

# Revisiting Ćirić type nonunique fixed point theorems via interpolation

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## ABSTRACT

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*In this paper, we aim to revisit some non-unique fixed point theorems that were initiated by Ćirić, first. We consider also some natural consequences of the obtained results. In addition, we provide a simple example to illustrate the validity of the main result.*

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## 1. INTRODUCTION AND PRELIMINARIES

The notion of "nonunique fixed point" was suggested and used efficiently by Ćirić [16] in 1974. Regarding the fact that Banach's fixed point theorem was abstracted from the papers of Liouville (1837) and Picard (1890), we underline the connection of the fixed point theorem and the solution of the differential equations. As well as the existence, the uniqueness of the solutions of differential equations is desired in most occasions. On the other hand, there are certain types of differential equations that have no unique solution. In connection with this fact, it is necessary to determine that non-unique fixed points are at least as significant as the unique ones. After the initial work of Ćirić [16], several authors have published nonunique fixed point results in various conditions in different abstract spaces, see e.g. [16, 37, 1, 21, 35, 36, 23, 24, 25].

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\*Dedicated to Professor Ljumbor Ćirić

On the other hand, recently, the notion of interpolative contraction was defined in [27] to revisit the well-known results of Kannan [22]. Following this pioneering result, several papers on the interpolative contraction have appeared in the literature, see e.g. [28, 19, 9, 4, 26, 8, 34].

One of the most interesting generalization of the metric space is the  $b$ -metric space, defined as follow:

**Definition 1.1** ([17, 14]). Let  $X$  be a nonempty set and let  $d : X \times X \rightarrow [0, \infty)$  satisfy the following conditions for all  $x, y, u \in X$ ,

$$(1.1) \quad \begin{aligned} (b1) \quad & d(x, y) = 0 \text{ if and only if } x = y \text{ (indistancy)} \\ (b2) \quad & d(x, y) = d(y, x) \text{ (symmetry)} \\ (b3) \quad & d(x, y) \leq s[d(x, u) + d(u, y)] \text{ (modified triangle inequality)}. \end{aligned}$$

Then, the map  $d$  is called a  $b$ -metric and the space  $(X, d)$  a  $b$ -metric space.

It is worthy to note that the notion of "b-metric" was announced also as "quasi-metric", see e.g. [13, 14]. It is also interesting to note that the notion of  $b$ -metric has a topology different from that of the standard metric. For example, closed ball is not a closed set. In the same way, the open ball does not form an open set. Besides, the  $b$ -metric needs not to be continuous. Considering the above-mentioned features of the  $b$ -metric, we can easily understand why so much research has been done on the  $b$ -metric, see e.g. [31, 20, 10, 11, 3, 30, 7, 2, 32, 33, 15, 6, 18, 5].

The following examples are not only standard, but also basic and interesting.

**Example 1.2** ([10, 11]). Let  $X = \mathbb{R}$ . Define

$$(1.2) \quad d(x, y) = |x - y|^p$$

for  $p > 1$ . Then  $d$  is a  $b$ -metric on  $\mathbb{R}$ . Clearly, the first two conditions hold. Since

$$|x - y|^p \leq 2^{p-1}[|x - z|^p + |z - y|^p],$$

the third condition holds with  $s = 2^{p-1}$ . Thus,  $(\mathbb{R}, d)$  is a  $b$ -metric space with a constant  $s = 2^{p-1}$ .

**Example 1.3** ([10, 11]). For  $p \in (0, 1)$ , take

$$X = l_p(\mathbb{R}) = \left\{ x = \{x_n\} \subset \mathbb{R} : \sum_{n=1}^{\infty} |x_n|^p < \infty \right\}.$$

Define

$$d(x, y) = \left( \sum_{n=1}^{\infty} |x_n - y_n|^p \right)^{1/p}.$$

Then  $(X, d)$  is a  $b$ -metric space with  $s = 2^{1/p}$ .

**Example 1.4** ([10, 11]). Let  $E$  be a Banach space and  $0_E$  be the zero vector of  $E$ . Let  $P$  be a cone in  $E$  with  $\text{int}(P) \neq \emptyset$  and  $\preceq$  be a partial ordering with respect to  $P$ . Let  $X$  be a non-empty set. Suppose the mapping  $d : X \times X \rightarrow E$  satisfies:

- (M1)  $0 \preceq d(x, y)$  for all  $x, y \in X$ ;
- (M2)  $d(x, y) = 0$  if and only if  $x = y$ ;
- (M3)  $d(x, y) \preceq d(x, z) + d(z, y)$ , for all  $x, y \in X$ ;
- (M4)  $d(x, y) = d(y, x)$  for all  $x, y \in X$ .

Then  $d$  is called a cone metric on  $X$ , and the pair  $(X, d)$  is called a cone metric space (CMS).

