^1|\mathcal{K}_2(s,\tau)|\left|\mathcal{N}_2(\widehat{\omega}(\tau;\alpha))-\mathcal{N}_2(\widehat{v}(\tau;\alpha))\right|d\tau\right]ds.$$

Taking the supremum on both sides, we get

$$\begin{aligned}&\left\|\Xi\widehat{\omega}(\eta,\alpha)-\Xi\widehat{v}(\eta,\alpha)\right\|_\infty\\ \leq\;&\frac{1-\mathfrak{q}}{\Psi(\mathfrak{q})}\left(\|\mathfrak{a}\|_\infty+\mathcal{K}_1^*c_1+\mathcal{K}_2^*c_2\right)\left\|\widehat{\omega}-\widehat{v}\right\|_\infty\\ &+\frac{\mathfrak{q}\eta^{\mathfrak{q}}}{\Psi(\mathfrak{q})\Gamma(\mathfrak{q}+1)}\left(\|\mathfrak{a}\|_\infty+\mathcal{K}_1^*c_1+\mathcal{K}_2^*c_2\right)\left\|\widehat{\omega}-\widehat{v}\right\|_\infty\\ =\;&\left(\frac{1-\mathfrak{q}}{\Psi(\mathfrak{q})}+\frac{\mathfrak{q}\eta^{\mathfrak{q}}}{\Psi(\mathfrak{q})\Gamma(\mathfrak{q}+1)}\right)\left(\|\mathfrak{a}\|_\infty+\mathcal{K}_1^*c_1+\mathcal{K}_2^*c_2\right)\left\|\widehat{\omega}-\widehat{v}\right\|_\infty.\end{aligned}$$

Due to Eq (3.7), we have

$$\left\|\Xi\widehat{\omega}(\eta,\alpha)-\Xi\widehat{v}(\eta,\alpha)\right\|_\infty\leq\Lambda\left\|\widehat{\omega}-\widehat{v}\right\|_\infty.$$

Hence Ξ is a contraction mapping. Thus, by Theorem 2.1, Ξ has a unique fixed point. Hence, the FFVFIE (1.1) has a unique fuzzy number solution $\widehat{\omega}(\eta;\alpha)$. □

4. Numerical approach

In this portion, we obtain approximation solutions to the problem defined by FFVFIE (1.1) as

$$\begin{cases}{}_0^{ABC}D_\eta^{\mathfrak{q}}\underline{\omega}(\eta;\alpha)=\mathcal{P}_1\left(\eta,\underline{\omega}(\eta;\alpha),\overline{\omega}(\eta;\alpha)\right)\\ {}_0^{ABC}D_\eta^{\mathfrak{q}}\overline{\omega}(\eta;\alpha)=\mathcal{P}_2\left(\eta,\underline{\omega}(\eta;\alpha),\overline{\omega}(\eta;\alpha)\right),\end{cases}\tag{4.1}$$

where $\mathcal{P}_1$ and $\mathcal{P}_2$ are two continuous functions. Applying the AB integral operator to both sides of the two equations in Eq (4.1), we get

$$\begin{cases}\underline{\omega}(\eta;\alpha)-\underline{\omega}(0;\alpha)=\frac{1-\mathfrak{q}}{\Psi(\mathfrak{q})}I_\eta^{\mathfrak{q}}\mathcal{P}_1\left(\eta,\underline{\omega}(\eta;\alpha),\overline{\omega}(\eta;\alpha)\right)\\ \qquad+\frac{\mathfrak{q}}{\Psi(\mathfrak{q})}\frac{1}{\Gamma(\mathfrak{q})}\int_0^\eta(\eta-\theta)^{\mathfrak{q}-1}\mathcal{P}_2\left(\theta,\underline{\omega}(\theta;\alpha),\overline{\omega}(\theta;\alpha)\right)d\theta,\\ \overline{\omega}(\eta;\alpha)-\overline{\omega}(0;\alpha)=\frac{1-\mathfrak{q}}{\Psi(\mathfrak{q})}I_\eta^{\mathfrak{q}}\mathcal{P}_2\left(\eta,\underline{\omega}(\eta;\alpha),\overline{\omega}(\eta;\alpha)\right)\\ \qquad+\frac{\mathfrak{q}}{\Psi(\mathfrak{q})}\frac{1}{\Gamma(\mathfrak{q})}\int_0^\eta(\eta-\theta)^{\mathfrak{q}-1}\mathcal{P}_2\left(\theta,\underline{\omega}(\theta;\alpha),\overline{\omega}(\theta;\alpha)\right)d\theta.\end{cases}\tag{4.2}$$

