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Some Generalizations of Novel $(\Delta\nabla)^{\Delta}$–Gronwall–Pachpatte Dynamic Inequalities on Time Scales with Applications

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Abstract: We established several novel inequalities of Gronwall–Pachpatte type on time scales. Our results can be used as handy tools to study the qualitative and quantitative properties of the solutions of the initial boundary value problem for a partial delay dynamic equation. The Leibniz integral rule on time scales has been used in the technique of our proof. Symmetry plays an essential role in determining the correct methods to solve dynamic inequalities.

Keywords: Gronwall's inequality; dynamic inequality; time scales; Leibniz integral rule on time scales

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1. Introduction

Stefan Hilger initiated the theory of time scales in his PhD thesis [1] in order to unify discrete and continuous analysis. Since then, this theory has received a lot of attention. The basic notion is to establish a result for a dynamic equation or a dynamic inequality where the domain of the unknown function is so-called time scale $\mathbb{T}$, which is an arbitrary closed subset of the reals $\mathbb{R}$. The three most common examples of calculus on time scales are continuous calculus, discrete calculus, and quantum calculus, i.e., when $\mathbb{T} = \mathbb{R}, \mathbb{T} = \mathbb{N}$ and $\mathbb{T} = \overline{q^{\mathbb{Z}}} = \{q^z : z \in \mathbb{Z}\} \cup \{0\}$ where $q > 1$. The books due to Bohner and Peterson [2,3] on the subject of time scales brief and organize much of time scales calculus.

Gronwall–Bellman-type inequalities, which have many applications in qualitative and quantitative behavior, have been developed by many mathematicians and several refinements and extensions have been made to the previous results. We refer the reader to the works [4–14].

Anderson [15] presented the following result on time scales.

$$\omega(u(t,s)) \le a(t,s) + c(t,s) \int_{t_0}^{t} \int_{s}^{\infty} \omega'(u(\tau,\eta))[d(\tau,\eta)w(u(\tau,\eta)) + b(\tau,\eta)]\nabla\eta\Delta\tau, \quad (1)$$

where u, a, c, and d are non-negative continuous functions defined for $(t,s) \in \mathbb{T} \times \mathbb{T}$ and b is a non-negative continuous function for $(t,s) \in [t_0,\infty)_{\mathbb{T}} \times [t_0,\infty)_{\mathbb{T}}$, and $\omega \in C^1(\mathbb{R}_+,\mathbb{R}_+)$ with $\omega' > 0$ for $u > 0$.

In [16], the authors discussed the following results:

$$\begin{aligned}\omega(u(\ell,t)) \quad &\le \quad a(\ell,t) + \int_0^{\theta(\ell)} \int_0^{\vartheta(t)} \Im_1(\varsigma,\eta)[f(\varsigma,\eta)\varpi(u(\varsigma,\eta)) \\ &+ \int_0^{\varsigma} \Im_2(\chi,\eta)\varpi(u(\chi,\eta))d\chi\Big]d\eta d\varsigma,\end{aligned}$$

$$\omega(u(\ell,t)) \leq a(\ell,t) + \int_0^{\theta(\ell)} \int_0^{\vartheta(t)} \Im_1(\varsigma,\eta)[f(\varsigma,\eta)\varpi(u(\varsigma,\eta))\eta(u(\varsigma,\eta)) + \int_0^{\varsigma} \Im_2(\chi,\eta)\varpi(u(\chi,\eta))d\chi\Big] d\eta d\varsigma,$$

and

$$\omega(u(\ell,t)) \leq a(\ell,t) + \int_0^{\theta(\ell)} \int_0^{\vartheta(t)} \Im_1(\varsigma,\eta)[f(\varsigma,\eta)\zeta(u(\varsigma,\eta))\varpi(u(\varsigma,\eta)) + \int_0^{\varsigma} \Im_2(\chi,\eta)\zeta(u(\chi,\eta))\varpi(u(\chi,\eta))d\chi\Big] d\eta d\varsigma,$$

where u, f, $\Im \in C(I_1 \times I_2, \mathbb{R}_+)$ and $a \in C(\zeta, \mathbb{R}_+)$ are nondecreasing functions, $I_1, I_2 \in \mathbb{R}$, $\theta \in C^1(I_1, I_1)$ and $\vartheta \in C^1(I_2, I_2)$ are nondecreasing with $\theta(\ell) \leq \ell$ on I_1, $\vartheta(t) \leq t$ on I_2, $\Im_1$, $\Im_2 \in C(\zeta, \mathbb{R}_+)$, and ω, ζ, $\varpi \in C(\mathbb{R}_+, \mathbb{R}_+)$ with $\{\omega, \zeta, \varpi\}(u) > 0$ for $u > 0$, and $\lim_{u \to +\infty} \omega(u) = +\infty$.

In this paper, by applying Leibniz integral rule on time scales, see Theorem 1 (iii) below, we established the delayed time scale version of the inequalities proved in [16]. Further, the results that are proved in this paper extended some known results in [17–19]. The paper is arranged as follows: In Section 2, we briefly presented the basic definitions and concepts related to the calculus of time scales. In Section 3, we proved the auxiliary results. In Section 4, we stated and proved the main results. In Section 5, we presented an application to discuss the boundedness of the solutions of an initial boundary value problem on time scales. In Section 6, we stated the conclusion. Symmetry plays an essential role in determining the correct methods to solve dynamic inequalities.

2. Preliminaries

We begin with the definition of time scale.

Definition 1. *A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the set of all real numbers $\mathbb{R}$.*

Now, we define two operators playing a central role in the analysis on time scales.

Definition 2. *If $\mathbb{T}$ is a time scale, then we define the forward jump operator $\sigma : \mathbb{T} \to \mathbb{T}$ and the backward jump operator $\rho : \mathbb{T} \to \mathbb{T}$ by*

$$\sigma(\xi) = \inf\{s \in \mathbb{T} : s > \xi\},$$

and

$$\rho(\xi) = \sup\{s \in \mathbb{T} : s < \xi\}.$$

In the above definitions, we put $\inf \emptyset = \sup \mathbb{T}$ (i.e., if ξ is the maximum of $\mathbb{T}$, then $\sigma(\xi) = \xi$) and $\sup \emptyset = \inf \mathbb{T}$ (i.e., if ξ is the minimum of $\mathbb{T}$, then $\rho(\xi) = \xi$), where $\emptyset$ is the empty set.

If $\mathbb{T} \in \{[a,b], [a,\infty), (-\infty, a], \mathbb{R}\}$, then $\sigma(\xi) = \rho(\xi) = \xi$. We note that $\sigma(\xi)$ and $\rho(\xi)$ in $\mathbb{T}$ when $\xi \in \mathbb{T}$ because $\mathbb{T}$ is a closed nonempty subset of $\mathbb{R}$.

Next, we define the graininess functions as follows:

Definition 3.

(i) *The forward graininess function $\mu : \mathbb{T} \to [0, \infty)$ is defined by*

$$\mu(\xi) = \sigma(\xi) - \xi.$$

(*ii*) *The backward graininess function* $\nu : \mathbb{T} \to [0, \infty)$ *is defined by*

$$\nu(\xi) = \xi - \rho(\xi).$$

With the operators defined above, we can begin to classify the points of any time scale depending on the proximity of their neighboring points in the following manner.

Definition 4. *Let* $\mathbb{T}$ *be a time scale. A point* $\xi \in \mathbb{T}$ *is said to be:*

(1) *Right-scattered if* $\sigma(\xi) > \xi$*;*
(2) *Left-scattered if* $\rho(\xi) < \xi$*;*
(3) *Isolated if* $\rho(\xi) < \xi < \sigma(\xi)$*;*
(4) *Right-dense if* $\sigma(\xi) = \xi$*;*
(5) *Left-dense if* $\rho(\xi) = \xi$*;*
(6) *Dense if* $\rho(\xi) = \xi = \sigma(\xi)$*.*

The closed interval on time scales is defined by

$$[a, b]_{\mathbb{T}} = [a, b] \cap \mathbb{T} = \{\xi \in \mathbb{T} : a \le \xi \le b\}.$$

Open intervals and half-open intervals are defined similarly.