Recall that a cone  $P$  in a Banach space  $(E, \|\cdot\|)$  is called normal if it there exist a real number  $K \geq 1$  satisfies the following condition:

$$x \preceq y \Rightarrow \|x\| \leq K\|y\| \text{ for all } x, y \in P.$$

Let  $E$  be a Banach space and  $P$  be a normal cone in  $E$  with the coefficient of normality denoted by  $K$ . Let  $X$  be a non-empty set and  $D : X \times X \rightarrow [0, \infty)$  be defined by  $D(x, y) = \|d(x, y)\|$ , where  $d : X \times X \rightarrow E$  is a cone metric space. Then  $(X, D)$  is a  $b$ -metric space with a constant  $s := K \geq 1$ .

In generalization of the contraction condition, several auxiliary functions were considered in the literature. Among them, we count the notion of comparison function which was defined by Rus [38].

**Definition 1.5** ([12, 38]). A function  $\phi : [0, \infty) \rightarrow [0, \infty)$  is called a comparison function if it is increasing and  $\phi^n(t) \rightarrow 0$  as  $n \rightarrow \infty$  for every  $t \in [0, \infty)$ , where  $\phi^n$  is the  $n$ -th iterate of  $\phi$ .

We refer [12, 38] for the basic features and interesting example for comparison functions. Among all, we recollect the following lemma that indicates the importance of the comparison functions.

**Lemma 1.6** ([12, 38]). *If  $\phi : [0, \infty) \rightarrow [0, \infty)$  is a comparison function, then*

- (1) *each iterate  $\phi^k$  of  $\phi$ ,  $k \geq 1$  is also a comparison function;*
- (2)  *$\phi$  is continuous at 0;*
- (3)  *$\phi(t) < t$  for all  $t > 0$ .*

**Definition 1.7** ([14]). Let  $s \geq 1$  be a real number. A function  $\phi : [0, \infty) \rightarrow [0, \infty)$  is called a  $(b)$ -comparison function if

- (1)  $\phi$  is increasing;
- (2) there exist  $k_0 \in \mathbb{N}$ ,  $a \in [0, 1)$  and a convergent nonnegative series  $\sum_{k=1}^{\infty} v_k$

such that

$$s^{k+1} \phi^{k+1}(t) \leq a s^k \phi^k(t) + v_k,$$

for  $k \geq k_0$  and any  $t \geq 0$ .

The collection of all  $(b)$ -comparison functions will be denoted by  $\Psi$ . Berinde [14] also proved the following important property of  $(b)$ -comparison functions.

**Lemma 1.8** ([14]). *Let  $\phi : [0, \infty) \rightarrow [0, \infty)$  be a  $(b)$ -comparison function. Then*

- (1) the series  $\sum_{k=0}^{\infty} s^k \phi^k(t)$  converges for any  $t \in [0, \infty)$ ;
- (2) the function  $b_s : [0, \infty) \rightarrow [0, \infty)$  defined as  $b_s = \sum_{k=0}^{\infty} s^k \phi^k(t)$  is increasing and is continuous at  $t = 0$ .

*Remark 1.9.* Any (b)-comparison function  $\phi$  satisfies  $\phi(t) < t$  and  $\lim_{n \rightarrow \infty} \phi^n(t) = 0$  for each  $t > 0$ .

In this paper, we shall reconsider some of well-known nonunique fixed point theorem via interpolation in the context of b-metric spaces.

## 2. NON-UNIQUE FIXED POINTS ON b-METRIC SPACE

We start this section by considering the analog of the notions, "orbitally continuous" and "orbitally complete", in the framework of b-metric space.

**Definition 2.1** (see [16]). Let  $(X, d)$  be a b-metric space and  $T$  be a self-map on  $X$ .

- (1)  $T$  is called orbitally continuous if

$$(2.1) \quad \lim_{i \rightarrow \infty} T^{n_i} x = z$$

implies

$$(2.2) \quad \lim_{i \rightarrow \infty} TT^{n_i} x = Tz$$

for each  $x \in X$ .

- (2)  $(X, d)$  is called orbitally complete if every Cauchy sequence of type  $\{T^{n_i} x\}_{i \in \mathbb{N}}$  converges with respect to  $\tau_d$ .

A point  $z$  is said to be a periodic point of a function  $T$  of period  $m$  if  $T^m z = z$ , where  $T^0 x = x$  and  $T^m x$  is defined recursively by  $T^m x = TT^{m-1} x$ .

### 2.1. Ćirić type non-unique fixed point results.

**Theorem 2.2.** For a nonempty set  $X$ , we suppose that the function  $d : X \times X \rightarrow [0, \infty)$  is a b-metric. We presume that a self-mapping  $T$  is orbitally continuous and  $(X, d, s)$  forms a  $T$ -orbitally complete b-metric space with  $s \geq 1$ . If there is  $\psi \in \Psi$  and  $\alpha \in (0, 1)$  such that

$$(2.3) \quad \min\{ (d^\alpha(Tx, Ty)d^{1-\alpha}(x, Tx)), (d^\alpha(Tx, Ty)d^{1-\alpha}(y, Ty)) \} - \min\{ d^\alpha(x, Ty), d^{1-\alpha}(Tx, y) \} \leq \psi(d(x, y)),$$

for all  $x, y \in X$ , then, for each  $x_0 \in X$  the sequence  $\{T^n x_0\}_{n \in \mathbb{N}}$  converges to a fixed point of  $T$ .