Set $\eta=\eta_{r+1}$ for $r=0,1,2,....$; it follows that

$$\begin{cases}\underline{\omega}(\eta_{r+1};\alpha)-\underline{\omega}(0;\alpha)=\frac{1-\mathfrak{q}}{\Psi(\mathfrak{q})}\mathcal{P}_1\left(\eta_r,\underline{\omega}(\eta_r;\alpha),\overline{\omega}(\eta_r;\alpha)\right)\\ \qquad+\frac{\mathfrak{q}}{\Psi(\mathfrak{q})}\frac{1}{\Gamma(\mathfrak{q})}\int_0^{\eta_{r+1}}(\eta_{r+1}-\theta)^{\mathfrak{q}-1}\mathcal{P}_2\left(\theta,\underline{\omega}(\theta;\alpha),\overline{\omega}(\theta;\alpha)\right)d\theta,\\ \overline{\omega}(\eta_{r+1};\alpha)-\overline{\omega}(0;\alpha)=\frac{1-\mathfrak{q}}{\Psi(\mathfrak{q})}\mathcal{P}_2\left(\eta_r,\underline{\omega}(\eta_r;\alpha),\overline{\omega}(\eta_r;\alpha)\right)\\ \qquad+\frac{\mathfrak{q}}{\Psi(\mathfrak{q})}\frac{1}{\Gamma(\mathfrak{q})}\int_0^{\eta_{r+1}}(\eta_{r+1}-\theta)^{\mathfrak{q}-1}\mathcal{P}_2\left(\theta,\underline{\omega}(\theta;\alpha),\overline{\omega}(\theta;\alpha)\right)d\theta,\end{cases}\tag{4.3}$$

which implies

$$\begin{cases} \underline{\omega}(\eta_{r+1};\alpha)-\underline{\omega}(0;\alpha)=\frac{1-\mathfrak{q}}{\Psi(\mathfrak{q})}\mathcal{P}_1\left(\eta_r,\underline{\omega}(\eta_r;\alpha),\overline{\omega}(\eta_r;\alpha)\right) \\ \quad +\frac{\mathfrak{q}}{\Psi(\mathfrak{q})}\frac{1}{\Gamma(\mathfrak{q})}\sum_{l=1}^{r}\int_{\eta_l}^{\eta_{l+1}}(\eta-\theta)^{\mathfrak{q}-1}\mathcal{P}_2\left(\theta,\underline{\omega}(\theta;\alpha),\overline{\omega}(\theta;\alpha)\right)d\theta, \\ \overline{\omega}(\eta_{r+1};\alpha)-\overline{\omega}(0;\alpha)=\frac{1-\mathfrak{q}}{\Psi(\mathfrak{q})}\mathcal{P}_2\left(\eta_r,\underline{\omega}(\eta_r;\alpha),\overline{\omega}(\eta_r;\alpha)\right) \\ \quad +\frac{\mathfrak{q}}{\Psi(\mathfrak{q})}\frac{1}{\Gamma(\mathfrak{q})}\sum_{l=1}^{r}\int_{\eta_l}^{\eta_{l+1}}(\eta_{r+1}-\theta)^{\mathfrak{q}-1}\mathcal{P}_2\left(\theta,\underline{\omega}(\theta;\alpha),\overline{\omega}(\theta;\alpha)\right)d\theta. \end{cases} \tag{4.4}$$

Now, we approximate the functions $\widehat{\mathcal{P}_i}\left(\theta,\underline{\omega}(\theta;\alpha),\overline{\omega}(\theta;\alpha)\right), i=1,2$ on $[\eta_l,\eta_{l+1}]$ through the use of the interpolation polynomial as follows, and as done in [48]

$$\begin{cases} \mathcal{P}_i\left(\theta,\underline{\omega}(\theta;\alpha),\overline{\omega}(\theta;\alpha)\right)\simeq\frac{\mathcal{P}_i\left(\eta_l,\underline{\omega}(\eta_l;\alpha),\overline{\omega}(\eta_l;\alpha)\right)}{h}(\eta-\eta_{l-1}) \\ \quad +\frac{\mathcal{P}_i\left(\eta_{l-1},\underline{\omega}(\eta_{l-1};\alpha),\overline{\omega}(\eta_{l-1};\alpha)\right)}{h}(\eta-\eta_l). \end{cases} \tag{4.5}$$