Two sets we need to consider are $\mathbb{T}^{\kappa}$ and $\mathbb{T}_{\kappa}$ which are defined as follows: $\mathbb{T}^{\kappa} = \mathbb{T} \setminus \{M\}$ if $\mathbb{T}$ has M as a left-scattered maximum and $\mathbb{T}^{\kappa} = \mathbb{T}$ otherwise. Similarly, $\mathbb{T}_{\kappa} = \mathbb{T} \setminus \{m\}$ if $\mathbb{T}$ has m as a right-scattered minimum and $\mathbb{T}_{\kappa} = \mathbb{T}$ otherwise. In fact, we can write

$$\mathbb{T}^{\kappa} = \begin{cases} \mathbb{T} \setminus \big(\rho(\sup \mathbb{T}), \sup \mathbb{T}\big], & \text{if} \quad \sup \mathbb{T} < \infty, \\ \mathbb{T}, & \text{if} \quad \sup \mathbb{T} = \infty, \end{cases}$$

and

$$\mathbb{T}_{\kappa} = \begin{cases} \mathbb{T} \setminus \big[\inf \mathbb{T}, \sigma(\inf \mathbb{T})\big), & \text{if} \quad \inf \mathbb{T} > -\infty, \\ \mathbb{T}, & \text{if} \quad \inf \mathbb{T} = -\infty. \end{cases}$$

Definition 5. *Let* $f : \mathbb{T} \to \mathbb{R}$ *be a function defined on a time scale* $\mathbb{T}$*. Then we define the function* $f^{\sigma} : \mathbb{T} \to \mathbb{R}$ *by*

$$f^{\sigma}(\xi) = (f \circ \sigma)(\xi) = f(\sigma(\xi)), \qquad \xi \in \mathbb{T},$$

and the function $f^{\rho} : \mathbb{T} \to \mathbb{R}$ *by*

$$f^{\rho}(\xi) = (f \circ \rho)(\xi) = f(\rho(\xi)), \qquad \xi \in \mathbb{T}.$$

Assume $f : \mathbb{T} \to \mathbb{R}$ is a function and $\xi \in \mathbb{T}^{\kappa}$. Then $f^{\Delta}(\xi) \in \mathbb{R}$ is said to be the delta derivative of f at ξ if for any $\varepsilon > 0$ there exists a neighborhood U of ξ such that, for every $s \in U$, we have

$$\big|[f(\sigma(\xi)) - f(s)] - f^{\Delta}(\xi)[\sigma(\xi) - s]\big| \le \varepsilon|\sigma(\xi) - s|.$$

Moreover, f is said to be delta differentiable on $\mathbb{T}^{\kappa}$ if it is delta differentiable at every $\xi \in \mathbb{T}^{\kappa}$.

Let $f, \varphi : \mathbb{T} \to \mathbb{R}$ be delta differentiable functions at $\xi \in \mathbb{T}^{\kappa}$. Then we have the following:

(i) $(f + \varphi)^{\Delta}(\xi) = f^{\Delta}(\xi) + \varphi^{\Delta}(\xi)$;
(ii) $(f\varphi)^{\Delta}(\xi) = f^{\Delta}(\xi)\varphi(\xi) + f(\sigma(\xi))\varphi^{\Delta}(\xi) = f(\xi)\varphi^{\Delta}(\xi) + f^{\Delta}(\xi)\varphi(\sigma(\xi))$;

(iii) $\left(\frac{f}{\varphi}\right)^{\Delta}(\xi) = \frac{f^{\Delta}(\xi)\varphi(\xi) - f(\xi)\varphi^{\Delta}(\xi)}{\varphi(\xi)\varphi(\sigma(\xi))}, \qquad \varphi(\xi)\varphi(\sigma(\xi)) \neq 0.$

A function $\varphi : \mathbb{T} \to \mathbb{R}$ is said to be right-dense continuous (rd-continuous) if φ is continuous at the right-dense points in $\mathbb{T}$ and its left-sided limits exist at all left-dense points in $\mathbb{T}$.

We say that a function $F : \mathbb{T} \to \mathbb{R}$ is a delta antiderivative of $f : \mathbb{T} \to \mathbb{R}$ if $F^{\Delta}(\xi) = f(\xi)$ for all $\xi \in \mathbb{T}^{\kappa}$. In this case, the definite delta integral of f is given by

$$\int_{\theta}^{\vartheta} f(\xi)\Delta\xi = F(\vartheta) - F(\theta) \qquad for\ all \quad \theta, \vartheta \in \mathbb{T}.$$

If $\varphi \in C_{rd}(\mathbb{T})$ and $\xi, \xi_0 \in \mathbb{T}$, then the definite integral $F(\xi) := \int_{\xi_0}^{\xi} \varphi(s)\Delta s$ exists, and $F^{\Delta}(\xi) = \varphi(\xi)$ holds.

Let $\theta, \vartheta, \gamma \in \mathbb{T}$, $c \in \mathbb{R}$, and f, φ be right-dense continuous functions on $[\theta, \vartheta]_{\mathbb{T}}$. Then

(i) $\int_{\theta}^{\vartheta} [f(\xi) + \varphi(\xi)]\Delta\xi = \int_{\theta}^{\vartheta} f(\xi)\Delta\xi + \int_{\theta}^{\vartheta} \varphi(\xi)\Delta\xi;$

(ii) $\int_{\theta}^{\vartheta} cf(\xi)\Delta\xi = c\int_{\theta}^{\vartheta} f(\xi)\Delta\xi;$

(iii) $\int_{\theta}^{\vartheta} f(\xi)\Delta\xi = \int_{\theta}^{\gamma} f(\xi)\Delta\xi + \int_{\gamma}^{\vartheta} f(\xi)\Delta\xi;$

(iv) $\int_{\theta}^{\vartheta} f(\xi)\Delta\xi = -\int_{\vartheta}^{\theta} f(\xi)\Delta\xi;$

(v) $\int_{\theta}^{\theta} f(\xi)\Delta\xi = 0;$

(vi) if $f(\xi) \geq \varphi(\xi)$ on $[\theta, b]_{\mathbb{T}}$, then $\int_{\theta}^{\vartheta} f(\xi)\Delta\xi \geq \int_{\theta}^{\vartheta} \varphi(\xi)\Delta\xi$.

We use the following crucial relations between calculus on time scales $\mathbb{T}$ and differential calculus on $\mathbb{R}$ and difference calculus on $\mathbb{Z}$. Note that:

(i) If $\mathbb{T} = \mathbb{R}$, then

$$\sigma(\xi) = \xi, \quad \mu(\xi) = 0, \quad f^{\Delta}(\xi) = f'(\xi), \quad \int_{\theta}^{\vartheta} f(\xi)\Delta\xi = \int_{\theta}^{\vartheta} f(\xi)d\xi. \tag{2}$$

(ii) If $\mathbb{T} = \mathbb{Z}$, then

$$\sigma(\xi) = \xi + 1, \quad \mu(\xi) = 1, \quad f^{\Delta}(\xi) = f(\xi+1) - f(\xi), \quad \int_{\theta}^{\vartheta} f(\xi)\Delta\xi = \sum_{\xi=\theta}^{\vartheta-1} f(\xi). \tag{3}$$

Theorem 1 ([10], Leibniz integral rule on time scales)**.** *In the following, by $\Psi^{\Delta}(r_1, r_2)$ we mean the delta derivative of $\Psi(r_1, r_2)$ with respect to r_1. Similarly, $\Psi^{\nabla}(r_1, r_2)$ is understood. If Ψ , Ψ^{Δ}, and Ψ^{∇} are continuous and $u, h : \mathbb{T} \to \mathbb{T}$ are delta-differentiable functions, then the following formulas hold $\forall r_1 \in \mathbb{T}^{\kappa}$:*

(i)

$$\left[\int_{u(r_1)}^{h(r_1)} \Psi(r_1, r_2)\Delta r_2\right]^{\Delta} = \int_{u(r_1)}^{h(r_1)} \Psi^{\Delta}(r_1, r_2)\Delta r_2 + h^{\Delta}(r_1)\Psi(\sigma(r_1), h(r_1)) - u^{\Delta}(r_1)\Psi(\sigma(r_1), u(r_1));$$

(ii)

$$\left[\int_{u(r_1)}^{h(r_1)} \Psi(r_1, r_2)\Delta r_2\right]^{\nabla} = \int_{u(r_1)}^{h(r_1)} \Psi^{\nabla}(r_1, r_2)\Delta r_2 + h^{\nabla}(r_1)\Psi(\rho(r_1), h(r_1)) - u^{\nabla}(r_1)\Psi(\rho(r_1), u(r_1));$$

(iii)

$$\left[\int_{u(r_1)}^{h(r_1)} \Psi(r_1, r_2)\nabla r_2\right]^{\Delta} = \int_{u(r_1)}^{h(r_1)} \Psi^{\Delta}(r_1, r_2)\nabla r_2 + h^{\Delta}(r_1)\Psi(\sigma(r_1), h(r_1)) - u^{\Delta}(r_1)\Psi(\sigma(r_1), u(r_1));$$

(iv)

$$\left[\int_{u(r_1)}^{h(r_1)} \Psi(r_1,r_2)\nabla r_2\right]^{\nabla} = \int_{u(r_1)}^{h(r_1)} \Psi^{\nabla}(r_1,r_2)\nabla r_2 + h^{\nabla}(r_1)\Psi(\rho(r_1),h(r_1)) - u^{\nabla}(r_1)\Psi(\rho(r_1),u(r_1)).$$

3. Auxiliary Result

We prove the following fundamental lemma that will be needed in our main results.

