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New insights for the fuzzy fractional partial differential equations pertaining to Katugampola generalized Hukuhara differentiability in the frame of Caputo operator and fixed point technique

Saima Rashid [a,b,*], Fahd Jarad [c,d], Hind Alamri [e]

[a] Department of Mathematics, Government College University, Faisalabad 38000, Pakistan
[b] Department of Computer Science and Mathematics, Lebanese American University, Beirut 11022801, Lebanon
[c] Department of Mathematics, Faculty of Arts and Science, Ćankaya Univ, 06790 Ankara, Türkiye
[d] Centre for Applied Mathematics and Bioinformatics, Gulf University for Science and Technology, Hawally Kuwait
[e] Department of Mathematics and Statistics, College of Science, Taif University, P.O. Box 11099, Taif 21944, Saudi Arabia

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ABSTRACT

In this article, we use the Caputo-Katugampola $\mathfrak{g}\mathbf{H}$-differentiability to solve a class of fractional PDE systems. With the aid of Caputo-Katugampola $\mathfrak{g}\mathbf{H}$-differentiability, we demonstrate the existence and uniqueness outcomes of two types of $\mathfrak{g}\mathbf{H}$-weak findings of the framework of fuzzy fractional coupled PDEs using Lipschitz assumptions and employing the Banach fixed point theorem with the mathematical induction technique. Moreover, owing to the entanglement in the initial value problems (IVPs), we establish the p Gronwall inequality of the matrix pattern and inventively explain the continuous dependence of the coupled framework's responses on the given assumptions and the ϵ-approximate solution of the coupled system. An illustrative example is provided to demonstrate that their existence and unique outcomes are accurate. Through experimentation, we demonstrate the efficacy of the suggested approach in resolving fractional differential equation algorithms under conditions of uncertainty found in engineering and physical phenomena. Additionally, comparisons are drawn for the computed outcomes. Ultimately, we make several suggestions for futuristic work.

1. Introduction

Fractional calculus has evolved into an effective instrument with more appropriate and productive findings in modeling numerous physical systems in multiple apparently multicultural and pervasive scientific disciplines [1–3]. It has received a significant amount of attention for solving fractional differential equations (DEs) and nonlinear fractional partial differential equations (FPDEs). Several more disciplines, including aeroelastic and monitoring strategies, data processing, biomedicine and health sciences. Fatoorehchi and Rach [4] described a method for inverting the Laplace transforms of two classes of rational transfer functions in control engineering. Aguiar et al. [5] expounded the fractional PID controller applied to a chemical plant with level and pH control. Recently, fractional differential equations and evolutionary algorithms have proven to be useful resources for modeling a wide range of manifestations in heat and mass transfer [6], magnetized micropolar fluid [7], nonlinear dynamics [8], mathematical modeling [9] and radiation Casson flow [10]. It has a wide range of applications in disciplines, including rheological behavior, thermal diffusivity in substances with recollection, and diffusive dynamic network approaches; see the fundamental monographs and the fascinating research for more information, see; [11–14].

Recently, Katugampola [15,16] invented an innovative notion of fractional integral/derivative, known as the Caputo-Katugampola fractional integral/derivative, that also generalizes the Riemann-Liouville (R-L) and Hadamard integral/derivative into a separate manifestation. Katugampola

* Corresponding author at: Department of Mathematics, Government College University, Faisalabad 38000, Pakistan.
E-mail addresses: saimarashid@gcuf.edu.pk (S. Rashid), fahd@cankaya.edu.tr (F. Jarad), h.amry@tu.edu.sa (H. Alamri).

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[17] investigated the system's existence and uniqueness (E-U) outcomes for fractional DEs of the C-K derivative, employing the Schauder's 2nd fixed point f_p theorem for additional results based on the novel fractional derivative. Almeida et al. [18] investigated the E-U formalism of an IVP for C-K fractional DEs, and an analytical formulation for resolving this challenge is indeed suggested. The researchers envisaged a discrete rendition of the C-K derivative and acquired a mathematical strategy for solving a linear fractional DE using the C-K fractional derivative in [19]. Hoa et al. [20] contemplated the fuzzified fractional DEs considering C-K fractional derivative scheme. Baleanu et al. [21] investigated the dynamic behavior and stabilization consequences of fractional DEs within the C-K derivative.

However, fuzzy interpretation and fuzzified DEs have been postulated to address uncertainties resulting from insufficient documentation in several computational or quantitative measurements of such deterministic real-world manifestations, see [22,23]. This hypothesis has been expanded and formed, and a broadening variety of uses are discussed in [24] and the references therein. Amane et al. [25] presented learning object analysis through fuzzy C-means clustering and web mining methods. Bhadane et al. [26] proposed the integrated framework for inclusive town planning using the fuzzy analytic hierarchy method for a semi urban town. Surono et al. [27] contemplated the implementation of Takagi Sugeno Kang fuzzy with rough set theory and mini-batch gradient descent uniform regularization.

The approach of fuzzified R-L type differentiability depending on Hukuhara differentiability (HD) was introduced in [28,29] and the researchers developed the presence of certain fuzzy integral equations employing adequate structural rigidity type environments utilizing the Hausdorff estimate of non-compactness. Diverse HD or generalized HD-type methodologies and techniques were then taken into account in a variety of publications (see; [30,31]) and we will now summarize most of these findings. The researchers of [32] identified several analyses to show the E-U of solutions to fuzzy fractional DEs via fractional R-L HD, whereas the researchers of [31,33] mentioned the generalized fractional R-L and Caputo HD of fuzzy-valued mappings. Rashid et al. [34] classified the configuration to a fuzzy fractional Swift Hohenberg equation using a hybrid transform within the Caputo generalized HD operator, whereas the scholars of [29,30] formed the E-U of a response to a fuzzified fractional DE via a Caputo type-II fuzzy fractional derivative and displayed a description of the Laplace transform of type-II fuzzy valued mappings. Arqub [35] explored novel findings by reproducing the kernel technique for generating solutions of fuzzy Fredholm-Volterra integrodifferential equations. Arqub [36] introduced a new formulation of series solutions of fuzzy DEs via strongly generalized differentiability. Mazandarani and Najariyan [37] investigated the E-U of solutions to fractional PDEs having uncertainty using the notions of generalized fractional R-L and Caputo HD of fuzzy-valued mappings, and the consistency characteristics of the solutions have been provided. Researchers [38] also presented and investigated the fuzzified hyperbolic Darboux problem with Caputo fractional $\mathfrak{g}\mathbf{H}$-type derivative:

$$ {}^{c}_{\mathfrak{g}\mathbf{H}}\mathcal{D}^{\hbar}_{\Bbbk}\mathbf{f}_1(\varphi,\eta)=\mathbf{h}_1(\varphi,\eta,\mathbf{f}_1(\varphi,\eta)),\quad \forall(\varphi,\eta)\in[0,\mathbf{c}]\times[0,\mathbf{d}],\quad \Bbbk=1,2 \tag{1.1}$$

subject to initial condition (ICs) $\mathbf{f}_1(\varphi,0)=\chi_1(\varphi)$ for every $\varphi\in[0,\mathbf{c}]$ and $\mathbf{f}_1(0,\eta)=\chi_2(\eta)$ for every $\eta\in[0,\mathbf{d}]$. Furthermore, the fractional-order $\hbar=(\hbar_1,\hbar_2)\in(0,1]\times(0,1]$ of Caputo $\mathfrak{g}\mathbf{H}$-type derivative formulation ${}^{c}_{\mathfrak{g}\mathbf{H}}\mathcal{D}^{\hbar}_{\Bbbk}$. Also, the E-U outcomes of two classifications of fuzzy solutions for (1.1) are provided with the help of the Banach and Schauder f_p theorems. It is worth noting that the operator ${}^{c}_{\mathfrak{g}\mathbf{H}}\mathcal{D}^{\hbar}_{\Bbbk}$ in (1.1) and the obtained findings of [38] both assume the presence of $\mathfrak{g}\mathbf{H}$-type and $\mathbf{H}$-difference. Authors [39] used a damping methodology for analytical simulation of fuzzified FPDEs using Caputo's $\mathfrak{g}\mathbf{H}$-type derivative; in this, temporal fractional Caputo derivative for fuzzified sets in the Hukuhara context was formally established. Furthermore, researchers [40] found fuzzified traveling approximate findings in a variety of particular contexts, including fuzzified condensation equations, fuzzified Klein-Gordon models and many others.

However, the most important feature of an ecological framework is its ecosystems. Even so, immediately preceding scientists concentrated on the advancement and preservation of a specific organism and ignored the competitive pressure triggered by the presence of various organisms. If two or more variables communicate and affect each other, this is referred to as a "coupling" connection [41]. Numerous scientists have expressed an interest in such widespread and challenging issues, claiming that a dynamic network and methodology could be represented by a solitary DE, as such coupled processes have garnered a lot of consideration. Dong et al. [42] used the following formula to demonstrate the E-U of solutions for a coupled scheme of dynamical implicit FDEs:

$$\begin{cases} {}^{c}\mathcal{D}^{\delta}\varphi(\mathbf{t})=\mathbf{h}_1(\mathbf{t},\eta(\mathbf{t}),{}^{c}\mathcal{D}^{\delta}\varphi(\mathbf{t})),\ \forall\ \mathbf{t}\in[0,1],\\ {}^{c}\mathcal{D}^{\gamma}\varphi(\mathbf{t})=\mathbf{h}_2(\mathbf{t},\varphi(\mathbf{t}),{}^{c}\mathcal{D}^{\gamma}\eta(\mathbf{t})),\ \forall\ \mathbf{t}\in[0,1], \end{cases} \tag{1.2}$$

supplemented with ICs $\varphi(0)=\varphi_0$ and $\eta(0)=\eta_0$. Definitely, one acknowledges that it is also a worthwhile research topic to use fuzzified FPDE processes in relation to coupling processes, and it is exceptionally beneficial and important to prolong the commensurate approaches to investigate the coupled structures for fuzzified FPDEs.

In summary, the major contributions of this paper are as follows:

• Inspired by the work of [38,42,43] and other pioneers, in this article, we utilize a novel fuzzy fractional-derivative notion and propose the E-U consequences for IVP of the coupled framework of C-K fuzzy FPDEs of the type: $\forall\ (\varphi,\eta)\in\mho=[0,\mathbf{c}]\times[0,\mathbf{d}]$ and $\Bbbk=1,2$,

$$\begin{cases} {}^{c}_{\mathfrak{g}\mathbf{H}}\mathcal{D}^{\delta;\rho}_{\Bbbk}\mathbf{f}_1(\varphi,\eta)=\mathbf{h}_1(\varphi,\eta,\mathbf{f}_2(\varphi,\eta)),\\ {}^{c}_{\mathfrak{g}\mathbf{H}}\mathcal{D}^{\gamma;\rho}_{\Bbbk}\mathbf{f}_2(\varphi,\eta)=\mathbf{h}_2(\varphi,\eta,\mathbf{f}_1(\varphi,\eta)),\\ \mathbf{f}_1(\varphi,0)=\omega_1(\varphi),\mathbf{f}_2(\varphi,0)=\chi_1(\varphi),\ \forall\ \varphi\in[0,\mathbf{c}],\\ \mathbf{f}_2(0,\eta)=\omega_2(\eta),\mathbf{f}_2(0,\eta)=\chi_2(\eta),\ \forall\ \eta\in[0,\mathbf{d}], \end{cases} \tag{1.3}$$

where $\delta=(\delta_1,\delta_2)$ and $\gamma=(\gamma_1,\gamma_2)\in(0,1]\times(0,1]$ represents the fractional-orders and $\rho>0$. The C-K $\mathfrak{g}\mathbf{H}$-type derivative operators ${}^{c}_{\mathfrak{g}\mathbf{H}}\mathcal{D}^{\delta;\rho}_{\Bbbk}$ and ${}^{c}_{\mathfrak{g}\mathbf{H}}\mathcal{D}^{\gamma;\rho}_{\Bbbk}$ are similar in sense as (1.1). This represents a major development with fuzzified hyperbolic governing equations.

