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Second-Order Multiparameter Problems Containing Complex Potentials

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Abstract: In this work, we provide some lower bounds for the number of squarly integrable solutions of some second-order multiparameter differential equations. To obtain the results, we use both Sims and Sleeman's ideas and the results are some generalization of the known results. To be more precise, we firstly construct the Weyl–Sims theory for the singular second-order differential equation with several spectral parameters. Then, we obtain some results for the several singular second-order differential equations with several spectral parameters.

Keywords: Weyl–Sims theory; Hilbert space; tensor product

MSC: 34B20; 46M05

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Citation: Erdal, I.; Uğurlu, E. Second-Order Multiparameter Problems Containing Complex Potentials. *Axioms* **2022**, *11*, 706. https://doi.org/10.3390/axioms11120706

Academic Editors: Jesús Martín Vaquero, Deolinda M. L. Dias Rasteiro, Araceli Queiruga-Dios and Fatih Yilmaz

Received: 9 October 2022
Accepted: 5 December 2022
Published: 8 December 2022

1. Introduction

The theory of multiparameter eigenvalue problems has been an attractive area since the first fundamental results in multiparameter theory were introduced by Atkinson [1]. Since every physical system contains parameters, many physical and engineering problems are modeled by systems of differential equations with several spectral parameters as seen in [2–4]. Moreover, in the literature a huge number of works exist that follow the results given in [1] (for example, see, [5–23]). Among other works, some papers contain singular multiparameter problems [24–30]. In particular, in [28] Sleeman considered the following $k-$differential equations:

$$-\frac{d^2\zeta_r}{dx_r^2}+\sum_{s=1}^{k}\{\rho_{rs}(x_r)\lambda_s+q_r(x_r)\}\zeta_r(x_r)=0,\ x_r\in[a_r,b_r), \tag{1}$$

where $1\le r\le k$, a_r, b_r are the regular point and singular point, respectively, for the r-th equation in (1), real valued functions ρ_{rs}, q_{rs} are continuous on $[a_r,b_r)$ with $\det\{\rho_{rs}(x_r)\}_{r,s=1}^{k}>0$ for all $(x_1,\dots,x_k)\in[a_1,b_1)\times\cdots\times[a_k,b_k)$, and λ_s are the spectral parameters . He proved that following inequality holds:

$$\int_{a_1}^{b_1}\cdots\int_{a_k}^{b_k}|\psi_1\psi_2\dots\psi_k|^2\det\{\rho_{rs}(x_r)\}_{r,s=1}^{k}dx_1\cdots dx_k<\infty,$$

where

$$\psi_r(x_r;\lambda_1,\dots,\lambda_k)=\vartheta_r(x_r;\lambda_1,\dots,\lambda_k)+\Phi_r(x_r;\lambda_1,\dots,\lambda_k)M_r(\lambda_1,\dots,\lambda_k)$$

is the solution of (1) such that ϑ_r and Φ_r are the solutions of (1) satisfying

$$\begin{aligned} \vartheta_r(a_r;\lambda_1,\ldots,\lambda_k) &= \cos\alpha_r, \quad \vartheta'_r(a_r;\lambda_1,\ldots,\lambda_k) = \sin\alpha_r, \\ \Phi_r(a_r;\lambda_1,\ldots,\lambda_k) &= \sin\alpha_r, \quad \Phi'_r(a_r;\lambda_1,\ldots,\lambda_k) = -\cos\alpha_r, \end{aligned}$$

$\alpha_r \in [0,\pi)$, $M(\lambda_1,\ldots,\lambda_k)$ is an analytic functions in each $\lambda_1,\ldots,\lambda_k$ and

$$\sum_{s=1}^{k} \operatorname{Im} \lambda_k \rho_{rs}(x_r)$$

is of one sign and nonzero for all $x_r \in [a_r, b_r)$. This result is the generalization of Weyl's result [31]; Weyl produced pioneering work for a second-order singular equation with a single spectral parameter. Note that Sleeman's results have been generalized by Uğurlu in [29] for the singular multiparameter dynamic equations with distributional potentials Refs. [30,32] for the singular Hamiltonian system of even-order with several spectral parameters, and Weyl's results have been generalized by Uğurlu in [33] for the fractional differential equations.

On the other hand, in 1957, Sims considered the following second-order equation [34]:

$$-y'' + q(x)y = \lambda y, \; x \in (a,b), \tag{2}$$

where a and b are the singular points for (2) and q is a complex-valued function on (a,b), and it is continuous on the same interval such that

$$q = q_1 + iq_2, \; x \in (a,b).$$

Sims introduced, in contrast to the classical Weyl theory, the notion that there may be three situations at a singular point:

(i) A limit-point case but only one square-integrable solution,
(ii) A limit-point case but two square-integrable solutions,
(iii) A limit-circle case and two square-integrable solutions.

Note that Weyl considered the second-order differential equation with a real valued potential function q. Since q is a real valued function, condition (ii) does not exist in the classical case. Sims's results have been generalized by Uğurlu in [35] for the fractional differential equations.

In this paper, our aim is to generalize the results of Sims as well as the results of Sleeman because Sleeman considered the multiparameter problem with some real valued potentials and Sims considered the problem with a complex potential. In this study, we collect these two problems. For this purpose, first of all, we consider a single second-order differential equation that has a complex-valued potential function with several spectral parameters. We construct the Weyl–Sims theory for this equation. After constructing the theory, we use the results for the several singular second-order differential equations having the complex-valued potential functions with several spectral parameters.

