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# SOME FURTHER PROPERTIES OF DISCRETE MUCKENHOUPT AND GEHRING WEIGHTS

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*Abstract.* The main objective of this paper is a further study of discrete Muckenhoupt and Gehring weights. We first restate monotonicity properties of Muckenhoupt and Gehring classes in terms of the corresponding norms. In addition, we establish some norm bounds for Muckenhoupt and Gehring weights. Next, we give a simple characterization of the weight belonging to both Muckenhoupt and Gehring class. Finally, we show that the transition functions, arising from inclusion problems between Muckenhoupt and Gehring classes, are decreasing. As an application, some particular examples of Muckenhoupt and Gehring power weights are also considered.

## 1. Introduction

In 1972, Muckenhoupt [27], established a characterization of Muckenhoupt $A_p$ class of weights in connection with the boundedness of the Hardy-Littlewood maximal operator in the space $L^p_w(\mathbb{R}_+)$, $p > 1$, where $w$ is the corresponding weight. Another important class of weights, the Gehring class $G_q$, $q > 1$, has been introduced by Gehring [14] in connection with local integrability properties of the gradient of quasi-conformal mappings. Due to the importance of these two classes in mathematical and harmonic analysis, their structure has been studied by numerous authors, and various results regarding the relationship between them and their applications have been established (see [2, 3, 9, 10, 12, 13, 14, 16, 17, 19, 20, 22, 23, 25, 27, 29, 30, 36] and the references therein).

Nowadays, the study of discrete analogues in harmonic analysis is quite active field of research. For example, the study of regularity and boundedness of discrete operators on $l^p$, higher summability theorems, as well as the study of structure of discrete Muckenhoupt and Gehring classes are research topics of several authors (see [4, 5, 6, 7, 8, 24, 31, 32, 34, 37] and the references therein). Although some results from Euclidean harmonic analysis admit an obvious variant in the discrete setting, others do not. The main objective of this paper is a study of some new fundamental properties of discrete Muckenhoupt and Gehring weights.

Throughout this paper, $\mathbb{N}$ stands for a set of positive integers i.e. $\mathbb{N} = \{1, 2, \dots, n\}$, while $I$ is an interval in $\mathbb{N}$. By interval $I$, we mean finite subset of $\mathbb{N}$ consisting of

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consecutive integers, i.e. $I=\{a+1,a+2,\ldots,a+n\}$, $a\in\mathbb{N}\cup\{0\}$, $n\in\mathbb{N}$, and $|I|$ stands for its cardinality. A discrete weight on $\mathbb{N}$ is a sequence $u=\{u(n)\}_{n=1}^{\infty}$ of nonnegative real numbers.

The weight $u$ belongs to a discrete Muckenhoupt class $\mathscr{A}_p$, $p>1$, if there exist a constant $C>1$ such that the inequality

$$\left(\frac{1}{|I|}\sum_{k\in I}u(k)\right)\left(\frac{1}{|I|}\sum_{k\in I}u^{-\frac{1}{p-1}}(k)\right)^{p-1}\leqslant C \tag{1}$$

holds for every interval $I\subset\mathbb{N}$. In addition, $\mathscr{A}_p(C)$ is a subclass of $\mathscr{A}_p$ consisting of weights $u$ satisfying (1) for a fixed constant $C>1$. The $\mathscr{A}_p$-norm of weight $u$ is defined by

$$\mathscr{A}_p(u)=\sup_{I\subset\mathbb{N}}\left(\frac{1}{|I|}\sum_{k\in I}u(k)\right)\left(\frac{1}{|I|}\sum_{k\in I}u^{-\frac{1}{p-1}}(k)\right)^{p-1}<\infty.$$

Due to the Hölder inequality, it follows that $\mathscr{A}_p(u)\geqslant 1$. Moreover, if $u\in\mathscr{A}_p(C)$, then $\mathscr{A}_p(u)\leqslant C$. A discrete weight $u$ is said to belong to a discrete Muckenhoupt class $\mathscr{A}_1$, if there exists a constant $A>1$ such that the inequality

$$\frac{1}{|I|}\sum_{k\in I}u(k)\leqslant A\inf_{k\in I}u(k) \tag{2}$$

holds for every interval $I\subset\mathbb{N}$. Similarly, $\mathscr{A}_1(A)$ is a subclass of $\mathscr{A}_1$ satisfying (2) with a fixed constant $A>1$. The $\mathscr{A}_1$-norm of $u\in\mathscr{A}_1$ is defined by

$$\mathscr{A}_1(u)=\sup_{I\subset\mathbb{N}}\frac{1}{|I|}\left(\frac{\sum_{k\in I}u(k)}{\inf_{k\in I}u(k)}\right)<\infty.$$

A class $\mathscr{A}_\infty$ consists of all weights $u$ such that their $\mathscr{A}_\infty(u)$-norm is finite, i.e.

$$\mathscr{A}_\infty(u)=\sup_{I\subset\mathbb{N}}\left(\frac{1}{|I|}\sum_{k\in I}u(k)\right)\left(\exp\frac{1}{|I|}\sum_{k\in I}\log\frac{1}{u(k)}\right)<\infty.$$

It should be noticed here that $\lim_{p\to\infty}\mathscr{A}_p(u)=\mathscr{A}_\infty(u)$ and $\lim_{p\to1^+}\mathscr{A}_p(u)=\mathscr{A}_1(u)$, due to the well-known limit value $\lim_{x\to0}(1+x)^{1/x}=e$.

Sometimes it is convenient to consider a symmetric form equivalent to (2). Namely, a discrete weight $u$ belongs to the discrete Muckenhoupt class $\mathscr{A}_2(A)$, $A>1$, if the inequality

$$\sum_{k\in I}u(k)\sum_{k\in I}u^{-1}(k)\leqslant A\,|I|^2 \tag{3}$$

holds for every subinterval $I\subset\mathbb{N}$. This class has been used in harmonic analysis by several authors. For example, Ariño and Muckenhoupt [1], proved that if $u$ is nonincreasing and satisfies (3), then the space $d(u^{-q^*/q},q^*)$ is the dual space of the classical discrete Lorentz space

$$d(v,q)=\left\{x:\|x\|_{v,q}=\left(\sum_{n=1}^{\infty}|x^{*}(n)|^{q}u(n)\right)^{1/q}<\infty\right\},$$

where $x^{*}(n)$ is the nonincreasing rearrangement of $|x(n)|$ and $q^{*}$ is the conjugate of $q$. In [28], Pavlov gave a full description of all complete interpolating sequences on the real line by using the integral form of (3). In particular, he proved that the real sequence $\lambda_n$ is a complete interpolating sequence if and only if the function $w=|F(x+iy)|^2$, $x,y\in\mathbb{R}$, satisfies the Muckenhoupt condition

$$\int_I w(t)dt\int_I w^{-1}(t)dt\leqslant A|I|^2, \tag{4}$$

for some constant $A>0$, $y\neq 0$, and for all intervals $I\subset\mathbb{N}$, where

$$F(z)=\lim_{R\to\infty}\prod_{|\lambda_n|<R}\left(1-\frac{z}{\lambda_n}\right).$$