*Proof.* Starting from an arbitrary  $x := x_0 \in X$ , we shall built a recursive sequence  $\{x_n\}$  in the following way:

$$(2.4) \quad x_0 := x \text{ and } x_n = Tx_{n-1} \text{ for all } n \in \mathbb{N}.$$

We presume that

$$(2.5) \quad x_n \neq x_{n-1} \text{ for all } n \in \mathbb{N}.$$

Indeed, if for some  $n \in \mathbb{N}$  we observe the inequality  $x_n = Tx_{n-1} = x_{n-1}$ , then, the proof is completed.

By replacing  $x = x_{n-1}$  and  $y = x_n$  in the inequality (2.3), we derive that

$$(2.6) \quad \begin{aligned} & \min\{(d^\alpha(Tx_{n-1}, Tx_n)d^{1-\alpha}(x_{n-1}, Tx_{n-1})), (d^\alpha(Tx_{n-1}, Tx_n)d^{1-\alpha}(x_n, Tx_n))\} \\ & - \min\{d^\alpha(x_{n-1}, Tx_n), d^{1-\alpha}(Tx_{n-1}, x_n)\} \\ & \leq \psi(d(x_{n-1}, x_n)). \end{aligned}$$

It yields that

$$(2.7) \quad \min\{d^{1-\alpha}(x_n, x_{n+1})d^\alpha(x_n, x_{n-1}), d(x_n, x_{n+1})\} \leq \psi(d(x_{n-1}, x_n)).$$

We shall prove that the sequence  $\{x_{n-1}, x_n\}$  is non-increasing. Suppose, on the contrary, that there is  $n_0$  such that  $d(x_{n_0}, x_{n_0+1}) > d(x_{n_0-1}, x_{n_0})$ . Since  $\psi(t) < t$  for all  $t > 0$ , for this case we get

$$d^{1-\alpha}(x_{n_0}, x_{n_0+1})d^\alpha(x_{n_0}, x_{n_0-1}) \leq \psi(d(x_{n_0-1}, x_{n_0})) < d(x_{n_0-1}, x_{n_0}),$$

which implies

$$d(x_{n_0}, x_{n_0-1}) \leq d^{1-\alpha}(x_{n_0}, x_{n_0+1})d^\alpha(x_{n_0}, x_{n_0-1}) \leq \psi(d(x_{n_0-1}, x_{n_0})) < d(x_{n_0-1}, x_{n_0}),$$

that is, a contradiction. Thus, we find that for all  $n \in \mathbb{N}$ ,

$$(2.8) \quad d(x_n, x_{n+1}) \leq \psi(d(x_{n-1}, x_n)) < d(x_{n-1}, x_n).$$

Recursively, we derive that

$$(2.9) \quad d(x_n, x_{n+1}) \leq \psi(d(x_{n-1}, x_n)) \leq \psi^2(d(x_{n-2}, x_{n-1})) \leq \dots \leq \psi^n(d(x_0, x_1)).$$

Taking (2.8) into account, we note that the sequence  $\{d(x_n, x_{n+1})\}$  is non-increasing.

In what follows, we shall prove that the sequence  $\{x_n\}$  is Cauchy. By using the triangle inequality (b3), we get

$$(2.10) \quad \begin{aligned} d(x_n, x_{n+k}) & \leq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+k})] \\ & \leq sd(x_n, x_{n+1}) + s\{s[d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+k})]\} \\ & = sd(x_n, x_{n+1}) + s^2d(x_{n+1}, x_{n+2}) + s^2d(x_{n+2}, x_{n+k}) \\ & \quad \vdots \\ & \leq sd(x_n, x_{n+1}) + s^2d(x_{n+1}, x_{n+2}) + \dots \\ & + s^{k-1}d(x_{n+k-2}, x_{n+k-1}) + s^{k-1}d(x_{n+k-1}, x_{n+k}) \\ & \leq sd(x_n, x_{n+1}) + s^2d(x_{n+1}, x_{n+2}) + \dots \\ & + s^{k-1}d(x_{n+k-2}, x_{n+k-1}) + s^k d(x_{n+k-1}, x_{n+k}), \end{aligned}$$

since  $s \geq 1$ . Combining (2.9) and (2.10), we derive that

$$\begin{aligned}
 (2.11) \quad d(x_n, x_{n+k}) &\leq s\psi^n(d(x_0, x_1)) + s^2\psi^{n+1}d(x_0, x_1) + \dots \\
 &+ s^{k-1}\psi^{n+k-2}(d(x_0, x_1)) + s^k\psi^{n+k-1}(d(x_0, x_1)) \\
 &= \frac{1}{s^{n-1}}[s^n\psi^n(d(x_0, x_1)) + s^{n+1}\psi^{n+1}d(x_0, x_1) + \dots \\
 &+ s^{n+k-2}\psi^{n+k-2}(d(x_0, x_1)) + s^{n+k-1}\psi^{n+k-1}(d(x_0, x_1))].
 \end{aligned}$$