2. Single Second-Order Equation

In this section, we shall consider the following multiparameter differential equation:

$$-y'' + q(x)y = \left\{ \sum_{k=1}^{n} \lambda_k w_k(x) \right\} y, \quad x \in [a,b), \tag{3}$$

where a, b are the regular point and singular point, respectively, for (3); λ_k, $k = 1,2,\ldots,n$ are spectral parameters; each real valued function w_k is continuous on $[a,b)$; and q is a complex-valued continuous function on $[a,b)$ such that

$$q(x) = q_1(x) + iq_2(x), \; x \in [a,b).$$

For further calculations, we need the following sets

$$\Lambda^{+} = \left\{ \boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{C}^n : \left(\sum_{k=1}^{n} \operatorname{Im} \lambda_k w_k \right) > 0, \quad x \in [a, b) \right\},$$

and

$$\Lambda^{-} = \left\{ \boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{C}^n : \left(\sum_{k=1}^{n} \operatorname{Im} \lambda_k w_k \right) < 0, \quad x \in [a, b) \right\}.$$

Throughout the paper, the bold letter parameter $\boldsymbol{\lambda}$ indicates that it contains $n-$tuple complex parameters such that $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_n) \in \mathbb{C}^n$. We should also note that the sets Λ^{+} and Λ^{-} are not empty.

Lemma 1. *If f and g are the solutions of following differential equations*

$$-y'' + q(x)y = \left\{ \sum_{k=1}^{n} \lambda_k w_k(x) \right\} y$$

and

$$-y'' + \overline{q(x)}y = \left\{ \sum_{k=1}^{n} \mu_k w_k(x) \right\} y,$$

respectively, then Green's formula can be notated as

$$\int_a^c \left\{ \sum_{k=1}^{n} (\mu_k - \lambda_k) w_k \right\} fg dx = -2i \int_a^c q_2 fg dx + [f, g](a) - [f, g](c), \tag{4}$$

where $c \in (a, b)$ and

$$[f, g](x) = f(x)g'(x) - f'(x)g(x)$$

for $x \in [a, b)$.

Proof. Using direct calculations, we obtain

$$\begin{aligned}
\int_a^c \left\{ \sum_{k=1}^{n} (\mu_k - \lambda_k) w_k \right\} fg dx &= \int_a^c \left(f \left\{ \sum_{k=1}^{n} \mu_k w_k \right\} g - g \left\{ \sum_{k=1}^{n} \lambda_k w_k \right\} f \right) dx \\
&= \int_a^c \left[f \left(-g'' + \overline{q(x)} g \right) - g \left(-f'' + q(x) f \right) \right] dx \\
&= \int_a^c \left[-f g'' + \overline{q(x)} f g + f'' g - q(x) f g \right] dx \\
&= -2i \int_a^c q_2 f g dx + \int_a^c (f'' g - f g'') dx \\
&= -2i \int_a^c q_2 f g dx + [f, g](a) - [f, g](c).
\end{aligned}$$

Then, the proof is completed. □

Corollary 1. *If one chooses* $\boldsymbol{\mu} = \overline{\boldsymbol{\lambda}}$, $g = \overline{f}$, *where* $\overline{\boldsymbol{\lambda}} = (\overline{\lambda}_1, \overline{\lambda}_2, \ldots, \overline{\lambda}_n)$, *in (4), then it is obtained from (4) that*

$$-2i \int_a^c \left(\sum_{k=1}^{n} \operatorname{Im} \lambda_k w_k - q_2 \right) |f|^2 dx = [f, \overline{f}](a) - [f, \overline{f}](c). \tag{5}$$

for $c \in [a, b)$.

Corollary 2. *Let f and g be the solutions of (3) corresponding to the same* $n-$*tuple parameter* $\boldsymbol{\lambda}$. *Then, it is obtained from (4) that*

$$[f, g](a) = [f, g](c)$$

for $c \in [a, b)$.

It is obvious that Equation (3) can be written as the following first-order equation:

$$\hat{y}' = A(x, \lambda)\hat{y}, \ \ x \in [a, b) \tag{6}$$

where

$$A(x; \lambda) = \begin{pmatrix} 0 & 1 \\ q(x) - \sum_{k=1}^{n} \lambda_k w_k(x) & 0 \end{pmatrix}, \ \hat{y}(x, \lambda) = \begin{pmatrix} y(x, \lambda) \\ y'(x, \lambda) \end{pmatrix}. \tag{7}$$

As is well known, (6) has an unique vector solution $\hat{y}$ for each fixed $\xi \in [a, b)$ satisfying

$$\hat{y}(\xi, \lambda) = \begin{pmatrix} y(\xi, \lambda) \\ y'(\xi, \lambda) \end{pmatrix} = \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix},$$

where η_1, η_2 are arbitrary complex numbers, due to the assumptions on q and w_k for $k = 1, 2, \ldots, n$ on $[a, b)$. Thus, we can construct a linearly independent set of solutions y_1, y_2 of (3) on $[a, b)$. For this purpose, we choose solutions $y_1$and y_2 satisfying the initial conditions

$$y_j^{(k-1)}(x) = w_{jk}, \ \ j, k = 1, 2, \ x \in [a, b)$$

where the determinant of the matrix $\left[w_{jk} \right]$ does not vanish. Then, a linearly independent set of solutions y_1, y_2 is called a fundamental system. Now, we can give the following lemma.

Lemma 2. *Every solution of (3) is a lineer combination of a fixed, arbitrarily chosen, fundamental system of solutions of (3).*

Proof. Let y_1, y_2 constitute a fundamental system of solutions of (3) and $\widetilde{y}$ be any solution of (3). We can choose constants c_1, c_2 at a fixed point x_0 of the interval $[a, b)$ such that

$$\begin{cases} \widetilde{y} = c_1 y_1 + c_2 y_2, \\ \widetilde{y}' = c_1 y_1' + c_2 y_2'. \end{cases} \tag{8}$$

The determinant of this system is the Wronskian of the fundamental system constructed by y_1, y_2 for $x = x_0$ and hence $[y_1, y_2](x_0) \neq 0$. On the other hand, (8) implies that the functions $\widetilde{y}$ and $c_1 y_1 + c_2 y_2$ are solutions of the (3) and yield the same initial conditions. Because of the uniqueness of such a solution,

$$y \equiv c_1 y_1 + c_2 y_2$$

on $[a, b)$. Thus, the set of all solutions of (3) constitutes a two-dimensional linear space that gives the proof . □

3. Nested Circles

In this section, we provide a geometric definition of the fractional transformation constructed by the solutions of (3) for $\boldsymbol{\lambda} \in \Lambda^+$ and $\boldsymbol{\lambda} \in \Lambda^-$. However, first of all, we want to provide the results for $\boldsymbol{\lambda} \in \Lambda^+$, and the results for $\boldsymbol{\lambda} \in \Lambda^-$ may then be given similarly.