By Eqs (4.4) and (4.5), we have

$$\begin{cases} \underline{\omega}(\eta_{r+1};\alpha)-\underline{\omega}(0;\alpha)=\frac{1-\mathfrak{q}}{\Psi(\mathfrak{q})}\mathcal{P}_1\left(\eta_r,\underline{\omega}(\eta_r;\alpha),\overline{\omega}(\eta_r;\alpha)\right) \\ \quad +\frac{\mathfrak{q}}{\Psi(\mathfrak{q})}\frac{1}{\Gamma(\mathfrak{q})}\sum_{l=1}^{r}\left(\frac{\mathcal{P}_1\left(\eta_l,\underline{\omega}(\eta_l;\alpha),\overline{\omega}(\eta_l;\alpha)\right)}{h}\mathcal{I}_{l-1,\mathfrak{q}}-\frac{\mathcal{P}_1\left(\eta_{l-1},\underline{\omega}(\eta_{l-1};\alpha),\overline{\omega}(\eta_{l-1};\alpha)\right)}{h}\mathcal{I}_{l,\mathfrak{q}}\right), \\ \overline{\omega}(\eta_{r+1};\alpha)-\overline{\omega}(0;\alpha)=\frac{1-\mathfrak{q}}{\Psi(\mathfrak{q})}\mathcal{P}_2\left(\eta_r,\underline{\omega}(\eta_r;\alpha),\overline{\omega}(\eta_r;\alpha)\right) \\ \quad +\frac{\mathfrak{q}}{\Psi(\mathfrak{q})}\frac{1}{\Gamma(\mathfrak{q})}\sum_{l=1}^{r}\left(\frac{\mathcal{P}_2\left(\eta_l,\underline{\omega}(\eta_l;\alpha),\overline{\omega}(\eta_l;\alpha)\right)}{h}\mathcal{I}_{l-1,\mathfrak{q}}-\frac{\mathcal{P}_2\left(\eta_{l-1},\underline{\omega}(\eta_{l-1};\alpha),\overline{\omega}(\eta_{l-1};\alpha)\right)}{h}\mathcal{I}_{l,\mathfrak{q}}\right), \end{cases} \tag{4.6}$$

where

$$\mathcal{I}_{l-1,\mathfrak{q}}=\int_{\eta_r}^{\eta_{r+1}}(\eta-\eta_{l-1})(\eta_{r+1}-\eta)^{\mathfrak{q}-1}d\eta,$$

and

$$\mathcal{I}_{l,\mathfrak{q}}=\int_{\eta_r}^{\eta_{r+1}}(\eta-\eta_l)(\eta_{r+1}-\eta)^{\mathfrak{q}-1}d\eta.$$

By simple calculations, we get

$$\begin{aligned}\mathcal{I}_{l-1,\mathfrak{q}} &= -\frac{1}{\mathfrak{q}}\left[(\eta_{l+1}-\eta_{l-1})(\eta_{r+1}-\eta_{l+1})^{\mathfrak{q}}-(\eta_l-\eta_{l-1})(\eta_{r+1}-\eta_l)^{\mathfrak{q}}\right] \\ &\quad -\frac{1}{\mathfrak{q}(\mathfrak{q}+1)}\left[(\eta_{r+1}-\eta_{l+1})^{\mathfrak{q}+1}-(\eta_{r+1}-\eta_l)^{\mathfrak{q}+1}\right],\end{aligned}$$

and

$$\begin{aligned}\mathcal{I}_{l,\mathfrak{q}} &= -\frac{1}{\mathfrak{q}}\left[(\eta_{l+1}-\eta_l)(\eta_{r+1}-\eta_{l+1})^{\mathfrak{q}}\right] \\ &\quad -\frac{1}{\mathfrak{q}(\mathfrak{q}+1)}\left[(\eta_{r+1}-\eta_{l+1})^{\mathfrak{q}+1}-(\eta_{r+1}-\eta_l)^{\mathfrak{q}+1}\right].\end{aligned}$$