• We will establish the existence and uniqueness of two types of $\mathfrak{g}\mathbf{H}$-weak solutions to (1.3), utilizing the mathematical inductive approach and the Banach fixed point theorem.

• The existence and uniqueness theorems are shown by a concrete example and numerical modeling of the (**(ii) (b)** mentioned in Definition 2.4) $\mathfrak{g}\mathbf{H}$-weak solution for (1.3) is suggested.

• The Gronwall inequality of the vector representation is determined, and the equivalency of (1.3) using a class of dynamical systems of the Volterra integro-differential equation is demonstrated.

• Additionally, the continuous dependency on the initial information and ε-approximate solution of (1.3) are creatively produced after varying the ICs, based on the proposed Gronwall inequality of the vector type, which is generated by the factor known as coupling in (1.3).

• To the best of our knowledge, the suggested method has not been extensively investigated for solving fuzzy mathematical frameworks in the context of C-K $\mathfrak{g}\mathbf{H}$-differentiability up to this point. The findings of the experiment show that the suggested approach is not only effective but also offers advantageous insights for a variety of tasks in engineering and physical processes like viscoelasticity.

• In a nutshell, a comparison analysis with the previous findings is conducted in order to show its efficacy. Also, the graphical illustrations have been presented in closed form with fuzzy solutions.

Remark 1.1. **(i)** When $\gamma = (1,1)$, then (1.3) having ICs reduces to an IVP as follows:

$$\begin{cases} {}^{c}_{\mathfrak{g}\mathbf{H}}\mathcal{D}^{\gamma;\rho}_{\Bbbk}\mathbf{f}_1(\varphi,\eta) = \mathbf{h}_1(\varphi,\eta,\mathbf{f}_2(\varphi,\eta)),\\ \frac{\partial^2\mathbf{f}_2(\varphi,\eta)}{\partial\varphi\partial\eta} = \mathbf{h}_2(\varphi,\eta,\mathbf{f}_1(\varphi,\eta)),\\ \mathbf{f}_1(\varphi,0) = \xi_1(\varphi),\ \mathbf{f}_1(0,\eta) = \xi_2(\eta),\\ \mathbf{f}_2(\varphi,0) = \chi_1(\varphi),\ \mathbf{f}_2(0,\eta) = \chi_2(\eta), \end{cases} \tag{1.4}$$

for every $(\varphi,\eta) \in [0,\mathbf{c}]\times[0,\mathbf{d}]$, $\Bbbk = 1,2$, where δ is similar to as defined in (1.3). Also, if $\delta = (1,1)$, then (1.4) changes to the format

$$\begin{cases} \frac{\partial^2\mathbf{f}_1(\varphi,\eta)}{\partial\varphi\partial\eta} = \mathbf{h}_1(\varphi,\eta,\mathbf{f}_2(\varphi,\eta)),\\ \frac{\partial^2\mathbf{f}_2(\varphi,\eta)}{\partial\varphi\partial\eta} = \mathbf{h}_2(\varphi,\eta,\mathbf{f}_1(\varphi,\eta)),\\ \mathbf{f}_1(\varphi,0) = \xi_1(\varphi),\ \mathbf{f}_1(0,\eta) = \xi_2(\eta),\\ \mathbf{f}_2(\varphi,0) = \chi_1(\varphi),\ \mathbf{f}_2(0,\eta) = \chi_2(\eta), \end{cases} \tag{1.5}$$

for every $(\varphi,\eta) \in [0,\mathbf{c}]\times[0,\mathbf{d}]$.

(ii) Riquier [44] evaluated a problem analogous to (1.5) more than 100 years ago, to the extent that we understand. Throughout the twentieth century, several European intellectuals published several articles on related topics (see, [45,46]). Kazakov [47,48] initially described the "Generalized Cauchy problem" as a PDE issue comprised of two formulae, where even the right side varies on an arbitrary mapping that is not distinguished in this formula, and both data points and boundary requirements are stipulated on both axes. In [44], author presented implementations of the generalized Cauchy IVP.

(iii) Whilst the (1.4) and (1.5) are structurally similar to Riquier [44] and author [47], they depend heavily on $\mathfrak{g}\mathbf{H}$-form formulations. As a result, we note that (1.4) and (1.5) are novel and haven't been documented in the literature.

The rest of the article is structured as follows: We outlined several essential ideas and other preliminary information related to fuzzy-valued fractional calculus in Section 2. In Section 3, we provide a numerical illustration that employs the Banach fixed point theorem to demonstrate the E-U of two types of $\mathfrak{g}\mathbf{H}$-weak solutions for (1.3). Furthermore, by altering the ICs, (1.3) will be comparable to a novel category of dynamic C-K type fractional order-coupled Volterra integro-differential frameworks is delivered in Section 3. The outcomes show that the solutions to (1.3) are continuously dependent on the initial values, and the ε-approximate solutions to (1.3) are provided. We present our numerical results and numerical experiments in Section 4 to illustrate the robustness of the suggested method. In order to assess our method for handling fuzzy real phenomena, we include fuzzy mathematical representations in the context of engineering. Section 5 provides a conclusion as well as suggestions for further investigation.

2. Basics on fuzzy-valued fractional calculus

To deal with (1.3), we first use several concepts introduced by Long et al. [38] and Hoa et al. [20] for fractional Caputo and C-K $\mathbf{h}_2\mathcal{H}$-derivative and integrals of fuzzified valued multivariate mappings, respectively.