Let $\varphi(x, \boldsymbol{\lambda})$ and $\psi(x, \boldsymbol{\lambda})$ be the solutions of (3) satisfying the initial conditions

$$\varphi(a, \boldsymbol{\lambda}) = \sin\alpha, \quad \psi(a, \boldsymbol{\lambda}) = \cos\alpha$$

$$\varphi'(a, \boldsymbol{\lambda}) = -\cos\alpha, \quad \psi'(a, \boldsymbol{\lambda}) = \sin\alpha,$$

where α is a complex number such that $\alpha = \alpha_1 + i\alpha_2$, and y' should be understood as $\frac{d}{dx}y$. With the help of Hartog's theorem on the separate analyticity [36], we may say that $\varphi(x, \boldsymbol{\lambda})$, $\psi(x, \boldsymbol{\lambda})$, $\varphi'(x, \boldsymbol{\lambda})$, and $\psi'(x, \boldsymbol{\lambda})$ are complete functions of the parameters $\lambda_1, \lambda_2, \ldots, \lambda_n$. Since

$$[\varphi, \psi](a) = 1$$

these solutions are linearly independent on the interval $[a, b)$. For this reason, we may consider the following solution of (3):

$$\chi(x, \boldsymbol{\lambda}) = \varphi(x, \boldsymbol{\lambda}) + m\psi(x, \boldsymbol{\lambda}).$$

Our aim is to determine the behavior of χ around the singular point b. Therefore, first of all, we shall impose the following regular boundary condition at a point:

$$y(c)\cos\gamma + y'(c)\sin\gamma = 0, \quad a < c < b, \tag{9}$$

where γ is a complex number such that $\gamma = \gamma_1 + i\gamma_2$.

For now, suppose that $\psi(c, \boldsymbol{\lambda}) \neq 0$ and $\psi'(c, \boldsymbol{\lambda}) \neq 0$ temporarily. In order for χ to satisfy the condition (9), we may write

$$m = m(c, \boldsymbol{\lambda}, z) = -\left\{\frac{\varphi(c, \boldsymbol{\lambda})z + \varphi'(c, \boldsymbol{\lambda})}{\psi(c, \boldsymbol{\lambda})z + \psi'(c, \boldsymbol{\lambda})}\right\}, \ z = \cot\gamma. \tag{10}$$

For the fixed choice of $\boldsymbol{\lambda}$ and c, (10) defines a linear fractional transformation from the complex $z-$plane to the complex $m-$plane. The inverse mapping is given by

$$z = z(c, \boldsymbol{\lambda}, m) = -\left\{\frac{\psi'(c, \boldsymbol{\lambda})m + \varphi'(c, \boldsymbol{\lambda})}{\psi(c, \boldsymbol{\lambda})m + \varphi(c, \boldsymbol{\lambda})}\right\}. \tag{11}$$

Since the critical point of transformation (10) is

$$z = -\frac{\psi'(c, \boldsymbol{\lambda})}{\psi(c, \boldsymbol{\lambda})},$$

with a direct computation using (5), the imaginary part of this point can be expressed as

$$\operatorname{Im}\left[-\frac{\psi'(c, \boldsymbol{\lambda})}{\psi(c, \boldsymbol{\lambda})}\right] = \frac{1}{|\psi(c, \boldsymbol{\lambda})|^2}\left[\int_a^c \left(\sum_{k=1}^{n} \operatorname{Im}\lambda_k w_k - q_2\right)|\psi(x, \boldsymbol{\lambda})|^2 dx - \frac{1}{2}\sinh 2\alpha_2\right]. \tag{12}$$

Thus, from the well known properties of the linear fractional transformation (10), the real axis of the $z-$plane has an image that is a boundary of a circle in the $m-$plane. Let us denote this circle corresponding to the point c and parameter $\boldsymbol{\lambda}$ where $\boldsymbol{\lambda} \in \Lambda^+$ by $D_c(\boldsymbol{\lambda})$.

From (12) one may see that in the case of $\boldsymbol{\lambda} \in \Lambda^+$, $q_2 \leq 0$, and $\alpha_2 \leq 0$, the critical point of mapping (10) lies in the upper half complex z-plane so that the lower half z-plane maps onto a circle $D_c(\boldsymbol{\lambda})$ in the complex m-plane. Therefore, a point m is in circle $D_c(\boldsymbol{\lambda})$ if and only if

$$\operatorname{Im} z(c, \boldsymbol{\lambda}, m) < 0$$

and is on the boundary of the circle $D_c(\boldsymbol{\lambda})$ if and only if

$$\operatorname{Im} z(c, \boldsymbol{\lambda}, m) = 0.$$

Note that, in case of $\boldsymbol{\lambda} \in \Lambda^-$, $q_2 \geq 0$ and $\alpha_2 \geq 0$, this critical point lies in the lower half of the $z-$plane so the upper half of $z-$plane maps onto a circle similarly.

