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Boundary value problem of weighted fractional derivative of a function with a respect to another function of variable order

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Abstract

This study aims to resolve weighted fractional operators of variable order in specific spaces. We establish an investigation on a boundary value problem of weighted fractional derivative of one function with respect to another variable order function. It is essential to keep in mind that the symmetry of a transformation for differential equations is connected to local solvability, which is synonymous with the existence of solutions. As a consequence, existence requirements for weighted fractional derivative of a function with respect to another function of constant order are necessary. Moreover, the stability with in Ulam–Hyers–Rassias sense is reviewed. The outcomes are derived using the Kuratowski measure of non-compactness. A model illustrates the trustworthiness of the observed results.

Mathematics Subject Classification: 26A33; 47H08; 34B15; 34A08; 37C25

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1 Introduction

The fractional calculus has gained prominence in recent decades due to the variety of applications in diverse areas of science and engineering [1, 10, 12, 19]. The Riemann–Liouville and Caputo fractional derivatives exist in the majority of commonly used fractional operators (with singular kernels). Nevertheless, there are additionally different kinds of fractional operators that help researchers in their endeavors to grasp many phenomena in the world around us, we refer to the ones in citations [6, 14–16, 18, 25]. Lately, fractional integration and derivation of variable orders has also been explored. See, for instance, [20, 29].

The solvability of differential equations represents one of the most important issues in differential equations. There are multiple techniques for analyzing the existence, such as Lie group symmetry [9, 24, 26]. Throughout this document, we use integral equivalence to confirm the existence result for the bvp ψ-wfd with variable order. Many authors have set up and studied bvps for numerous forms of fractional differential equations [2, 21].

While many other research works on the existence of solutions to fractional constant order problems have been carried, the existence of solutions to variable-order problems is infrequently mentioned in the literature, and there have been only a few research papers on the stability of solutions; we refer to [13, 22, 23, 27, 29]. As a result of investigating this intriguing special research topic, our findings are novel and notable.

The weighted fractional differential of a function with constant order operators have recently gained popularity. Refs [3, 4, 17]. In this paper, we will study the boundary value problem for ψ-wfd of variable order (Bvpwfdvo)

$$\begin{cases} \mathfrak{D}_w^{\sigma(\zeta)} h(\zeta) = f(t, h(\zeta), \mathfrak{I}_w^{\sigma(\zeta)} h(\zeta)), & \zeta \in L, \\ h(0) = h(\epsilon) = 0, \end{cases} \tag{Bvpwfdvo}$$

where $L = [0, \epsilon]$, $0 < \epsilon < \infty$, $\sigma(\zeta) : L \to (1, 2]$ is the variable order of the fractional derivative equation, $f : L \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a given function, and $\mathfrak{I}_w^{\sigma(\zeta)}$ and $\mathfrak{D}_w^{\sigma(\zeta)}$ are the left ψ-wfi and ψ-wfd, respectively, of variable order $\sigma(\zeta)$ for function $h(\zeta)$.

The ψ-wfi of variable order $\sigma(\zeta) : L \to (n-1, n]$ for a function f has the form

$$\mathfrak{I}_w^{\sigma(\zeta)} f(\zeta) = \frac{w^{-1}(\zeta)}{\Gamma(\sigma(\zeta))} \int_0^\zeta \bigl(\psi(\zeta) - \psi(s)\bigr)^{\sigma(\zeta)-1} w(s) f(s) \psi'(s)\, ds, \quad \zeta > 1. \tag{1.1}$$

The corresponding derivative in Riemann–Liouville settings is

$$\mathfrak{D}_w^{\sigma(\zeta)} f(\zeta) = \frac{w^{-1}(\zeta)}{\Gamma(n - \sigma(\zeta))} \left(\frac{\mathfrak{D}_\zeta}{\psi'(\zeta)} \right)^n \left(w(\zeta) \int_0^\zeta \bigl(\psi(\zeta) - \psi(s)\bigr)^{n-\sigma(\zeta)-1} w(s) f(s) \psi'(s)\, ds \right), \tag{1.2}$$

where the weight $w(\zeta) > 0$ is a continuous function, $w^{-1}(\zeta) = \frac{1}{w(\zeta)}$ and $\psi \in \mathcal{C}^1(L, \mathbb{R}^+)$ satisfied $\psi'(\zeta) > 0$, for all $\zeta \in L$.

2 Preliminaries

Before we begin, let us notate and make some abbreviation to avoid repetition.

K-mnc–Kuratowski measure of non-compactness; ψ-wfd–weighted fractional differential equation of function with respect to function ψ, ψ-wf–weighted fractional integral equation of function with respect to function ψ; bvp–boundary value problem; and UHRs stads for Ulam–Hyers–Rassias stable.

In this section, we begin by introducing several terms and conceptual results, which will be employed across the document.

Let $L = [1, \epsilon]$ be a compact interval and denote by $\mathcal{C}(L, \mathbb{R})$ the Banach space of continuous functions $y : L \to R$ with the usual norm

$$\|y\| = \sup\bigl\{ |y(\zeta)|, \zeta \in L \bigr\}.$$

We define the weighted Banach space

$$\mathcal{C}_w(L, \mathbb{R}) = \bigl\{ y \in \mathcal{C}(L, \mathbb{R}) / w(\zeta) y(\zeta) \in \mathcal{C}(L, \mathbb{R}) \bigr\},$$

equipped with norm

$$\|y\|_w = \sup\{|w(\zeta)y(\zeta)|, \zeta \in L\}.$$

Remark 2.1 It is worth noting that the semigroup property is satisfied for a standard ψ-wfd for constant orders, but not for the general case with variable orders $\sigma(\zeta)$, $\varrho(\zeta)$, i.e.,

$$\mathfrak{I}_w^{\sigma(\zeta)}\big(\mathfrak{I}_w^{\varrho(\zeta)}\big)f(\zeta) \neq \mathfrak{I}_w^{\sigma(\zeta)+\varrho(\zeta)}f(\zeta).$$

In what follow, for all $\delta \in [0,1]$ and $\zeta, s \in (0,\epsilon]$ with $\zeta \geq s$, we pose

$$\psi_\delta(\zeta,s) := \big(\psi(\zeta) - \psi(s)\big)^\delta.$$

Lemma 2.2 *If $\sigma \in \mathcal{C}(L,(1,2])$ and there exists a number $\delta \in [0,1]$ such that $h \in \mathcal{C}_w(L,R)$, then the fractional integral variable order $\mathfrak{I}_w^{\sigma(\zeta)}h$ exists for $\zeta \in L$.*

Proof The function $\Gamma(\sigma(\zeta))$ is continuous non-zero function on L, denoted $\Gamma^\star = \sup_{\zeta\in L}\frac{1}{\Gamma(\sigma(\zeta))}$ and $w^\star = \sup_{\zeta\in L}\frac{1}{w(\zeta)}$.

$$\begin{aligned}&\text{If } \psi(\zeta,s) < 1, \quad \text{then } \psi_{\sigma(t)-1}(\zeta,s) \leq 1,\\ &\text{If } \psi(\zeta,s) \geq 1, \quad \text{then } \psi_{\sigma(t)-1}(\zeta,s) \leq \psi_\delta(\epsilon,0)\end{aligned}$$

and

$$\psi_{\sigma(\zeta)-1}(\zeta,s) \leq \psi^\star = \sup\{1,\psi_\delta(\epsilon,0)\}.$$

Let $\zeta \in L$. From the definition (1.1), applying that the function $\psi_\delta(\cdot,0)$ is an increasing function on L for $\delta \in (0,1]$, we obtain

$$\begin{aligned}\big|\mathfrak{I}_w^{\sigma(\zeta)}h(\zeta)\big| &= \frac{w^{-1}(\zeta)}{\Gamma(\sigma(\zeta))}\int_1^\zeta \psi_{\sigma(\zeta)-1}(\zeta,s)w(s)\big|h(s)\big|\psi'(s)\,ds\\ &\leq \Gamma^\star w^\star\psi^\star\psi_\delta(\epsilon,0)\|h\|_w\int_1^\zeta \psi_{-\delta}(s,0)\psi'(s)\,ds\\ &\leq \Gamma^\star w^\star\psi^\star\frac{\psi_1(\epsilon,0)}{1-\delta}\|h\|_w < \infty,\end{aligned}$$

which confirms that the ψ-wfi of variable order for the function h ($\mathfrak{I}_w^{\sigma(\zeta)}h$) exists for any $\zeta \in L$. □

Proposition 2.3 ([17]) (1) *For $\sigma > 0$ and $\varrho > 0$, we have*

$$\big(\mathfrak{I}_w^\sigma\big(w^{-1}(\zeta)\psi_{\varrho-1}(\zeta,0)\big)\big)(\zeta) = \frac{\Gamma(\varrho)}{\Gamma(\varrho+\sigma)}w^{-1}(\zeta)\psi_{\varrho+\sigma-1}(\zeta,0). \tag{2.1}$$

(2) *For $\sigma > n$ and $\varrho > 0$, we have*

$$\big(\mathfrak{D}_w^\sigma\big(w^{-1}(\zeta)\psi_{\varrho-1}(\zeta,0)\big)\big)(\zeta) = \frac{\Gamma(\varrho)}{\Gamma(\varrho-\sigma)}w^{-1}(\zeta)\psi_{\varrho-\sigma-1}(\zeta,0). \tag{2.2}$$

Theorem 2.4 ([17]) *Let* $\sigma > 0$. *Then, we have*

$$\left(\mathfrak{D}_w^\sigma \mathfrak{I}_w^\sigma\right) f = f.$$

Theorem 2.5 ([17]) *Let* $\sigma > 0$, $n = -[-\sigma]$. *Then*

$$\left(\mathfrak{I}_w^\sigma \mathfrak{D}_w^\sigma f\right)(\zeta) = f(\zeta) - w^{-1}(\zeta) \sum_{k=1}^{n} a_k \psi_{\sigma-k}(\zeta, 0).$$

Definition 2.6 ([5, 28, 29]) Let the set $I \subset \mathbb{R}$.