Further, Lyubarskii and Seip [21], showed that condition (4) can be replaced by the corresponding discrete version (3) and proved that the real sequence $\lambda_n$ is a complete interpolating sequence if and only if there is a relatively dense subsequence $\lambda_{n_k}$ such that the numbers $d(k)=\left|F^{'}(\lambda_{n_k})\right|^2$ satisfy the discrete Muckenhoupt condition

$$\sum_{k\in I}d(k)\sum_{k\in I}d^{-1}(k)\leqslant A|I|^2, \tag{5}$$

for some constant $A>0$ and for all intervals $I\subset\mathbb{N}$. It seems that checking the Muckenhoupt condition (4) for a function $F$ given by an infinite product is quite hard. At the first sight, condition (5) is easier to verify since it involves only countably many sets $I$ instead of all finite intervals. In addition, in the case of the Lorentz sequence spaces $d(v,q)$ one can have a better feeling of the behavior of multiplication, composition operators and inducing sequences, than in the case of the classical Lorentz spaces $L^{p,q}$ as well, as in Banach spaces (see, e. g. [18]).

These observations lead us to study the structure of the discrete Muckenhoupt classes in [32], where we have proved that if $v$ is a nonincreasing sequence satisfying (2), then for $p\in[1,\ C/(C-1))$ the inequality

$$\frac{1}{|I|}\sum_{k\in I}v^{p}(k)\leqslant A\left(\frac{1}{|I|}\sum_{k\in I}v(k)\right)^{p} \tag{6}$$

holds for every interval $I\subset\mathbb{N}$. Relation (6) shows that any Muckenhoupt weight $\mathscr{A}_1$ belongs to certain Gehring classes of weights satisfying the reverse Hölder inequality. More precisely, the weight $u$ belongs to a discrete Gehring class $\mathscr{G}_q$, $q>1$, if there exist a constant $K>1$ such that the relation

$$\left(\frac{1}{|I|}\sum_{k\in I}u^{q}(k)\right)^{\frac{1}{q}}\leqslant K\frac{1}{|I|}\sum_{k\in I}u(k) \tag{7}$$

is satisfied for any interval $I \subset \mathbb{N}$. Moreover, $\mathscr{G}_q(K)$ is a subclass of $\mathscr{G}_q$ consisting of weights $u$ satisfying (7) with a fixed constant $K > 1$. The $\mathscr{G}_q$-norm of $u \in \mathscr{G}_q$ is defined by

$$\mathscr{G}_q(u) = \sup_{I \subset \mathbb{N}} \left[ \frac{|I|}{\sum_{k\in I} u(k)} \left( \frac{1}{|I|} \sum_{k\in I} u^q(k) \right)^{\frac{1}{q}} \right]^{\frac{q}{q-1}} < \infty.$$

Due to Hölder's inequality it follows that $\mathscr{G}_q(u) \geqslant 1$, while for $u \in \mathscr{G}_q(K)$ we have $\mathscr{G}_q(u) \leqslant K^{q/q-1}$. Finally, we give specific definitions of classes $\mathscr{G}_1$ and $\mathscr{G}_\infty$. A discrete class $\mathscr{G}_1$ consists of all weights $u$ with finite $\mathscr{G}_1(u)$-norm, i.e.

$$\mathscr{G}_1(u) = \sup_{I \subset \mathbb{N}} \exp \left( \frac{1}{|I|} \sum_{k\in I} \frac{u(k)}{u_I} \log \frac{u(k)}{u_I} \right) < \infty,$$

where $u_I = (1/|I|) \sum_{k\in I} u(k)$. Similarly, Gehring class $\mathscr{G}_\infty$ consists of all weights with finite $\mathscr{G}_\infty(u)$-norm, i.e.

$$\sup_{I \subset \mathbb{N}} \frac{\sup_{k\in I} u(k)}{\frac{1}{|I|} \sum_{k\in I} u(k)} < \infty.$$

Recently, Saker et al. [34] (see also [32]), established several interesting properties of discrete classes $\mathscr{A}_p$ and $\mathscr{G}_q$, as well as some relationships between them. Those results are discrete extensions of the previously known integral results established in the above mentioned references. In particular, they proved that Muckenhoupt classes satisfy the following properties:

(A1) $\mathscr{A}_1 \subset \mathscr{A}_p \subset \mathscr{A}_q \subset \mathscr{A}_\infty$, $1 < p < q$,

(A2) $\mathscr{A}_\infty = \bigcup_{p>1} \mathscr{A}_p$.

Similarly, they also proved the corresponding properties for the Gehring classes:

(G1) $\mathscr{G}_\infty \subset \mathscr{G}_q \subset \mathscr{G}_p \subset \mathscr{G}_1$, $1 < p < q$,

(G2) $\mathscr{G}_1 = \bigcup_{q>1} \mathscr{G}_q$.

Saker et al. [35] proved the discrete result due to Korenovskii [19] and established the so called self-improving property of Muckenhoupt weights. More precisely, they proved that if $u \in \mathscr{A}_p(A) \subset \mathscr{A}_p$, $p > 1$, $A > 1$, then $u \in \mathscr{A}_q$, for $q \in [p_0, p)$, where $p_0$ is the unique positive root of the equation

$$(Ap_0)^{\frac{1}{q-1}} \left( \frac{q - p_0}{q - 1} \right) = 1.$$

Similarly, Saker et al. [33] proved the discrete version of the results due to D'Apuzzo and Sbordone [11] and established the self-improving properties of the weighted Muckenhoupt and Gehring classes. In particular, they proved that if $v \in \mathscr{G}_q(K) \subset \mathscr{G}_q$, $q > 1$,

$K > 1$, is a nonincreasing sequence, then $v \in \mathscr{G}_p$ for $p \in [q, q^*)$, where $q^*$ is a unique positive solution of the equation

$$\left(\frac{x-1}{x}\right)\left(\frac{x}{x-q}\right)^{\frac{1}{q}} = K.$$

To prove the main results in [35] and [33] the authors used the assumptions and terminologies used in the classical setting and proved some new discrete inequalities. This has been done by applying some algebraic inequalities to overcome the nonexistence of the power rules as well as the chain rules which are the main tools used in the proof of the classical results. For the reader's convenience, properties (A1) and (G1) will be referred to as monotonicity properties of Muckenhoupt and Gehring classes. These classes of weights are closely connected. Namely, it has been proved in [34] that $\mathscr{A}_\infty = \mathscr{G}_1$, so every Muckenhoupt class belongs to $\mathscr{G}_1$, while every Gehring class belongs to $\mathscr{A}_\infty$. Therefore, one of the most interesting problems in connection with these classes is inclusion of Muckenhoupt classes into Gehring classes and vice versa. Although there are lots of results for this topic in integral case (see, e.g. [22, 23]), such problems are still unsolved in the discrete case, since techniques that have been used in integral case do not have a discrete counterpart.