Lemma 1. *Suppose $\mathbb{T}_1$ and $\mathbb{T}_2$ are two time scales and $a \in C(\Omega = \mathbb{T}_1 \times \mathbb{T}_2, \mathbb{R}_+)$ is nondecreasing with respect to $(\varrho,t) \in \Omega$. Assume that τ, κ, $f \in C(\Omega,\mathbb{R}_+)$, $\ell_1 \in C^1(\mathbb{T}_1,\mathbb{T}_1)$, and $\ell_2 \in C^1(\mathbb{T}_2,\mathbb{T}_2)$ are nondecreasing functions with $\ell_1(\varrho) \le \varrho$ on $\mathbb{T}_1$ and $\ell_2(t) \le t$ on $\mathbb{T}_2$. Furthermore, suppose Λ, $\zeta \in C(\mathbb{R}_+,\mathbb{R}_+)$ are nondecreasing functions with $\{\Lambda,\zeta\}(\kappa) > 0$ for $\kappa > 0$ and $\lim_{\kappa\to+\infty}\Lambda(\kappa) = +\infty$. If $\kappa(\varrho,t)$ satisfies*

$$\Lambda(\kappa(\varrho,t)) \le a(\varrho,t) + \int_{\varrho_0}^{\ell_1(\varrho)}\int_{t_0}^{\ell_2(t)} \tau(\varsigma,\eta)f(\varsigma,\eta)\zeta(\kappa(\varsigma,\eta))\Delta\eta\nabla\varsigma \tag{4}$$

for $(\varrho,t) \in \Omega$, then

$$\kappa(\varrho,t) \le \Lambda^{-1}\left\{\Upsilon^{-1}\left[\Upsilon(a(\varrho,t)) + \int_{\varrho_0}^{\ell_1(\varrho)}\int_{t_0}^{\ell_2(t)} \tau(\varsigma,\eta)f(\varsigma,\eta)\Delta\eta\Delta\varsigma\right]\right\} \tag{5}$$

for $0 \le \varrho \le \varrho_1$ and $0 \le t \le t_1$, where

$$\Upsilon(v) = \int_{v_0}^{v} \frac{\Delta\varsigma}{\zeta(\Lambda^{-1}(\varsigma))}, v \ge v_0 > 0,\ \Upsilon(+\infty) = \int_{v_0}^{+\infty} \frac{\Delta\varsigma}{\zeta(\Lambda^{-1}(\varsigma))} = +\infty \tag{6}$$

and $(\varrho_1,t_1) \in \Omega$ is chosen so that

$$\left(\Upsilon(a(\varrho,t)) + \int_{\varrho_0}^{\ell_1(\varrho)}\int_{t_0}^{\ell_2(t)} \tau_1(\varsigma,\eta)f(\varsigma,\eta)\Delta\eta\Delta\varsigma\right) \in \mathrm{Dom}\left(\Upsilon^{-1}\right).$$

Proof. Suppose that $a(\varrho,t) > 0$. Fixing an arbitrary $(\varrho_0,t_0) \in \Omega$, we define a positive and nondecreasing function $\psi(\varrho,t)$ by

$$\psi(\varrho,t) = a(\varrho_0,t_0) + \int_{\varrho_0}^{\ell_1(\varrho)}\int_{t_0}^{\ell_2(t)} \tau(\varsigma,\eta)f(\varsigma,\eta)\zeta(\kappa(\varsigma,\eta))\Delta\eta\nabla\varsigma \tag{7}$$

for $0 \le \varrho \le \varrho_0 \le \varrho_1$ and $0 \le t \le t_0 \le t_1$. Then, $\psi(\varrho_0,t) = \psi(\varrho,t_0) = a(\varrho_0,t_0)$ and

$$\kappa(\varrho,t) \le \Lambda^{-1}(\psi(\varrho,t)). \tag{8}$$

Takingthe Δ-derivative for (7) with employing Theorem 1(*iii*), we have

$$\begin{aligned}
\psi^{\Delta_\varrho}(\varrho,t) &= \ell_1^{\Delta}(\varrho)\int_{t_0}^{\ell_2(t)} \tau(\ell_1(\varrho),\eta)f(\ell_1(\varrho),\eta)\zeta(\kappa(\ell_1(\varrho),\eta))\Delta\eta \\
&\le \ell_1^{\Delta}(\varrho)\int_{t_0}^{\ell_2(t)} \tau(\ell_1(\varrho),\eta)f(\ell_1(\varrho),\eta)\zeta\left(\Lambda^{-1}(\psi(\ell_1(\varrho),\eta))\right)\Delta\eta \\
&\le \zeta\left(\Lambda^{-1}(\psi(\ell_1(\varrho),\ell_2(t)))\right)\ell_1^{\Delta}(\varrho)\int_{t_0}^{\ell_2(t)} \tau(\ell_1(\varrho),\eta)f(\ell_1(\varrho),\eta)\Delta\eta.
\end{aligned} \tag{9}$$

The inequality (9) can be written in the form

$$\frac{\psi^{\Delta_\varrho}(\varrho,t)}{\zeta(\Lambda^{-1}(\psi(\varrho,t)))} \le \ell_1^{\Delta}(\varrho)\int_{t_0}^{\ell_2(t)} \tau(\ell_1(\varrho),\eta)f(\ell_1(\varrho),\eta)\Delta\eta. \tag{10}$$

Taking the Δ-integral for inequality (10) we obtain

$$\begin{aligned}\Upsilon(\psi(\varrho,t)) &\leq \Upsilon(\psi(\varrho_0,t)) + \int_{\varrho_0}^{\ell_1(\varrho)}\int_{t_0}^{\ell_2(t)} \tau(\varsigma,\eta)f(\varsigma,\eta)\Delta\eta\Delta\varsigma \\ &\leq \Upsilon(a(\varrho_0,t_0)) + \int_{\varrho_0}^{\ell_1(\varrho)}\int_{t_0}^{\ell_2(t)} \tau(\varsigma,\eta)f(\varsigma,\eta)\Delta\eta\Delta\varsigma.\end{aligned}$$

Since $(\varrho_0,t_0) \in \Omega$ is chosen arbitrarily,

$$\psi(\varrho,t) \leq \Upsilon^{-1}\left[\Upsilon(a(\varrho,t)) + \int_{\varrho_0}^{\ell_1(\varrho)}\int_{t_0}^{\ell_2(t)} \tau(\varsigma,\eta)f(\varsigma,\eta)\Delta\eta\Delta\varsigma\right]. \tag{11}$$

From (11) and (8), we obtain the desired result (5). We carry out the above procedure with $\epsilon > 0$ instead of $a(\varrho,t)$ when $a(\varrho,t) = 0$ and subsequently let $\epsilon \to 0$. □

Remark 1. *If we take $\mathbb{T} = \mathbb{R}$, $\varrho_0 = 0$, and $t_0 = 0$ in Lemma 1, then, inequality (4) becomes the inequality obtained in ([16], Lemma 2.1).*

4. Main Results

In the following theorems, with the help of the Leibniz integral rule on time scales and Theorem 1 (item (iii)) and employing Lemma 1, we establish some new dynamic inequalities of the Gronwall–Bellman–Pachpatte type of time scale.

Theorem 2. *Let κ, a, f, ℓ_1, and ℓ_2 be as in Lemma 1. Let also $\tau_1,\tau_2 \in C(\Omega,\mathbb{R}_+)$. If $\kappa(\varrho,t)$ satisfies*

$$\begin{aligned}\Lambda(\kappa(\varrho,t)) \leq\; & a(\varrho,t) + \int_{\varrho_0}^{\ell_1(\varrho)}\int_{t_0}^{\ell_2(t)} \tau_1(\varsigma,\eta)[f(\varsigma,\eta)\zeta(\kappa(\varsigma,\eta)) \\ & + \int_{\varrho_0}^{\varsigma} \tau_2(\chi,\eta)\zeta(\kappa(\chi,\eta))\Delta\chi\Big]\Delta\eta\nabla\varsigma\end{aligned} \tag{12}$$

for $(\varrho,t) \in \Omega$, then

$$\kappa(\varrho,t) \leq \Lambda^{-1}\left\{\Upsilon^{-1}\left(p(\varrho,t) + \int_{\varrho_0}^{\ell_1(\varrho)}\int_{t_0}^{\ell_2(t)} \tau_1(\varsigma,\eta)f(\varsigma,\eta)\Delta\eta\Delta\varsigma\right)\right\} \tag{13}$$

for $0 \leq \varrho \leq \varrho_1$ and $0 \leq t \leq t_1$, where Υ is defined by (6) and

$$p(\varrho,t) = \Upsilon(a(\varrho,t)) + \int_{\varrho_0}^{\ell_1(\varrho)}\int_{t_0}^{\ell_2(t)} \tau_1(\varsigma,\eta)\left(\int_{\varrho_0}^{\varsigma} \tau_2(\chi,\eta)\Delta\chi\right)\Delta\eta\nabla\varsigma \tag{14}$$

and $(\varrho_1,t_1) \in \Omega$ is chosen so that

$$\left(p(\varrho,t) + \int_{\varrho_0}^{\ell_1(\varrho)}\int_{t_0}^{\ell_2(t)} \tau_1(\varsigma,\eta)f(\varsigma,\eta)\Delta\eta\Delta\varsigma\right) \in \mathrm{Dom}\left(\Upsilon^{-1}\right).$$

Proof. By the same steps of the proof of Lemma 1, we can obtain (13), with suitable changes. □

Remark 2. *If we take $\tau_2(\varrho,t) = 0$, then Theorem 2 reduces to Lemma 1.*