For the sake of brevity, assume that there are fuzzy numbers $\mathbf{F}_{\mathfrak{n}}$ spaces $\bar{\mathcal{W}}_1,\ \bar{\mathcal{W}}_2 : \mathbb{R} \mapsto [0,1]$, the functions that they are normal, fuzzy convex, upper semi-continuous and compact. Introducing υ_1-level sets of $\mathbf{F}_{\mathfrak{n}}$ $\varkappa : \mathbb{R} \mapsto [0,1]$

$$[\varkappa]^{\upsilon_1} = \begin{cases} \{\varphi \in \mathbb{R} : \varkappa(\varphi) \geq \upsilon_1\},\ if\ \upsilon_1 \in (0,1],\\ cl\{\varphi \in \mathbb{R}^{\mathfrak{n}} | \varkappa(\varphi) > 0\},\ if\ \ \upsilon_1 = 0, \end{cases}$$

where the closure of the set is denoted by cl. For each $\varkappa \in \bar{\mathcal{W}}_\imath$, $\imath = 1,2$ and $\upsilon_1 \in [0,1]$. Thus, the υ_1-level set of the $\mathbf{F}_{\mathfrak{n}}$ of $\varkappa$ is the closed bounded interval $[\varkappa^-_{\upsilon_1}, \varkappa^+_{\upsilon_1}]$, where $\varkappa^-_{\upsilon_1}$ and $\varkappa^+_{\upsilon_1}$ presents the leftmost endpoint and rightmost end point of $\varkappa$, and suppose $[\varkappa]^{\upsilon_1} = \varkappa^+_{\upsilon_1} - \varkappa^-_{\upsilon_1}$ denotes the diameter of the υ_1-level sets of $\varkappa$. For $\imath = 1,2$ the supremum metric on $\bar{\mathcal{W}}_\imath$ is classified as

$$\bar{d}_\infty(\varkappa,\lambda) = \sup_{\upsilon_1\in[0,1]} \max\left\{|\varkappa^-_{\upsilon_1} - \lambda^-_{\upsilon_1}|, |\varkappa^+_{\upsilon_1} - \lambda^+_{\upsilon_1}|\right\}, \qquad \forall \varkappa,\lambda \in \bar{\mathcal{W}}_\imath.$$

For every $\varkappa,\lambda \in \bar{\mathcal{W}}_1$, $\upsilon_1 \in [0,1]$, we have

$$[\varkappa + \lambda]^{\upsilon_1} = [\varkappa]^{\upsilon_1} + [\lambda]^{\upsilon_1} \tag{2.1}$$

and if $\varkappa \ominus \lambda$ exists. Now the $\mathbf{H}$-difference is described as

$$[\varkappa \ominus \lambda]^{\upsilon_1} = [\varkappa^-_{\upsilon_1} - \lambda^-_{\upsilon_1}, \varkappa^+_{\upsilon_1} - \lambda^+_{\upsilon_1}]. \tag{2.2}$$

Lemma 2.1. *([49]) For every $\mu,\varpi,\lambda,\mathbf{e} \in \bar{\mathcal{W}}_\imath$, $(\imath = 1,2)$, the respective features are as follows:*

(i) $\bar{d}_\infty(\mu+\varpi,\lambda+\mathbf{e})\le \bar{d}_\infty(\mu,\lambda)+\bar{d}_\infty(\varpi,\mathbf{e})$.
(ii) *If* $\mu\ominus\varpi$ *and* $\lambda\ominus\mathbf{e}$ *hold, then* $\bar{d}_\infty(\mu\ominus\varpi,\lambda\ominus\mathbf{e})\le \bar{d}_\infty(\mu,\lambda)+\bar{d}_\infty(\varpi,\mathbf{e})$.

Remark 2.1. The outcomes of Lemma 2.1 **(ii)** are predicated on the presence of $\mathbf{H}$-difference, that will be employed to demonstrate the key findings.

Definition 2.1. ([50]) Suppose there is a mapping $\lambda\in(\mho,\bar{\mathcal{W}}_1)$ defined to be $\mathfrak{g}\mathbf{H}$-type differentiable in accordance to φ at $(\varphi_0,\eta_0)\in\mho$, if $\exists$ a component $\frac{\partial\lambda(\varphi_0,\eta_0)}{\partial\varphi}\in\bar{\mathcal{W}}_1$ such that $(\varphi_0+\hbar,\eta_0)\in\mathbf{J}$ exists for every relatively small $\hbar$, $\lambda(\varphi_0+\hbar,\eta_0)\ominus_{\mathfrak{g}\mathbf{H}}\lambda(\varphi_0+\hbar,\eta_0)$ and

$$\lim_{\hbar\mapsto 0}\frac{\lambda(\varphi_0+\hbar,\eta_0)\ominus_{\mathfrak{g}\mathbf{H}}\lambda(\varphi_0+\hbar,\eta_0)}{\hbar}=\frac{\partial\lambda(\varphi_0,\eta_0)}{\partial\varphi},$$

where $\lambda\ominus_{\mathfrak{g}\mathbf{H}}\varkappa$ represents the $\mathfrak{g}\mathbf{H}$-type difference of $\lambda\in\bar{\mathcal{W}}_1$ and $\varkappa\in\mathbf{E}_1$, which has the $\mathbf{F}_\mathfrak{n}$ of form $\mathbf{f}_2$ if it exists such that

$$\lambda\ominus_{\mathfrak{g}\mathbf{H}}\varkappa=\mu \iff \begin{cases}\textbf{(a)}\ \lambda=\varkappa+\mu\ \ or\\ \textbf{(b)}\ \varkappa=\lambda+(-1)\mu.\end{cases} \tag{2.3}$$

With this, $\frac{\partial\lambda(\varphi_0,\eta_0)}{\partial\varphi}\in\bar{\mathcal{W}}_1$ is termed as the $\mathfrak{g}\mathbf{H}$ -type derivative of λ at (φ_0,η_0) with regard to φ as long as the leftmost limit holds.