• The set I is called a generalized interval if it is either an interval or a point or the empty set.

• The finite set $\mathcal{P}$ of generalized intervals is called a partition of I if each x in I lies in exactly one of the generalized intervals E in $\mathcal{P}$.

• The function $g : I \to \mathbb{R}$ is called a piecewise constant with respect to partition $\mathcal{P}$ of I if for any $E \in \mathcal{P}, g$ is constant on E.

In the following, we recall some important and necessary information about the K-mnc.

Definition 2.7 ([7])

Let $\mathcal{M}_X$ be the bounded subsets of a Banach space X. The K-mnc ϑ is a mapping $\vartheta : \mathcal{M}_X \to [0, \infty]$ initially derived from a construction as laid out in the following format

$$\vartheta(D) = \inf\left\{ \epsilon > 0 : D\ (\in \mathcal{M}_X) \subseteq \bigcup_{\iota=1}^{n} D_\iota, \operatorname{diam}(D_\iota) \leq \epsilon \right\},$$

where

$$\operatorname{diam}(D_\iota) = \sup\left\{ \|x - y\| : x, y \in D_\iota \right\}.$$

The K-mnc satisfied the following properties:

Proposition 2.8 ([7, 8]). *Let* D, D_1, D_2 *be a bounded subsets of a Banach space* X, *then*:

1. $\vartheta(D) = 0 \Longleftrightarrow D$ *is relatively compact.*
2. $\vartheta(\phi) = 0$.
3. $\vartheta(D) = \vartheta(\overline{D}) = \vartheta(\operatorname{conv} D)$.
4. $D_1 \subset D_2 \Longrightarrow \vartheta(D_1) \leq \vartheta(D_2)$.
5. $\vartheta(D_1 + D_2) \leq \vartheta(D_1) + \vartheta(D_2)$.
6. $\vartheta(\Pi D) = |\Pi| \vartheta(D)$, $\Pi \in \mathbb{R}$.
7. $\vartheta(D_1 \cup D_2) = \operatorname{Max}\{\vartheta(D_1), \vartheta(D_2)\}$.
8. $\vartheta(D_1 \cap D_2) = \operatorname{Min}\{\vartheta(D_1), \vartheta(D_2)\}$.
9. $\vartheta(D + a_0) = \vartheta(D)$ *for any* $a_0 \in X$.

Lemma 2.9 ([11]) *Let* X *be a Banach space. If* U *is a bounded and equicontinuous subset of the the space* $C(L, X)$ *of continuous functions, then*:

($\mathcal{I}_1$) *$\vartheta(U(.)) \in C(L,\mathbb{R}_+)$, means that the function $\vartheta(U(\zeta))$ is a continuous function for $\zeta \in L$, and*

$$\widehat{\vartheta}(U) = \sup_{\zeta\in L} \vartheta\big(U(\zeta)\big),$$

where $\widehat{\vartheta}(U)$ is the K-mnc on the space $C(L,X)$.

($\mathcal{I}_2$) *$\vartheta(\int_0^\epsilon x(\theta)\,d\theta : x \in U) \le \int_0^\epsilon \vartheta(U(\theta))\,d\theta$, where*

$$U(\zeta) = \big\{x(\zeta) : x \in U\big\}, \quad \zeta \in L.$$

Theorem 2.10 ([7] (*DFPT*)) *If Δ is nonempty, bounded, convex and closed subset of a Banach space X, and $\Phi : \Delta \longrightarrow \Delta$ is a continuous operator satisfying*

$$\vartheta\big(\Phi(\Lambda)\big) \le k\vartheta(\Lambda), \quad \forall \Lambda \ne \emptyset \subset \Delta, k \in [0,1),$$

i.e., Φ is k-set contractions, then Φ has at least one fixed point in Δ.

Definition 2.11 Let the function $\rho \in C(L,\mathbb{R}_+)$. The (Bvpwfdvo) is UHRs with respect to ρ if there exists a constant $c_f > 0$ such that for any $\varepsilon > 0$ and for every $z \in C(L,\mathbb{R})$ such that

$$\big|\mathfrak{D}_w^{\sigma(\zeta)} z(\zeta) - f\big(\zeta, z(\zeta), I_w^{\sigma(\zeta)} z(\zeta)\big)\big| \le \varepsilon\rho(\zeta), \quad \zeta \in L, \tag{2.3}$$

there exists a solution $h \in C(L,\mathbb{R})$ for (Bvpwfdvo) satisfying

$$\big|z(\zeta) - h(\zeta)\big| \le c_f \varepsilon\rho(\zeta), \quad \zeta \in L.$$

3 Existence solutions of (Bvpwfdvo)

Let us proceed with the following assumption:

Hypothesis 1 (H1) Let $n \in \mathbb{N}$ be such an integer and a finite point sequence $\{\zeta_j\}_{j=0}^n$ be given in such a way $0 = \zeta_0 < \zeta_j < \zeta_n = \epsilon$, $j = 1,\dots,n-1$.

Denote $L_j := (\zeta_{j-1},\zeta_j]$, $j = 1,2,\dots,n$. Then $\mathcal{P} = \bigcup_{j=1}^n L_j$ is a partition of the interval L.

For each $l = 1,2,\dots,n$, the symbol $E_l = \mathcal{C}_w(L_l,\mathbb{R})$ indicates the weighted Banach space of continuous functions $x : L_l \to \mathbb{R}$ equipped with the norm

$$\|x\|_{E_l} = \sup_{\zeta\in L_l}\big|w(\zeta)x(\zeta)\big|.$$

Let $\sigma(\zeta) : L \to (1,2]$ be a piecewise constant function with respect to $\mathcal{P}$, i.e., $\sigma(\zeta) = \sum_{l=1}^n \mathbb{1}_l(\zeta)$, where $1 < \sigma_l \le 2$ are constants and $\mathbb{1}_l$ is the indicator of the interval $L_l, l = 1,2,\dots,n$:

$$\mathbb{1}_l(\zeta) = \begin{cases} 1, & \text{for } \zeta \in L_l, \\ 0, & \text{elsewhere.} \end{cases}$$

Then, for any $\zeta \in L_l$, $l = 1,2,\dots,n$, the ψ-wfd of variable order $\sigma(\zeta)$ for function $h \in \mathcal{C}_w(L,\mathbb{R})$, defined by (1.2), could be presented as a sum of ψ-wfd constant orders $\sigma_j, j = 1,2,\dots,l$.

$$\begin{aligned}\mathfrak{D}_w^{\sigma(\zeta)}h(\zeta) &= \frac{w^{-1}(\zeta)}{\Gamma(2-\sigma(\zeta))}\left(\frac{\mathfrak{D}_\zeta}{\psi'(\zeta)}\right)^2\left(w(\zeta)\int_1^\zeta \psi_{1-\sigma(\zeta)}(\zeta,s)w(s)h(s)\psi'(s)\,ds\right)\\ &= \frac{w^{-1}(\zeta)}{\Gamma(2-\sigma(\zeta))}\left[\sum_{j=1}^{l-1}\left(\frac{\mathfrak{D}_\zeta}{\psi'(\zeta)}\right)^2\left(w(\zeta)\int_{\zeta_{j-1}}^{\zeta_j} \psi_{1-\sigma_j}(\zeta,s)w(s)h(s)\psi'(s)\,ds\right)\right.\\ &\quad\left.+\left(\frac{\mathfrak{D}_\zeta}{\psi'(\zeta)}\right)^2\left(w(\zeta)\int_{\zeta_{l-1}}^{\zeta} \psi_{1-\sigma_l}(\zeta,s)w(s)h(s)\psi'(s)\,ds\right)\right].\end{aligned}$$