Inevitably, we derive

$$(2.12) \quad d(x_n, x_{n+k}) \leq \frac{1}{s^{n-1}} [P_{n+k-1} - P_{n-1}], \quad n \geq 1, k \geq 1,$$

where  $P_n = \sum_{j=0}^n s^j \psi^j(d(x_0, x_1))$ ,  $n \geq 1$ . From Lemma 1.8, the series  $\sum_{j=0}^{\infty} s^j \psi^j(d(x_0, x_1))$  is convergent and since  $s \geq 1$ , upon taking limit  $n \rightarrow \infty$  in (2.39), we observe

$$(2.13) \quad \lim_{n \rightarrow \infty} d(x_n, x_{n+k}) \leq \lim_{n \rightarrow \infty} \frac{1}{s^{n-1}} [P_{n+k-1} - P_{n-1}] = 0.$$

We deduce that the sequence  $\{x_n\}$  is Cauchy in  $(X, d)$ .

Taking into account the  $T$ -orbitally completeness, we note that there is  $z \in X$  such that  $x_n \rightarrow z$ . Owing to the orbital continuity of  $T$ , we conclude that  $x_n \rightarrow Tz$ . Consequently, we find  $z = Tz$  which terminates the proof.  $\square$

**Example 2.3.** Let the set  $X = \{a, b, c, g, e\}$  and  $d : X \times X \rightarrow [0, \infty)$  be a  $b$ -metric (with  $s = 2$ ) defined as follows

$d(x, y)$	$a$	$b$	$c$	$g$	$e$
$a$	0	1	9	25	16
$b$	1	0	4	16	9
$c$	9	4	0	4	1
$g$	25	16	4	0	1
$e$	16	9	1	1	0

Let also the mapping  $T : X \rightarrow X$  be given as

$x$	$a$	$b$	$c$	$g$	$e$
$Tx$	$b$	$b$	$g$	$e$	$e$

Thus, we have

$d(Tx, Ty)$	$Ta$	$Tb$	$Tc$	$Tg$	$Te$	and	$d(x, Ty)$	$a$	$b$	$c$	$g$	$e$
$Ta = b$	0	0	16	9	9		$Ta = b$	1	0	4	16	9
$Tb = b$	0	0	16	9	9		$Tb = b$	1	0	4	16	9
$Tc = g$	16	16	0	1	1		$Tc = g$	25	16	4	0	1
$Tg = e$	9	9	1	0	0		$Tg = e$	16	9	1	1	0
$Te = e$	9	9	1	0	0		$Te = e$	16	9	1	1	0

We choose  $\alpha = \frac{1}{2}$  and  $\phi : [0, \infty) \rightarrow [0, \infty)$  as  $\phi(t) = \frac{t}{2}$ . We shall denote,

$$m_1(x, y) = \min\{(d^{1/2}(Tx, Ty)d^{1/2}(x, Tx)), (d^{1/2}(Tx, Ty)d^{1/2}(y, Ty))\}$$

$$m_2(x, y) = \min\{d^{1/2}(x, Ty), d^{1/2}(Tx, y)\}.$$

Then we have to consider the following cases:

(1) For  $x = a, y = b$  and  $x = g, y = e$ , we have  $d(Tx, Ty) = 0$  and obviously, (2.3) holds.

(2) For  $x = a, y = c$ , we have

$$m_1(a, c) = \min\{d^{1/2}(Ta, Tc)d^{1/2}(a, Ta), (d^{1/2}(Ta, Tc)d^{1/2}(c, Tc))\} = \min\{4 \cdot 1, 4 \cdot 2\} = 4$$

$$m_2(a, c) = \min\{5, 2\}.$$

$$\text{Thus, } m_1(a, c) - m_2(a, c) = 2 < \frac{9}{2} = \phi(d(a, c)).$$

(3) For  $x = a, y = g$ , we have

$$m_1(a, g) = \min\{d^{1/2}(Ta, Tg)d^{1/2}(a, Ta), (d^{1/2}(Ta, Tg)d^{1/2}(g, Tg))\} = \min\{3 \cdot 1, 3 \cdot 1\} = 1$$

$$m_2(a, g) = \min\{4, 4\},$$

and obviously (2.3) holds.

(4) For  $x = a, y = e$ , we have

$$m_1(a, e) = \min\{d^{1/2}(Ta, Te)d^{1/2}(a, Ta), (d^{1/2}(Ta, Te)d^{1/2}(e, Te))\} = \min\{3 \cdot 1, 3 \cdot 0\} = 0$$

$$m_2(a, e) = \min\{4, 3\} = 3,$$

and (2.3) holds.

(5) For  $x \in \{b, e\}$  and  $y \in X$ , since  $d(x, Tx) = 0$ , we have  $m_1(x, y) = 0$  and then (2.3) holds.