Corollary 1. *Let the functions κ, f, τ_1, τ_2, a, ℓ_1, and ℓ_2 be as in Theorem 2. Further, suppose that $q > p > 0$ are constants. If $\kappa(\varrho, t)$ satisfies*

$$\begin{aligned} \kappa^q(\varrho,t) \quad &\leq \quad a(\varrho,t) + \frac{q}{q-p} \int_{\varrho_0}^{\ell_1(\varrho)} \int_{t_0}^{\ell_2(t)} \tau_1(\varsigma,\eta) [f(\varsigma,\eta)\kappa^p(\varsigma,\eta) \\ &\quad + \int_{\varrho_0}^{\varsigma} \tau_2(\chi,\eta)\kappa^p(\chi,\eta)\Delta\chi \Big] \Delta\eta\nabla\varsigma \end{aligned} \tag{15}$$

for $(\varrho, t) \in \Omega$, then

$$\kappa(\varrho,t) \leq \left\{ p(\varrho,t) + \int_{\varrho_0}^{\ell_1(\varrho)} \int_{t_0}^{\ell_2(t)} \tau_1(\varsigma,\eta) f(\varsigma,\eta)\Delta\eta\Delta\varsigma \right\}^{\frac{1}{q-p}} \tag{16}$$

where

$$p(\varrho,t) = (a(\varrho,t))^{\frac{q-p}{q}} + \int_{\varrho_0}^{\ell_1(\varrho)} \int_{t_0}^{\ell_2(t)} \tau_1(\varsigma,\eta) \left(\int_{\varrho_0}^{\varsigma} \tau_2(\chi,\eta)\Delta\chi \right) \Delta\eta\nabla\varsigma.$$

Proof. Applying Theorem 2, by letting $\Lambda(\kappa) = \kappa^q$ and $\zeta(\kappa) = \kappa^p$, we have

$$\Upsilon(v) = \int_{v_0}^{v} \frac{\Delta\varsigma}{\zeta(\Lambda^{-1}(\varsigma))} = \int_{v_0}^{v} \frac{\Delta\varsigma}{\varsigma^{\frac{p}{q}}} \geq \frac{q}{q-p} \left(v^{\frac{q-p}{q}} - v_0^{\frac{q-p}{q}} \right), v \geq v_0 > 0$$

and

$$\Upsilon^{-1}(v) \geq \left\{ v_0^{\frac{q-p}{q}} + \frac{q-p}{q} v \right\}^{\frac{1}{q-p}}$$

and we obtain the inequality (16). □

Theorem 3. *Under the hypotheses of Theorem 2, suppose $\Lambda, \zeta, \varpi \in C(\mathbb{R}_+, \mathbb{R}_+)$ are nondecreasing functions with $\{\Lambda, \zeta, \varpi\}(\kappa) > 0$ for $\kappa > 0$ and $\kappa(\varrho, t)$ satisfies*

$$\begin{aligned} \Lambda(\kappa(\varrho,t)) \quad &\leq \quad a(\varrho,t) + \int_{\varrho_0}^{\ell_1(\varrho)} \int_{t_0}^{\ell_2(t)} \tau_1(\varsigma,\eta) [f(\varsigma,\eta)\zeta(\kappa(\varsigma,\eta))\varpi(\kappa(\varsigma,\eta)) \\ &\quad + \int_{\varrho_0}^{\varsigma} \tau_2(\chi,\eta)\zeta(\kappa(\chi,\eta))\Delta\chi \Big] \Delta\eta\nabla\varsigma \end{aligned} \tag{17}$$

for $(\varrho, t) \in \Omega$. Then,

$$\kappa(\varrho,t) \leq \Lambda^{-1} \left\{ \Upsilon^{-1} \left(F^{-1} \left[F(p(\varrho,t)) + \int_{\varrho_0}^{\ell_1(\varrho)} \int_{t_0}^{\ell_2(t)} \tau_1(\varsigma,\eta) f(\varsigma,\eta)\Delta\eta\Delta\varsigma \right] \right) \right\} \tag{18}$$

for $0 \leq \varrho \leq \varrho_1$ and $0 \leq t \leq t_1$, where Υ and p are as in (6) and (14), respectively, and

$$F(v) = \int_{v_0}^{v} \frac{\Delta\varsigma}{\varpi(\Lambda^{-1}(\Upsilon^{-1}(\varsigma)))}, v \geq v_0 > 0, \qquad F(+\infty) = +\infty \tag{19}$$

and $(\varrho_1, t_1) \in \Omega$ is chosen so that

$$\left[F(p(\varrho,t)) + \int_{\varrho_0}^{\ell_1(\varrho)} \int_{t_0}^{\ell_2(t)} \tau_1(\varsigma,\eta) f(\varsigma,\eta)\Delta\eta\Delta\varsigma \right] \in \mathrm{Dom}\left(F^{-1}\right).$$

Proof. Assume that $a(\varrho,t) > 0$. Fixing an arbitrary $(\varrho_0,t_0) \in \Omega$, we define a positive and nondecreasing function $\psi(\varrho,t)$ by

$$\begin{aligned}\psi(\varrho,t) &= a(\varrho_0,t_0) + \int_{\varrho_0}^{\ell_1(\varrho)} \int_{t_0}^{\ell_2(t)} \tau_1(\varsigma,\eta)[f(\varsigma,\eta)\zeta(\kappa(\varsigma,\eta))\varpi(\kappa(\varsigma,\eta)) \\ &\quad + \int_{\varrho_0}^{\varsigma} \tau_2(\chi,\eta)\zeta(\kappa(\chi,\eta))\Delta\chi\Big]\Delta\eta\nabla\varsigma \end{aligned} \tag{20}$$

for $0 \le \varrho \le \varrho_0 \le \varrho_1$, and $0 \le t \le t_0 \le t_1$. Then, $\psi(\varrho_0,t) = \psi(\varrho,t_0) = a(\varrho_0,t_0)$ and

$$\kappa(\varrho,t) \le \Lambda^{-1}(\psi(\varrho,t)) \tag{21}$$

Taking the Δ-derivative for (20) with employing Theorem 1 (*iii*) gives

$$\begin{aligned}\psi^{\Delta_\varrho}(\varrho,t) &= \ell_1^{\Delta}(\varrho)\int_{t_0}^{\ell_2(t)} \tau_1(\ell_1(\varrho),\eta)[f(\ell_1(\varrho),\eta)\zeta(\kappa(\ell_1(\varrho),\eta))\varpi(\kappa(\ell_1(\varrho),\eta)) \\ &\quad + \int_{\varrho_0}^{\ell_1(\varrho)} \tau_2(\chi,\eta)\zeta(\kappa(\chi,\eta))\Delta\chi\Big]\Delta\eta \\ &\le \ell_1^{\Delta}(\varrho)\int_{t_0}^{\ell_2(t)} \tau_1(\ell_1(\varrho),\eta)\Big[f(\ell_1(\varrho),\eta)\zeta\Big(\Lambda^{-1}(\psi(\ell_1(\varrho),\eta))\Big)\varpi\Big(\Lambda^{-1}(\psi(\ell_1(\varrho),\eta))\Big) \\ &\quad + \int_{\varrho_0}^{\ell_1(\varrho)} \tau_2(\chi,\eta)\zeta\Big(\Lambda^{-1}(\psi(\chi,\eta))\Big)\Delta\chi\Big]\Delta\eta \\ &\le \ell_1^{\Delta}(\varrho).\zeta\Big(\Lambda^{-1}(\psi(\ell_1(\varrho),\ell_2(t)))\Big) \times \\ &\quad \int_{t_0}^{\ell_2(t)} \tau_1(\ell_1(\varrho),\eta)\Big[f(\ell_1(\varrho),\eta)\varpi\Big(\Lambda^{-1}(\psi(\ell_1(\varrho),\eta))\Big) + \int_{\varrho_0}^{\ell_1(\varrho)} \tau_2(\chi,\eta)\Delta\chi\Big]\Delta\eta\end{aligned} \tag{22}$$

From (22), we have

$$\begin{aligned}\frac{\psi^{\Delta_\varrho}(\varrho,t)}{\zeta(\Lambda^{-1}(\psi(\varrho,t)))} &\le \ell_1^{\Delta}(\varrho)\int_{t_0}^{\ell_2(t)} \tau_1(\ell_1(\varrho),\eta)\Big[f(\ell_1(\varrho),\eta)\varpi\Big(\Lambda^{-1}(\psi(\ell_1(\varrho),\eta))\Big) \\ &\quad + \int_{\varrho_0}^{\ell_1(\varrho)} \tau_2(\chi,\eta)\Delta\chi\Big]\Delta\eta.\end{aligned} \tag{23}$$

Taking the Δ-integral for (23) gives

$$\begin{aligned}\Upsilon(\psi(\varrho,t)) &\le \Upsilon(\psi(\varrho_0,t)) + \int_{\varrho_0}^{\ell_1(\varrho)} \int_{t_0}^{\ell_2(t)} \tau_1(\varsigma,\eta)\Big[f(\varsigma,\eta)\varpi\Big(\Lambda^{-1}(\psi(\varsigma,\eta))\Big) \\ &\quad + \int_{\varrho_0}^{\varsigma} \tau_2(\chi,\eta)\Delta\chi\Big]\Delta\eta\Delta\varsigma \\ &\le \Upsilon(a(\varrho_0,t_0)) + \int_{\varrho_0}^{\ell_1(\varrho)} \int_{t_0}^{\ell_2(t)} \tau_1(\varsigma,\eta)\Big[f(\varsigma,\eta)\varpi\Big(\Lambda^{-1}(\psi(\varsigma,\eta))\Big) \\ &\quad + \int_{\varrho_0}^{\varsigma} \tau_2(\chi,\eta)\Delta\chi\Big]\Delta\eta\Delta\varsigma.\end{aligned}$$

Since $(\varrho_0,t_0) \in \Omega$ is chosen arbitrarily, the last inequality can be rewritten as

$$\Upsilon(\psi(\varrho,t)) \le p(\varrho,t) + \int_{\varrho_0}^{\ell_1(\varrho)} \int_{t_0}^{\ell_2(t)} \tau_1(\varsigma,\eta)f(\varsigma,\eta)\varpi\Big(\Lambda^{-1}(\psi(\varsigma,\eta))\Big)\Delta\eta\Delta\varsigma. \tag{24}$$

Since $p(\varrho,t)$ is a nondecreasing function, an application of Lemma 1 to (24) gives us

$$\psi(\varrho,t) \leq \Upsilon^{-1}\left(F^{-1}\left[F(p(\varrho,t)) + \int_{\varrho_0}^{\ell_1(\varrho)}\int_{t_0}^{\ell_2(t)} \tau_1(\varsigma,\eta)f(\varsigma,\eta)\Delta\eta\Delta\varsigma\right]\right). \tag{25}$$

From (21) and (25), we obtain the desired inequality (18).