Therefore, we need a new approach to establish the corresponding discrete results. In particular, it will be interesting to find indices $s^* = s^*(p, C)$, $r^* = r^*(q, K)$ and the corresponding constants $K_s = K_s(p, C)$, $C_r = C_r(q, K)$, such that the following inclusions hold:

$$\mathscr{A}_p(C) \subset \mathscr{G}_s(K_s), \text{ for all } s < s^*,$$

$$\mathscr{G}_q(K) \subset \mathscr{A}_r(C_r), \text{ for all } r > r^*.$$

The main objective of the present paper is a further study of discrete Muckenhoupt and Gehring classes. We first restate monotonicity properties (A1) and (A2) in terms of the corresponding norms. We will show that these norms show similar behavior as the corresponding classes. Some particular examples of Muckenhoupt and Gehring power weights are also considered to illustrate the difficulties arising from discrete setting, since there are no power rules as in the integral case. Next, we establish some norm estimates for Muckenhoupt and Gehring weights. We also give a simple characterization of the weight belonging to both classes $\mathscr{A}_p$ and $\mathscr{G}_q$. Finally, motivated by the above inclusion problems, we define the so called transition functions between Muckenhoupt and Gehring classes and show that they are decreasing.

## 2. Main results

The main tool in establishing our results will be the Hölder and the Jensen inequalities. Recall that the Hölder inequality asserts that

$$\sum_{k=1}^{n} u(k)v(k) \leqslant \left(\sum_{k=1}^{n} u^p(k)\right)^{\frac{1}{p}} \left(\sum_{k=1}^{n} v^q(k)\right)^{\frac{1}{q}}, \tag{8}$$

where $1/p+1/q=1$, $p>1$, and $\{u(k)\}_{k=1}^{\infty}$, $\{v(k)\}_{k=1}^{\infty}$ are nonnegative sequences. If $0<p<1$, then the sign of inequality (8) is reversed. The Jensen inequality asserts that if $f:[a,b]\to\mathbb{R}$ is a convex function and $(p_1,p_2,\dots,p_n)$ is a nonnegative $n$-tuple such that $\sum_{k=1}^{n}p_k=1$, then the inequality

$$f\left(\sum_{k=1}^{n}p_kx_k\right)\leqslant\sum_{k=1}^{n}p_kf(x_k) \tag{9}$$

holds for any $n$-tuple $(x_1,x_2,\dots,x_n)\in[a,b]^n$. If $f$ is concave function, then the sign of (9) is reversed. An important consequence of the Jensen inequality is the generalized mean inequality. Recall that a weighted power mean $M_r(x_1,x_2,\dots,x_n)$ is defined by

$$M_r(x_1,x_2,\dots,x_n)=\left(\sum_{k=1}^{n}p_kx_k^r\right)^{\frac{1}{r}},$$

where $\sum_{k=1}^{n}p_k=1$, $p_k>0$, and $(x_1,x_2,\dots,x_n)$ is a positive $n$-tuple. The nonweighted means correspond to the setting $p_k=1/n$, $k=1,2,\dots,n$. Recall that for $p=1,0,-1$ we obtain respectively, the arithmetic, geometric and harmonic mean. In addition, $M_{-\infty}(x_1,x_2,\dots,x_n)=\min\{x_1,x_2,\dots,x_n\}$ and $M_{\infty}(x_1,x_2,\dots,x_n)=\max\{x_1,x_2,\dots,x_n\}$. The generalized mean inequality asserts that if $r<s$, then

$$M_r(x_1,x_2,\dots,x_n)\leqslant M_s(x_1,x_2,\dots,x_n). \tag{10}$$

Inequality (10) is true for real values of $r$ and $s$, as well as for positive and negative infinity values. For more details about the Jensen inequality and means inequalities, the reader is referred to [15, 26].

The results that follow can be considered as discrete versions of integral results established in [10, 13, 30] with modifications in the proofs, in order to overcame the lack of the appropriate tools on the discrete space like power rules, chain rules, etc. Our first result refers to monotonicity properties (A1) and (G1) of Muckenhoupt and Gehring classes. In fact, our intention is to restate these properties in terms of the corresponding Muckenhoupt and Gehring norms. The following proposition asserts that these norms show similar behavior as the corresponding classes.

PROPOSITION 1. *Let $u$ be a nonnegative weight and let $1\leqslant r<p$ be real numbers.*

*(i) If $u\in\mathscr{A}_r$, then $\mathscr{A}_{\infty}(u)\leqslant\mathscr{A}_p(u)\leqslant\mathscr{A}_r(u)$.*

*(ii) If $u\in\mathscr{G}_p$, then $\mathscr{G}_r^{1-\frac{1}{r}}(u)\leqslant\mathscr{G}_p^{1-\frac{1}{p}}(u)$. In addition, if $u\in\mathscr{G}_{\infty}$, then $\mathscr{G}_p^{1-\frac{1}{p}}(u)\leqslant\mathscr{G}_{\infty}(u)$.*

*Proof.* (i) Since $u\in\mathscr{A}_r$, we know that $u\in\mathscr{A}_p\subset\mathscr{A}_{\infty}$. We consider two cases depending on whether $r>1$ or $r=1$. If $r>1$, then, utilizing the generalized mean inequality (10), it follows that

$$\left(\frac{1}{|I|}\sum_{k\in I}u^{-\frac{1}{p-1}}(k)\right)^{1-p} \geqslant \left(\frac{1}{|I|}\sum_{k\in I}u^{-\frac{1}{r-1}}(k)\right)^{1-r},$$

since $-1/(p-1) > -1/(r-1)$, and consequently,

$$\left(\frac{1}{|I|}\sum_{k\in I}u^{-\frac{1}{p-1}}(k)\right)^{p-1} \leqslant \left(\frac{1}{|I|}\sum_{k\in I}u^{-\frac{1}{r-1}}(k)\right)^{r-1}.$$