(6) For  $x = c$  and  $y = g$

$$m_1(c, g) = \min\{d^{1/2}(Tc, Tg)d^{1/2}(c, Tc), (d^{1/2}(Tc, Tg)d^{1/2}(g, Tg))\} = \min\{1 \cdot 2, 1 \cdot 1\} = 1$$

$$m_2(c, g) = \min\{1, 0\} = 0.$$

Therefore,

$$m_1(c, g) - m_2(c, g) = 1 < 2 = \phi(d(c, g)).$$

(7) For  $x = c$  and  $y = e$

$$m_1(c, e) = \min\{d^{1/2}(Tc, Te)d^{1/2}(c, Tc), (d^{1/2}(Tc, Te)d^{1/2}(e, Te))\} = \min\{1 \cdot 2, 1 \cdot 0\} = 0$$

$$m_2(c, e) = \min\{1, 1\} = 1.$$

Therefore,

$$m_1(c, e) - m_2(c, e) = -1 < \frac{1}{2} = \phi(d(c, e)).$$

Then the conditions of Theorem 2.2 hold and clearly,  $T$  has two fixed points,  $x = b$  and  $x = e$ .

In the next corollaries we give some consequences of the Theorem 2.2.

**Corollary 2.4.** *For a nonempty set  $X$ , we suppose that the function  $d : X \times X \rightarrow [0, \infty)$  is a  $b$ -metric. We presume that a self-mapping  $T$  on  $X$  is orbitally*

continuous and  $(X, d, s)$  forms a  $T$ -orbitally complete  $b$ -metric space with  $s \geq 1$ . If there is  $q \in [0, \frac{1}{s})$  and  $\alpha \in (0, 1)$  such that

$$(2.14) \quad \min\{(d^\alpha(Tx, Ty)d^{1-\alpha}(x, Tx)), (d^\alpha(Tx, Ty)d^{1-\alpha}(y, Ty))\} - \min\{d^\alpha(x, Ty), d^{1-\alpha}(Tx, y)\} \leq qd(x, y),$$

for all  $x, y \in X$ , then for each  $x_0 \in X$  the sequence  $\{T^n x_0\}_{n \in \mathbb{N}}$  converges to a fixed point of  $T$ .

*Proof.* It is sufficient to take  $\psi(t) = qt$ , where  $q \in [0, \frac{1}{s})$ , in Theorem 2.2.  $\square$

**Corollary 2.5.** Let  $T$  be an orbitally continuous self-map on the  $T$ -orbitally complete metric space  $(X, d)$ . If there is a comparison function  $\psi$  and  $\alpha \in (0, 1)$  such that

$$(2.15) \quad \min\{(d^\alpha(Tx, Ty)d^{1-\alpha}(x, Tx)), (d^\alpha(Tx, Ty)d^{1-\alpha}(y, Ty))\} - \min\{d^\alpha(x, Ty), d^{1-\alpha}(Tx, y)\} \leq \psi(d(x, y)),$$

for all  $x, y \in X$ , then for each  $x_0 \in X$  the sequence  $\{T^n x_0\}_{n \in \mathbb{N}}$  converges to a fixed point of  $T$ .

*Proof.* It is sufficient to take  $s = 1$  in Theorem 2.2.  $\square$

**Corollary 2.6.** Let  $T$  be an orbitally continuous self-map on the  $T$ -orbitally complete metric space  $(X, d)$ . If there is  $q \in [0, 1)$  and  $\alpha \in (0, 1)$  such that

$$(2.16) \quad \min\{(d^\alpha(Tx, Ty)d^{1-\alpha}(x, Tx)), (d^\alpha(Tx, Ty)d^{1-\alpha}(y, Ty))\} - \min\{d^\alpha(x, Ty), d^{1-\alpha}(Tx, y)\} \leq qd(x, y),$$

for all  $x, y \in X$ , then for each  $x_0 \in X$  the sequence  $\{T^n x_0\}_{n \in \mathbb{N}}$  converges to a fixed point of  $T$ .

*Proof.* It is sufficient to take  $\psi(t) = qt$ , where  $q \in [0, 1)$ , in Corollary 2.6.  $\square$

## 2.2. Pachpatte type non-unique fixed point results [37].

**Theorem 2.7.** For a nonempty set  $X$ , we suppose that the function  $d : X \times X \rightarrow [0, \infty)$  is a  $b$ -metric. We presume that a self-mapping  $T$  is orbitally continuous and  $(X, d, s)$  forms a  $T$ -orbitally complete  $b$ -metric space with  $s \geq 1$ . If there exists  $\psi \in \Psi$  and  $\alpha \in (0, 1)$  such that

$$(2.17) \quad m(x, y) - n(x, y) \leq \psi(d^\alpha(x, Tx)d^{1-\alpha}(y, Ty)),$$

for all  $x, y \in X$ , where

$$\begin{aligned} m(x, y) &= \min\{d(Tx, Ty), d^\alpha(x, y)d^{1-\alpha}(Tx, Ty), d(y, Ty)\}, \\ n(x, y) &= \min\{d^\alpha(x, Tx)d^{1-\alpha}(y, Ty), d^\alpha(x, Ty)d^{1-\alpha}(y, Tx)\}, \end{aligned}$$

then, for each  $x_0 \in X$  the sequence  $\{T^n x_0\}_{n \in \mathbb{N}}$  converges to a fixed point of  $T$ .