Suppose that $a(\varrho,t) = 0$ for some $(\varrho,t) \in \Omega$. Let $a_\epsilon(\varrho,t) = a(\varrho,t) + \epsilon$, for all $(\varrho,t) \in \Omega$, where $\epsilon > 0$ be arbitrary. Then, $a_\epsilon(\varrho,t) > 0$ and $a_\epsilon(\varrho,t) \in C(\Omega,\mathbb{R}_+)$ are nondecreasing with respect to $(\varrho,t) \in \Omega$. We carry out the above procedure with $a_\epsilon(\varrho,t) > 0$ instead of $a(\varrho,t)$, and we obtain

$$\kappa(\varrho,t) \leq \Lambda^{-1}\left\{\Upsilon^{-1}\left(F^{-1}\left[F(p_\epsilon(\varrho,t)) + \int_{\varrho_0}^{\ell_1(\varrho)}\int_{t_0}^{\ell_2(t)} \tau_1(\varsigma,\eta)f(\varsigma,\eta)\Delta\eta\Delta\varsigma\right]\right)\right\}$$

where

$$p_\epsilon(\varrho,t) = \Upsilon(a_\epsilon(\varrho,t)) + \int_{\varrho_0}^{\ell_1(\varrho)}\int_{t_0}^{\ell_2(t)} \tau_1(\varsigma,\eta)\left(\int_{\varrho_0}^{\varsigma} \tau_2(\chi,\eta)\Delta\chi\right)\Delta\eta\nabla\varsigma.$$

Letting $\epsilon \to 0^+$, we obtain (18). The proof is complete. □

Remark 3. *If we take $\mathbb{T} = \mathbb{R}$, $\varrho_0 = 0$, and $t_0 = 0$ in Theorem 3, then the inequality (17) becomes the inequality obtained in ([16], Theorem 2.2(A_2)).*

Corollary 2. *Let the functions κ, a, f, τ_1, τ_2, ℓ_1, and ℓ_2 be as in Theorem 2. Further suppose that q, p, and r are constants with $p > 0$, $r > 0$, and $q > p + r$. If $\kappa(\varrho,t)$ satisfies*

$$\begin{aligned}\kappa^q(\varrho,t) \leq\; & a(\varrho,t) + \int_{\varrho_0}^{\ell_1(\varrho)}\int_{t_0}^{\ell_2(t)} \tau_1(\varsigma,\eta)[f(\varsigma,\eta)\kappa^p(\varsigma,\eta)\kappa^r(\varsigma,\eta) \\ & + \int_{\varrho_0}^{\varsigma} \tau_2(\chi,\eta)\kappa^p(\chi,\eta)\Delta\chi\Big]\Delta\eta\nabla\varsigma\end{aligned} \tag{26}$$

for $(\varrho,t) \in \Omega$, then

$$\kappa(\varrho,t) \leq \left\{[p(\varrho,t)]^{\frac{q-p-r}{q-p}} + \frac{q-p-r}{q}\int_{\varrho_0}^{\ell_1(\varrho)}\int_{t_0}^{\ell_2(t)} \tau_1(\varsigma,\eta)f(\varsigma,\eta)\Delta\eta\Delta\varsigma\right\}^{\frac{1}{q-p-r}} \tag{27}$$

where

$$p(\varrho,t) = (a(\varrho,t))^{\frac{q-p}{q}} + \frac{q-p}{q}\int_{\varrho_0}^{\ell_1(\varrho)}\int_{t_0}^{\ell_2(t)} \tau_1(\varsigma,\eta)\left(\int_{\varrho_0}^{\varsigma} \tau_2(\chi,\eta)\Delta\chi\right)\Delta\eta\nabla\varsigma$$

Proof. An application of Theorem 3 with $\Lambda(\kappa) = \kappa^q, \zeta(\kappa) = \kappa^p$, and $\varpi(\kappa) = \kappa^r$ yields the desired inequality (27). □

Theorem 4. *Under the hypotheses of Theorem 3, if $\kappa(\varrho,t)$ satisfies*

$$\begin{aligned}\Lambda(\kappa(\varrho,t)) \leq\; & a(\varrho,t) + \int_{\varrho_0}^{\ell_1(\varrho)}\int_{t_0}^{\ell_2(t)} \tau_1(\varsigma,\eta)[f(\varsigma,\eta)\zeta(\kappa(\varsigma,\eta))\varpi(\kappa(\varsigma,\eta)) \\ & + \int_{\varrho_0}^{\varsigma} \tau_2(\chi,\eta)\zeta(\kappa(\chi,\eta))\varpi(\kappa(\chi,\eta))\Delta\chi\Big]\Delta\eta\nabla\varsigma\end{aligned} \tag{28}$$

for $(\varrho,t) \in \Omega$, then

$$\kappa(\varrho,t) \leq \Lambda^{-1}\left\{\Upsilon^{-1}\left(F^{-1}\left[p_0(\varrho,t) + \int_{\varrho_0}^{\ell_1(\varrho)}\int_{t_0}^{\ell_2(t)} \tau_1(\varsigma,\eta)f(\varsigma,\eta)\Delta\eta\Delta\varsigma\right]\right)\right\} \tag{29}$$

for $0 \leq \varrho \leq \varrho_1$ *and* $0 \leq t \leq t_1$, *where*

$$p_0(\varrho,t) = F(\Upsilon(a(\varrho,t))) + \int_{\varrho_0}^{\ell_1(\varrho)} \int_{t_0}^{\ell_2(t)} \tau_1(\varsigma,\eta) \left(\int_{\varrho_0}^{\varsigma} \tau_2(\chi,\eta)\Delta\chi \right) \Delta\eta \nabla\varsigma$$

and $(\varrho_1, t_1) \in \Omega$ *is chosen so that*

$$\left[p_0(\varrho,t) + \int_{\varrho_0}^{\ell_1(\varrho)} \int_{t_0}^{\ell_2(t)} \tau_1(\varsigma,\eta) f(\varsigma,\eta)\Delta\eta\Delta\varsigma \right] \in \mathrm{Dom}\left(F^{-1}\right).$$

Proof. Assume that $a(\varrho,t) > 0$. Fixing an arbitrary $(\varrho_0, t_0) \in \Omega$, we define a positive and nondecreasing function $\psi(\varrho,t)$ by

$$\begin{aligned} \psi(\varrho,t) &= a(\varrho_0,t_0) + \int_{\varrho_0}^{\ell_1(\varrho)} \int_{t_0}^{\ell_2(t)} \tau_1(\varsigma,\eta)[f(\varsigma,\eta)\zeta(\kappa(\varsigma,\eta))\varpi(\kappa(\varsigma,\eta)) \\ &\quad + \int_{\varrho_0}^{\varsigma} \tau_2(\chi,\eta)\zeta(\kappa(\chi,\eta))\varpi(\kappa(\chi,\eta))\Delta\chi \Big] \Delta\eta\nabla\varsigma \end{aligned}$$

for $0 \leq \varrho \leq \varrho_0 \leq \varrho_1$ and $0 \leq t \leq t_0 \leq t_1$. Then, $\psi(\varrho_0,t) = \psi(\varrho,t_0) = a(\varrho_0,t_0)$, and

$$\kappa(\varrho,t) \leq \Lambda^{-1}(\psi(\varrho,t)). \tag{30}$$

By the same steps as in the proof of Theorem 3, we obtain

$$\begin{aligned} \psi(\varrho,t) &\leq \Upsilon^{-1}\Bigg\{ \Upsilon(a(\varrho_0,t_0)) + \int_{\varrho_0}^{\ell_1(\varrho)} \int_{t_0}^{\ell_2(t)} \tau_1(\varsigma,\eta) \Big[f(\varsigma,\eta)\varpi\Big(\Lambda^{-1}(\psi(\varsigma,\eta))\Big) \\ &\quad + \int_{\varrho_0}^{\varsigma} \tau_2(\chi,\eta)\varpi\Big(\Lambda^{-1}(\psi(\chi,\eta))\Big)\Delta\chi \Big] \Delta\eta\Delta\varsigma \Bigg\}. \end{aligned}$$

We define a non-negative and nondecreasing function $v(\varrho,t)$ by

$$\begin{aligned} v(\varrho,t) &= \Upsilon(a(\varrho_0,t_0)) + \int_{\varrho_0}^{\ell_1(\varrho)} \int_{t_0}^{\ell_2(t)} \tau_1(\varsigma,\eta) \Big[\Big[f(\varsigma,\eta)\varpi\Big(\Lambda^{-1}(\psi(\varsigma,\eta))\Big) \Big] \\ &\quad + \int_{\varrho_0}^{\varsigma} \tau_2(\chi,\eta)\varpi\Big(\Lambda^{-1}(\psi(\chi,\eta))\Big)\Delta\chi \Big] \Delta\eta\nabla\varsigma \end{aligned}$$