Therefore we obtain

$$\begin{aligned}\mathscr{A}_p(u) &= \sup_{I\subset\mathbb{N}}\left(\frac{1}{|I|}\sum_{k\in I}u(k)\right)\left(\frac{1}{|I|}\sum_{k\in I}u^{-\frac{1}{p-1}}(k)\right)^{p-1}\\ &\leqslant \sup_{I\subset\mathbb{N}}\left(\frac{1}{|I|}\sum_{k\in I}u(k)\right)\left(\frac{1}{|I|}\sum_{k\in I}u^{-\frac{1}{r-1}}(k)\right)^{r-1} = \mathscr{A}_r(u),\end{aligned}$$

as claimed. The previous inequality also holds for $r=1$. Namely, yet another use of the generalized mean inequality implies

$$\left(\frac{1}{|I|}\sum_{k\in I}u^{-\frac{1}{p-1}}(k)\right)^{1-p} \geqslant \inf_{k\in I}u(k),$$

where from we obtain

$$\begin{aligned}\mathscr{A}_p(u) &= \sup_{I\subset\mathbb{N}}\left(\frac{1}{|I|}\sum_{k\in I}u(k)\right)\left(\frac{1}{|I|}\sum_{k\in I}u^{-\frac{1}{p-1}}(k)\right)^{p-1}\\ &\leqslant \sup_{I\subset\mathbb{N}}\left(\frac{1}{|I|}\frac{\sum_{k\in I}u(k)}{\inf_{k\in I}u(k)}\right) = \mathscr{A}_1(u).\end{aligned}$$

It remains to prove that $\mathscr{A}_\infty(u) \leqslant \mathscr{A}_p(u)$. More precisely, applying (9) with $n=|I|$, $p_k = 1/|I|$, $k=1,2,\ldots,|I|$, $f(u) = \exp\left[(1/(p-1))u\right]$ and $x_k = \log(1/u(k))$, it follows that

$$\begin{aligned}&\left(\exp\left[\frac{1}{|I|}\sum_{k\in I}\log\frac{1}{u(k)}\right]\right)^{\frac{1}{p-1}}\\ &= \exp\left[\frac{1}{p-1}\left(\frac{1}{|I|}\sum_{k\in I}\log\frac{1}{u(k)}\right)\right]\\ &\leqslant \frac{1}{|I|}\sum_{k\in I}\exp\left[\frac{1}{p-1}\log\frac{1}{u(k)}\right] = \frac{1}{|I|}\sum_{k\in I}u^{-\frac{1}{p-1}}(k),\end{aligned}$$

and consequently,

$$\begin{aligned}\mathscr{A}_\infty(u) &= \sup_{I\subset\mathbb{N}}\left(\frac{1}{|I|}\sum_{k\in I}u(k)\right)\exp\left[\frac{1}{|I|}\sum_{k\in I}\log\frac{1}{u(k)}\right]\\ &\leqslant \sup_{I\subset\mathbb{N}}\left(\frac{1}{|I|}\sum_{k\in I}u(k)\right)\left(\frac{1}{|I|}\sum_{k\in I}u^{-\frac{1}{p-1}}(k)\right)^{p-1}=\mathscr{A}_p(u),\end{aligned}$$

which proves our assertion.

(ii) Let $r>1$ and let $u\in\mathscr{G}_p$. Then, $u\in\mathscr{G}_r$, so utilizing the generalized mean inequality we obtain

$$\begin{aligned}\mathscr{G}_r^{1-\frac{1}{r}}(u) &= \sup_{I\subset\mathbb{N}}\frac{|I|}{\sum_{k\in I}u(k)}\left(\frac{1}{|I|}\sum_{k\in I}u^r(k)\right)^{\frac{1}{r}}\\ &\leqslant \sup_{I\subset\mathbb{N}}\frac{|I|}{\sum_{k\in I}u(k)}\left(\frac{1}{|I|}\sum_{k\in I}u^p(k)\right)^{\frac{1}{p}}=\mathscr{G}_p^{1-\frac{1}{p}}(u).\end{aligned}$$

Obviously, the last inequality also holds for $r=1$. It remains to prove that $\mathscr{G}_p^{1-\frac{1}{p}}(u)\leqslant\mathscr{G}_\infty(u)$, provided that $u\in\mathscr{G}_\infty$. Clearly, this also holds due to the weighted mean inequality since

$$\begin{aligned}\mathscr{G}_p^{1-\frac{1}{p}}(u) &= \sup_{I\subset\mathbb{N}}\frac{|I|}{\sum_{k\in I}u(k)}\left(\frac{1}{|I|}\sum_{k\in I}u^p(k)\right)^{\frac{1}{p}}\\ &\leqslant \sup_{I\subset\mathbb{N}}\frac{|I|\sup_{k\in I}u(k)}{\sum_{k\in I}u(k)}=\mathscr{G}_\infty(u). \quad\square\end{aligned}$$

Now, our intention is to consider some particular examples of Muckenhoupt and Gehring weights. In order to simplify our further discussion, from now on we study classes $\mathscr{A}_p$ and $\mathscr{G}_q$ for real indices $p,\ q>1$.

EXAMPLE 1. A bounded weight $u$ such that $0<m\leqslant u(k)\leqslant M$, $k\in\mathbb{N}$, belongs to every Muckenhoupt class $\mathscr{A}_p$. Namely, utilizing the generalized mean inequality (10), it follows that

$$\left(\frac{1}{|I|}\sum_{k\in I}u(k)\right)\left(\frac{1}{|I|}\sum_{k\in I}u^{-\frac{1}{p-1}}(k)\right)^{p-1}\leqslant\frac{\sup_{k\in I}u(k)}{\inf_{k\in I}u(k)}\leqslant\frac{M}{m},$$

which implies that $\mathscr{A}_p(u)<\infty$. However, Muckenhoupt weight does not have to be a bounded function. The weight $u$ defined by $u(n)=(n+a)^\alpha$ belongs to a class $\mathscr{A}_p$ if $-1<\alpha<p-1$. Clearly, the case $\alpha=0$ is trivial. We will show that $u\in\mathscr{A}_p$ for

$0<\alpha<p-1$, the proof for $-1<\alpha<0$ is similar and it is left to the reader. Hence, let $I=\{a+1,a+2,\dots,a+n\}$, $a\in\mathbb{N}\cup\{0\}$, $n\in\mathbb{N}$, and let

$$L_I(u)=\frac{\sum_{k=1}^n (a+k)^\alpha}{n}\left(\frac{\sum_{k=1}^n (a+k)^{-\frac{\alpha}{p-1}}}{n}\right)^{p-1}.$$

Our aim is to find an upper bound for $L_I(u)$, not dependent on interval $I$. By the generalized mean inequality we have

$$\frac{\sum_{k=1}^n (a+k)^\alpha}{n}\leqslant (n+a)^\alpha.$$

On the other hand, considering $\sum_{k=1}^n (a+k)^{-\frac{\alpha}{p-1}}$ as a lower Darboux sum of decreasing function $f(x)=(x+a)^{-\frac{\alpha}{p-1}}$ on interval $[0,n]$, we obtain the following estimate

$$\sum_{k=1}^n (a+k)^{-\frac{\alpha}{p-1}}\leqslant\int_0^n (x+a)^{-\frac{\alpha}{p-1}}dx\leqslant\frac{p-1}{p-1-\alpha}\left[(n+a)^{\frac{p-1-\alpha}{p-1}}-a^{\frac{p-1-\alpha}{p-1}}\right].$$

Therefore, the above two estimates provide the inequality

$$\begin{aligned} L_I^{\frac{1}{p-1}}(u) &\leqslant \frac{p-1}{p-1-\alpha}\cdot\frac{n+a-a\left(\frac{n+a}{a}\right)^{\frac{\alpha}{p-1}}}{n}\\ &\leqslant \frac{p-1}{p-1-\alpha}\cdot\frac{n+a-a}{n}=\frac{p-1}{p-1-\alpha}, \end{aligned}$$

and consequently, $A_p(u)=\sup_{I\subset\mathbb{N}}L_I(u)\leqslant\left(\frac{p-1}{p-1-\alpha}\right)^{p-1}<\infty$, i.e. $u\in\mathscr{A}_p$.