Analogously, the $\mathfrak{g}\mathbf{H}$-type derivative of λ at (φ_0,η_0) with regard to η rightmost limit holds.

Remark 2.2. From Definition 2.1, the presence of the $\mathfrak{g}\mathbf{H}$-type difference is a fundamental requirement for the $\mathfrak{g}\mathbf{H}$-type derivative of $\mathbf{F}_\mathfrak{n}$ of λ having reverence to φ or η, that would confirm the idea of C-K $\mathfrak{g}\mathbf{H}$-type derivative in (1.3) and correlating outcomes described in this article.

In accordance with the research of [20], we provide the additional interpretations and terminologies for the space $\mathbb{C}(\mho,\bar{\mathcal{W}}_\iota)$ of all fuzzified-valued continuous mappings and the space $\mathcal{L}_1(\mho,\bar{\mathcal{W}}_\iota)$ of Lebesque integrable fuzzified-valued mappings on $\mho=[0,\mathbf{c}]\times[0,\mathbf{d}]$; *here* $\iota=1,2$.

Definition 2.2. Assume that $\mho=[0,\mathbf{c}]\times[0,\mathbf{d}]$, $\delta=(\delta_1,\delta_2)$, $\gamma=(\gamma_1,\gamma_2)\in(0,1]\times(0,1]$, $\mathbf{f}_1\in\mathbb{C}(\mho,\bar{\mathcal{W}}_1)\cap\mathbb{L}^1(\mho,\bar{\mathcal{W}}_1)$, $\mathbf{f}_2\in\mathbb{C}(\mho,\bar{\mathcal{W}}_2)\cap\mathbb{L}^1(\mho,\bar{\mathcal{W}}_2)$, $[\mathbf{f}_1(\varphi,\eta)]^{\upsilon_1}=[\mathbf{f}^-_{1\,\upsilon_1}(\varphi,\eta),\mathbf{f}^+_{1\upsilon_1}(\varphi,\eta)]$ and $[\mathbf{f}_2(\varphi,\eta)]^{\tau_1}=[\mathbf{f}^-_{2\,\tau_1}(\varphi,\eta),\mathbf{f}^+_{2\tau_1}(\varphi,\eta)]$. Afterwards, depending on the level set, proceed as follows:

$$\left[{}^{RL}_{F}\mathcal{I}^{\delta,\rho}_{0^+}\mathbf{f}_1(\varphi,\eta)\right]^{\upsilon_1}=\left[{}^{RL}_{F}\mathcal{I}^{\delta,\rho}_{0^+}\mathbf{f}^-_{1\upsilon_1}(\varphi,\eta),\ {}^{RL}_{F}\mathcal{I}^{\delta,\rho}_{0^+}\mathbf{f}^+_{1\upsilon_1}(\varphi,\eta)\right]$$

and

$$\left[{}^{RL}_{F}\mathcal{I}^{\gamma,\rho}_{0^+}\mathbf{f}_2(\varphi,\eta)\right]^{\upsilon_1}=\left[{}^{RL}_{F}\mathcal{I}^{\gamma,\rho}_{0^+}\mathbf{f}^-_{2\upsilon_1}(\varphi,\eta),\ {}^{RL}_{F}\mathcal{I}^{\gamma,\rho}_{0^+}\mathbf{f}^+_{2\upsilon_1}(\varphi,\eta)\right],$$

the fuzzy R-L generalized fractional integral of orders δ and γ for fuzzy-valued multivariable mappings $\mathbf{f}_1(\varphi,\eta)$ and $\mathbf{f}_2(\varphi,\eta)$, respectively, is characterized by

$${}^{RL}_{F}\mathcal{I}^{\delta,\rho}_{0^+}\mathbf{f}_1(\varphi,\eta)=\frac{\rho^{2-\delta_1-\delta_2}}{\Gamma(\delta_1)\Gamma(\delta_2)}\int_0^{\varphi}\int_0^{\eta}\mathbf{s}^{\rho-1}(\varphi^\rho-\mathbf{s}^\rho)^{\delta_1-1}\mathbf{t}^{\rho-1}(\varphi^\rho-\mathbf{t}^\rho)^{\delta_2-1}\mathbf{f}_1(\mathbf{s},\mathbf{t})d\mathbf{t}d\mathbf{s} \tag{2.4}$$

and

$${}^{RL}_{F}\mathcal{I}^{\gamma,\rho}_{0^+}\mathbf{f}_1(\varphi,\eta)=\frac{\rho^{2-\gamma_1-\gamma_2}}{\Gamma(\gamma_1)\Gamma(\gamma_2)}\int_0^{\varphi}\int_0^{\eta}\mathbf{s}^{\rho-1}(\varphi^\rho-\mathbf{s}^\rho)^{\gamma_1-1}\mathbf{t}^{\rho-1}(\varphi^\rho-\mathbf{t}^\rho)^{\gamma_2-1}\mathbf{f}_1(\mathbf{s},\mathbf{t})d\mathbf{t}d\mathbf{s}. \tag{2.5}$$