Thus, the equation of the bvp of ψ-wfd of variable order can be written for any $\zeta \in L_l$, $l = 1,2,\dots,n$ in the form

$$\begin{aligned}&\frac{w^{-1}(\zeta)}{\Gamma(2-\sigma(\zeta))}\left[\sum_{j=1}^{l-1}\left(\frac{\mathfrak{D}_\zeta}{\psi'(\zeta)}\right)^2\left(w(\zeta)\int_{\zeta_{j-1}}^{\zeta_j} \psi_{1-\sigma_j}(\zeta,s)w(s)h(s)\psi'(s)\,ds\right)\right.\\ &\quad\left.+\left(\frac{\mathfrak{D}_\zeta}{\psi'(\zeta)}\right)^2\left(w(\zeta)\int_{\zeta_{l-1}}^{\zeta} \psi_{1-\sigma_l}(\zeta,s)w(s)h(s)\psi'(s)\,ds\right)\right] = f\left(t,h(\zeta),\mathfrak{I}_w^{\sigma(\zeta)}h(\zeta)\right).\end{aligned} \tag{3.1}$$

Let the function $\tilde{h} \in E_\ell$ be such that $\tilde{h}(\zeta) \equiv 0$ on $\zeta \in [1,\zeta_{\ell-1}]$ and it solves integral Equation (3.1). Then (3.1) is reduced to

$$_{\zeta_{\ell-1}}\mathfrak{D}_w^{\sigma_\ell}\tilde{h}(\zeta) = f\left(\zeta,\tilde{h}(\zeta),\,_{\zeta_{\ell-1}}\mathfrak{I}_w^{\sigma_\ell}\tilde{h}(\zeta)\right),\quad \zeta \in L_\ell.$$

Taking into account the above for any $\ell = 1,2,\dots,n$, we consider the following auxiliary bvp for ψ-wfd of constant order

$$\begin{cases} _{\zeta_{\ell-1}}\mathfrak{D}_w^{\sigma_\ell}h(\zeta) = f(\zeta,h(\zeta),\,_{\zeta_{\ell-1}}\mathfrak{I}_w^{\sigma_\ell}h(\zeta)), & \zeta \in L_\ell,\\ h(\zeta_{\ell-1}) = 0, \qquad h(\zeta_\ell) = 0. \end{cases} \tag{Bvpwfdco}$$

Lemma 3.1 *Let $\ell \in \{1,2\dots,n\}$ be a natural number, $f \in \mathcal{C}_w(L_\ell \times \mathbb{R}\times\mathbb{R},\mathbb{R})$, and there exists a number $\delta \in (0,1)$ such that $(\psi(\zeta)-\psi(1))^\delta f(\zeta) \in \mathcal{C}_w(L_\ell \times \mathbb{R}\times\mathbb{R},\mathbb{R})$.*

Then the function $h_\ell \in E_\ell$ is a solution of (Bvpwfdco) *if and only if h_ℓ solves the integral equation*

$$\begin{aligned}h(\zeta) &= -\frac{w(\zeta_\ell)\psi_{1-\sigma_\ell}(\zeta_\ell,\zeta_{\ell-1})}{w(\zeta)\psi_{1-\sigma_\ell}(\zeta,\zeta_{\ell-1})}\,_{\zeta_{\ell-1}}\mathfrak{I}_w^{\sigma_\ell}\left(f\left(\zeta,h(\zeta),\,_{\zeta_{\ell-1}}\mathfrak{I}_w^{\sigma_\ell}h(\zeta)\right)\right)_{\zeta=\zeta_\ell}\\ &\quad+\,_{\zeta_{\ell-1}}\mathfrak{I}_w^{\sigma_\ell}\left(f\left(\zeta,h(\zeta),\,_{\zeta_{\ell-1}}\mathfrak{I}_w^{\sigma_\ell}h(\zeta)\right)\right).\end{aligned} \tag{3.2}$$

Proof Let $h_\ell \in E_\ell$ be a solution of the problem (Bvpwfdco). Using the operator $_{\zeta_{\ell-1}}\mathfrak{I}_w^{\sigma_\ell}$ to both sides of the equation in the problem (Bvpwfdco), we find (see Theorem 2.5)

$$\begin{aligned}h_\ell(\zeta) &= -a_1w^{-1}(\zeta)\psi_{\sigma_\ell-1}(\zeta,\zeta_{\ell-1}) - a_2w^{-1}(\zeta)\psi_{\sigma_\ell-2}(\zeta,\zeta_{\ell-1})\\ &\quad+\,_{\zeta_{\ell-1}}\mathfrak{I}_w^{\sigma_\ell}\left(f\left(\zeta,h(\zeta),\,_{\zeta_{\ell-1}}\mathfrak{I}_w^{\sigma_\ell}h(\zeta)\right)\right),\end{aligned}$$

where a_1, a_2 are two constants.

Based on the operating environment h as well as the boundary condition $h(\zeta_{\ell-1}) = 0$, we conclude that $a_2 = 0$.

Based on the boundary condition $h(\zeta_\ell) = 0$, we obtain

$$a_1 = w(\zeta_\ell)\psi_{1-\sigma_\ell}(\zeta_\ell,\zeta_{\ell-1})\, {}_{\zeta_{\ell-1}}\Im_w^{\sigma_\ell}\big(f\big(\zeta,h(\zeta),\, {}_{\zeta_{\ell-1}}\Im_w^{\sigma_\ell}h(\zeta)\big)\big)_{\zeta=\zeta_\ell}.$$

Then, we find h_ℓ solves integral Equation (3.2).

In contrast, suppose $h_\ell \in E_\ell$ be a solution of integral Equation (3.2). In respect of the continuity $w(\zeta)\psi_\delta(\zeta,0)f(\zeta)$, we deduce that h_ℓ is the solution of problem (Bvpwfdco). □

Theorem 3.2 *Let the conditions of Lemma 3.1 be satisfied and there are constants $V, W > 0$ such that*

$$\psi_\delta(\zeta,0)\big|f(t,x_1,y_1) - f(t,x_2,y_2)\big| \le V|x_1 - x_2| + W|y_1 - y_2|,$$

where $x_i, y_i \in \mathbb{R}$, $i = 1,2$, $t \in L_\ell$, and the inequality

$$d < 1 \tag{3.3}$$

holds, where

$$d = \frac{2\psi_{\sigma_\ell-1}(\zeta_\ell,\zeta_{\ell-1})(\psi_{1-\delta}(\zeta_\ell,0) - \psi_{1-\delta}(\zeta_{\ell-1},0))}{(1-\delta)\Gamma(\sigma_\ell)}\left(V + W\frac{\psi_{\sigma_\ell}(\zeta_\ell,\zeta_{\ell-1})}{\Gamma(\sigma_\ell+1)}\right).$$

Then, the (Bvpwfdco) does have at least one solution in E_ℓ.

Proof Let $r_l = \frac{2f_w^\star\psi_{\sigma_\ell}(\zeta_\ell,\zeta_{\ell-1})}{(1-d)\Gamma(\sigma_\ell+1)}$ with $f_w^\star = \sup_{\zeta\in L_\ell}|w(\zeta)f(\zeta,0,0)|$. Consider the set

$$B_\ell = \big\{h \in E_\ell, \|h\|_{E_\ell} \le r_\ell\big\}.$$

It is clear that the set B_ℓ is a nonempty, bounded, closed convex subset of E_ℓ, $\forall \ell \in \{1,2,\dots,n\}$.

We introduce the operator $\mathcal{F}$ defined on E_ℓ by

$$\begin{aligned}\mathcal{F}h(\zeta) = {}&-\frac{w^{-1}(\zeta)\psi_{1-\sigma_\ell}(\zeta_\ell,\zeta_{\ell-1})}{\Gamma(\sigma_\ell)\psi_{1-\sigma_\ell}(\zeta,\zeta_{\ell-1})}\int_{\zeta_{\ell-1}}^{\zeta_\ell}\psi_{\sigma_\ell-1}(\zeta_\ell,s)w(s)\psi'(s)f\big(s,h(s),{}_{\zeta_{\ell-1}}\Im_w^{\sigma_\ell}h(s)\big)\,ds\\ &+\frac{w^{-1}(\zeta)}{\Gamma(\sigma_\ell)}\int_{\zeta_{\ell-1}}^{\zeta}\psi_{\sigma_\ell-1}(\zeta,s)w(s)\psi'(s)f\big(s,h(s),\,{}_{\zeta_{\ell-1}}\Im_w^{\sigma_\ell}h(s)\big)\,ds.\end{aligned} \tag{3.4}$$

Out from qualities of fractional integrals and from the continuity of function $\psi_\delta(\cdot,0)w(\cdot)f(\cdot)$, the above operator $\mathcal{F}: E_\ell \longrightarrow E_\ell$ is clearly defined.