Setting $\eta_l=lh$, we get

$$\mathcal{I}_{l-1,\mathfrak{q}}=\frac{h^{\mathfrak{q}+1}}{\mathfrak{q}(\mathfrak{q}+1)}\left[(r+1-l)^{\mathfrak{q}}(r-l+2+\mathfrak{q})-(r-l)^{\mathfrak{q}}(r-l+2+2\mathfrak{q})\right], \tag{4.7}$$

and

$$\mathcal{I}_{l,\mathfrak{q}} = \frac{h^{\mathfrak{q}+1}}{\mathfrak{q}(\mathfrak{q}+1)}\left[(r+1-l)^{\mathfrak{q}+1} - (r-l)^{\mathfrak{q}}(r-l+1+\mathfrak{q})\right]. \tag{4.8}$$

Substituting Eqs (4.7) and (4.8) into Eq (4.6), we get

$$\underline{\omega}(\eta_{r+1};\alpha) = \begin{cases} \underline{\omega}(0;\alpha) + \frac{1-\mathfrak{q}}{\Psi(\mathfrak{q})}\mathcal{P}_1\left(\eta_r, \underline{\omega}(\eta_r;\alpha), \overline{\omega}(\eta_r;\alpha)\right) + \frac{\mathfrak{q}}{\Psi(\mathfrak{q})}\sum_{l=1}^{r} \\ \left(\frac{\mathcal{P}_1\left(\eta_l, \underline{\omega}(\eta_l;\alpha), \overline{\omega}(\eta_l;\alpha)\right)}{\Gamma(\mathfrak{q}+2)} h^{\mathfrak{q}}\left[(r+1-l)^{\mathfrak{q}}(r-l+2+\mathfrak{q}) - (r-l)^{\mathfrak{q}}(r-l+2+2\mathfrak{q})\right] \right. \\ \left. -\frac{\mathcal{P}_1\left(\eta_{l-1}, \underline{\omega}(\eta_{l-1};\alpha), \overline{\omega}(\eta_{l-1};\alpha)\right)}{\Gamma(\mathfrak{q}+2)} h^{\mathfrak{q}}\left[(r+1-l)^{\mathfrak{q}+1} - (r-l)^{\mathfrak{q}}(r-l+1+\mathfrak{q})\right]\right), \end{cases}$$

$$\overline{\omega}(\eta_{r+1};\alpha) = \begin{cases} \overline{\omega}(0;\alpha) + \frac{1-\mathfrak{q}}{\Psi(\mathfrak{q})}\mathcal{P}_2\left(\eta_r, \underline{\omega}(\eta_r;\alpha), \overline{\omega}(\eta_r;\alpha)\right) + \frac{\mathfrak{q}}{\Psi(\mathfrak{q})}\sum_{l=1}^{r} \\ \left(\frac{\mathcal{P}_2\left(\eta_l, \underline{\omega}(\eta_l;\alpha), \overline{\omega}(\eta_l;\alpha)\right)}{\Gamma(\mathfrak{q}+2)} h^{\mathfrak{q}}\left[(r+1-l)^{\mathfrak{q}}(r-l+2+\mathfrak{q}) - (r-l)^{\mathfrak{q}}(r-l+2+2\mathfrak{q})\right] \right. \\ \left. -\frac{\mathcal{P}_2\left(\eta_{l-1}, \underline{\omega}(\eta_{l-1};\alpha), \overline{\omega}(\eta_{l-1};\alpha)\right)}{\Gamma(\mathfrak{q}+2)} h^{\mathfrak{q}}\left[(r+1-l)^{\mathfrak{q}+1} - (r-l)^{\mathfrak{q}}(r-l+1+\mathfrak{q})\right]\right). \end{cases}$$

5. Examples

Example 5.1. *Consider the following fuzzy problem*

$$\begin{cases} {}_{0}^{ABC}D_{\eta}^{\frac{1}{4}}\widehat{\omega}(\eta,\alpha) = -\frac{(1-\eta^2)}{2}\widehat{\omega}(\eta,\alpha) + \int_0^{\eta} e^{\eta} s\widehat{\omega}(s,\alpha)ds \\ \qquad + \int_0^1 \left(1-\eta^2\right) s\widehat{\omega}(s,\alpha)ds \\ \widehat{\omega}(0,\alpha) = \left[\underline{\omega}(0,\alpha), \overline{\omega}(0,\alpha)\right] = [\alpha-1, 1-\alpha]. \end{cases} \tag{5.1}$$