EXAMPLE 2. Similarly to Example 1, it is obvious that a bounded weight $v$ such that $0<m\leqslant v(k)\leqslant M$, $k\in\mathbb{N}$, belongs to every Gehring class $\mathscr{G}_q$. Further, the weight $v(n)=(n+a)^\beta$ belongs to a class $\mathscr{G}_q$ if $\beta>-1/q$. We will prove that $v\in\mathscr{G}_q$ for $-1/q<\beta<0$, the rest is proved similarly and it is left to the reader. Therefore, let

$$L_I(v)=\frac{n}{\sum_{k=1}^n (a+k)^\beta}\left(\frac{\sum_{k=1}^n (a+k)^{\beta q}}{n}\right)^{\frac{1}{q}},$$

where $I=\{a+1,a+2,\dots,a+n\}$, $a\in\mathbb{N}\cup\{0\}$, $n\in\mathbb{N}$. Now, since $-1/q<\beta<0$, the generalized mean inequality yields

$$\frac{n}{\sum_{k=1}^n (a+k)^\beta}\leqslant (n+a)^{-\beta}.$$

In addition, considering $\sum_{k=1}^n (a+k)^{\beta q}$ as a lower Darboux sum of decreasing function $f(x)=(x+a)^{\beta q}$ on interval $[0,n]$, we obtain

$$\sum_{k=1}^n (a+k)^{\beta q}\leqslant\int_0^n (a+x)^{\beta q}dx=\frac{1}{\beta q+1}\left[(n+a)^{\beta q+1}-a^{\beta q+1}\right].$$

Thus, combining the previous two estimates, we obtain

$$
\begin{aligned}
L_I^q(v) &\leqslant \frac{1}{\beta q+1} \cdot \frac{n+a-a\left(\frac{a}{n+a}\right)^{\beta q}}{n} \\
&\leqslant \frac{1}{\beta q+1} \cdot \frac{n+a-a}{n} = \frac{1}{\beta q+1},
\end{aligned}
$$

and consequently, $\mathscr{G}_q(v) = \sup_{I\subset\mathbb{N}} L_I^{\frac{q}{q-1}}(v) \leqslant (\beta q+1)^{\frac{1}{1-q}} < \infty$, i.e. $v \in \mathscr{G}_q$.

It should be noticed here that Muckenhoupt and Gehring power weights discussed in the previous two examples are in accordance to monotonicity properties and Proposition 1.

Our next result provides a characterization of a nonnegative weight belonging to both Muckenhoupt and Gehring classes $\mathscr{A}_p$ and $\mathscr{G}_s$. We show that such weight can be characterized via the corresponding Muckenhoupt class.

THEOREM 1. *Let $u$ be a nonnegative weight and let $p,q,s$ be real numbers such that $p,\ s>1$ and $q=s(p-1)+1$. Then, $u\in\mathscr{A}_p\cap\mathscr{G}_s$ if and only if $u^s\in\mathscr{A}_q$.*

*Proof.* We first prove that if $u\in\mathscr{A}_p\cap\mathscr{G}_s$, then $u^s\in\mathscr{A}_q$. Namely, since $u\in\mathscr{A}_p\cap\mathscr{G}_s$, then there exist positive constants $A,B>1$ such that

$$
\left(\frac{1}{|I|}\sum_{k\in I} u(k)\right)\left(\frac{1}{|I|}\sum_{k\in I} u^{-\frac{1}{p-1}}(k)\right)^{p-1} \leqslant A
$$

and

$$
\left(\frac{1}{|I|}\sum_{k\in I} u^s(k)\right)^{\frac{1}{s}} \leqslant B\left(\frac{1}{|I|}\sum_{k\in I} u(k)\right).
$$

Now, since $q-1=s(p-1)$, the first inequality yields

$$
\left(\frac{1}{|I|}\sum_{k\in I} (u^s)^{-\frac{1}{q-1}}(k)\right)^{q-1} \leqslant A^s \frac{1}{\left(\frac{1}{|I|}\sum_{k\in I} u(k)\right)^s},
$$

while from the second one we obtain

$$
\frac{1}{|I|}\sum_{k\in I} u^s(k) \leqslant B^s\left(\frac{1}{|I|}\sum_{k\in I} u(k)\right)^s.
$$

Clearly, by multiplying the last two inequalities, we have

$$
\begin{aligned}
&\left(\frac{1}{|I|}\sum_{k\in I} u^s(k)\right)\left(\frac{1}{|I|}\sum_{k\in I} (u^s)^{-\frac{1}{q-1}}(k)\right)^{q-1} \\
&\leqslant A^s B^s \left(\frac{1}{|I|}\sum_{k\in I} u(k)\right)^s \frac{1}{\left(\frac{1}{|I|}\sum_{k\in I} u(k)\right)^s} = (AB)^s,
\end{aligned}
$$

which implies that $u^s \in \mathscr{A}_q$. It remains to prove the opposite direction. Since $u^s \in \mathscr{A}_q$, there exist a constant $D > 0$ such that

$$\left(\frac{1}{|I|}\sum_{k\in I} u^s(k)\right)\left(\frac{1}{|I|}\sum_{k\in I} (u^s)^{-\frac{1}{q-1}}(k)\right)^{q-1} \leqslant D,$$

and consequently,

$$\left(\frac{1}{|I|}\sum_{k\in I} u^s(k)\right)^{\frac{1}{s}}\left(\frac{1}{|I|}\sum_{k\in I} u^{-\frac{1}{p-1}}(k)\right)^{p-1} \leqslant D^{\frac{1}{s}}, \tag{11}$$

since $q-1 = s(p-1)$. In addition, due to the generalized mean inequality we have

$$\left(\frac{1}{|I|}\sum_{k\in I} u^s(k)\right)^{\frac{1}{s}} \geqslant \frac{1}{|I|}\sum_{k\in I} u(k),$$

which together with (11) yields

$$\left(\frac{1}{|I|}\sum_{k\in I} u(k)\right)\left(\frac{1}{|I|}\sum_{k\in I} u^{-\frac{1}{p-1}}(k)\right)^{p-1} \leqslant D^{\frac{1}{s}}.$$