From the definition of the operator $\mathcal{F}$ and Lemma 3.1, we perceive that the fixed points of $\mathcal{F}$ are solutions of problem (Bvpwfdco). For this reason, it suffices to verify the axioms of Theorem 2.10, it is done in four steps.

Step 1. $\mathcal{F}(B_\ell) \subseteq B_\ell$. Let $\in B_\ell$ using (H1), we have

$$\begin{aligned}&\big|w(\zeta)\mathcal{F}(h)\big|\\ &\quad\le -\frac{\psi_{1-\sigma_\ell}(\zeta_\ell,\zeta_{\ell-1})}{\Gamma(\sigma_\ell)\psi_{1-\sigma_\ell}(\zeta,\zeta_{\ell-1})}\int_{\zeta_{\ell-1}}^{\zeta_\ell}\psi_{\sigma_\ell-1}(\zeta_\ell,s)w(s)\psi'(s)\big|f\big(s,h(s),\,{}_{\zeta_{\ell-1}}\Im_w^{\sigma_\ell}h(s)\big)\big|\,ds\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(\sigma_\ell)}\int_{\zeta_{\ell-1}}^{\zeta}\psi_{\sigma_\ell-1}(\zeta,s)w(s)\psi'(s)\left|f\left(s,h(s),\,{}_{\zeta_{\ell-1}}\Im_w^{\sigma_\ell}h(s)\right)\right|ds\\
&\le\frac{2}{\Gamma(\sigma_\ell)}\int_{\zeta_{\ell-1}}^{\zeta_\ell}\psi_{\sigma_\ell-1}(\zeta_\ell,s)w(s)\psi'(s)\left|f\left(s,h(s),\,{}_{\zeta_{\ell-1}}\Im_w^{\sigma_\ell}h(s)\right)\right|ds\\
&\le\frac{2}{\Gamma(\sigma_\ell)}\int_{\zeta_{\ell-1}}^{\zeta_\ell}\psi_{\sigma_\ell-1}(\zeta_\ell,s)w(s)\psi'(s)\left|f\left(s,h(s),\,{}_{\zeta_{\ell-1}}\Im_w^{\sigma_\ell}h(s)\right)-f(s,0,0)\right|ds\\
&+\frac{2}{\Gamma(\sigma_\ell)}\int_{\zeta_{\ell-1}}^{\zeta_\ell}\psi_{\sigma_\ell-1}(\zeta_\ell,s)w(s)\psi'(s)\left|f(s,0,0)\right|ds\\
&\le\frac{2}{\Gamma(\sigma_\ell)}\int_{\zeta_{\ell-1}}^{\zeta_\ell}\psi_{\sigma_\ell-1}(\zeta_\ell,s)\psi'(s)\psi_{-\delta}(s,0)\left(V\left|w(s)h(s)\right|+W\left|w(s)\,{}_{\zeta_{\ell-1}}\Im_w^{\sigma_\ell}h(s)\right|\right)ds\\
&+\frac{2f_w^\star}{\Gamma(\sigma_\ell)}\int_{\zeta_{\ell-1}}^{\zeta_\ell}\psi_{\sigma_\ell-1}(\zeta_\ell,s)\psi'(s)\,ds\\
&\le\frac{2\psi_{\sigma_\ell-1}(\zeta_\ell,\zeta_{\ell-1})}{\Gamma(\sigma_\ell)}\left(V\|h\|_{E_\ell}+W\|{}_{\zeta_{\ell-1}}\Im_w^{\sigma_\ell}h\|_{E_\ell}\right)\int_{\zeta_{\ell-1}}^{\zeta_\ell}\psi'(s)\psi_{-\delta}(s,0)\,ds\\
&+\frac{2f_w^\star}{\Gamma(\sigma_\ell+1)}\psi_{\sigma_\ell}(\zeta_\ell,\zeta_{\ell-1})\\
&\le dr_\ell+\frac{2f_w^\star}{\Gamma(\sigma_\ell+1)}\psi_{\sigma_\ell}(\zeta_\ell,\zeta_{\ell-1})\\
&=r_\ell,
\end{aligned}
$$

which means that $\mathcal{F}(B_\ell)\subseteq B_\ell$.

Step 2. $\mathcal{F}$ is continuous.

Let $h_k\in E_\ell$, $k=1,2,\dots$ Presume the sequence $\{h_k\}_{k=1}^{\infty}$ is convergent to $h\in E_\ell$. Then for any $k=1,2,\dots$ we have

$$
\begin{aligned}
&w(\zeta)\left|\mathcal{F}h_k(\zeta)-\mathcal{F}h(\zeta)\right|\\
&\le-\frac{\psi_{1-\sigma_\ell}(\zeta_\ell,\zeta_{\ell-1})}{\Gamma(\sigma_\ell)\psi_{1-\sigma_\ell}(\zeta,\zeta_{\ell-1})}\int_{\zeta_{\ell-1}}^{\zeta_\ell}\psi_{\sigma_\ell-1}(\zeta_\ell,s)w(s)\psi'(s)\\
&\times\left|f\left(s,h_k(s),\,{}_{\zeta_{\ell-1}}\Im_w^{\sigma_\ell}h_k(s)\right)-f\left(s,h(s),\,{}_{\zeta_{\ell-1}}\Im_w^{\sigma_\ell}h(s)\right)\right|ds\\
&+\frac{1}{\Gamma(\sigma_\ell)}\int_{\zeta_{\ell-1}}^{\zeta}\psi_{\sigma_\ell-1}(\zeta,s)w(s)\psi'(s)\\
&\times\left|f\left(s,h_k(s),\,{}_{\zeta_{\ell-1}}\Im_w^{\sigma_\ell}h_k(s)\right)-f\left(s,h(s),\,{}_{\zeta_{\ell-1}}\Im_w^{\sigma_\ell}h(s)\right)\right|ds\\
&\le\frac{2\psi_{\sigma_\ell-1}(\zeta,\zeta_{\ell-1})}{\Gamma(\sigma_\ell)}\int_{\zeta_{\ell-1}}^{\zeta_\ell}w(s)\psi'(s)\left|f\left(s,h_k(s),\,{}_{\zeta_{\ell-1}}\Im_w^{\sigma_\ell}h_k(s)\right)-f\left(s,h(s),\,{}_{\zeta_{\ell-1}}\Im_w^{\sigma_\ell}h(s)\right)\right|ds\\
&\le\frac{2\psi_{\sigma_\ell-1}(\zeta,\zeta_{\ell-1})}{\Gamma(\sigma_\ell)}\\
&\times\int_{\zeta_{\ell-1}}^{\zeta_\ell}\psi_{-\delta}(s,0)w(s)\psi'(s)\left(V\left|h_k(s)-h(s)\right|+W\,{}_{\zeta_{\ell-1}}\Im_w^{\sigma_\ell}\left|h_k(s)-h(s)\right|\right)ds\\
&\le\frac{2\psi_{\sigma_\ell-1}(\zeta,\zeta_{\ell-1})}{\Gamma(\sigma_\ell)}\left(V+W\frac{\psi_{\sigma_\ell}(\zeta,\zeta_{\ell-1})}{\Gamma(\sigma_\ell+1)}\right)\int_{\zeta_{\ell-1}}^{\zeta_\ell}\psi_{-\delta}(s,0)\psi'(s)\,ds\|h_k-h\|_{E_\ell}\\
&\le\frac{2\psi_{\sigma_\ell-1}(\zeta_\ell,\zeta_{\ell-1})(\psi_{1-\delta}(\zeta_\ell,0)-\psi_{1-\delta}(\zeta_{\ell-1},0))}{(1-\delta)\Gamma(\sigma_\ell)}\left(V+W\frac{\psi_{\sigma_\ell}(\zeta,\zeta_{\ell-1})}{\Gamma(\sigma_\ell+1)}\right)\|h_k-h\|_{E_\ell}
\end{aligned}
$$

i.e., we acquire

$$\|\mathcal{F}h_k - \mathcal{F}h\|_{E_\ell} \longrightarrow 0 \quad \text{as } k \longrightarrow \infty$$

As a result, the operator $\mathcal{F}$ is continuous on E_ℓ.

Step 3. $\mathcal{F}$ is bounded and equicontinuous.