The equivalent form of this problem for the Case (2.1) is given by

$$\begin{cases} \begin{cases} {}_{0}^{ABC}D_{\eta}^{\frac{1}{4}}\underline{\omega}(\eta,\alpha) = -\frac{(1-\eta^2)}{2}\underline{\omega}(\eta,\alpha) + \int_0^{\eta} e^{\eta} s\underline{\omega}(s,\alpha)ds \\ \qquad + \int_0^1 \left(1-\eta^2\right) s\underline{\omega}(s,\alpha)ds \\ \underline{\omega}(0,\alpha) = \alpha - 1, \end{cases} \\ \begin{cases} {}_{0}^{ABC}D_{\eta}^{\frac{1}{4}}\overline{\omega}(\eta,\alpha) = -\frac{(1-\eta^2)}{2}\overline{\omega}(\eta,\alpha) + \int_0^{\eta} e^{\eta} s\overline{\omega}(s,\alpha)ds \\ \qquad + \int_0^1 \left(1-\eta^2\right) s\overline{\omega}(s,\alpha)ds \\ \overline{\omega}(0,\alpha) = 1 - \alpha. \end{cases} \end{cases} \tag{5.2}$$

Here, $\mathfrak{g}(\eta) = 0, \mathfrak{a}(\eta) = -\frac{(1-\eta^2)}{2}$.

We noted that Conditions (H_1)–(H_3) *are satisfied and* $\Omega_1 \simeq 0.62 < 1$. *Thus all conditions in Theorem 3.1 are satisfied and hence the nonlinear FFVFIE given by Eq (5.2) has at least one solution. Also, using the given data we can easily confirm that the following inequality holds*

$$\Lambda = \left(\frac{1-\frac{1}{4}}{\Psi(\frac{1}{4})} + \frac{\frac{1}{4}}{\Psi(\frac{1}{4})\Gamma(\frac{1}{4}+1)}\right)(\|\mathfrak{a}\|_{\infty} + \mathcal{K}_1^* c_1 + \mathcal{K}_2^* c_2) < 1.$$

Now, consider $\underline{\omega}(\eta,\alpha)$ *and apply* ${}_{0}^{ABC}I_{\eta}^{\frac{1}{4}}$ *to both sides of the equation given by Eq (5.2); then, we get*

$$\underline{\omega}(\eta,\alpha) \ = \ \alpha - 1 + \frac{1-\frac{1}{4}}{\Psi(\frac{1}{4})}$$

$$\left[-\frac{\left(1-\eta^2\right)}{2}\underline{\omega}(\eta,\alpha)+\int_0^\eta e^\eta s\underline{\omega}(s,\alpha)ds+\int_0^1\left(1-\eta^2\right)s\underline{\omega}(s,\alpha)ds\right]$$
$$+\frac{\frac{1}{4}}{\Psi(\frac{1}{4})\Gamma(\frac{1}{4})}\int_0^\eta(\eta-s)^{\frac{1}{4}-1}$$
$$\left[-\frac{\left(1-s^2\right)}{2}\underline{\omega}(s,\alpha)+\int_0^s e^s\tau\underline{\omega}(\tau,\alpha)d\tau+\int_0^1\left(1-\eta^2\right)\tau\underline{\omega}(\tau,\alpha)d\tau\right]ds,$$

and

$$\begin{aligned}\overline{\omega}(\eta,\alpha) &= 1-\alpha+\frac{1-\frac{1}{4}}{\Psi(\frac{1}{4})}\\ &\left[-\frac{\left(1-\eta^2\right)}{2}\overline{\omega}(\eta,\alpha)+\int_0^\eta e^\eta s\overline{\omega}(s,\alpha)ds+\int_0^1\left(1-\eta^2\right)s\overline{\omega}(s,\alpha)ds\right]\\ &+\frac{\frac{1}{4}}{\Psi(\frac{1}{4})\Gamma(\frac{1}{4})}\int_0^\eta(\eta-s)^{\frac{1}{4}-1}\\ &\left[-\frac{\left(1-s^2\right)}{2}\overline{\omega}(s,\alpha)+\int_0^s e^s\tau\overline{\omega}(\tau,\alpha)d\tau+\int_0^1\left(1-\eta^2\right)\tau\overline{\omega}(\tau,\alpha)d\tau\right]ds\end{aligned}$$

Here, we provide the plot of the fuzzy solutions at different values of uncertainty for the given problem in Figure 1.