Then, $v(\varrho_0,t) = v(\varrho,t_0) = \Upsilon(a(\varrho_0,t_0))$,

$$\psi(\varrho,t) \leq \Upsilon^{-1}[v(\varrho,t)] \tag{31}$$

and then, employing Theorem 1 (*iii*), we have

$$\begin{aligned} v^{\Delta\varrho}(\varrho,t) &\leq \ell_1^{\Delta}(\varrho) \int_{t_0}^{\ell_2(t)} \tau_1(\ell_1(\varrho),\eta) \Big[f(\ell_1(\varrho),\eta)\varpi\Big(\Lambda^{-1}\Big(\Upsilon^{-1}(v(\ell_1(\varrho),t))\Big)\Big) \\ &\quad + \int_{\varrho_0}^{\ell_1(\varrho)} \tau_2(\chi,\eta)\varpi\Big(\Lambda^{-1}\Big(\Upsilon^{-1}(v(\chi,t))\Big)\Big)\Delta\chi \Big] \Delta\eta \\ &\leq \ell_1^{\Delta}(\varrho)\varpi\Big(\Lambda^{-1}\Big(\Upsilon^{-1}(v(\ell_1(\varrho),\ell_2(t)))\Big)\Big) \int_{t_0}^{\ell_2(t)} \tau_1(\ell_1(\varrho),\eta)[f(\ell_1(\varrho),\eta) \\ &\quad + \int_{\varrho_0}^{\ell_1(\varrho)} \tau_2(\chi,\eta)\Delta\chi \Big] \Delta\eta \end{aligned}$$

or

$$\frac{v^{\Delta\varrho}(\varrho,t)}{\varpi(\Lambda^{-1}(\Upsilon^{-1}(v(\varrho,t))))} \leq \ell_1^{\Delta}(\varrho)\int_{t_0}^{\ell_2(t)}\tau_1(\ell_1(\varrho),\eta)[f(\ell_1(\varrho),\eta) + \int_{\varrho_0}^{\ell_1(\varrho)}\tau_2(\chi,\eta)\Delta\chi\Big]\Delta\eta.$$

Taking the Δ-integral for the above inequality gives

$$F(v(\varrho,t)) \leq F(v(\varrho_0,t)) + \int_{\varrho_0}^{\ell_1(\varrho)}\int_{t_0}^{\ell_2(t)}\tau_1(\varsigma,\eta)\Big[f(\varsigma,\eta)+\int_{\varrho_0}^{\varsigma}\tau_2(\chi,\eta)\Delta\chi\Big]\Delta\eta\Delta\varsigma$$

or

$$v(\varrho,t) \leq F^{-1}\Big\{F(\Upsilon(a(\varrho_0,t_0))) + \int_{\varrho_0}^{\ell_1(\varrho)}\int_{t_0}^{\ell_2(t)}\tau_1(\varsigma,\eta)[f(\varsigma,\eta) + \int_{\varrho_0}^{\varsigma}\tau_2(\chi,\eta)\Delta\chi\Big]\Delta\eta\Delta\varsigma\Big\}. \quad (32)$$

From (30)–(32), and since $(\varrho_0,t_0)\in\Omega$ is chosen arbitrarily, we obtain the desired inequality (29). If $a(\varrho,t)=0$, we carry out the above procedure with $\epsilon>0$ instead of $a(\varrho,t)$ and subsequently let $\epsilon\to 0$. The proof is complete. □

Remark 4. *If we take $\mathbb{T}=\mathbb{R}$ and $\varrho_0=0$ and $t_0=0$ in Theorem 4, then, inequality (28) becomes the inequality obtained in ([16], Theorem 2.2(A_3)).*

Corollary 3. *Under the hypotheses of Corollary 2, if $\kappa(\varrho,t)$ satisfies*

$$\kappa^q(\varrho,t) \leq a(\varrho,t) + \int_{\varrho_0}^{\ell_1(\varrho)}\int_{t_0}^{\ell_2(t)}\tau_1(\varsigma,\eta)[f(\varsigma,\eta)\kappa^p(\varsigma,\eta)\kappa^r(\varsigma,\eta) + \int_{\varrho_0}^{\varsigma}\tau_2(\chi,\eta)\kappa^p(\chi,\eta)\kappa^r(\chi,\eta)\Delta\chi\Big]\Delta\eta\nabla\varsigma \quad (33)$$

for $(\varrho,t)\in\Omega$, then

$$\kappa(\varrho,t) \leq \Big\{p_0(\varrho,t) + \frac{q-p-r}{q}\int_{\varrho_0}^{\ell_1(\varrho)}\int_{t_0}^{\ell_2(t)}\tau_1(\varsigma,\eta)f(\varsigma,\eta)\Delta\eta\Delta\varsigma\Big\}^{\frac{1}{q-p-r}} \quad (34)$$

where

$$p_0(\varrho,t) = (a(\varrho,t))^{\frac{q-p-r}{q}} + \frac{q-p-r}{q}\int_{\varrho_0}^{\ell_1(\varrho)}\int_{t_0}^{\ell_2(t)}\tau_1(\varsigma,\eta)\Big(\int_{\varrho_0}^{\varsigma}\tau_2(\chi,\eta)\Delta\chi\Big)\Delta\eta\nabla\varsigma$$

Proof. An application of Theorem 4 with $\Lambda(\kappa)=\kappa^q, \zeta(\kappa)=\kappa^p$, and $\varpi(\kappa)=\kappa^r$ yields the desired inequality (34). □

Theorem 5. *Under the hypotheses of Theorem 3, if $\kappa(\varrho,t)$ satisfies*

$$\Lambda(\kappa(\varrho,t)) \leq a(\varrho,t) + \int_{\varrho_0}^{\ell_1(\varrho)}\int_{t_0}^{\ell_2(t)}\tau_1(\varsigma,\eta)\varpi(\kappa(\varsigma,\eta))\times \Big[f(\varsigma,\eta)\zeta(\kappa(\varsigma,\eta)) + \int_{\varrho_0}^{\varsigma}\tau_2(\chi,\eta)\Delta\chi\Big]\Delta\eta\nabla\varsigma \quad (35)$$

for $(\varrho, t) \in \Omega$, *then*

$$\kappa(\varrho, t) \leq \Lambda^{-1}\left\{\Upsilon_1^{-1}\left(F_1^{-1}\left[F_1(p_1(\varrho, t)) + \int_{\varrho_0}^{\ell_1(\varrho)} \int_{t_0}^{\ell_2(t)} \tau_1(\varsigma, \eta) f(\varsigma, \eta) \Delta\eta \Delta\varsigma\right]\right)\right\} \tag{36}$$

for $0 \leq \varrho \leq \varrho_2$ *and* $0 \leq t \leq t_2$, *where*

$$\Upsilon_1(v) = \int_{v_0}^{v} \frac{\Delta\varsigma}{\varpi(\Lambda^{-1}(\varsigma))}, v \geq v_0 > 0, \Upsilon_1(+\infty) = \int_{v_0}^{+\infty} \frac{\Delta\varsigma}{\varpi(\Lambda^{-1}(\varsigma))} = +\infty \tag{37}$$

$$F_1(v) = \int_{v_0}^{v} \frac{\Delta\varsigma}{\zeta\left[\Lambda^{-1}\left(\Upsilon_1^{-1}(\varsigma)\right)\right]}, v \geq v_0 > 0, F_1(+\infty) = +\infty \tag{38}$$

$$p_1(\varrho, t) = \Upsilon_1(a(\varrho, t)) + \int_{\varrho_0}^{\ell_1(\varrho)} \int_{t_0}^{\ell_2(t)} \tau_1(\varsigma, \eta)\left(\int_{\varrho_0}^{\varsigma} \tau_2(\chi, \eta)\Delta\chi\right)\Delta\eta\nabla\varsigma \tag{39}$$

and $(\varrho_2, t_2) \in \Omega$ *is chosen so that*

$$\left[F_1(p_1(\varrho, t)) + \int_{\varrho_0}^{\ell_1(\varrho)} \int_{t_0}^{\ell_2(t)} \tau_1(\varsigma, \eta) f(\varsigma, \eta)\Delta\eta\Delta\varsigma\right] \in \mathrm{Dom}\left(F_1^{-1}\right).$$