From (11), we have

$$\operatorname{Im}\{z(c, \boldsymbol{\lambda}, m)\} = -\operatorname{Im}\left\{ \frac{\psi'(c, \boldsymbol{\lambda})m + \varphi'(c, \boldsymbol{\lambda})}{\psi(c, \boldsymbol{\lambda})m + \varphi(c, \boldsymbol{\lambda})} \right\}. \tag{13}$$

By using (5), we obtain

$$\begin{aligned} [\chi, \overline{\chi}](c) =& 2i \int_a^c \left(\sum_{k=1}^n \operatorname{Im} \lambda_k w_k - q_2 \right) |\chi|^2 dx - i(1 + |m|^2)(\sinh 2\alpha_2) \\ & - 2i \operatorname{Im} m \cosh 2\alpha_2 \end{aligned} \tag{14}$$

and it also follows from (13) that

$$[\chi, \overline{\chi}](c) = 2i|\chi(c, \boldsymbol{\lambda})|^2 \operatorname{Im} z. \tag{15}$$

Hence, we can write

$$\begin{aligned} |\chi(c, \boldsymbol{\lambda})|^2 \operatorname{Im} z = & \int_a^c \left(\sum_{k=1}^n \operatorname{Im} \lambda_k w_k - q_2 \right) |\chi|^2 dx \\ & - \frac{1}{2}(1 + |m|^2)(\sinh 2\alpha_2) - \operatorname{Im} m \cosh 2\alpha_2. \end{aligned} \tag{16}$$

Now, (16) indicates that m is in or on the circle $D_c(\boldsymbol{\lambda})$ if and only if

$$\int_a^c \left(\sum_{k=1}^n \operatorname{Im} \lambda_k w_k - q_2 \right) |\chi|^2 dx \leq \frac{1}{2}(1 + |m|^2) \sinh 2\alpha_2 + \operatorname{Im} m(\cosh 2\alpha_2). \tag{17}$$

Moreover, the center of $D_c(\boldsymbol{\lambda})$ corresponds the conjugate of critical point of the transformation given in (10). In other words, the center of $D_c(\boldsymbol{\lambda})$ is equal to

$$p_c(\boldsymbol{\lambda}) = m\left(c, \boldsymbol{\lambda}, -\frac{\overline{\psi'}(c, \boldsymbol{\lambda})}{\overline{\psi}(c, \boldsymbol{\lambda})} \right) = -\frac{[\overline{\psi}, \varphi](c)}{[\overline{\psi}, \psi](c)}. \tag{18}$$

Since the image of $z = 0$ in the $m-$plane is on the boundary of the circle $D_c(\boldsymbol{\lambda})$, that is,

$$m = -\frac{\varphi'(c, \boldsymbol{\lambda})}{\psi'(c, \boldsymbol{\lambda})}$$

is a point on the boundary of $D_c(\boldsymbol{\lambda})$, then we can introduce the radius $r_c(\boldsymbol{\lambda})$ of the circle $D_c(\boldsymbol{\lambda})$ as

$$r_c(\boldsymbol{\lambda}) = \left| \frac{[\varphi, \psi](c)}{[\overline{\psi}, \psi](c)} \right|.$$

Then, from Corollary 2, since $[\varphi, \psi](c) = 1$, we can rewrite the radius $r_c(\boldsymbol{\lambda})$ of the circle $D_c(\boldsymbol{\lambda})$ as

$$r_c(\boldsymbol{\lambda}) = \frac{1}{|[\overline{\psi}, \psi](c)|}$$

or alternatively

$$r_c(\boldsymbol{\lambda}) = \left[2\int_a^c \left(\sum_{k=1}^n \operatorname{Im}\lambda_k w_k - q_2\right)|\psi|^2 dx - \sinh 2\alpha_2\right]^{-1}. \tag{19}$$

Now, as we know it well, (10) brings the real axis of the z-plane to the boundary of the circle $D_c(\boldsymbol{\lambda})$ in the m-plane, and therefore the inverse mapping (11) transforms the circle $D_c(\boldsymbol{\lambda})$ into the real axis of z-plane. Thus, its critical point

$$m_0 = -\frac{\varphi(c,\boldsymbol{\lambda})}{\psi(c,\boldsymbol{\lambda})}$$

must be on the boundary of $D_c(\boldsymbol{\lambda})$. Furthermore, since the points $m = -\frac{\varphi'(c,\boldsymbol{\lambda})}{\psi'(c,\boldsymbol{\lambda})}$ and $m_0 = -\frac{\varphi(c,\boldsymbol{\lambda})}{\psi(c,\boldsymbol{\lambda})}$ are on the boundary of the circle $D_c(\boldsymbol{\lambda})$, we can write

$$\left|-\frac{\varphi'(c,\boldsymbol{\lambda})}{\psi'(c,\boldsymbol{\lambda})} + \frac{\varphi(c,\boldsymbol{\lambda})}{\psi(c,\boldsymbol{\lambda})}\right| \leq 2r_c(\boldsymbol{\lambda})$$

and with the help of (19), we obtain

$$2\int_a^c \left(\sum_{k=1}^n \operatorname{Im}\lambda_k w_k - q_2\right)|\psi|^2 dx - \sinh 2\alpha_2 \leq 2|\psi'(c,\boldsymbol{\lambda})||\psi(c,\boldsymbol{\lambda})|. \tag{20}$$

Up to the present, we have continued under the condition that $\psi(c,\boldsymbol{\lambda})$ and $\psi'(c,\boldsymbol{\lambda})$ are not equal to zero. Now, we can investigate this situation in detail. If for $c \in [a,b)$ there exists $\boldsymbol{\lambda}'$ such that $\psi(c,\boldsymbol{\lambda}') = 0$, then neighborhood $N(\boldsymbol{\lambda}')$ of $\boldsymbol{\lambda}'$ exists such that $\psi(c,\boldsymbol{\lambda}) \neq 0$ for $\boldsymbol{\lambda} \neq \boldsymbol{\lambda}'$ and $\boldsymbol{\lambda} \in N(\boldsymbol{\lambda}')$. However, if for $\boldsymbol{\lambda} \to \boldsymbol{\lambda}'$, then the right hand side of (20) approaches to zero, which is impossible due to the fact the left hand side of (20) is exactly positive because of the restrictions on the $\boldsymbol{\lambda}$, q, and α. Similarly, we deduce that $\psi'(c,\boldsymbol{\lambda}) \neq 0$. Hence, the assumptions behind the solution $\psi(c,\boldsymbol{\lambda})$ and $\psi'(c,\boldsymbol{\lambda})$can be removed.