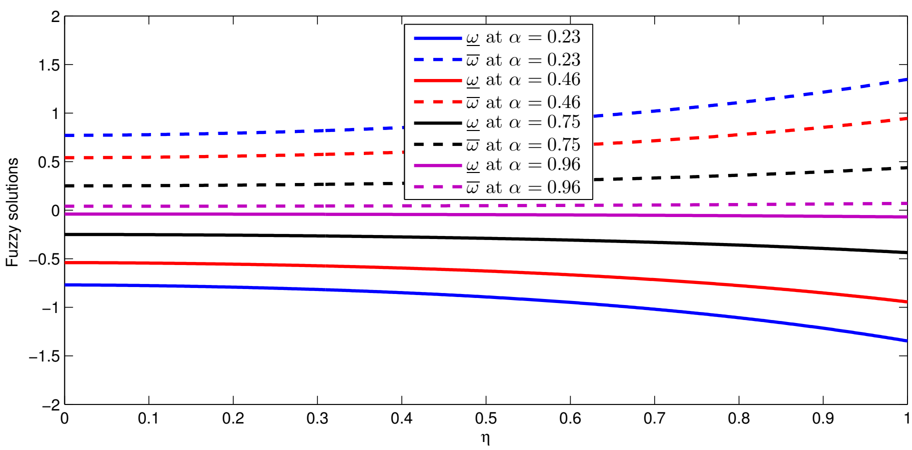

Figure 1. Graphical presentation of fuzzy approximate solutions upto initial three terms at different values of uncertainty α for Example 5.1.

Further, we present the surface plot of the fuzzy solutions in Figure 2.

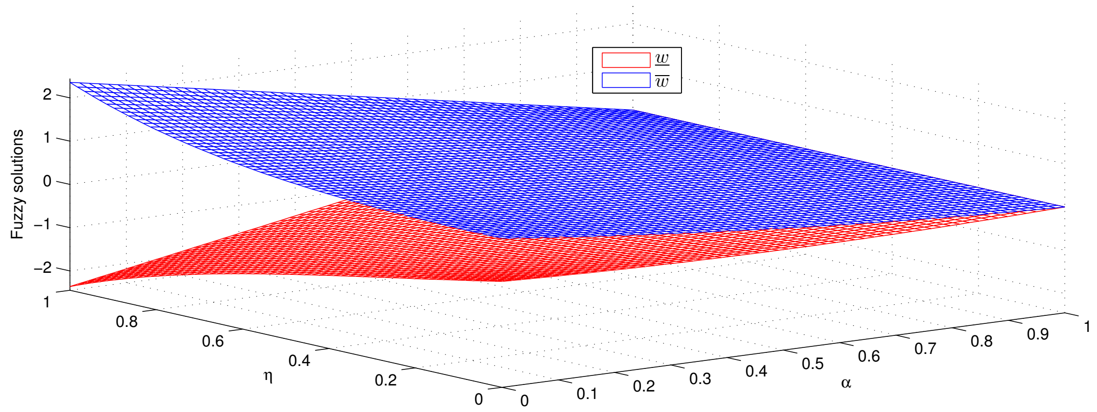

Figure 2. Surface plot of fuzzy approximate solutions for the initial three terms corresponding to different values of uncertainty α and space variable η in Example 5.1.

Example 5.2. *Consider the following FDE with fuzzy number initial values:*

$$\begin{cases} {}_{0}^{ABC}D_{\eta}^{\frac{1}{3}}\widehat{\omega}(\eta,\alpha) = \frac{\eta^{\frac{1}{2}}}{\Gamma(\frac{1}{2})} - \frac{\eta^2}{2} + \frac{\eta^2 e^{\eta}}{2}\widehat{\omega}(\eta,\alpha) + \int_0^{\eta} e^{\eta} s\widehat{\omega}(s,\alpha)ds \\ \qquad + \int_0^1 \eta^2 s\widehat{\omega}(s,\alpha)ds \\ \widehat{\omega}(0,\alpha) = \left[\underline{\omega}(0,\alpha), \overline{\omega}(0,\alpha)\right] = [\alpha, 3-2\alpha]. \end{cases} \tag{5.3}$$