Definition 2.3. Suppose for every $\varepsilon>0$, $\exists\ \wp_1,\wp_2>0$ such that for every $(\varphi,\eta,\mathbf{f}_1)\in\mho\times\mathbb{C}(\mho,\bar{\mathcal{W}}_1)$ and $(\varphi,\eta,\mathbf{f}_2)\in\mho\times\mathbb{C}(\mho,\bar{\mathcal{W}}_2)$ having $|\varphi-\varphi_0|+|\eta-\eta_0|+\bar{d}_{1\infty}(\mathbf{f}_1,\Psi)<\wp_1$ and $|\varphi-\varphi_0|+|\eta-\eta_0|+\bar{d}_{1\infty}(\mathbf{f}_2,\Psi)<\wp_2$, $\bar{d}_\infty(\mathbf{h}_1(\varphi,\eta,\mathbf{f}_2),\mathbf{h}_1(\varphi_0,\eta_0,\Phi))<\varepsilon$ and $\bar{d}_\infty(\mathbf{h}_2(\varphi,\eta,\mathbf{f}_1),\mathbf{h}_2(\varphi_0,\eta_0,\Psi))<\varepsilon$, then the functions $\mathbf{h}_1:\mho\times\mathbb{C}(\mho,\bar{\mathcal{W}}_2)\mapsto\bar{\mathcal{W}}_1$ and $\mathbf{h}_2:\mho\times\mathbb{C}(\mho,\bar{\mathcal{W}}_1)\mapsto\bar{\mathcal{W}}_2$ termed as jointly continuous at point $(\varphi_0,\eta_0,\Phi)\in\mho\times\mathbb{C}(\mho,\bar{\mathcal{W}}_2)$ and $(\varphi_0,\eta_0,\Psi)\in\mho\times\mathbb{C}(\mho,\bar{\mathcal{W}}_1)$, respectively.

For all $(\varphi,\eta)\in\mho=[0,\mathbf{c}]\times[0,\mathbf{d}]$, suppose

$$\Psi(\varphi,\eta)=\omega_2(\eta)+[\omega_1(\varphi)\ominus\omega_1(0)], \tag{2.6}$$

$$\Phi(\varphi,\eta)=\chi_2(\eta)+[\chi_1(\varphi)\ominus\chi_1(0)], \tag{2.7}$$

where $\omega_1\in\mathbb{C}([0,\mathbf{c}],\bar{\mathcal{W}}_1)$, $\omega_1\in\mathbb{C}([0,\mathbf{c}],\bar{\mathcal{W}}_2)$, $\omega_2\in\mathbb{C}([0,\mathbf{d}],\bar{\mathcal{W}}_1)$ and $\chi_2\in\mathbb{C}([0,\mathbf{d}],\bar{\mathcal{W}}_2)$ are the provided mappings such that $\omega_2(\eta)\ominus\omega_1(0)$ and $\chi_2(\eta)\ominus\chi_1(0)$ hold, respectively. Then, we have

$$\hat{\mathbb{C}}^{\mathbf{h}_1}_{\Psi}(\mho,\bar{\mathcal{W}}_2)=\left\{\mathbf{f}_2\in\mathbb{C}(\mho,\bar{\mathcal{W}}_2):\Psi(\varphi,\eta)\ominus(-1)\,{}^{RL}_{F}\mathcal{I}^{\delta;\rho}_{0^+}\mathbf{h}_1(\varphi,\eta,\mathbf{f}_2(\varphi,\eta))\ exists\ \forall(\varphi,\eta)\in\mho\right\}, \tag{2.8}$$

$$\hat{\mathbb{C}}^{\mathbf{h}_2}_{\Phi}(\mho,\bar{\mathcal{W}}_1)=\left\{\mathbf{f}_1\in\mathbb{C}(\mho,\bar{\mathcal{W}}_1):\Phi(\varphi,\eta)\ominus(-1)\,{}^{RL}_{F}\mathcal{I}^{\gamma;\rho}_{0^+}\mathbf{h}_2(\varphi,\eta,\mathbf{f}_1(\varphi,\eta))\ exists\ \forall(\varphi,\eta)\in\mho\right\}, \tag{2.9}$$

where $\Psi(.,.)$ and $\Phi(.,.)$ are stated in (2.6) and (2.7), respectively. Also, stating $\mathbb{C}_{\mathbb{J}}(\mho,\bar{\mathcal{W}}_{m_1},\bar{\mathcal{W}}_\mathfrak{n})=\{\hbar:\mho\times\mathbb{C}(\mho,\bar{\mathcal{W}}_{m_1})\mapsto\bar{\mathcal{W}}_\mathfrak{n}|\hbar\ as\ jointly\ continuous\}$ for every $m_1,\mathfrak{n}=1,2$ $(m_1\neq\mathfrak{n})$ and for $\Bbbk,\jmath=0,1,2$ and $\iota=1,2$, $\mathbb{C}^{\Bbbk,\jmath}_{\mathfrak{g}\mathbf{H}}(\mho,\bar{\mathcal{W}}_\iota)$ by a collection of mappings $\Phi:\mho\subset\mathbb{R}^2\mapsto\bar{\mathcal{W}}_\iota$, contains partial $\mathfrak{g}\mathbf{H}$ derivatives onward to order $\Bbbk$ in regard to φ and onward to order $\jmath$ in regard to η in $\mho$. In $\mathbb{C}(\mho,\bar{\mathcal{W}}_\iota)$, assume the supremum metrics θ stated as

$$\theta(\mathbf{f}_1,\mathbf{f}_2)=\sup_{(\varphi,\eta)\in\mho}\bar{d}_\infty\big(\mathbf{f}_1(\varphi,\eta),\mathbf{f}_2(\varphi,\eta)\big), \tag{2.10}$$

and specify the weighted metric $\bar{d}_{r_1}$ for $\bar{r}=(r_1,r_2)\in[0,1]\times[0,1]$ as follows

$$\bar{d}_{r_1}(\varphi,\eta)=\sup_{(\varphi,\eta)\in\mho}\big\{\varphi^{r_1}\eta^{r_2}\bar{d}_\infty(\Phi(\varphi,\eta),\mathbf{f}_2(\varphi,\eta))\big\}. \tag{2.11}$$