Clearly, the last relation implies that $u \in \mathscr{A}_p$. On the other hand, yet another application of the generalized mean inequality yields relation

$$\left(\frac{1}{|I|}\sum_{k\in I} u(k)\right)^{-1} \leqslant \left(\frac{1}{|I|}\sum_{k\in I} u^{-\frac{1}{p-1}}(k)\right)^{p-1},$$

which together with (11) implies

$$\left(\frac{1}{|I|}\sum_{k\in I} u(k)\right)^{-1}\left(\frac{1}{|I|}\sum_{k\in I} u^s(k)\right)^{\frac{1}{s}} \leqslant D^{\frac{1}{s}},$$

respectively,

$$\left(\frac{1}{|I|}\sum_{k\in I} u^s(k)\right)^{\frac{1}{s}} \leqslant D^{\frac{1}{s}}\left(\frac{1}{|I|}\sum_{k\in I} u(k)\right).$$

Finally, the last inequality implies that $u \in \mathscr{G}_s$. Hence, we have $u \in \mathscr{A}_p \cap \mathscr{G}_s$ and the proof is complete. □

The above theorem will be utilized in establishing some norm estimates for Muckenhoupt and Gehring weights. In particular, we obtain mutual bounds for the norm of Gehring weight $u \in \mathscr{G}_p$, expressed in terms of suitable Muckenhoupt norms.

THEOREM 2. *Let $u$ be nonnegative weight and let $p,q>1$. If $u \in \mathscr{A}_{\frac{q+p-1}{p}} \cap \mathscr{G}_p$, then $u^p \in \mathscr{A}_q$ and*

$$\mathscr{A}_q(u^p)\left[\mathscr{A}_{\frac{q+p-1}{p}}(u)\right]^{-p} \leqslant [\mathscr{G}_p(u)]^{p-1} \leqslant \mathscr{A}_q(u^p). \tag{12}$$

*If $u \in \mathscr{A}_q$ and $u^{-\frac{1}{p-1}} \in \mathscr{A}_{1+\frac{q-1}{p-1}}$, then $u \in \mathscr{A}_p$ and*

$$\mathscr{A}_p(u) \leqslant \mathscr{A}_q(u)\left[\mathscr{A}_{1+\frac{q-1}{p-1}}(u^{-\frac{1}{p-1}})\right]^{p-1}. \tag{13}$$

*Proof.* We prove (12) first. Since $u \in \mathscr{A}_{\frac{q+p-1}{p}} \cap \mathscr{G}_p$, we conclude by Theorem 1 that $u^p \in \mathscr{A}_q$. Furthermore, taking into account definition of $\mathscr{G}_p(u)$, it follows that

$$\frac{1}{|I|}\sum_{k\in I} u^p(k) \leqslant [\mathscr{G}_p(u)]^{p-1}\left(\frac{1}{|I|}\sum_{k\in I} u(k)\right)^p,$$

and so,

$$\begin{aligned}
&\left(\frac{1}{|I|}\sum_{k\in I} u^p(k)\right)\left(\frac{1}{|I|}\sum_{k\in I} u^{-\frac{p}{q-1}}(k)\right)^{q-1} \\
&\leqslant [\mathscr{G}_p(u)]^{p-1}\left[\left(\frac{1}{|I|}\sum_{k\in I} u(k)\right)\left(\frac{1}{|I|}\sum_{k\in I} u^{-\frac{p}{q-1}}(k)\right)^{\frac{q-1}{p}}\right]^p \\
&= [\mathscr{G}_p(u)]^{p-1}\left[\left(\frac{1}{|I|}\sum_{k\in I} u(k)\right)\left(\frac{1}{|I|}\sum_{k\in I} u^{-\frac{1}{\frac{q+p-1}{p}-1}}(k)\right)^{\frac{q+p-1}{p}-1}\right]^p \\
&\leqslant [\mathscr{G}_p(u)]^{p-1}\left[\mathscr{A}_{\frac{q+p-1}{p}}(u)\right]^p.
\end{aligned}$$

This means that $\mathscr{A}_q(u^p) \leqslant \left[\mathscr{G}_p(u)\right]^{p-1}\left[\mathscr{A}_{\frac{q+p-1}{p}}(u)\right]^p$, i.e. we obtain the first inequality in (12). On the other hand, applying the generalized mean inequality, it follows that

$$\left(\frac{1}{|I|}\sum_{k\in I} u^{-\frac{p}{q-1}}(k)\right)^{\frac{1-q}{p}} \leqslant \frac{1}{|I|}\sum_{k\in I}\left[u^{-\frac{p}{q-1}}(k)\right]^{\frac{1-q}{p}} = \frac{1}{|I|}\sum_{k\in I} u(k),$$

and consequently,

$$\left(\frac{1}{|I|}\sum_{k\in I} u(k)\right)\left(\frac{1}{|I|}\sum_{k\in I} u^{-\frac{p}{q-1}}(k)\right)^{\frac{q-1}{p}} \geqslant 1.$$

Therefore we have

$$\begin{aligned}
&\frac{1}{\frac{1}{|I|}\sum_{k\in I}u(k)}\left(\frac{1}{|I|}\sum_{k\in I}u^p(k)\right)^{\frac{1}{p}}\\
&=\frac{\left[\left(\frac{1}{|I|}\sum_{k\in I}u^p(k)\right)\left(\frac{1}{|I|}\sum_{k\in I}u^{-\frac{p}{q-1}}(k)\right)^{q-1}\right]^{\frac{1}{p}}}{\left(\frac{1}{|I|}\sum_{k\in I}u(k)\right)\left(\frac{1}{|I|}\sum_{k\in I}u^{-\frac{p}{q-1}}(k)\right)^{\frac{q-1}{p}}}\\
&\leqslant\left[\left(\frac{1}{|I|}\sum_{k\in I}u^p(k)\right)\left(\frac{1}{|I|}\sum_{k\in I}u^{-\frac{p}{q-1}}(k)\right)^{q-1}\right]^{\frac{1}{p}}\leqslant\left[\mathscr{A}_q(u^p)\right]^{\frac{1}{p}},
\end{aligned}$$

which implies that $\left[\mathscr{G}_p(u)\right]^{p-1}\leqslant\mathscr{A}_q(u^p)$, as claimed.