The equivalent form of this problem for the Case (2.1) is given by

$$\begin{cases} \begin{cases} {}_{0}^{ABC}D_{\eta}^{\frac{1}{3}}\underline{\omega}(\eta,\alpha) = \frac{\eta^{\frac{1}{3}}}{\Gamma(\frac{1}{3})} - \frac{\eta^2}{2} + \frac{\eta^2 e^{\eta}}{2}\underline{\omega}(\eta,\alpha) + \int_0^{\eta} e^{\eta} s\underline{\omega}(s,\alpha)ds \\ \qquad + \int_0^1 \eta^2 s\underline{\omega}(s,\alpha)ds \\ \underline{\omega}(0,\alpha) = \alpha, \end{cases} \\ \begin{cases} {}_{0}^{ABC}D_{\eta}^{\frac{1}{3}}\overline{\omega}(\eta,\alpha) = -\frac{\eta^{\frac{1}{3}}}{\Gamma(\frac{1}{3})} - \frac{\eta^2}{2} + \frac{\eta^2 e^{\eta}}{2}\overline{\omega}(\eta,\alpha) + \int_0^{\eta} e^{\eta} s\overline{\omega}(s,\alpha)ds \\ \qquad + \int_0^1 \eta^2 s\overline{\omega}(s,\alpha)ds \\ \overline{\omega}(0,\alpha) = 3-2\alpha. \end{cases} \end{cases}$$

Here $\mathfrak{g}(\eta) = \frac{\eta^{\frac{1}{2}}}{\Gamma(\frac{1}{2})} - \frac{\eta^2}{2}$ *and* $\mathfrak{a}(\eta) = \frac{\eta^2 e^{\eta}}{2}$. *Thus, the functions* $\mathfrak{a}$ *and* $\mathfrak{g}$ *are continuous. Also, Conditions* (H_1)*–*(H_3) *are satisfied and* $\Omega_1 < 1$. *Thus, all conditions in Theorem 3.1 are satisfied and the nonlinear FFVFIE given by Eq (5.3) has at least one solution. Also, using the given data, we can easily confirm that the following inequality holds*

$$\left(\frac{1-\frac{1}{3}}{\Psi(\frac{1}{3})} + \frac{\frac{1}{3}}{\Psi(\frac{1}{3})\Gamma(\frac{1}{3}+1)}\right)(\|\mathfrak{a}\|_{\infty} + \mathcal{K}_1^* c_1 + \mathcal{K}_2^* c_2) < 1.$$

Now, consider $\underline{\omega}(\eta,\alpha)$ *and apply* ${}_{0}^{ABC}I_{\eta}^{\frac{1}{3}}$ *to both sides of the above equation; then, we get*

$$\underline{\omega}(\eta,\alpha) \;=\; \alpha + \frac{1-\frac{1}{3}}{\Psi(\frac{1}{3})}$$

$$\left[\frac{\eta^{\frac{1}{3}}}{\Gamma(\frac{1}{3})} - \frac{\eta^2}{2} + \frac{\eta^2 e^\eta}{2}\underline{\omega}(\eta,\alpha) + \int_0^\eta e^\eta s\underline{\omega}(s,\alpha)ds + \int_0^1 \eta^2 s\underline{\omega}(s,\alpha)ds\right]$$
$$+\frac{\frac{1}{3}}{\Psi(\frac{1}{3})\Gamma(\frac{1}{3})}\int_0^\eta (\eta-s)^{\frac{1}{3}-1}$$
$$\left[\frac{s^{\frac{1}{3}}}{\Gamma(\frac{1}{3})} - \frac{s^2}{2} + \frac{s^2 e^s}{2}\underline{\omega}(s,\alpha) + \int_0^\eta e^s\tau\underline{\omega}(\tau,\alpha)d\tau + \int_0^1 s^2\tau\underline{\omega}(\tau,\alpha)d\tau\right]ds$$

and

$$\begin{aligned}\overline{\omega}(\eta,\alpha) &= 3-2\alpha+\frac{1-\frac{1}{3}}{\Psi(\frac{1}{3})}\\ &\left[\frac{\eta^{\frac{1}{3}}}{\Gamma(\frac{1}{3})} - \frac{\eta^2}{2} + \frac{\eta^2 e^\eta}{2}\overline{\omega}(\eta,\alpha) + \int_0^\eta e^\eta s\overline{\omega}(s,\alpha)ds + \int_0^1 \eta^2 s\overline{\omega}(s,\alpha)ds\right]\\ &+\frac{\frac{1}{3}}{\Psi(\frac{1}{3})\Gamma(\frac{1}{3})}\int_0^\eta (\eta-s)^{\frac{1}{3}-1}\\ &\left[\frac{s^{\frac{1}{3}}}{\Gamma(\frac{1}{3})} - \frac{s^2}{2} + \frac{s^2 e^s}{2}\overline{\omega}(s,\alpha) + \int_0^\eta e^s\tau\overline{\omega}(\tau,\alpha)d\tau + \int_0^1 s^2\tau\overline{\omega}(\tau,\alpha)d\tau\right]ds.\end{aligned}$$

Here, we provide the plot of the fuzzy solutions at different values of uncertainty for the given problem in Figure 3.