Finally, for $a < c' < c < b$, if $m(c,\lambda)$ is in or on $D_c(\boldsymbol{\lambda})$, then $m(c,\lambda)$ is also inside $D_{c'}(\boldsymbol{\lambda})$ from (17). This means that if $c' < c$ then, $D_{c'}(\boldsymbol{\lambda})$ contains $D_c(\boldsymbol{\lambda})$. Therefore, as $c \to b$ the circles $D_c(\boldsymbol{\lambda})$ converge either to a limit-circle or to a limit-point. In both cases, there is a point inside all the circles $D_c(\boldsymbol{\lambda})$ such that if $M(b,\boldsymbol{\lambda})$ is any point on the limit-circle or is a limit-point, then it follows from (17) that

$$\int_a^b \left(\sum_{k=1}^n \operatorname{Im}\lambda_k w_k - q_2\right)|\varphi + M\psi|^2 dx < \infty.$$

Then, we may summarize the results as the following Theorem.

Theorem 1. *Let $[a,b)$ be a semi-open interval, where a is the regular and b is the singular point, and complex-valued $q(x)$ and real-valued w_k are continuous functions on $[a,b)$ for $k = 1,2,\ldots,n$. If $M(b,\boldsymbol{\lambda})$ is any point inside the all circles $D_c(\boldsymbol{\lambda})$, then*

$$\chi(x,\boldsymbol{\lambda}) = \varphi(x,\boldsymbol{\lambda}) + M(b,\boldsymbol{\lambda})\psi(x,\boldsymbol{\lambda}).$$

is a solution of (3) such that

$$-\infty < \int_a^b \left(\sum_{k=1}^n \operatorname{Im}\lambda_k w_k(x) - q_2\right)|\chi(x,\boldsymbol{\lambda})|^2 dx < \infty,$$

in the case $\boldsymbol{\lambda} \in \Lambda^+$, $q_2 \leq 0$, $\operatorname{Im}\alpha \leq 0$ or $\boldsymbol{\lambda} \in \Lambda^-$, $q_2 \geq 0$, $\operatorname{Im}\alpha \geq 0$.

Now, it is obvious that either a limit-circle or limit-point case prevails at $x = b$. If a limit-circle situation occurs at $x = b$, then we obtain

$$\left| \int_a^b \left(\sum_{k=1}^n \operatorname{Im} \lambda_k w_k - q_2 \right) |\psi|^2 dx \right| < \infty$$

from (19). Thus, the last inequality and previous theorem indicate that two linearly independent solutions $\chi(x, \boldsymbol{\lambda})$, $\psi(x, \boldsymbol{\lambda})$ of (3) exist and each of them is squarly integrable in $[a, b)$ with respect to the weight function $\left(\sum_{k=1}^n \operatorname{Im} \lambda_k w_k - q_2 \right)$. If a limit-point case prevails at $x = b$, then only one point $M(b, \boldsymbol{\lambda})$ exists inside all of the circles $D_c(\boldsymbol{\lambda})$. Hence, only one solution $\chi(x, \boldsymbol{\lambda})$ is squarly integrable in $[a, b)$ with respect to the weight function $\left(\sum_{k=1}^n \operatorname{Im} \lambda_k w_k - q_2 \right)$. Here, it should be remarked that from (19), we obtain $\psi(x, \boldsymbol{\lambda}) \notin L_2(a, b)$. That is, from (19) we can write

$$\int_a^b \left(\sum_{k=1}^n \operatorname{Im} \lambda_k w_k - q_2 \right) |\psi|^2 dx = \infty.$$

However, it may be that

$$\int_a^b |\psi|^2 dx < \infty.$$

This situation is a special limit-point case that has no analog in the classical limit-point and limit-circle theory, and in [34], Sims gave two examples for clarifying this case for one parameter case. In the next section, we show that the limit-point and limit-circle theory are independent of the n-tuple parameter $\boldsymbol{\lambda}$ so that this special case exists for the multiparameter case.

Briefly, the following cases occur at singular point b.

(I) The limit-point case: there are linearly independent solutions of (3) satisfying

$$\left| \int_a^b \left(\sum_{k=1}^n \operatorname{Im} \lambda_k w_k - q_2 \right) |\chi|^2 dx \right| < \infty$$

and

$$\int_a^b \left(\sum_{k=1}^n \operatorname{Im} \lambda_k w_k - q_2 \right) |\psi|^2 dx = \infty, \quad \int_a^b |\psi|^2 dx = \infty.$$

(II) The limit-point case: there exist linearly independent solutions of (3) satisfying

$$\left| \int_a^b \left(\sum_{k=1}^n \operatorname{Im} \lambda_k w_k - q_2 \right) |\chi|^2 dx \right| < \infty.$$

and

$$\int_a^b \left(\sum_{k=1}^n \operatorname{Im} \lambda_k w_k - q_2 \right) |\psi|^2 dx = \infty, \quad \int_a^b |\psi|^2 dx < \infty.$$

(III) The limit-circle case: there exist linearly independent solutions of (3) satisfying

$$\left| \int_a^b \left(\sum_{k=1}^n \operatorname{Im} \lambda_k w_k - q_2 \right) |\chi|^2 dx \right| < \infty$$

and

$$\left| \int_a^b \left(\sum_{k=1}^n \operatorname{Im} \lambda_k w_k - q_2 \right) |\psi|^2 dx \right| < \infty.$$

4. Independence of the Theory from the Parameters

In this section, we will show that the Weyl–Sims theory for the multiparameter single eigenvalue problem is independent of the n-tuple parameter $\boldsymbol{\lambda}$.