Definition 2.4. Assume that $\delta=(\delta_1,\delta_2),\ \gamma=(\gamma_1,\gamma_2)\in(0,1]\times(0,1],\ \mathbf{f}_1\in\mathbb{C}^{2,2}_{\mathfrak{g}\mathbf{H}}(\mho,\bar{\mathcal{W}}_1)$ and $\mathbf{f}_2\in\mathbb{C}^{2,2}_{\mathfrak{g}\mathbf{H}}(\mho,\bar{\mathcal{W}}_2)$. Defining the C-K $\mathfrak{g}\mathbf{H}$-type derivative of order δ in regard of φ and η of the mappings $\mathbf{f}_1$ as

$$\begin{aligned}{}^{c}_{\mathfrak{g}\mathbf{H}}\mathcal{D}^{\delta;\rho}_{0^+}\mathbf{f}_1(\varphi,\eta) &= {}^{RL}_{F}\mathcal{I}^{1-\delta;\rho}_{0^+}\Big(\frac{\partial^2\mathbf{f}_1(\varphi,\eta)}{\partial\varphi\partial\eta}\Big)\\ &=\frac{\rho^{\delta_1+\delta_2}}{\Gamma(1-\delta_1)\Gamma(1-\delta_2)}\int_0^\eta(\varphi^\rho-\mathbf{s}^\rho)^{-\delta_1}(\eta^\rho-\mathbf{t}^\rho)^{-\delta_2}\frac{\partial^2\mathbf{f}_1(\mathbf{s},\mathbf{t})}{\partial\mathbf{s}\partial\mathbf{t}}d\mathbf{t}d\mathbf{s}\end{aligned} \tag{2.12}$$

as well as create the C-K $\mathfrak{g}\mathbf{H}$-type derivative of order γ in regard of φ and η of the mappings $\mathbf{f}_1$ as

$$\begin{aligned}{}^{c}_{\mathfrak{g}\mathbf{H}}\mathcal{D}^{\gamma;\rho}_{0^+}\mathbf{f}_2(\varphi,\eta) &= {}^{RL}_{F}\mathcal{I}^{1-\gamma;\rho}_{0^+}\Big(\frac{\partial^2\mathbf{f}_2(\varphi,\eta)}{\partial\varphi\partial\eta}\Big)\\ &=\frac{\rho^{\gamma_1+\gamma_2}}{\Gamma(1-\gamma_1)\Gamma(1-\gamma_2)}\int_0^\eta(\varphi^\rho-\mathbf{s}^\rho)^{-\gamma_1}(\eta^\rho-\mathbf{t}^\rho)^{-\gamma_2}\frac{\partial^2\mathbf{f}_2(\mathbf{s},\mathbf{t})}{\partial\mathbf{s}\partial\mathbf{t}}d\mathbf{t}d\mathbf{s},\end{aligned} \tag{2.13}$$

if the rightmost representations are described, where $1-\delta=(1-\delta_1,1-\delta_2),\ (1-\gamma)=(1-\gamma_1,1-\gamma_2)\in[0,1)\times[0,1)$. We identify different instances that are homologous to **(a)** and **(b)** in (2.3) and $\mathbf{f}_1\in\bar{\mathcal{W}}_1$ is termed

(i) (a) C-K $\mathfrak{g}\mathbf{H}$-differentiable of order δ in regard to φ and η, that represents ${}^{c}_{\mathfrak{g}\mathbf{H}}\mathcal{D}^{\delta;\rho}_{1}\mathbf{f}_1(\varphi,\eta)$ if $\frac{\partial^2\mathbf{f}_1}{\partial\varphi\partial\eta}(.,.)$ as a $\mathfrak{g}\mathbf{H}$-type derivative in type I (that is., $\Bbbk=1$ in (1.3)) at (φ,η).

(ii) (b) C-K $\mathfrak{g}\mathbf{H}$-differentiable of order δ in regard to φ and η, that represents ${}^{c}_{\mathfrak{g}\mathbf{H}}\mathcal{D}^{\delta;\rho}_{2}\mathbf{f}_1(\varphi,\eta)$ if $\frac{\partial^2\mathbf{f}_1}{\partial\varphi\partial\eta}(.,.)$ as a $\mathfrak{g}\mathbf{H}$-type derivative in type II (that is., $\Bbbk=1$ in (1.3)) at (φ,η).

Remark 2.3. When $\delta=\gamma=(1,1)$, in Definition 2.4, then we have

$${}^{c}_{\mathfrak{g}\mathbf{H}}\mathcal{D}^{\delta;\rho}=\frac{\partial^2\mathbf{f}_1}{\partial\varphi\partial\eta}(\varphi,\eta),\quad {}^{c}_{\mathfrak{g}\mathbf{H}}\mathcal{D}^{\gamma;\rho}=\frac{\partial^2\mathbf{f}_2}{\partial\varphi\partial\eta}(\varphi,\eta),\quad\forall\ (\varphi,\eta)\in\mho.$$

Lemma 2.2. *Assume that* $\Psi(.,.)$ *and* $\Phi(.,.)$ *are defined analogously in* (2.6) *and* (2.7), *and* $\bar{z}_\imath(\varphi,\eta)\in\mathbb{C}(\mho,\bar{\mathcal{W}}_1)$ *is continuous for* $\imath=1,2$. *Then the fuzzified mappings*

$$\bar{\mathbf{Z}}_1(\varphi,\eta)=\Psi(\varphi,\eta)+{}^{RL}_{F}\mathcal{I}^{\delta,\rho}_{0^+}z_1(\varphi,\eta) \tag{2.14}$$

and

$$\bar{\mathbf{Z}}_2(\varphi,\eta)=\Phi(\varphi,\eta)\ominus{}^{RL}_{F}\mathcal{I}^{\delta,\rho}_{0^+}z_2(\varphi,\eta) \tag{2.15}$$

*are **(a)** C-K $\mathfrak{g}\mathbf{H}$-differentiable and **(b)** C-K $\mathfrak{g}\mathbf{H}$-differentiable (if it exists), respectively. Also,*

$${}^{c}_{\mathfrak{g}\mathbf{H}}\mathcal{D}^{\delta;\rho}_{1}\bar{\mathbf{Z}}_1(\varphi,\eta)=z_1(\varphi,\eta) \tag{2.16}$$

and

$${}^{c}_{\mathfrak{g}\mathbf{H}}\mathcal{D}^{\delta;\rho}_{1}\bar{\mathbf{Z}}_2(\varphi,\eta)=-z_1(\varphi,\eta). \tag{2.17}$$