By the first step for $h \in B_\ell$, we obtain $\|\mathcal{F}h\|_{E_\ell} \le r_\ell$, which confirm that $\mathcal{F}(B_\ell)$ is bounded. Rest to prove that $\mathcal{F}(B_\ell)$ is equicontinuous. Let $\zeta_1 < \zeta_2 \in L_\ell$ and $h \in B_\ell$. Then

$$\begin{aligned}
& w(\zeta)\bigl|\mathcal{F}h(\zeta_1) - \mathcal{F}h(\zeta_2)\bigr| \\
&\quad \le \frac{\psi_{1-\sigma_\ell}(\zeta_\ell,\zeta_{\ell-1})}{\Gamma(\sigma_\ell)}\bigl(\psi_{\sigma_\ell-1}(\zeta_2,\zeta_{\ell-1}) - \psi_{\sigma_\ell-1}(\zeta_1,\zeta_{\ell-1})\bigr) \\
&\qquad \times \int_{\zeta_{\ell-1}}^{\zeta_\ell} \psi_{\sigma_\ell-1}(\zeta_\ell,s)w(s)\psi'(s)\bigl|f\bigl(s,h(s),\, {}_{\zeta_{\ell-1}}\Im_w^{\sigma_\ell}h(s)\bigr)\bigr|\,ds \\
&\qquad + \frac{1}{\Gamma(\sigma_\ell)}\int_{\zeta_{\ell-1}}^{\zeta_1}\bigl(\psi_{\sigma_\ell-1}(\zeta_2,s) - \psi_{\sigma_\ell-1}(\zeta_1,s)\bigr)w(s)\psi'(s)\bigl|f\bigl(s,h(s),\, {}_{\zeta_{\ell-1}}\Im_w^{\sigma_\ell}h(s)\bigr)\bigr|\,ds \\
&\qquad + \frac{1}{\Gamma(\sigma_\ell)}\int_{\zeta_1}^{\zeta_2}\psi_{\sigma_\ell-1}(\zeta_2,s)w(s)\psi'(s)\bigl|f\bigl(s,h(s),\, {}_{\zeta_{\ell-1}}\Im_w^{\sigma_\ell}h(s)\bigr)\bigr|\,ds \\
&\quad \le -\frac{\psi_{1-\sigma_\ell}(\zeta_\ell,\zeta_{\ell-1})}{\Gamma(\sigma_\ell)}\bigl(\psi_{\sigma_\ell-1}(\zeta_2,\zeta_{\ell-1}) - \psi_{\sigma_\ell-1}(\zeta_1,\zeta_{\ell-1})\bigr) \\
&\qquad \times \int_{\zeta_{\ell-1}}^{\zeta_\ell}\psi_{\sigma_\ell-1}(\zeta_\ell,s)\psi'(s)\psi_{-\delta}(s,0)\bigl(Vw(s)\bigl|h(s)\bigr| + Ww(s)\bigl|{}_{\zeta_{\ell-1}}\Im_w^{\sigma_\ell}h(s)\bigr|\bigr)\,ds \\
&\qquad + -\frac{f_w^\star\psi_{1-\sigma_\ell}(\zeta_\ell,\zeta_{\ell-1})}{\Gamma(\sigma_\ell)}\bigl(\psi_{\sigma_\ell-1}(\zeta_2,\zeta_{\ell-1}) - \psi_{\sigma_\ell-1}(\zeta_1,\zeta_{\ell-1})\bigr)\int_{\zeta_{\ell-1}}^{\zeta_\ell}\psi_{\sigma_\ell-1}(\zeta_\ell,s)\psi'(s)\,ds \\
&\qquad + \frac{1}{\Gamma(\sigma_\ell)}\int_{\zeta_{\ell-1}}^{\zeta_1}\psi_{\sigma_\ell-1}(\zeta_2,\zeta_1)\psi'(s)\psi_{-\delta}(s,0)\bigl(Vw(s)\bigl|h(s)\bigr| + Ww(s)\bigl|{}_{\zeta_{\ell-1}}\Im_w^{\sigma_\ell}h(s)\bigr|\bigr)\,ds \\
&\qquad + \frac{f_w^\star}{\Gamma(\sigma_\ell)}\int_{\zeta_{\ell-1}}^{\zeta_1}\psi_{\sigma_\ell-1}(\zeta_2,\zeta_1)\psi'(s)\,ds + \frac{f_w^\star}{\Gamma(\sigma_\ell)}\int_{\zeta_1}^{\zeta_2}\psi_{\sigma_\ell-1}(\zeta_2,s)\psi'(s)\,ds \\
&\qquad + \frac{1}{\Gamma(\sigma_\ell)}\int_{\zeta_1}^{\zeta_2}\psi_{\sigma_\ell-1}(\zeta_2,s)\psi'(s)\psi_{-\delta}(s,0)\bigl(Vw(s)\bigl|h(s)\bigr| + Ww(s)\bigl|{}_{\zeta_{\ell-1}}\Im_w^{\sigma_\ell}h(s)\bigr|\bigr)\,ds \\
&\quad \le \frac{\psi_{1-\sigma_\ell}(\zeta_\ell,\zeta_{\ell-1})}{\Gamma(\sigma_\ell)}\bigl(\psi_{\sigma_\ell-1}(\zeta_2,\zeta_{\ell-1}) - \psi_{\sigma_\ell-1}(\zeta_1,\zeta_{\ell-1})\bigr)\psi_{\sigma_\ell-1}(\zeta_\ell,\zeta_{\ell-1}) \\
&\qquad \times \left(V + W\frac{\psi_{\sigma_\ell}(\zeta,\zeta_{\ell-1})}{\Gamma(\sigma_\ell+1)}\right)\|h\|_{E_\ell}\int_{\zeta_{\ell-1}}^{\zeta_\ell}\psi'(s)\psi_{-\delta}(s,0)\,ds \\
&\qquad + \frac{f_w^\star\psi_1(\zeta_\ell,\zeta_{\ell-1})}{\Gamma(\sigma_\ell+1)}\bigl(\psi_{\sigma_\ell-1}(\zeta_2,\zeta_{\ell-1}) - \psi_{\sigma_\ell-1}(\zeta_1,\zeta_{\ell-1})\bigr) \\
&\qquad + \frac{\psi_{\sigma_\ell-1}(\zeta_2,\zeta_1)}{(1-\delta)\Gamma(\sigma_\ell)}\bigl(\psi_{1-\delta}(\zeta_1,0) - \psi_{1-\delta}(\zeta_{\ell-1},0)\bigr)\left(V + W\frac{\psi_{\sigma_\ell}(\zeta,\zeta_{\ell-1})}{\Gamma(\sigma_\ell+1)}\right)\|h\|_{E_\ell} \\
&\qquad + \frac{\psi_1(\zeta_1,\zeta_{\ell-1})f_w^\star}{\Gamma(\sigma_\ell+1)}\psi_{\sigma_\ell-1}(\zeta_2,\zeta_1) + \frac{f_w^\star}{\Gamma(\sigma_\ell+1)}\psi_{\sigma_\ell}(\zeta_2,\zeta_1) \\
&\qquad + \frac{\psi_{\sigma_\ell-1}(\zeta_2,\zeta_1)}{(1-\delta)\Gamma(\sigma_\ell)}\left(V + W\frac{\psi_{\sigma_\ell}(\zeta,\zeta_{\ell-1})}{\Gamma(\sigma_\ell+1)}\right)\|h\|_{E_\ell}\bigl(\psi_{1-\delta}(\zeta_2,0) - \psi_{1-\delta}(\zeta_1,0)\bigr).
\end{aligned}$$

As an outcome, we acquire

$$\begin{aligned}
&\left|w(\zeta)\mathcal{F}h(\zeta_1)-\mathcal{F}h(\zeta_2)\right|\\
&\quad\leq\left[\frac{(\psi_{1-\delta}(\zeta_\ell,0)-\psi_{1-\delta}(\zeta_{\ell-1},0))}{(1-\delta)\Gamma(\sigma_\ell)}\left(V+W\frac{\psi_{\sigma_\ell}(\zeta,\zeta_{\ell-1})}{\Gamma(\sigma_\ell+1)}\right)\|h\|_{E_\ell}+\frac{f_w^\star\psi_1(\zeta_\ell,\zeta_{\ell-1})}{\Gamma(\sigma_\ell+1)}\right]\\
&\qquad\times\left(\psi_{\sigma_\ell-1}(\zeta_2,\zeta_{\ell-1})-\psi_{\sigma_\ell-1}(\zeta_1,\zeta_{\ell-1})\right)\\
&\qquad+\left[\frac{2(\psi_{1-\delta}(\zeta_1,0)-\psi_{1-\delta}(\zeta_{\ell-1},0))}{(1-\delta)\Gamma(\sigma_\ell)}\left(V+W\frac{\psi_{\sigma_\ell}(\zeta,\zeta_{\ell-1})}{\Gamma(\sigma_\ell+1)}\right)\|h\|_{E_\ell}+\frac{\psi_1(\zeta_1,\zeta_{\ell-1})f_w^\star}{\Gamma(\sigma_\ell+1)}\right]\\
&\qquad\times\psi_{\sigma_\ell-1}(\zeta_2,\zeta_1)+\frac{f_w^\star}{\Gamma(\sigma_\ell+1)}\psi_{\sigma_\ell}(\zeta_2,\zeta_1).
\end{aligned}$$

Hence $|\mathcal{F}h(\zeta 2)-\mathcal{F}h(\zeta 2)|\longrightarrow 0$ as $|\zeta_2-\zeta_1|\longrightarrow 0$. It signifies that $\mathcal{F}(B_\ell)$ is equicontinuous.