Firstly, we shall define a set of real numbers such that

$$\boldsymbol{\Lambda}^* = \left\{ \boldsymbol{\beta} = (\beta_1, \beta_2, \ldots, \beta_m) : \sum_{k=1}^{n} \beta_k w_k > 0,\ x \in [a,b) \right\}.$$

Theorem 2. *If for some n-tuple complex number $\boldsymbol{\lambda}' \in \Lambda^+$ and every $\boldsymbol{\beta} \in \Lambda^*$*

$$\int_a^b \left(\sum_{k=1}^{n} \beta_k w_k - q_2 \right) \left| \varphi(x, \boldsymbol{\lambda}') \right|^2 dx < \infty$$

and

$$\int_a^b \left(\sum_{k=1}^{n} \beta_k w_k - q_2 \right) \left| \psi(x, \boldsymbol{\lambda}') \right|^2 dx < \infty$$

then, for all other $\boldsymbol{\lambda} \in \Lambda^+$ satisfying

$$\operatorname{Re}(\boldsymbol{\lambda} - \boldsymbol{\lambda}'),\ \operatorname{Im}(\boldsymbol{\lambda} - \boldsymbol{\lambda}') \in \Lambda^*,$$

the following inequalities hold:

$$\int_a^b \left(\sum_{k=1}^{n} \beta_k w_k - q_2 \right) \left| \varphi(x, \boldsymbol{\lambda}) \right|^2 dx < \infty,$$

$$\int_a^b \left(\sum_{k=1}^{n} \beta_k w_k - q_2 \right) \left| \psi(x, \boldsymbol{\lambda}) \right|^2 dx < \infty.$$

Proof. Let us consider the following nonhomogeneous multiparameter differential equation:

$$- y'' + q(x)y - \left\{ \sum_{k=1}^{n} \lambda_k' w_k \right\} y = \left\{ \sum_{k=1}^{n} (\lambda_k w_k - \lambda_k' w_k) \right\} y. \tag{21}$$

Using two linearly independent solutions $\varphi(x, \boldsymbol{\lambda}')$ and $\psi(x, \boldsymbol{\lambda}')$ of the homogeneous multiparameter differential equation

$$-y'' + q(x)y - \left\{ \sum_{k=1}^{n} \lambda_k' w_k \right\} y = 0,$$

the general solution of (21) can be written as

$$y(x, \boldsymbol{\lambda}) = c_1(x)\varphi(x, \boldsymbol{\lambda}') + c_2(x)\psi(x, \boldsymbol{\lambda}').$$

By the help of variation of parameters, we obtain that

$$c_1(x) = \int_a^x \left\{ \sum_{k=1}^{n} (\lambda_k - \lambda_k') w_k \right\} y(t)\psi(t, \boldsymbol{\lambda}') dt$$

and

$$c_2(x) = \int_a^x \left\{ \sum_{k=1}^{n} (\lambda_k' - \lambda_k) w_k \right\} y(t)\varphi(t, \boldsymbol{\lambda}') dt.$$

Since $\psi(x,\boldsymbol{\lambda})$ is a solution of (21), we obtain

$$\psi(x,\boldsymbol{\lambda}) = \psi(x,\boldsymbol{\lambda}') + \varphi(x,\boldsymbol{\lambda}') \int_a^x \left[\sum_{k=1}^{n} (\lambda_k - \lambda_k')w_k\right] \psi(t,\boldsymbol{\lambda})\psi(t,\boldsymbol{\lambda}')dt \quad (22)$$
$$- \psi(x,\boldsymbol{\lambda}') \int_a^x \left[\sum_{k=1}^{n} (\lambda_k - \lambda_k')w_k\right] \psi(t,\boldsymbol{\lambda})\varphi(t,\boldsymbol{\lambda}')dt.$$

From the well known inequality

$$|x_1 + x_2 + x_3|^2 \leq 3\Big(|x_1|^2 + |x_2|^2 + |x_3|^2\Big), \quad (23)$$

we obtain

$$|\psi(x,\boldsymbol{\lambda})|^2 \leq 3\big|\psi(x,\boldsymbol{\lambda}')\big|^2 + 3\big|\varphi(x,\boldsymbol{\lambda}')\big|^2 \left|\int_a^x \left[\sum_{k=1}^{n} (\lambda_k - \lambda_k')w_k\right] \psi(t,\boldsymbol{\lambda})\psi(t,\boldsymbol{\lambda}')dt\right|^2 \quad (24)$$
$$3\big|\psi(x,\boldsymbol{\lambda}')\big|^2 \left|\int_a^x \left[\sum_{k=1}^{n} (\lambda_k - \lambda_k')w_k\right] \psi(t,\boldsymbol{\lambda})\varphi(t,\boldsymbol{\lambda}')dt\right|^2.$$

Let $\lambda_k - \lambda_k' = \beta_k + i\delta_k$ for $\beta, \delta \in \boldsymbol{\Lambda}^*$. Then, it follows from (23) that

$$|\psi(x,\boldsymbol{\lambda})|^2 \leq 3\big|\psi(x,\boldsymbol{\lambda}')\big|^2 + 6\big|\varphi(x,\boldsymbol{\lambda}')\big|^2 \left|\int_a^x \left[\sum_{k=1}^{n} \beta_k w_k\right] \psi(t,\boldsymbol{\lambda})\psi(t,\boldsymbol{\lambda}')dt\right|^2$$
$$+ 6\big|\varphi(x,\boldsymbol{\lambda}')\big|^2 \left|\int_a^x \left[\sum_{k=1}^{n} \delta_k w_k\right] \psi(t,\boldsymbol{\lambda})\psi(t,\boldsymbol{\lambda}')dt\right|^2 \quad (25)$$
$$+ 6\big|\psi(x,\boldsymbol{\lambda}')\big|^2 \left|\int_a^x \left[\sum_{k=1}^{n} \beta_k w_k\right] \psi(t,\boldsymbol{\lambda})\varphi(t,\boldsymbol{\lambda}')dt\right|^2$$
$$+ 6\big|\psi(x,\boldsymbol{\lambda}')\big|^2 \left|\int_a^x \left[\sum_{k=1}^{n} \delta_k w_k\right] \psi(t,\boldsymbol{\lambda})\varphi(t,\boldsymbol{\lambda}')dt\right|^2.$$