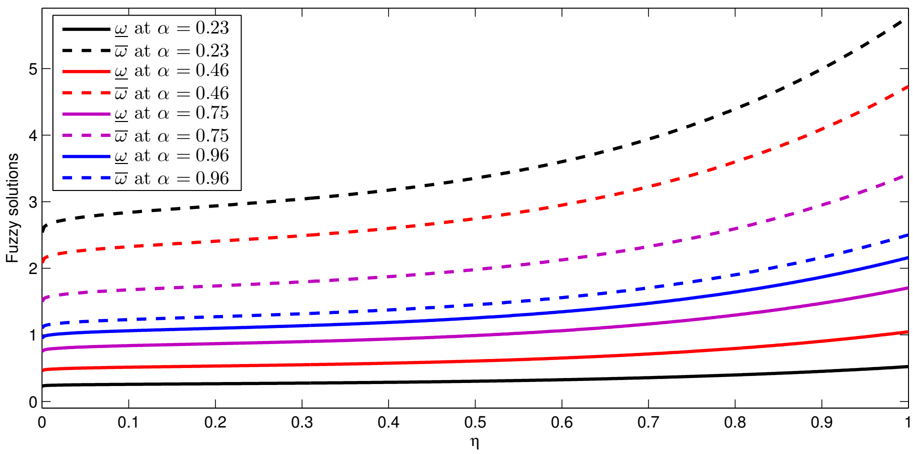

Figure 3. Graphical presentation of fuzzy approximate solutions for up to initial three terms at different values of uncertainty α for Example 5.2.

In Figure 4, we present the surface plot for Example 5.2.

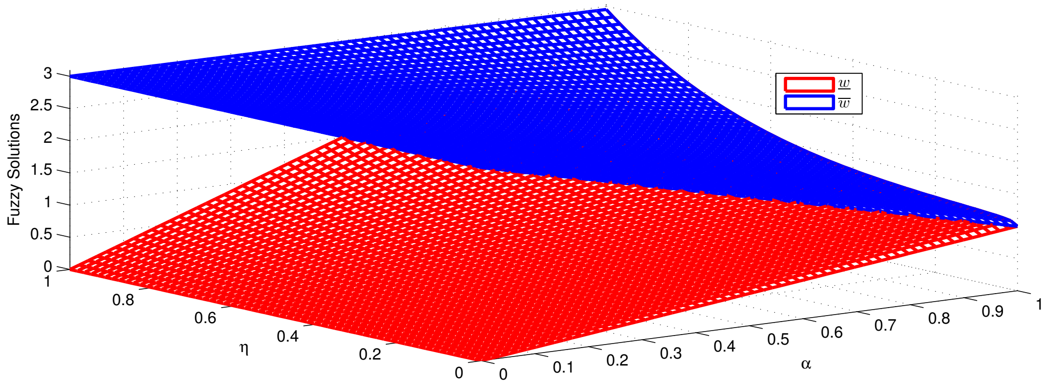

Figure 4. Surface plot of fuzzy approximate solutions for the initial three terms corresponding to different values of uncertainty α and space variable η in Example 5.2.

6. Conclusions

In this paper, we considered FFVFIE with the ABC fractional derivative. Also, we showed the existence and uniqueness of Eq (3.1) by using the fixed point techniques. We implemented the numerical examples to better grasp the FABM and its enforcement. We presented the curves and surface plots of the fuzzy approximate solutions for up to the initial three terms under the conditions of the given fractional order and took various values of uncertainty. The concerned plots provided us with information about two fuzzy solutions upper and lower. In the concluding observations, addressing the solution of FDEs with uncertainty is an extremely difficult issue, essentially, in the case of advanced differentiability like the fuzzy ABC fractional derivative. This is because the acquired systems, which are described as parametric coupled systems are solved more diligently than classical fuzzy differential equations. Regarding the significance of these systems, we have already found the solutions to the uncertain systems through the use of the ABC fractional derivative sense. It is clear that the solution at each point for every level is an interval, implying that our solutions are fuzzy number functions in each point of the domain.

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Conflict of interest

The authors declare no conflict of interest.

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