Step 4. $\mathcal{F}$ is k-set contraction.

For $H\in B_\ell$. We denote by ϑ_w the K-mnc on E_ℓ, by utilizing Lemma 2.9 and the third step, we get

$$\vartheta_w(\mathcal{F}H)=\sup_{\zeta\in L_\ell}\vartheta\left(w(\zeta)\mathcal{F}H(\zeta)\right),$$

where $H(\zeta)=\{h(\zeta),h\in H\}$.

$$\begin{aligned}
&\vartheta\left(w(\zeta)\mathcal{F}H(\zeta)\right)\\
&\quad=\vartheta\left(w(\zeta)\mathcal{F}h(\zeta),h\in H\right)\\
&\quad\leq\vartheta\Bigg\{-\frac{\psi_{1-\sigma_\ell}(\zeta_\ell,\zeta_{\ell-1})}{\Gamma(\sigma_\ell)\psi_{1-\sigma_\ell}(\zeta,\zeta_{\ell-1})}\int_{\zeta_{\ell-1}}^{\zeta_\ell}\psi_{\sigma_\ell-1}(\zeta_\ell,s)\psi'(s)\vartheta w(s)\\
&\qquad\times f\left(s,h(s),{}_{\zeta_{\ell-1}}\Im_w^{\sigma_\ell}h(s)\right)ds+\frac{1}{\Gamma(\sigma_\ell)}\int_{\zeta_{\ell-1}}^{\zeta}\psi_{\sigma_\ell-1}(\zeta,s)\psi'(s)\vartheta w(s)\\
&\qquad\times f\left(s,h(s),{}_{\zeta_{\ell-1}}\Im_w^{\sigma_\ell}h(s)\right)ds,h\in H\Bigg\}\\
&\quad\leq-\frac{\psi_{1-\sigma_\ell}(\zeta_\ell,\zeta_{\ell-1})}{\Gamma(\sigma_\ell)\psi_{1-\sigma_\ell}(\zeta,\zeta_{\ell-1})}\\
&\qquad\times\int_{\zeta_{\ell-1}}^{\zeta_\ell}\psi_{\sigma_\ell-1}(\zeta_\ell,s)\psi'(s)\psi_{-\delta}(s,0)\left[V\vartheta_w(H)+W\frac{\psi_{\sigma_\ell}(\zeta,\zeta_{\ell-1})}{\Gamma(\sigma_\ell+1)}\vartheta_w(H)\right]ds\\
&\qquad+\frac{1}{\Gamma(\sigma_\ell)}\psi_{\sigma_\ell-1}(\zeta,\zeta_{\ell-1})\\
&\qquad\times\int_{\zeta_{\ell-1}}^{\zeta}\psi'(s)\psi_{-\delta}(s,0)\left[V\vartheta_w(H)+W\frac{\psi_{\sigma_\ell}(\zeta,\zeta_{\ell-1})}{\Gamma(\sigma_\ell+1)}\vartheta_w(H)\right]ds\\
&\quad\leq\frac{2[\psi_{1-\delta}(\zeta_\ell,0)-\psi_{1-\delta}(\zeta_{\ell-1},0)]}{(1-\delta)\Gamma(\sigma_\ell)\psi_{1-\sigma_\ell}(\zeta,\zeta_{\ell-1})}\left[V+W\frac{\psi_{\sigma_\ell}(\zeta,\zeta_{\ell-1})}{\Gamma(\sigma_\ell+1)}\right]\vartheta_w(H),
\end{aligned}$$

thus

$$\vartheta_w(\mathcal{F}H)\leq\frac{2[\psi_{1-\delta}(\zeta_\ell,0)-\psi_{1-\delta}(\zeta_{\ell-1},0)]}{(1-\delta)\Gamma(\sigma_\ell)\psi_{1-\sigma_\ell}(\zeta,\zeta_{\ell-1})}\left(V+W\frac{\psi_{\sigma_\ell}(\zeta,\zeta_{\ell-1})}{\Gamma(\sigma_\ell+1)}\right)\vartheta_w(H).$$

According to Inequality (3.3), $\mathcal{F}$ is a k-set contraction.

As a matter of fact, all Theorem 2.10 requirements have been met, so as side effect $\mathcal{F}$ admits a fixed point $\mathcal{F}(\tilde{h_\ell}) = h\ell$, where $\tilde{h} \in B_\ell$, which is a solution of the bvp for ψ-wfd of constant order. Since $B_\ell \subset E_\ell$, the claim of Theorem 3.2 is established. □

We are now going to demonstrate the existence of (Bvpwfdvo).

Consider the following hypothesis:

Hypothesis 2 (H2) Let $f \in C(L \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and there exists a number $\delta \in (0,1)$ such that $w(\zeta)(\psi(\zeta) - \psi(1))^\delta f(\zeta) \in C(L \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and there are constants $V, W > 0$ such that

$$\psi_\delta(\zeta,0)\left|f(\zeta,x_1,y_1) - f(\zeta,x_2,y_2)\right| \le V|x_1 - x_2| + W|y_1 - y_2|,$$

where $x_i, y_i \in \mathbb{R}$, $i = 1,2$, $\zeta \in L$.

Theorem 3.3 *Let the conditions* (H1), (H2), *and Inequality* (3.3) *be satisfied for all* $\ell \in \{1,2,\dots,n\}$. *Then the* (Bvpwfdvo) *incorporates at least one solution in* $C(L,\mathbb{R})$.

Proof For any $\ell \in \{1,2,\dots,n\}$, according to Theorem 2.10 the (Bvpwfdco) possesses at least one solution $\tilde{h}_\ell \in E_\ell$. For any $\ell \in \{1,2,\dots,n\}$, we define the function

$$h_\ell = \begin{cases} 0, & \zeta \in [0,\zeta_{\ell-1}], \\ \tilde{h}_\ell, & \zeta \in L_\ell. \end{cases}$$

Thus, the function $h_\ell \in C([0,\zeta_\ell],\mathbb{R})$ solves the integral Equation (3.2) for $\zeta \in L_\ell$, which means that $h_\ell(1) = 0$, $h_\ell(\zeta_\ell) = \widetilde{h}_\ell(\zeta_\ell) = 0$ and solves (3.2) for $\zeta \in L_\ell$, $\ell \in \{1,2,\dots,n\}$.

Then the function

$$h(\zeta) = \begin{cases} h_1(\zeta), & \zeta \in L_1, \\ h_2(t), & \zeta \in L_2, \\ \dots, & \\ h_n(\zeta), & t \in L_n = [0,\epsilon], \end{cases}$$

is a solution of the (Bvpwfdvo) in $C(L,\mathbb{R})$. □

4 Ulam–Hyers–Rassias stability of (Bvpwfdvo)

We present the underlying assertion:

Hypothesis 3 (H3) The function $\rho \in C(L,\mathbb{R}_+)$ is increasing and there exists $\lambda_\rho > 0$ such that

$${}_{\zeta_{\ell-1}}I_w^{\sigma_\ell}\rho(\zeta) \le \lambda_\rho \rho(\zeta), \quad \text{for } \zeta \in L_\ell, \ell = 1,2,\dots,n.$$

Theorem 4.1 *Let the conditions* (H1), (H2), (H3), *and Inequality* (3.3) *be satisfied. Then, the* (Bvpwfdvo) *is UHRs with respect to* ρ.

Proof Let $\varepsilon > 0$ be an arbitrary number and the function $z(\zeta)$ from $C(L,\mathbb{R})$ satisfy Inequality (2.3).

For any $\ell \in \{1,2,\dots,n\}$, we define the functions $z_1(\zeta) \equiv z(\zeta)$, $\zeta \in [0,\zeta_1]$ and for $\ell = 2,3,\dots,n$

$$z_\ell(\zeta) = \begin{cases} 0, & \zeta \in [0,\zeta_{\ell-1}], \\ z(\zeta), & \zeta \in L_\ell. \end{cases}$$

For any $\ell \in \{1,2,\dots,n\}$, according to Equality (1.2), for $\zeta \in L_\ell$, we obtain

$$\mathfrak{D}_w^{\sigma(\zeta)} z_\ell(\zeta) = \frac{w^{-1}(\zeta)}{\Gamma(n-\sigma(\zeta))}\left(\frac{\mathfrak{D}_\zeta}{\psi'(\zeta)}\right)^n \left(w(\zeta)\int_0^\zeta \left(\psi_{n-\sigma(\zeta)-1}(\zeta,s)w(s)z(s)\psi'(s)\,ds\right)\right.$$