By the Schwarz inequality, we have

$$|\psi(x,\boldsymbol{\lambda})|^2 \leq 3\big|\psi(x,\boldsymbol{\lambda}')\big|^2$$
$$+ 6\big|\varphi(x,\boldsymbol{\lambda}')\big|^2 \int_a^x \left[\sum_{k=1}^{n} \beta_k w_k\right] |\psi(t,\boldsymbol{\lambda})|^2 dt \int_a^x \left[\sum_{k=1}^{n} \beta_k w_k\right] \big|\psi(t,\boldsymbol{\lambda}')\big|^2 dt$$
$$+ 6\big|\varphi(x,\boldsymbol{\lambda}')\big|^2 \int_a^x \left[\sum_{k=1}^{n} \delta_k w_k\right] |\psi(t,\boldsymbol{\lambda})|^2 dt \int_a^x \left[\sum_{k=1}^{n} \delta_k w_k\right] \big|\psi(t,\boldsymbol{\lambda}')\big|^2 dt \quad (26)$$
$$+ 6\big|\psi(x,\boldsymbol{\lambda}')\big|^2 \int_a^x \left[\sum_{k=1}^{n} \beta_k w_k\right] |\psi(t,\boldsymbol{\lambda})|^2 dt \int_a^x \left[\sum_{k=1}^{n} \beta_k w_k\right] \big|\varphi(t,\boldsymbol{\lambda}')\big|^2 dt$$
$$+ 6\big|\psi(x,\boldsymbol{\lambda}')\big|^2 \int_a^x \left[\sum_{k=1}^{n} \delta_k w_k\right] |\psi(t,\boldsymbol{\lambda})|^2 dt \int_a^x \left[\sum_{k=1}^{n} \delta_k w_k\right] \big|\varphi(t,\boldsymbol{\lambda}')\big|^2 dt.$$

Let K be the maximum value of the second, fourth, sixth, and eighth integrals in (26) as $x \to b$ and $\beta_k + \delta_k = \eta_k$ for $k = 1, 2, \ldots, n$. Then, from (26), we find

$$|\psi(x,\boldsymbol{\lambda})|^2 \leq 3\big|\psi(x,\boldsymbol{\lambda}')\big|^2 \quad (27)$$
$$+6\Big[\big|\varphi(x,\boldsymbol{\lambda}')\big|^2 + \big|\psi(x,\boldsymbol{\lambda}')\big|^2\Big] K \int_a^x \left[\sum_{k=1}^{n} \eta_k w_k\right] |\psi(t,\boldsymbol{\lambda})|^2 dt.$$

For $a < c' < c < b$, the multiplication of both sides of (27) by $\left[\sum_{k=1}^{n} \eta_k w_k - q_2\right]$ and the integration of both sides from c' to c implies

$$\int_{c'}^{c} \left[\sum_{k=1}^{n} \eta_k w_k - q_2\right] |\psi(t,\lambda)|^2 dt \leq 3 \int_{c'}^{c} \left[\sum_{k=1}^{n} \eta_k w_k - q_2\right] |\psi(t,\lambda')|^2 dt \tag{28}$$

$$+ 6K \int_{c'}^{c} \left[\sum_{k=1}^{n} \eta_k w_k - q_2\right] \left\{ |\varphi(t,\lambda')|^2 + |\psi(t,\lambda')|^2 \right\} \int_{a}^{c} \left[\sum_{k=1}^{n} \eta_k w_k\right] |\psi(t,\lambda)|^2 dt.$$

If we choose c' sufficiently large so that the inequality

$$6K \int_{c'}^{c} \left[\sum_{k=1}^{n} \eta_k w_k - q_2\right] \left\{ |\varphi(t,\lambda')|^2 + |\psi(t,\lambda')|^2 \right\} < \frac{1}{2}$$

holds then (28) gives

$$\int_{c'}^{c} \left[\sum_{k=1}^{n} \eta_k w_k - q_2\right] |\psi(t,\lambda)|^2 dt \leq 6 \int_{c'}^{c} \left[\sum_{k=1}^{n} \eta_k w_k - q_2\right] |\psi(t,\lambda')|^2 dt \tag{29}$$

$$+ \int_{a}^{c'} \left[\sum_{k=1}^{n} \eta_k w_k\right] |\psi(t,\lambda)|^2 dt.$$

The right side of (29) is independent of c. So, taking the limit as $c \to b$, we finally obtain

$$\int_{c'}^{b} \left[\sum_{k=1}^{n} \eta_k w_k - q_2\right] |\psi(t,\lambda)|^2 dt < \infty.$$

A similar treatment is valid for $\varphi(x,\lambda)$. This completes the proof. □