It remains to prove (13). Let $q^*=q/(q-1)$ and $p^*=p/(p-1)$. Then holds the identity

$$\begin{aligned}
&\left(\frac{1}{|I|}\sum_{k\in I}u(k)\right)\left(\frac{1}{|I|}\sum_{k\in I}u^{1-p^*}(k)\right)^{p-1}\\
&=\left(\frac{1}{|I|}\sum_{k\in I}u(k)\right)\left(\frac{1}{|I|}\sum_{k\in I}u^{1-q^*}(k)\right)^{q-1}\\
&\quad\times\left[\left(\frac{1}{|I|}\sum_{k\in I}u^{1-p^*}(k)\right)\left(\frac{1}{|I|}\sum_{k\in I}u^{1-q^*}(k)\right)^{\frac{1-q}{p-1}}\right]^{p-1}. \qquad (14)
\end{aligned}$$

Further, employing the Cauchy-Schwarz inequality, we have

$$|I|^2=\left(\sum_{k\in I}\left(u^{1-q^*}(k)\right)^{1/2}\left(u^{1-q^*}(k)\right)^{-1/2}\right)^2\leqslant\left(\sum_{k\in I}u^{1-q^*}(k)\right)\left(\sum_{k\in I}u^{q^*-1}(k)\right),$$

and consequently,

$$\left(\frac{1}{|I|}\sum_{k\in I}u^{1-q^*}(k)\right)^{\frac{1-q}{p-1}}\leqslant\left(\frac{1}{|I|}\sum_{k\in I}u^{q^*-1}(k)\right)^{\frac{q-1}{p-1}},$$

since $p,q>1$. Now, the previous inequality implies the following relation:

$$\begin{aligned}
&\left[\left(\frac{1}{|I|}\sum_{k\in I}u^{1-p^*}(k)\right)\left(\frac{1}{|I|}\sum_{k\in I}u^{1-q^*}(k)\right)^{\frac{1-q}{p-1}}\right]^{p-1}\\
&\leqslant\left[\left(\frac{1}{|I|}\sum_{k\in I}u^{1-p^*}(k)\right)\left(\frac{1}{|I|}\sum_{k\in I}u^{q^*-1}(k)\right)^{\frac{q-1}{p-1}}\right]^{p-1}
\end{aligned}$$

$$= \left[ \left( \frac{1}{|I|} \sum_{k \in I} u^{1-p^*}(k) \right) \left( \frac{1}{|I|} \sum_{k \in I} \left( u^{1-p^*} \right)^{\frac{q^*-1}{1-p^*}} (k) \right)^{\frac{q-1}{p-1}} \right]^{p-1}$$

$$= \left[ \left( \frac{1}{|I|} \sum_{k \in I} u^{1-p^*}(k) \right) \left( \frac{1}{|I|} \sum_{k \in I} \left( u^{1-p^*} \right)^{\frac{1-p}{q-1}} (k) \right)^{\frac{q-1}{p-1}} \right]^{p-1}.$$

Finally, since $1-p^* = -1/(p-1)$, taking into account the last inequality and (14), we obtain

$$\begin{aligned}
&\left( \frac{1}{|I|} \sum_{k \in I} u(k) \right) \left( \frac{1}{|I|} \sum_{k \in I} u^{1-p^*}(k) \right)^{p-1} \\
&\leqslant \left( \frac{1}{|I|} \sum_{k \in I} u(k) \right) \left( \frac{1}{|I|} \sum_{k \in I} u^{1-q^*}(k) \right)^{q-1} \\
&\quad \times \left[ \left( \frac{1}{|I|} \sum_{k \in I} u^{1-p^*}(k) \right) \left( \frac{1}{|I|} \sum_{k \in I} \left( u^{1-p^*} \right)^{\frac{1-p}{q-1}} (k) \right)^{\frac{q-1}{p-1}} \right]^{p-1} \\
&= \left( \frac{1}{|I|} \sum_{k \in I} u(k) \right) \left( \frac{1}{|I|} \sum_{k \in I} u^{1-q^*}(k) \right)^{q-1} \\
&\quad \times \left[ \left( \frac{1}{|I|} \sum_{k \in I} u^{1-p^*}(k) \right) \left( \frac{1}{|I|} \sum_{k \in I} \left[ u^{1-p^*}(k) \right]^{-\frac{1}{\left(1+\frac{q-1}{p-1}\right)-1}} \right)^{\left(1+\frac{q-1}{p-1}\right)-1} \right]^{p-1} \\
&\leqslant \mathscr{A}_q(u) \left[ \mathscr{A}_{\frac{q-1}{p-1}+1}\left(u^{1-p^*}\right) \right]^{p-1},
\end{aligned}$$

which means that $u \in \mathscr{A}_p$ and $\mathscr{A}_p(u) \leqslant \mathscr{A}_q(u) \left[ \mathscr{A}_{1+\frac{q-1}{p-1}}\left(u^{-\frac{1}{p-1}}\right) \right]^{p-1}$. The proof is now complete. □

Now, we consider a nonnegative weight of the form $u = u_1^{\theta} u_2^{1-\theta}$, $0 \leqslant \theta \leqslant 1$, where $u_1$ and $u_2$ are arbitrary Muckenhoupt weights. We will show that $u$ also belongs to appropriate Muckenhoupt class and we will establish an upper bound for its norm.

THEOREM 3. *Let $p_1, p_2 > 1$ be real numbers and let $u_1 \in \mathscr{A}_{p_1}$, $u_2 \in \mathscr{A}_{p_2}$ be nonnegative weights. If $0 \leqslant \theta \leqslant 1$, then $u_1^{\theta} u_2^{1-\theta} \in \mathscr{A}_{p_\theta}$ and*

$$\mathscr{A}_{p_\theta}\left(u_1^{\theta} u_2^{1-\theta}\right) \leqslant \left(\mathscr{A}_{p_1}(u_1)\right)^{\theta} \left(\mathscr{A}_{p_2}(u_2)\right)^{1-\theta}, \tag{15}$$

*where $p_\theta = \theta p_1 + (1-\theta) p_2$.*

*Proof.* Let

$$
\begin{aligned}
L &= \left(\frac{1}{|I|}\sum_{k\in I} u_1^{\theta}(k)u_2^{1-\theta}(k)\right)\left(\frac{1}{|I|}\sum_{k\in I}\left[u_1^{\theta}(k)u_2^{1-\theta}(k)\right]^{-\frac{1}{p_\theta-1}}\right)^{p_\theta-1} \\
&= \left(\frac{1}{|I|}\sum_{k\in I} u_1^{\theta}(k)u_2^{1-\theta}(k)\right)\left(\frac{1}{|I|}\sum_{k\in I} u_1^{-\frac{\theta}{p_\theta-1}}(k)u_2^{-\frac{1-\theta}{p_\theta-1}}(k)\right)^{p_\theta-1}.
\end{aligned}
$$

Applying the Hölder inequality to the first factor of the previous expression, we have

$$
\sum_{k\in I} u_1^{\theta}(k)u_2^{1-\theta}(k) \leqslant \left(\sum_{k\in I} u_1(k)\right)^{\theta}\left(\sum_{k\in I} u_2(k)\right)^{1-\theta}.
$$