Taking the ${}_{\zeta_{\ell-1}}I_w^{\sigma_\ell}$ of both sides of the Inequality (2.3) and applying (H3), we obtain

$$\begin{aligned}
&\left| w(\zeta)z_\ell(\zeta) - \frac{\psi_{1-\sigma_\ell}(\zeta_\ell,\zeta_{\ell-1})}{\Gamma(\sigma_\ell)\psi_{1-\sigma_\ell}(\zeta,\zeta_{\ell-1})}\int_{\zeta_{\ell-1}}^{\zeta_\ell} \psi_{\sigma_\ell-1}(\zeta_\ell,s)w(s)\psi'(s)f\left(s,z_\ell(s),{}_{\zeta_{\ell-1}}\mathfrak{I}_w^{\sigma_\ell} z_\ell(s)\right)ds \right. \\
&\quad \left. + \frac{1}{\Gamma(\sigma_\ell)}\int_{\zeta_{\ell-1}}^{\zeta} \psi_{\sigma_\ell-1}(\zeta,s)w(s)\psi'(s)f\left(s,z_\ell(s),{}_{\zeta_{\ell-1}}\mathfrak{I}_w^{\sigma_\ell} z_\ell(s)\right)ds \right| \\
&\le \varepsilon\, {}_{\zeta_{\ell-1}}I_w^{\sigma_\ell}\rho(\zeta) \le \varepsilon\lambda_\rho\rho(\zeta).
\end{aligned}$$

According to Theorem 3.3, the (Bvpwfdvo) has a solution $h \in C(L,\mathbb{R})$ defined by $h(\zeta) = h_\ell(\zeta)$ for $\zeta \in L_\ell$, $\ell = 1,2,\dots,n$, where

$$h_\ell = \begin{cases} 0, & \zeta \in [0,\zeta_{\ell-1}], \\ \tilde{h}_\ell, & \zeta \in L_\ell, \end{cases}$$

and $\tilde{h}_\ell \in E_\ell$ is a solution of (Bvpwfdco). According to Lemma 3.1, the integral equation

$$\begin{aligned}
\tilde{h}_\ell(\zeta) &= -\frac{w(\zeta_\ell)\psi_{1-\sigma_\ell}(\zeta_\ell,\zeta_{\ell-1})}{w(\zeta)\psi_{1-\sigma_\ell}(\zeta,\zeta_{\ell-1})}\, {}_{\zeta_{\ell-1}}\mathfrak{I}_w^{\sigma_\ell}\left(f\left(\zeta,\tilde{h}_\ell(\zeta),{}_{\zeta_{\ell-1}}\mathfrak{I}_w^{\sigma_\ell}\tilde{h}_\ell(\zeta)\right)\right)_{\zeta=\zeta_\ell} \\
&\quad + {}_{\zeta_{\ell-1}}\mathfrak{I}_w^{\sigma_\ell}\left(f\left(\zeta,\tilde{h}_\ell(\zeta),{}_{\zeta_{\ell-1}}\mathfrak{I}_w^{\sigma_\ell}\tilde{h}_\ell(\zeta)\right)\right),
\end{aligned}$$

holds. Let $\zeta \in L_\ell$, where $\ell \in \{1,2,\dots,n\}$. Then by Equations (3.3) and (3.4), we obtain

$$\begin{aligned}
&w(\zeta)\left|z(\zeta)-h(\zeta)\right| \\
&\quad = w(\zeta)\left|z(\zeta)-h_\ell(\zeta)\right| = w(\zeta)\left|z_\ell(\zeta)-\tilde{h}_\ell(\zeta)\right| \\
&\quad \le \left| w(\zeta)z_\ell(\zeta) + \frac{\psi_{1-\sigma_\ell}(\zeta_\ell,\zeta_{\ell-1})}{\Gamma(\sigma_\ell)\psi_{1-\sigma_\ell}(\zeta,\zeta_{\ell-1})}\right. \\
&\qquad \times \int_{\zeta_{\ell-1}}^{\zeta_\ell} \psi_{\sigma_\ell-1}(\zeta_\ell,s)w(s)\psi'(s)f\left(s,z_\ell(s),{}_{\zeta_{\ell-1}}\mathfrak{I}_w^{\sigma_\ell} z_\ell(s)\right)ds \\
&\qquad \left. - \frac{1}{\Gamma(\sigma_\ell)}\int_{\zeta_{\ell-1}}^{\zeta} \psi_{\sigma_\ell-1}(\zeta,s)w(s)\psi'(s)f\left(s,z_\ell(s),{}_{\zeta_{\ell-1}}\mathfrak{I}_w^{\sigma_\ell} z_\ell(s)\right)ds\right| \frac{\psi_{1-\sigma_\ell}(\zeta_\ell,\zeta_{\ell-1})}{\Gamma(\sigma_\ell)\psi_{1-\sigma_\ell}(\zeta,\zeta_{\ell-1})} \\
&\qquad \times \int_{\zeta_{\ell-1}}^{\zeta_\ell} \psi_{\sigma_\ell-1}(\zeta_\ell,s)w(s)\psi'(s)\left|f\left(s,z_\ell(s),{}_{\zeta_{\ell-1}}\mathfrak{I}_w^{\sigma_\ell} z_\ell(s)\right) - f\left(s,\tilde{h}_\ell(s),{}_{\zeta_{\ell-1}}\mathfrak{I}_w^{\sigma_\ell}\tilde{h}_\ell(s)\right)\right| ds \\
&\qquad + \frac{1}{\Gamma(\sigma_\ell)}\int_{\zeta_{\ell-1}}^{\zeta} \psi_{\sigma_\ell-1}(\zeta,s)w(s)\psi'(s)\left|f\left(s,z_\ell(s),{}_{\zeta_{\ell-1}}\mathfrak{I}_w^{\sigma_\ell} z_\ell(s)\right)\right.
\end{aligned}$$

$$
\begin{aligned}
&-f\big(s,\tilde{h}_\ell(s), {}_{\zeta_{\ell-1}}\mathfrak{I}_w^{\sigma_\ell}\tilde{h}_\ell(s)\big)\big|\,ds \\
&\le \varepsilon\lambda_\rho\rho(\zeta) + \frac{\psi_{1-\sigma_\ell}(\zeta_\ell,\zeta_{\ell-1})}{\Gamma(\sigma_\ell)\psi_{1-\sigma_\ell}(\zeta,\zeta_{\ell-1})} \\
&\quad\times \int_{\zeta_{\ell-1}}^{\zeta_\ell}\psi_{\sigma_\ell-1}(\zeta_\ell,s)\psi_{-\delta}(s,0)\psi'(s)\big(Vw(s)\big|z_\ell(s)-\tilde{h}_\ell(s)\big| \\
&\quad+ Ww(s)\,{}_{\zeta_{\ell-1}}\mathfrak{I}_w^{\sigma_\ell}\big|z_\ell(s)-\tilde{h}_\ell(s)\big|\big)\,ds \\
&\quad+ \frac{1}{\Gamma(\sigma_\ell)}\int_{\zeta_{\ell-1}}^{\zeta}\psi_{\sigma_\ell-1}(\zeta,s)\psi_{-\delta}(s,0)\psi'(s)\big(Vw(s)\big|z_\ell(s)-\tilde{h}_\ell(s)\big| \\
&\quad+ Ww(s)\,{}_{\zeta_{\ell-1}}\mathfrak{I}_w^{\sigma_\ell}\big|z_\ell(s)-\tilde{h}_\ell(s)\big|\big)\,ds \\
&\le \varepsilon\lambda_\rho\rho(\zeta) + \frac{\psi_{\sigma_\ell-1}(\zeta,\zeta_{\ell-1})}{\Gamma(\sigma_\ell)} \\
&\quad\times \int_{\zeta_{\ell-1}}^{\zeta_\ell}\psi_{-\delta}(s,0)\psi'(s)\big(Vw(s)\big|z_\ell(s)-\tilde{h}_\ell(s)\big| + Ww(s)\,{}_{\zeta_{\ell-1}}\mathfrak{I}_w^{\sigma_\ell}\big|z_\ell(s)-\tilde{h}_\ell(s)\big|\big)\,ds \\
&\quad+ \frac{\psi_{\sigma_\ell-1}(\zeta,\zeta_{\ell-1})}{\Gamma(\sigma_\ell)}\int_{\zeta_{\ell-1}}^{\zeta}\psi_{-\delta}(s,0)\psi'(s)\big(Vw(s)\big|z_\ell(s)-\tilde{h}_\ell(s)\big| \\
&\quad+ Ww(s)\,{}_{\zeta_{\ell-1}}\mathfrak{I}_w^{\sigma_\ell}\big|z_\ell(s)-\tilde{h}_\ell(s)\big|\big)\,ds \\
&\le \varepsilon\lambda_\rho\rho(\zeta) + \frac{(\psi_{1-\delta}(\zeta_\ell,0)-\psi_{1-\delta}(\zeta_{\ell-1},0))\psi_{\sigma_\ell-1}(\zeta,\zeta_{\ell-1})}{(1-\delta)\Gamma(\sigma_\ell)} \\
&\quad\times\left(V\big\|z_\ell(s)-\tilde{h}_\ell(s)\big\|_{E_\ell} + W\frac{\psi_{\sigma_\ell}(\zeta,\zeta_{\ell-1})}{\Gamma(\sigma_\ell+1)}\|z_\ell-\tilde{h}_\ell\|_{E_\ell}\right) \\
&\quad+ \frac{(\psi_{1-\delta}(\zeta,0)-\psi_{1-\delta}(\zeta_{\ell-1},0))\psi_{\sigma_\ell-1}(\zeta,\zeta_{\ell-1})}{(1-\delta)\Gamma(\sigma_\ell)} \\
&\quad\times\left(V\big\|z_\ell(s)-\tilde{h}_\ell(s)\big\|_{E_\ell} + W\frac{\psi_{\sigma_\ell}(\zeta,\zeta_{\ell-1})}{\Gamma(\sigma_\ell+1)}\|z_\ell-\tilde{h}_\ell\|_{E_\ell}\right) \\
&\le \varepsilon\lambda_\rho\rho(\zeta) + d\|z-h\|_w.
\end{aligned}
$$