5. Several Second-Order Equations

In this section, we will generalize the previous results to some several second-order differential equations. Namely, we will consider the following $(n+1)-$equations with unique $n-$spectral parameters:

$$-y_r''(x_r) + q_r(x_r) y_r(x_r) = \left\{ \sum_{k=1}^{n} \lambda_k w_{rk}(x_r) \right\} y_r(x_r), \quad x_r \in [a_r, b_r), \tag{30}$$

where q_r is a complex-valued continuous function such that $q_r = q_{r_1} + i q_{r_2}$, $q_{r_2} \leq 0$, and w_{rk} is a real-valued continuous function on $[a_r, b_r)$; also, a_r is the regular point, and b_r is the singular point for $r-$th equation in (30), where $r = 1, 2, \ldots, n+1$. We assume that,

$$\det \begin{bmatrix} w_{11}(x_1) & w_{12}(x_1) & \cdots & w_{1n}(x_1) & -q_{12} \\ w_{21}(x_2) & w_{22}(x_2) & \cdots & w_{2n}(x_1) & -q_{22} \\ \vdots & \vdots & & \vdots & \vdots \\ w_{n+1,1}(x_{n+1}) & w_{n+1,2}(x_{n+1}) & \cdots & w_{n+1,n}(x_{n+1}) & -q_{n+1,2} \end{bmatrix} > 0 \tag{31}$$

for $x = (x_1, x_2, \ldots, x_{n+1}) \in I$ where I is the Cartesian product of the $(n+1)-$intervals $[a_r, b_r)$, $r = 1, 2, \ldots, n+1$ such that

$$I = I_1 \times I_2 \times ... \times I_{n+1}$$

and let

$$f(x_1, x_2, \ldots, x_{n+1}) = \prod_{k=1}^{n+1} f_k(x_k), \quad g(x_1, x_2, \ldots, x_{n+1}) = \prod_{k=1}^{n+1} g_k(x_k)$$

for $(x_1, x_2, ..., x_{n+1}) \in I$. Now, for Hilbert space H, a suitable inner product is given for functions $f, g \in H$ by

$$\langle f, g \rangle = \int_{a_1}^{b_1} \cdots \int_{a_n}^{b_n} \prod_{k=1}^{n+1} f_k(x_k) \prod_{k=1}^{n+1} \overline{g_k(x_k)} \det \begin{bmatrix} w_{11}(x_1) & \cdots & -q_{12} \\ \vdots & & \vdots \\ w_{n+1,1}(x_{n+1}) & \cdots & -q_{n+1,2} \end{bmatrix} dx_1 ... dx_{n+1}. \tag{32}$$

Theorem 3. *If (31) holds and each equation in (30) is in the limit-circle case, that is, each solution of (30) satisfies the following inequality:*

$$\left| \int_{a_r}^{b_r} w_{rk}(x_r) |y_r(x_r, \boldsymbol{\lambda})|^2 dx_r \right| < \infty, \left| \int_{a_r}^{b_r} q_{r2} |y_r(x_r, \boldsymbol{\lambda})|^2 dx_r \right| < \infty, \tag{33}$$

for some $\boldsymbol{\lambda} \in \Lambda^+, q_{r2} \leq 0$ or $\boldsymbol{\lambda} \in \Lambda^-, q_{r2} \geq 0$ and $k = 1, 2, \ldots, n$, $r = 1, 2, \ldots, n+1$, then the inner product $\langle y, y \rangle$ is a constant.

Proof. Assume that each equation is in the limit-circle case in (30), i.e., $2(n+1)$-linearly independent solutions hold (33). From (p. 210, [4]), it can be seen that (32) is equal to

$$\langle y, y \rangle = \det \begin{bmatrix} \int\limits_{a_1}^{b_1} w_{11}(x_1)|y_1(x_1)|^2 dx_1 & \cdots & -\int\limits_{a_1}^{b_1} q_{12}(x_1)|y_1(x_1)|^2 dx_1 \\ \vdots & & \vdots \\ \int\limits_{a_{n+1}}^{b_{n+1}} w_{(n+1)1}(x_{n+1})|y_{n+1}(x_{n+1})|^2 dx_{n+1} & \cdots & -\int\limits_{a_{n+1}}^{b_{n+1}} q_{(n+1)1}(x_{n+1})|y_{n+1}(x_{n+1})|^2 dx_{n+1} \end{bmatrix}. \tag{34}$$

Hence, (33) and (34) complete the proof. □

In the view of Theorem 3, the next theorem can be given as the main result.

Theorem 4. *If the $r-th$ equation in* (30) *has l_r linearly independent solutions satifying* (33)*, for some $\boldsymbol{\lambda} \in \Lambda^+$, $q_{r2} \leq 0$ or $\boldsymbol{\lambda} \in \Lambda^-$, $q_{r2} \geq 0$ and $1 \leq l_r \leq 2$, then linearly independent products $y(x_1, x_2, \ldots, x_{n+1})$ of solutions of equation* (30) *satisfying*

$$\langle y, y \rangle = const, \tag{35}$$

are not less than

$$\prod_{p=1}^{n+1} l_r.$$

6. Conclusions and Discussion

In 1957, Sims [34] generalized the results of Weyl [31] by considering the potential function $q(x)$ as a complex-valued function on the given interval. The most important part of Sims's result is that the limit point case may occur even if one of the linearly independent soutions can be squarly integrable on the given interval. Moreover, Sims gave two examples relating to this unexpected result. On the other side, in 1972 Sleeman [28] generalized the results of Weyl by considering several spectral parameters rather than considering one spectral parameter.

In this work, we have collected these two ideas in one, and hence the results of this paper are a generalization of both the results of Sims and Sleeman (and of Weyl). Indeed, in this study, we initially have considered a singular multiparameter second-order differential equation containing a complex-valued potential. Then, we give a geometric meaning of the limit point and limit circle situations and show that the theory for multiparameter problems is independent of the $n-$tuple complex parameters $\boldsymbol{\lambda}$. In the last part, we use the results that we obtained while constructing the theory for several multiparameter singular second-order differential equations that have complex valued potential functions. We shall also note that the results of this paper can also be a generalization of the results of [29] when the time scale is considered as a subset of the real line.

Author Contributions: Investigation, I.E. and E.U.; data curation, I.E.; writing—original draft, I.E. and E.U.; writing—review and editing, I.E.; visualization, E.U. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Institutional Review Board Statement: Not applicable.

Informed Consent Statement: Not applicable.

Data Availability Statement: Not applicable.

Conflicts of Interest: The authors declare no conflict of interest.

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