Proof. Suppose that $a(\varrho, t) > 0$. Fixing an arbitrary $(\varrho_0, t_0) \in \Omega$, we define a positive and nondecreasing function $\psi(\varrho, t)$ by

$$\begin{aligned}\psi(\varrho, t) \quad &= \quad a(\varrho_0, t_0) + \int_{\varrho_0}^{\ell_1(\varrho)} \int_{t_0}^{\ell_2(t)} \tau_1(\varsigma, \eta)\varpi(\kappa(\varsigma, \eta))[f(\varsigma, \eta)\zeta(\kappa(\varsigma, \eta)) \\ &\quad + \int_{\varrho_0}^{\varsigma} \tau_2(\chi, \eta)\Delta\chi\Big]\Delta\eta\nabla\varsigma\end{aligned}$$

for $0 \leq \varrho \leq \varrho_0 \leq \varrho_2$ and $0 \leq t \leq t_0 \leq t_2$. Then, $\psi(\varrho_0, t) = \psi(\varrho, t_0) = a(\varrho_0, t_0)$, and

$$\kappa(\varrho, t) \leq \Lambda^{-1}(\psi(\varrho, t)). \tag{40}$$

Employing Theorem 1 (*iii*),

$$\begin{aligned}\psi^{\Delta_\varrho}(\varrho, t) \quad &\leq \quad \ell_1^{\Delta}(\varrho) \int_{t_0}^{\ell_2(t)} \tau_1(\ell_1(\varrho), \eta)\eta\left[\Lambda^{-1}(\psi(\ell_1(\varrho), \eta))\right]\left[f(\ell_1(\varrho), \eta)\zeta\left(\Lambda^{-1}(\psi(\ell_1(\varrho), \eta))\right)\right. \\ &\quad \left. + \int_{\varrho_0}^{\ell_1(\varrho)} \tau_2(\chi, \eta)\Delta\chi\right]\Delta\eta \\ &\leq \quad \ell_1^{\Delta}(\varrho)\eta\left[\Lambda^{-1}(\psi(\ell_1(\varrho), \ell_2(t)))\right] \int_{t_0}^{\ell_2(t)} \tau_1(\ell_1(\varrho), \eta)\left[f(\ell_1(\varrho), \eta)\zeta\left(\Lambda^{-1}(\psi(\ell_1(\varrho), \eta))\right)\right. \\ &\quad \left. + \int_{\varrho_0}^{\ell_1(\varrho)} \tau_2(\chi, \eta)\Delta\chi\right]\Delta\eta\end{aligned}$$

Then,

$$\begin{aligned}\frac{\psi^{\Delta_\varrho}(\varrho, t)}{\eta[\Lambda^{-1}(\psi(\varrho, t))]} \quad &\leq \quad \ell_1^{\Delta}(\varrho) \int_{t_0}^{\ell_2(t)} \tau_1(\ell_1(\varrho), \eta)\left[f(\ell_1(\varrho), \eta)\zeta\left(\Lambda^{-1}(\psi(\ell_1(\varrho), \eta))\right)\right. \\ &\quad \left. + \int_{\varrho_0}^{\ell_1(\varrho)} \tau_2(\chi, \eta)\Delta\chi\right]\Delta\eta.\end{aligned}$$

Taking the Δ-integral for the above inequality gives

$$\begin{aligned}\Upsilon_1(\psi(\varrho,t)) \quad &\leq \quad \Upsilon_1(\psi(0,t)) + \int_{\varrho_0}^{\ell_1(\varrho)}\int_{t_0}^{\ell_2(t)} \tau_1(\varsigma,\eta)\Big[f(\varsigma,\eta)\zeta\Big(\Lambda^{-1}(\psi(\varsigma,\eta))\Big) \\ &\quad + \int_{\varrho_0}^{\varsigma} \tau_2(\chi,\eta)\Delta\chi\Big]\Delta\eta\Delta\varsigma\end{aligned}$$

Then,

$$\begin{aligned}\Upsilon_1(\psi(\varrho,t)) \quad &\leq \quad \Upsilon_1(a(\varrho_0,t_0)) + \int_{\varrho_0}^{\ell_1(\varrho)}\int_{t_0}^{\ell_2(t)} \tau_1(\varsigma,\eta)\Big[f(\varsigma,\eta)\zeta\Big(\Lambda^{-1}(\psi(\varsigma,\eta))\Big) \\ &\quad + \int_{\varrho_0}^{\varsigma} \tau_2(\chi,\eta)\Delta\chi\Big]\Delta\eta\Delta\varsigma.\end{aligned}$$

Since $(\varrho_0,t_0) \in \Omega$ is chosen arbitrarily, the last inequality can be restated as

$$\Upsilon_1(\psi(\varrho,t)) \leq p_1(\varrho,t) + \int_{\varrho_0}^{\ell_1(\varrho)}\int_{t_0}^{\ell_2(t)} \tau_1(\varsigma,\eta)f(\varsigma,\eta)\zeta\Big(\Lambda^{-1}(\psi(\varsigma,\eta))\Big)\Delta\eta\Delta\varsigma \tag{41}$$

It is easy to observe that $p_1(\varrho,t)$ be a positive and nondecreasing function for all $(\varrho,t) \in \Omega$. Then, an application of Lemma 1 to (41) yields the inequality

$$\psi(\varrho,t) \leq \Upsilon_1^{-1}\left(F_1^{-1}\left[F_1(p_1(\varrho,t)) + \int_{\varrho_0}^{\ell_1(\varrho)}\int_{t_0}^{\ell_2(t)} \tau_1(\varsigma,\eta)f(\varsigma,\eta)\Delta\eta\Delta\varsigma\right]\right). \tag{42}$$

From (42) and (40), we obtain the desired inequality (36).

If $a(\varrho,t) = 0$, we carry out the above procedure with $\epsilon > 0$ instead of $a(\varrho,t)$ and subsequently let $\epsilon \to 0$. The proof is complete. □

Remark 5. *If we take $\mathbb{T} = \mathbb{R}$ and $\varrho_0 = 0$ and $t_0 = 0$ in Theorem 5, then, inequality (35) becomes the inequality obtained in ([16], Theorem 2.7).*

Theorem 6. *Under the hypotheses of Theorem 3, let p be a non-negative constant. If $\kappa(\varrho,t)$ satisfies*

$$\begin{aligned}\Lambda(\kappa(\varrho,t)) \quad &\leq \quad a(\varrho,t) + \int_{\varrho_0}^{\ell_1(\varrho)}\int_{t_0}^{\ell_2(t)} \tau_1(\varsigma,\eta)\kappa^p(\varsigma,\eta) \times \\ &\quad \left[f(\varsigma,\eta)\zeta(\kappa(\varsigma,\eta)) + \int_{\varrho_0}^{\varsigma} \tau_2(\chi,\eta)\Delta\chi\right]\Delta\eta\nabla\varsigma\end{aligned} \tag{43}$$

for $(\varrho,t) \in \Omega$, then

$$\kappa(\varrho,t) \leq \Lambda^{-1}\left\{\Upsilon_1^{-1}\left(F_1^{-1}\left[F_1(p_1(\varrho,t)) + \int_{\varrho_0}^{\ell_1(\varrho)}\int_{t_0}^{\ell_2(t)} \tau_1(\varsigma,\eta)f(\varsigma,\eta)\Delta\eta\Delta\varsigma\right]\right)\right\} \tag{44}$$

for $0 \leq \varrho \leq \varrho_2$ and $0 \leq t \leq t_2$, where

$$\Upsilon_1(v) = \int_{v_0}^{v} \frac{\Delta\varsigma}{[\Lambda^{-1}(\varsigma)]^p}, v \geq v_0 > 0, \Upsilon_1(+\infty) = \int_{v_0}^{+\infty} \frac{\Delta\varsigma}{[\Lambda^{-1}(\varsigma)]^p} = +\infty \tag{45}$$

and F_1 and p_1 are as in Theorem 5 and $(\varrho_2,t_2) \in \Omega$ is chosen so that

$$\left[F_1(p_1(\varrho,t)) + \int_{\varrho_0}^{\ell_1(\varrho)}\int_{t_0}^{\ell_2(t)} \tau_1(\varsigma,\eta)f(\varsigma,\eta)\Delta\eta\Delta\varsigma\right] \in \mathrm{Dom}\Big(F_1^{-1}\Big).$$

Proof. An application of Theorem 5, with $\varpi(\kappa) = \kappa^p$, yields the desired inequality (44). □

Remark 6. *Take* $\mathbb{T} = \mathbb{R}$*, the inequality established in Theorem 6 generalizes ([18], Theorem 1) (with* $p = 1$*,* $a(\varrho,t) = b(\varrho) + c(t)$*,* $\varrho_0 = 0$*,* $t_0 = 0$*,* $\tau_1(\varsigma,\eta)f(\varsigma,\eta) = h(\varsigma,\eta)$*, and* $\tau_1(\varsigma,\eta)\left(\int_{\varrho_0}^{\varsigma}\tau_2(\chi,\eta)\Delta\chi\right) = g(\varsigma,\eta)$*).*

Corollary 4. *Under the hypotheses of Theorem 6, let* $q > p > 0$ *be constants. If* $\kappa(\varrho,t)$ *satisfies*

$$\begin{aligned}\kappa^q(\varrho,t) \quad &\leq \quad a(\varrho,t) + \frac{p}{p-q}\int_{\varrho_0}^{\ell_1(\varrho)}\int_{t_0}^{\ell_2(t)}\tau_1(\varsigma,\eta)\kappa^p(\varsigma,\eta)\times \\ &\left[f(\varsigma,\eta)\zeta(\kappa(\varsigma,\eta)) + \int_{\varrho_0}^{\varsigma}\tau_2(\chi,\eta)\Delta\chi\right]\Delta\eta\nabla\varsigma \end{aligned} \tag{46}$$

for $(\varrho,t)\in\Omega$*, then*

$$\kappa(\varrho,t) \leq \left\{F_1^{-1}\left[F_1(p_1(\varrho,t)) + \int_{\varrho_0}^{\ell_1(\varrho)}\int_{t_0}^{\ell_2(t)}\tau_1(\varsigma,\eta)f(\varsigma,\eta)\Delta\eta\nabla\varsigma\right]\right\}^{\frac{1}{q-p}} \tag{47}$$

for $0 \leq \varrho \leq \varrho_2$ *and* $0 \leq t \leq t_2$*, where*

$$p_1(\varrho,t) = [a(\varrho,t)]^{\frac{q-p}{q}} + \int_{\varrho_0}^{\ell_1(\varrho)}\int_{t_0}^{\ell_2(t)}\tau_1(\varsigma,\eta)\left(\int_{\varrho_0}^{\varsigma}\tau_2(\chi,\eta)\Delta\chi\right)\Delta\eta\nabla\varsigma$$

and F_1 *is defined in Theorem 5.*

Proof. An application of Theorem 6 with $\Lambda(\kappa(\varrho,t)) = \kappa^p$ to (46) yields the inequality (47); to save space we omit the details. □