*Proof.* By verbatim, following the initial lines of the proof of the Theorem 2.2, we shall set-up a recursive sequence  $\{x_n = Tx_{n-1}\}_{n \in \mathbb{N}}$ , by starting from an arbitrary initial value  $x_0 := x \in X$ .

Replacing in the inequality (2.17)  $x = x_{n-1}$  and  $y = x_n$ , we obtain that

$$(2.18) \quad m(x_{n-1}, x_n) - n(x_{n-1}, x_n) \leq \psi(d^\alpha(x_{n-1}, Tx_{n-1})d^{1-\alpha}(x_n, Tx_n)),$$



where

$$\begin{aligned} m(x_{n-1}, x_n) &= \min\{d(Tx_{n-1}, Tx_n), d^\alpha(x_{n-1}, x_n)d^{1-\alpha}(Tx_{n-1}, Tx_n), d(x_n, Tx_n)\}, \\ n(x_{n-1}, x_n) &= \min\{d^\alpha(x_{n-1}, Tx_{n-1})d^{1-\alpha}(x_n, Tx_n), d^\alpha(x_{n-1}, Tx_n)d^{1-\alpha}(x_n, Tx_{n-1})\}. \end{aligned}$$

By simplifying the above inequality, we get

$$(2.19) \quad m(x_{n-1}, x_n) \leq \psi(d^\alpha(x_{n-1}, x_n)d^{1-\alpha}(x_n, x_{n+1})),$$

where

$$m(x_{n-1}, x_n) = \min\{d(x_n, x_{n+1}), d^\alpha(x_{n-1}, x_n)d^{1-\alpha}(x_n, x_{n+1})\}.$$

It is clear that the case

$$m(x_{n-1}, x_n) = d^\alpha(x_{n-1}, x_n)d^{1-\alpha}(x_n, x_{n+1})$$

is not possible for any  $n \in \mathbb{N}$ . If it would be the case, the inequality (2.19) turns into

$$(2.20) \quad \begin{aligned} d^\alpha(x_{n-1}, x_n)d^{1-\alpha}(x_n, x_{n+1}) &\leq \psi(d^\alpha(x_{n-1}, x_n)d^{1-\alpha}(x_n, x_{n+1})) \\ &< d^\alpha(x_{n-1}, x_n)d^{1-\alpha}(x_n, x_{n+1}), \end{aligned}$$

which is a contradiction since  $\psi(t) < t$  for all  $t > 0$ . Consequently, we derive

$$(2.21) \quad \begin{aligned} d(x_n, x_{n+1}) &\leq \psi(d^\alpha(x_{n-1}, x_n)d^{1-\alpha}(x_n, x_{n+1})) \\ &< d^\alpha(x_{n-1}, x_n)d^{1-\alpha}(x_n, x_{n+1}), \end{aligned}$$

which yields

$$(2.22) \quad d(x_n, x_{n+1}) < d(x_{n-1}, x_n).$$

On account of the fact that the comparison function  $\psi$  is nondecreasing, together with the inequalities (2.21) and (2.22), we find that

$$(2.23) \quad d(x_n, x_{n+1}) \leq \psi(d^\alpha(x_{n-1}, x_n)d^{1-\alpha}(x_n, x_{n+1})) < \psi(d(x_{n-1}, x_n)),$$

Recursively, we obtain that

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \psi(d(x_{n-1}, x_n)) \\ &\leq \psi^2(d(x_{n-2}, x_{n-1})) \leq \dots \leq \psi^n(d(x_0, x_1)). \end{aligned}$$

Hence, we conclude that

$$\lim_{n \rightarrow \infty} d(x_{n+1}, x_n) = 0.$$

The remaining part of the proof is verbatim repetition of the related lines in the proof of Theorem 2.2, so we omit it.  $\square$

Below, we deduce some consequences of the Theorem 2.7 for particular choice of the comparison function and the constant  $s$ . In case of  $\psi(t) = qt$  in Theorem 2.7 we deduce the following result.

**Corollary 2.8.** *For a nonempty set  $X$ , we suppose that the function  $d : X \times X \rightarrow [0, \infty)$  is a  $b$ -metric. We presume that a self-mapping  $T$  is orbitally continuous and  $(X, d, s)$  forms a  $T$ -orbitally complete  $b$ -metric space with  $s \geq 1$ . Assume that there exists  $q \in [0, \frac{1}{s})$  and  $\alpha \in (0, 1)$ , such that*

$$(2.24) \quad m(x, y) - n(x, y) \leq qd^\alpha(x, Tx)d^{1-\alpha}(y, Ty),$$

for all  $x, y \in X$ , where  $m(x, y)$  and  $n(x, y)$  are defined as in Theorem 2.7. Then, for each  $x_0 \in X$  the sequence  $\{T^n x_0\}_{n \in \mathbb{N}}$  converges to a fixed point of  $T$ .

If the statements of Theorem 2.7 are considered in the context of standard metric space instead of a  $b$ -metric space, we shall obtain the following consequence.