Then,

$$
\|z-h\|_w(1-d) \le \varepsilon\lambda_\rho\rho(\zeta),
$$

which implies that for any $\zeta\in L$, we have

$$
\big|z(\zeta)-h(\zeta)\big| \le \|z-h\|_w \le \frac{\varepsilon\lambda_\rho}{(1-d)}\rho(\zeta).
$$

Then the (Bvpwfdvo) is UHRs. □

5 Example

Let $L := [0,2]$, $\eta = 0$, $\eta_1 = 1$, $\eta_2 = 2$. Consider the scalar (Bvpwfdvo)

$$
\begin{cases}
\mathfrak{D}_{0^+}^{\sigma(\zeta)}h(\zeta) = \frac{3}{17}\psi_{\sigma(\zeta)}(\zeta,0) + \psi_{-\frac{1}{5}}(\zeta,0)\frac{h(\zeta)}{\zeta+7} + \frac{\psi(\zeta,0)}{\zeta^3+2}\mathfrak{I}_{0^+}^{\sigma(\zeta)}h(\zeta), \quad t\in L, \\
h(0)=0, \qquad h(2)=0,
\end{cases}
\tag{5.1}
$$

where $w(\zeta) = 1 + t^2$, $\psi(\zeta) = -\arctan\frac{1}{1+\zeta}$, this implies that $\psi'(\zeta) = \frac{1}{1+(1+\zeta)^2}$ and

$$\sigma(\zeta) = \begin{cases} 1.4, & \zeta \in L_1 := [0,1], \\ 1.8, & \zeta \in L_1 :=]1,2]. \end{cases} \tag{5.2}$$

Denote

$$f(\zeta,h,z) = \frac{3}{17}\psi_{\sigma(\zeta)}(\zeta,0) + \psi_{-\frac{1}{5}}(\zeta,0)\frac{h}{\zeta+7} + \frac{\psi(\zeta,0)}{\zeta^3+2})z, \quad (\zeta,h,z) \in [0,2]\times\mathbb{R}\times\mathbb{R}.$$

For $\delta = \frac{1}{5}$, $V = \frac{1}{7}$, and $W = \frac{1}{2}$, the assumption (H2) holds. Indeed,

$$\begin{aligned} \bigl|f(\eta,h_1,z_1) - f(\eta,h_2,z_2)\bigr| &= \left|\frac{h_1}{\zeta+7} + \frac{\psi_{\frac{6}{5}}(\zeta,0)}{\zeta^3+2}z_1 - \frac{h_2}{\zeta+7} - \frac{\psi_{\frac{6}{5}}(\zeta,0)}{\zeta^3+2}z_2\right| \\ &\le \frac{1}{\zeta+7}|h_1-h_2| + \frac{\psi_{\frac{6}{5}}(\zeta,0)}{\zeta^3+2}|z_1-z_2| \\ &\le \frac{1}{7}|h_1-h_2| + \frac{1}{2}|z_1-z_2|. \end{aligned}$$

By (5.2), according to (Bvpwfdco), we consider two auxiliary bvps of ψ-wfd of constant order

$$\begin{cases} \mathfrak{D}_{0^+}^{1.4}h(\zeta) = \frac{3}{17}\psi_{1.4}(\zeta,0) + \psi_{-\frac{1}{5}}(\zeta,0)\frac{h(\zeta)}{\zeta+7} + \frac{\psi(\zeta,0)}{\zeta^3+2}\mathfrak{I}_{0^+}^{1.4}h(\zeta), & t \in L_1, \\ h(1) = 0, \qquad h(2) = 0, \end{cases} \tag{5.3}$$

and

$$\begin{cases} \mathfrak{D}_{0^+}^{1.8}h(\zeta) = \frac{3}{17}\psi_{1.8}(\zeta,0) + \psi_{-\frac{1}{5}}(\zeta,0)\frac{h(\zeta)}{\zeta+7} + \frac{\psi(\zeta,0)}{\zeta^3+2}\mathfrak{I}_{0^+}^{1.8}h(\zeta), & t \in L_2, \\ h(1) = 0, \qquad h(2) = 0, \end{cases} \tag{5.4}$$

Secondly, we demonstrate that the requirement (3.3) is satisfied for $\ell = 1$. Consequently,

$$\begin{aligned} &\frac{2\psi_{\sigma_1-1}(\zeta_1,\zeta_0)(\psi_{1-\delta}(\zeta_1,0) - \psi_{1-\delta}(\zeta_0,0))}{(1-\delta)\Gamma(\sigma_1)}\left(V + W\frac{\psi_{\sigma_1}(\zeta_1,\zeta_0)}{\Gamma(\sigma_1+1)}\right) \\ &\quad \simeq 0.162691784641 < 1. \end{aligned}$$

Let $\rho(\zeta) = \psi_{\frac{3}{5}}(\zeta,0)$. Then we attain

$$\begin{aligned} \mathfrak{I}_{0^+}^{1.4}\rho(\zeta) &= \frac{1}{(\zeta^2+1)\Gamma(1.4)}\int_0^\zeta \psi_{0.4}(\zeta,s)\bigl(\zeta^2+1\bigr)\psi_{\frac{3}{5}}(s,0)\psi'(s)\,ds \\ &= \frac{5\psi_{\frac{3}{5}}(t,0)}{\Gamma(1.4)}\int_0^\zeta \psi'(s)\psi_{0.4}(\zeta,s)\,ds \\ &\le \frac{1.03}{\Gamma(2.4)}\psi_{\frac{3}{5}}(t,0) = \lambda_\rho\rho(\zeta), \end{aligned}$$

where $\lambda_\rho = \frac{1.03}{\Gamma(2.4)}$. Then, assumption (H3) is satisfied.

By Theorem 2.10, the bvp (5.3) has a solution $\tilde{h}_1 \in E_1$. We demonstrate that the Requirement (3.3) is satisfied for $\ell = 2$. Consequently,

$$\frac{2\psi_{\sigma_2-1}(\zeta_2,\zeta_1)(\psi_{1-\delta}(\zeta_2,0)-\psi_{1-\delta}(\zeta_1,0))}{(1-\delta)\Gamma(\sigma_2)}\left(V+W\frac{\psi_{\sigma_2}(\zeta_2,\zeta_1)}{\Gamma(\sigma_2+1)}\right)$$
$$\simeq 0.0117027930094 < 1.$$

As a result, the Condition (3.3) is satisfied. We also attain

$$\begin{aligned}\mathfrak{I}_{0^+}^{1.8}\rho(\zeta) &= \frac{1}{(\zeta^2+1)\Gamma(1.8)}\int_1^\zeta \psi_{0.8}(\zeta,s)\big(\zeta^2+1\big)\psi_{\frac{3}{5}}(s,0)\psi'(s)\,ds\\ &= \frac{5\psi_{\frac{3}{5}}(t,0)}{\Gamma(1.8)}\int_1^\zeta \psi'(s)\psi_{0.8}(\zeta,s)\,ds\\ &\le \frac{1.03}{\Gamma(2.8)}\psi_{\frac{3}{5}}(t,0)=\lambda_\rho\rho(\zeta),\end{aligned}$$

where $\lambda_\rho = \frac{1.26}{\Gamma(2.8)}$. Then, assumption (H3) is satisfied.

By Theorem 2.10, the bvp (5.4) has a solution $\tilde{h}_2 \in E_2$.

Hence, Theorem 3.3 provides a solution for the bvp (5.2).

$$h(\zeta)=\begin{cases}\tilde{h}_1(\zeta), & \zeta\in L_1,\\ h_2(\zeta), & \zeta\in L_2,\end{cases}$$

where

$$h_2(\zeta)=\begin{cases}0, & \zeta\in L_1,\\ \tilde{h}_2(\zeta), & \zeta\in L_2.\end{cases}$$

According to Theorem 4.1, the bvp for ψ-wfd (5.2) is UHRs with respect to ρ.

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Author contributions

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