Remark 7. *Taking* $\mathbb{T} = \mathbb{R}$*,* $\varrho_0 = 0$*,* $t_0 = 0$*,* $a(\varrho,t) = b(\varrho) + c(t)$*,* $\tau_1(\varsigma,\eta)f(\varsigma,\eta) = h(\varsigma,\eta)$*, and* $\tau_1(\varsigma,\eta)\left(\int_{\varrho_0}^{\varsigma}\tau_2(\chi,\eta)\Delta\chi\right) = g(\varsigma,\eta)$ *in Corollary 4, we obtain ([20], Theorem 1).*

Remark 8. *Taking* $\mathbb{T} = \mathbb{R}$*,* $\varrho_0 = 0$*,* $t_0 = 0$*,* $a(\varrho,t) = c^{\frac{p}{p-q}}$*,* $\tau_1(\varsigma,\eta)f(\varsigma,\eta) = h(\eta)$*, and* $\tau_1(\varsigma,\eta)\left(\int_{\varrho_0}^{\varsigma}\tau_2(\chi,\eta)\Delta\chi\right) = g(\eta)$ *and keeping t fixed in Corollary 4, we obtain ([21], Theorem 2.1).*

5. Application

In the following, we discuss the boundedness of the solutions of the initial boundary value problem for a partial delay dynamic equation of the form

$$(\Xi^q)^{\Delta_\varrho\nabla_t}(\varrho,t) = A\left(\varrho,t,\Xi(\varrho-h_1(\varrho),t-h_2(t)),\int_{\varrho_0}^{\varrho}B(\varsigma,t,\Xi(\varsigma-h_1(\varsigma),t))\Delta\varsigma\right) \tag{48}$$

$$\Xi(\varrho,t_0) = a_1(\varrho),\Xi(\varrho_0,t) = a_2(t),a_1(\varrho_0) = a_{t_0}(0) = 0$$

for $(\varrho,t)\in\Omega$, where $\Xi, b \in C(\Omega,\mathbb{R}_+), A \in C(\Omega\times R^2,R), B \in C(\zeta\times R,R)$, and $h_1 \in C^1(\mathbb{T}_1,\mathbb{R}_+)$ and $h_2 \in C^1(\mathbb{T}_2,\mathbb{R}_+)$ are nondecreasing functions such that $h_1(\varrho) \leq \varrho$ on $\mathbb{T}_1$, $h_2(t) \leq t$ on $\mathbb{T}_2$, and $h_1^{\Delta}(\varrho) < 1$ and $h_2^{\Delta}(t) < 1$.

Theorem 7. *Assume that the functions* a_1*,* a_2*,* A*, and* B *in (48) satisfy the conditions*

$$|a_1(\varrho) + a_2(t)| \leq a(\varrho,t), \tag{49}$$

$$|A(\varsigma,\eta,\Xi,\kappa)| \leq \frac{q}{q-p}\tau_1(\varsigma,\eta)\left[f(\varsigma,\eta)|\Xi|^p + |\kappa|\right], \tag{50}$$

$$|B(\chi,\eta,\Xi)| \leq \tau_2(\chi,\eta)|\Xi|^p, \tag{51}$$

where $a(\varrho,t), \tau_1(\varsigma,\eta), f(\varsigma,\eta)$, and $\tau_2(\chi,\eta)$ are as in Theorem 2 and $q > p > 0$ are constants. If $\Xi(\varrho,t)$ satisfies (48), then

$$|\Xi(\varrho,t)| \leq \left\{ p(\varrho,t) + M_1M_2 \int_{\varrho_0}^{\ell_1(\varrho)} \int_{t_0}^{\ell_2(t)} \bar{\tau}_1(\varsigma,\eta)\bar{f}(\varsigma,\eta)\Delta\eta\Delta\varsigma \right\}^{\frac{1}{q-p}} \quad (52)$$

where

$$\begin{aligned} p(\varrho,t) &= (a(\varrho,t))^{\frac{q-p}{q}} \\ &+ M_1M_2 \int_{\varrho_0}^{\ell_1(\varrho)} \int_{t_0}^{\ell_2(t)} \bar{\tau}_1(\varsigma,\eta)\left(M_1 \int_{\varrho_0}^{\varsigma} \bar{\tau}_2(\chi,\eta)\Delta\chi\right)\Delta\eta\nabla\varsigma \end{aligned}$$

and

$$M_1 = \underset{\varrho\in I_1}{Max}\frac{1}{1-h_1^{\Delta}(\varrho)}, \qquad M_2 = \underset{t\in I_2}{Max}\frac{1}{1-h_2^{\Delta}(t)}$$

and $\bar{\tau}_1(\gamma,\xi) = \tau_1(\gamma + h_1(\varsigma), \xi + h_2(\eta)), \bar{\tau}_2(\mu,\xi) = \tau_2(\mu, \xi + h_2(\eta))$, and $\bar{f}(\gamma,\xi) = f(\gamma + h_1(\varsigma), \xi + h_2(\eta))$.

Proof. If $\Xi(\varrho,t)$ is any solution of (48), then

$$\begin{aligned} &\Xi^q(\varrho,t) = a_1(\varrho) + a_2(t) \\ &+ \int_{\varrho_0}^{\varrho}\int_{t_0}^{t} A\left(\varsigma,\eta,\Xi(\varsigma - h_1(\varsigma),\eta - h_2(\eta)), \int_{\varrho_0}^{\varsigma} B(\chi,\eta,\Xi(\chi - h_1(\chi),\eta))\Delta\chi\right)\Delta\eta\nabla\varsigma. \end{aligned} \quad (53)$$

Using the conditions (49)–(51) in (53), we obtain

$$\begin{aligned} |\Xi(\varrho,t)|^q &\leq a(\varrho,t) + \frac{q-p}{q}\int_{\varrho_0}^{\varrho}\int_{t_0}^{t}\tau_1(\varsigma,\eta)\big[f(\varsigma,\eta)|\Xi(\varsigma - h_1(\varsigma),\eta - h_2(\eta))|^p \\ &+ \int_{\varrho_0}^{\varsigma}\tau_2(\chi,\eta)|\Xi(\chi,\eta)|^p\Delta\chi\Big]\Delta\eta\nabla\varsigma. \end{aligned} \quad (54)$$

Now, making a change in variables on the right side of (54), $\varsigma - h_1(\varsigma) = \gamma, \eta - h_2(\eta) = \xi$, $\varrho - h_1(\varrho) = \ell_1(\varrho)$ for $\varrho \in \mathbb{T}_1$, and $t - h_2(t) = \ell_2(t)$ for $t \in \mathbb{T}_2$. We obtain the inequality

$$\begin{aligned} |\Xi(\varrho,t)|^q &\leq a(\varrho,t) + \frac{q-p}{q}M_1M_2\int_{\varrho_0}^{\ell_1(\varrho)}\int_{t_0}^{\ell_2(t)}\bar{\tau}_1(\gamma,\xi)\Big[\bar{f}(\gamma,\xi)|\Xi(\gamma,\xi)|^p \\ &+ M_1\int_{\varrho_0}^{\gamma}\bar{\tau}_2(\mu,\xi)|\Xi(\mu,\eta)|^p\Delta\mu\Big]\Delta\xi\Delta\gamma. \end{aligned} \quad (55)$$

We can rewrite the inequality (55) as follows:

$$\begin{aligned} |\Xi(\varrho,t)|^q &\leq a(\varrho,t) + \frac{q-p}{q}M_1M_2\int_{\varrho_0}^{\ell_1(\varrho)}\int_{t_0}^{\ell_2(t)}\bar{\tau}_1(\varsigma,\eta)\Big[\bar{f}(\varsigma,\eta)|\Xi(\varsigma,\eta)|^p \\ &+ M_1\int_{\varrho_0}^{\varsigma}\bar{\tau}_2(\chi,\eta)|\Xi(\chi,\eta)|^p\Delta\chi\Big]\Delta\eta\Delta\varsigma. \end{aligned} \quad (56)$$

As an application of Corollary 1 to (56) with $\kappa(\varrho,t) = |\Xi(\varrho,t)|$, we obtain the desired inequality (52). □

6. Conclusions

In this work, by employing the Leibniz integral rule on time scales, we studied further extensions of the delay dynamic inequalities proved in [15,16] and generalized a few of those inequalities to a generic time scale. We also looked at the qualitative characteristics of various different dynamic equations' time scale solutions. Furthermore, as future work, we intend to give more generalizations of these results in other directions by using the (q, ω)-Hahn difference operator.

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