**Corollary 2.9.** *Let  $T$  be an orbitally continuous self-map on the  $T$ -orbitally complete metric space  $(X, d)$ . Suppose that there exist a comparison function  $\psi$  and  $\alpha \in (0, 1)$  such that*

$$(2.25) \quad m(x, y) - n(x, y) \leq \psi(d^\alpha(x, Tx)d^{1-\alpha}(y, Ty)),$$

for all  $x, y \in X$ , where  $m(x, y)$  and  $n(x, y)$  are defined as in Theorem 2.7. Then for each  $x_0 \in X$  the sequence  $\{T^n x_0\}_{n \in \mathbb{N}}$  converges to a fixed point of  $T$ .

For  $\psi(t) = qt$  in Corollary 2.9, the following results is derived.

**Corollary 2.10.** *Let  $T$  be an orbitally continuous self-map on the  $T$ -orbitally complete standard metric space  $(X, d)$ . Suppose that there exists  $q \in [0, 1)$  and  $\alpha \in (0, 1)$  such that*

$$(2.26) \quad m(x, y) - n(x, y) \leq qd^\alpha(x, Tx)d^{1-\alpha}(y, Ty),$$

for all  $x, y \in X$ , where  $m(x, y)$  and  $n(x, y)$  are defined as in Theorem 2.7. Then, for each  $x_0 \in X$  the sequence  $\{T^n x_0\}_{n \in \mathbb{N}}$  converges to a fixed point of  $T$ .

**2.3. K-type non-unique fixed point results [23].** The following theorem is inspired by the main theorem of [23]

**Theorem 2.11.** *For a nonempty set  $X$ , we suppose that the function  $d : X \times X \rightarrow [0, \infty)$  is a  $b$ -metric. We presume that a self-mapping  $T$  is orbitally continuous and  $(X, d, s)$  forms a  $T$ -orbitally complete  $b$ -metric space with  $s \geq 1$ . Assume that there exists real numbers  $\alpha, \beta, \gamma \in (0, 1)$  with  $\alpha + \beta + \gamma < 1$  and  $\psi \in \Psi$ . If the following inequality*

$$(2.27) \quad d^\alpha(Tx, Ty)d^\beta(x, Tx)d^\gamma(y, Ty) \left[ \frac{d(y, Tx) + d(x, Ty)}{2s} \right]^{1-\alpha-\beta-\gamma} \leq \psi(d(x, y))$$

holds for all  $x, y \in X$ , then,  $T$  has at least one fixed point.

*Proof.* Starting from an arbitrary point  $x = x_0 \in X$ , we shall construct a sequence  $\{x_n\}$  as follows:

$$(2.28) \quad x_{n+1} := Tx_n \quad n = 0, 1, 2, \dots$$

Letting  $x = x_n$  and  $y = x_{n+1}$ , in the inequality (2.27) yields

$$(2.29) \quad d^\alpha(Tx_n, Tx_{n+1})d^\beta(x_n, Tx_n)d^\gamma(x_{n+1}, Tx_{n+1}) \left[ \frac{d(x_{n+1}, Tx_n) + d(x_n, Tx_{n+1})}{2s} \right]^{1-\alpha-\beta-\gamma} \leq \psi(d(x_n, x_{n+1})).$$

On account of (2.28), the statement (2.29) turns into

$$(2.30) \quad d^\alpha(x_{n+1}, x_{n+2})d^\beta(x_n, x_{n+1})d^\gamma(x_{n+1}, x_{n+2})\left[\frac{d(x_{n+1}, x_{n+1})+d(x_n, x_{n+2})}{2s}\right]^{1-\alpha-\beta-\gamma} \leq \psi(d(x_n, x_{n+1})).$$

By elementary calculation and simplification, we derive

$$(2.31) \quad d^\alpha(x_{n+1}, x_{n+2})d^\beta(x_n, x_{n+1})d^\gamma(x_{n+1}, x_{n+2})\left[\frac{d(x_n, x_{n+1})+d(x_{n+1}, x_{n+2})}{2}\right]^{1-\alpha-\beta-\gamma} \leq \psi(d(x_n, x_{n+1})).$$

Suppose that  $d(x_n, x_{n+1}) < d(x_{n+1}, x_{n+2})$ . Then the inequality (2.31) turns into

$$(2.32) \quad d^\alpha(x_{n+1}, x_{n+2})d^\beta(x_n, x_{n+1})d^\gamma(x_{n+1}, x_{n+2})[d(x_n, x_{n+1})]^{1-\alpha-\beta-\gamma} \\ \leq d^\alpha(x_{n+1}, x_{n+2})d^\beta(x_n, x_{n+1})d^\gamma(x_{n+1}, x_{n+2})\left[\frac{d(x_n, x_{n+1})+d(x_{n+1}, x_{n+2})}{2s}\right]^{1-\alpha-\beta-\gamma} \\ \leq \psi(d(x_n, x_{n+1})).$$

Then we get

$$(2.33) \quad d^{1-\alpha-\gamma}(x_n, x_{n+1})d^{1-\beta}(x_{n+1}, x_{n+2}) \leq \psi(d(x_n, x_{n+1})) < d(x_n, x_{n+1}).$$

The above inequality can be expressed as

$$(2.34) \quad d^{1-\alpha-\gamma}(x_n, x_{n+1}) < d^{1-\alpha-\gamma}(x_n, x_{n+1}),$$

which is a contradiction. Hence, we conclude that

$$d(x_n, x_{n+1}) \geq d(x_{n+1}, x_{n+2}).$$

So, the inequality above together with (2.32) yields that

$$(2.35) \quad d(x_{n+1}, x_{n+2}) \leq \psi(d(x_n, x_{n+1})) < d(x_n, x_{n+1})$$

Thus, the sequence  $\{d(x_n, x_{n+1})\}$  is non-increasing.