Similarly, since $p_\theta = \theta p_1 + (1-\theta)p_2 \geqslant \min\{p_1,p_2\} > 1$, applying the Hölder inequality with parameters $(p_\theta-1)/[\theta(p_1-1)] > 1$ and $(p_\theta-1)/[(1-\theta)(p_2-1)] > 1$, to the second factor of $L$, we have

$$
\begin{aligned}
&\sum_{k\in I} u_1^{-\frac{\theta}{p_\theta-1}}(k)u_2^{-\frac{(1-\theta)}{p_\theta-1}}(k) \\
&\leqslant \left(\sum_{k\in I} u_1^{-\frac{1}{p_1-1}}(k)\right)^{\frac{\theta(p_1-1)}{p_\theta-1}}\left(\sum_{k\in I} u_2^{-\frac{1}{p_2-1}}(k)\right)^{\frac{(1-\theta)(p_2-1)}{p_\theta-1}}.
\end{aligned}
$$

Therefore, utilizing the previous two estimates, as well as the fact that $u_1 \in \mathscr{A}_{p_1}$, $u_2 \in \mathscr{A}_{p_2}$, we obtain the following upper bound for $L$:

$$
\begin{aligned}
L &\leqslant \left(\frac{1}{|I|}\sum_{k\in I} u_1(k)\right)^{\theta}\left(\frac{1}{|I|}\sum_{k\in I} u_2(k)\right)^{1-\theta} \\
&\quad\times\left(\left(\frac{1}{|I|}\sum_{k\in I} u_1^{-\frac{1}{p_1-1}}(k)\right)^{\frac{\theta(p_1-1)}{p_\theta-1}}\left(\frac{1}{|I|}\sum_{k\in I} u_2^{-\frac{1}{p_2-1}}(k)\right)^{\frac{(1-\theta)(p_2-1)}{p_\theta-1}}\right)^{p_\theta-1} \\
&= \left(\frac{1}{|I|}\sum_{k\in I} u_1(k)\right)^{\theta}\left(\frac{1}{|I|}\sum_{k\in I} u_2(k)\right)^{1-\theta} \\
&\quad\times\left(\frac{1}{|I|}\sum_{k\in I} u_1^{-\frac{1}{p_1-1}}(k)\right)^{\theta(p_1-1)}\left(\frac{1}{|I|}\sum_{k\in I} u_2^{-\frac{1}{p_2-1}}(k)\right)^{(1-\theta)(p_2-1)}
\end{aligned}
$$

$$
\begin{aligned}
&= \left[\left(\frac{1}{|I|}\sum_{k\in I} u_1(k)\right)\left(\frac{1}{|I|}\sum_{k\in I} u_1^{-\frac{1}{p_1-1}}(k)\right)^{p_1-1}\right]^{\theta} \\
&\quad\times\left[\left(\frac{1}{|I|}\sum_{k\in I} u_2(k)\right)\left(\frac{1}{|I|}\sum_{k\in I} u_2^{-\frac{1}{p_2-1}}(k)\right)^{p_2-1}\right]^{1-\theta} \\
&\leqslant \left[\mathscr{A}_{p_1}(u_1)\right]^{\theta}\left[\mathscr{A}_{p_2}(u_2)\right]^{1-\theta}.
\end{aligned}
$$

Consequently, $u_1^{\theta}u_2^{1-\theta} \in \mathscr{A}_{p_\theta}$ and (15) holds, as claimed. □

REMARK 1. In particular, if $u_1, u_2 \in \mathscr{A}_p$, then Theorem 3 implies that the weight $u = u_1^{\theta}u_2^{1-\theta}$, $0 \leqslant \theta \leqslant 1$, also belongs to a class $\mathscr{A}_p$. Moreover, since $\mathscr{A}_p(1) = 1$, it follows by Theorem 3 that if $u \in \mathscr{A}_p$, then $u^{\theta} \in \mathscr{A}_p$, for $0 \leqslant \theta \leqslant 1$. In addition, due to (15), we obtain the following estimate: $\mathscr{A}_p\left(u^{\theta}\right) \leqslant \left(\mathscr{A}_p(u)\right)^{\theta}$, $0 \leqslant \theta \leqslant 1$.

We have already discussed that every Muckenhoupt class belongs to $\mathscr{G}_1$ since $\mathscr{A}_\infty = \mathscr{G}_1$. In the same way, every Gehring class belongs to $\mathscr{A}_\infty$. Therefore, taking into account monotonicity properties (A1) and (G1), it turns out that the following functions are well defined for $p \geqslant 1$:

$$
\begin{aligned}
\sigma(p) &= \sup\left\{\sigma \geqslant 1 : \mathscr{A}_p \subset \mathscr{G}_\sigma\right\}, \\
\theta(p) &= \inf\left\{\theta \geqslant 1 : \mathscr{G}_p \subset \mathscr{A}_\theta\right\}.
\end{aligned} \tag{16}
$$

The above functions $\sigma$ and $\theta$ are referred to as transition functions between Muckenhoupt and Gehring classes, and vice versa. Our last result shows that $\sigma(p)$ and $\theta(p)$ are decreasing functions with respect to argument $p \geqslant 1$.

THEOREM 4. *Let $\sigma(p)$ and $\theta(p)$ be defined by (16). If $p_1 < p_2$, then $\sigma(p_1) \geqslant \sigma(p_2)$ and $\theta(p_1) \geqslant \theta(p_2)$.*

*Proof.* Let us denote $S_p = \left\{\sigma \geqslant 1 : \mathscr{A}_p \subset \mathscr{G}_\sigma\right\}$. Now, let $p_1 < p_2$ and let $\sigma_0 \in S_{p_2}$. This means that $\mathscr{A}_{p_2} \subset \mathscr{G}_{\sigma_0}$. Further, due to monotonicity property of Muckenhoupt classes, we have $\mathscr{A}_{p_1} \subset \mathscr{A}_{p_2}$, which implies that $\mathscr{A}_{p_1} \subset \mathscr{G}_{\sigma_0}$, i.e $\sigma_0 \in S_{p_1}$. Therefore, $S_{p_2} \subset S_{p_1}$ and consequently,

$$
\sigma(p_2) = \sup S_{p_2} \leqslant \sup S_{p_1} = \sigma(p_1),
$$

as claimed.

Similarly, let $T_p = \left\{\theta \geqslant 1 : \mathscr{G}_p \subset \mathscr{A}_\theta\right\}$ and let $\theta_0 \in T_{p_1}$. Then, $\mathscr{G}_{p_1} \subset \mathscr{A}_{\theta_0}$. Moreover, since $p_2 > p_1$, it follows that $\mathscr{G}_{p_2} \subset \mathscr{G}_{p_1} \subset \mathscr{A}_{\theta_0}$, i.e. $\theta_0 \in T_{p_2}$. Finally, $T_{p_1} \subset T_{p_2}$ and

$$
\sigma(p_1) = \inf T_{p_1} \geqslant \inf T_{p_2} = \sigma(p_2),
$$

which completes the proof. □

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