Recursively, we find that

$$(2.36) \quad d(x_n, x_{n+1}) \leq \psi(d(x_{n-1}, x_n)) \leq \psi^2(d(x_{n-2}, x_{n-1})) \leq \cdots \leq \psi^n(d(x_0, x_1)).$$

As a next step, we shall show that the sequence  $\{x_n\}$  is Cauchy. By employing the triangle inequality (b3), we get

$$\begin{aligned}
 (2.37) \quad d(x_n, x_{n+k}) &\leq s[d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+k})] \\
 &\leq sd(x_n, x_{n+1}) + s\{s[d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+k})]\} \\
 &= sd(x_n, x_{n+1}) + s^2d(x_{n+1}, x_{n+2}) + s^2d(x_{n+2}, x_{n+k}) \\
 &\quad \vdots \\
 &\leq sd(x_n, x_{n+1}) + s^2d(x_{n+1}, x_{n+2}) + \dots \\
 &\quad + s^{k-1}d(x_{n+k-2}, x_{n+k-1}) + s^{k-1}d(x_{n+k-1}, x_{n+k}) \\
 &\leq sd(x_n, x_{n+1}) + s^2d(x_{n+1}, x_{n+2}) + \dots \\
 &\quad + s^{k-1}d(x_{n+k-2}, x_{n+k-1}) + s^k d(x_{n+k-1}, x_{n+k}),
 \end{aligned}$$

since  $s \geq 1$ . On account of (2.37) and (2.36), we deduce that

$$\begin{aligned}
 (2.38) \quad d(x_n, x_{n+k}) &\leq s\psi^n(d(x_0, x_1)) + s^2\psi^{n+1}d(x_0, x_1) + \dots \\
 &\quad + s^{k-1}\psi^{n+k-2}(d(x_0, x_1)) + s^k\psi^{n+k-1}(d(x_0, x_1)) \\
 &= \frac{1}{s^{n-1}}[s^n\psi^n(d(x_0, x_1)) + s^{n+1}\psi^{n+1}d(x_0, x_1) \\
 &\quad + \dots + s^{n+k-2}\psi^{n+k-2}(d(x_0, x_1)) + s^{n+k-1}\psi^{n+k-1}(d(x_0, x_1))].
 \end{aligned}$$

Consequently, we have

$$(2.39) \quad d(x_n, x_{n+k}) \leq \frac{1}{s^{n-1}} [P_{n+k-1} - P_{n-1}], \quad n \geq 1, k \geq 1,$$

where  $P_n = \sum_{j=0}^n s^j \psi^j(d(x_0, x_1))$ ,  $n \geq 1$ . From Lemma 1.8, the series  $\sum_{j=0}^{\infty} s^j \psi^j(d(x_0, x_1))$

is convergent and since  $s \geq 1$ , upon taking limit  $n \rightarrow \infty$  in (2.39), we obtain

$$(2.40) \quad \lim_{n \rightarrow \infty} d(x_n, x_{n+k}) \leq \lim_{n \rightarrow \infty} \frac{1}{s^{n-1}} [P_{n+k-1} - P_{n-1}] = 0.$$

We deduce that the sequence  $\{x_n\}$  is Cauchy in  $(X, d)$ .

The remaining part of the proof is verbatim repetition of the related lines in the proof of Theorem 2.2. □

Finally, we state the following consequence of Theorem 2.11.

**Corollary 2.12.** *Let  $T$  be an orbitally continuous self-map on the  $T$ -orbitally complete  $b$ -metric space  $(X, d, s)$  with  $s \geq 1$ . Suppose there exist real numbers*

$q \in [0, \frac{1}{s})$  and  $\alpha, \beta, \gamma \in (0, 1)$  with  $\alpha + \beta + \gamma < 1$ . If  
(2.41)

$$d^\alpha(Tx, Ty)d^\beta(x, Tx)d^\gamma(y, Ty) \left[ \frac{d(y, Tx) + d(x, Ty)}{2s} \right]^{1-\alpha-\beta-\gamma} \leq qd(x, y)$$

holds for all  $x, y \in X$ , then,  $T$  has at least one fixed point.

*Proof.* Take  $\psi(t) = qt$  in the proof of Theorem 2.11, where  $q \in [0, 1)$ .  $\square$

Notice also that the above theorem and corollary of this section are valid in the setting of standard metric space.

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