Proof. Implementing operator ${}^{c}_{\mathfrak{g}\mathbf{H}}\mathcal{D}^{\delta;\rho}_{1}$ to both sides of (2.14), focused on the ${}^{c}_{\mathfrak{g}\mathbf{H}}\mathcal{D}^{\delta;\rho}_{1}\mathbf{f}_1(\varphi,\eta)$ interpretations in the particular instance **(i)** of Definition 2.4 for $\mathbf{f}_1\in\mathbb{C}^{2,2}_{\mathfrak{g}\mathbf{H}}(\mho,\bar{\mathcal{W}}_1)$, next, according to [20] and (2.1) that

$$\begin{aligned}\big[{}^{c}_{\mathfrak{g}\mathbf{H}}\mathcal{D}^{\delta;\rho}_{1}\bar{\mathbf{Z}}_1(\varphi,\eta)\big]^{\upsilon_1} &=\Bigg[{}^{RL}_{F}\mathcal{I}^{1-\delta;\rho}_{0^+}\Bigg(\frac{\partial^2\big(\Psi(\varphi,\eta)^-_{\upsilon_1}+{}^{RL}_{F}\mathcal{I}^{\delta;\rho}_{0^+}\mathbf{z}_{1}{}^-_{\upsilon_1}(\varphi,\eta)\big)}{\partial\varphi\partial\eta}\Bigg),\\ &\qquad {}^{RL}_{F}\mathcal{I}^{1-\delta;\rho}_{0^+}\Bigg(\frac{\partial^2\big(\Psi(\varphi,\eta)^+_{\upsilon_1}+{}^{RL}_{F}\mathcal{I}^{\delta;\rho}_{0^+}\mathbf{z}_{1}{}^+_{\upsilon_1}(\varphi,\eta)\big)}{\partial\varphi\partial\eta}\Bigg)\Bigg]\\ &=\Bigg[{}^{RL}_{F}\mathcal{I}^{1-\delta;\rho}_{0^+}\Bigg(\frac{\partial^2\big({}^{RL}_{F}\mathcal{I}^{\delta;\rho}_{0^+}\mathbf{z}_1(\varphi,\eta)\big)}{\partial\varphi\partial\eta}\Bigg)\Bigg]^{\upsilon_1}\\ &=\big[\mathbf{z}_1(\varphi,\eta)\big]^{\upsilon_1}.\end{aligned} \tag{2.18}$$

Furthermore, ${}^{c}_{\mathfrak{g}\mathbf{H}}\mathcal{D}^{\delta;\rho}_{1}\bar{\mathbf{Z}}_1(\varphi,\eta)=\mathbf{z}_1(\varphi,\eta)$.

Analogously, implement operator ${}^{c}_{\mathfrak{g}\mathbf{H}}\mathcal{D}_2^{\delta;\rho}$ on both sides of (2.15). After which, predicated on Definition 2.4 particular instance **(ii)** and utilizing the idea of [20] and (2.2), we have ${}^{c}_{\mathfrak{g}\mathbf{H}}\mathcal{D}_2^{\delta;\rho}\bar{\mathbf{Z}}_1(\varphi,\eta) = -\mathbf{z}_2(\varphi,\eta)$, which is the required result. □

Lemma 2.3. *Assume that* $\Psi(.,.)$ *and* $\Phi(.,.)$ *are stated before in* (2.6) *and* (2.7)*, respectively, suppose that there be continuous mappings* $\mathbf{h}_1 \in \mathbb{C}_{\mathbb{J}}(\mho, \bar{\mathcal{W}}_2, \bar{\mathcal{W}}_1)$ *and* $\mathbf{h}_2 \in \mathbb{C}_{\mathbb{J}}(\mho, \bar{\mathcal{W}}_1, \bar{\mathcal{W}}_2)$*, and suppose the mappings are fuzzy valued* $\mathbf{f}_1 \in \mathbb{C}^{2,2}_{\mathfrak{g}\mathbf{H}}(\mho, \bar{\mathcal{W}}_1)$ *and* $\mathbf{f}_2 \in \mathbb{C}^{2,2}_{\mathfrak{g}\mathbf{H}}(\mho, \bar{\mathcal{W}}_2)$*. Then,* (1.3) *with ICs is identical to the subsequent nonlinear fractional coupled Volterra integro-differential system. For every* $(\varphi,\eta) \in \mho$,

$$\begin{cases} \mathbf{f}_1(\varphi,\eta) = \Psi(\varphi,\eta) + {}^{RL}_{F}\mathcal{I}^{\delta;\rho}_{0^+}\mathbf{h}_1(\varphi,\eta,\mathbf{f}_2(\varphi,\eta)) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad f\,or\ \Bbbk = 1 \\ \mathbf{f}_2(\varphi,\eta) = \Phi(\varphi,\eta) + {}^{RL}_{F}\mathcal{I}^{\delta;\rho}_{0^+}\mathbf{h}_2(\varphi,\eta,\mathbf{f}_1(\varphi,\eta)) \end{cases} \tag{2.19}$$

or

$$\begin{cases} \mathbf{f}_1(\varphi,\eta) = \Psi(\varphi,\eta) \ominus (-1)\, {}^{RL}_{F}\mathcal{I}^{\delta;\rho}_{0^+}\mathbf{h}_1(\varphi,\eta,\mathbf{f}_2(\varphi,\eta)) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad f\,or\ \Bbbk = 2. \\ \mathbf{f}_2(\varphi,\eta) = \Phi(\varphi,\eta) \ominus (-1)\, {}^{RL}_{F}\mathcal{I}^{\delta;\rho}_{0^+}\mathbf{h}_2(\varphi,\eta,\mathbf{f}_1(\varphi,\eta)) \end{cases} \tag{2.20}$$

Proof. Considering $\mathbf{f}_1 \in \mathbb{C}^{2,2}_{\mathfrak{g}\mathbf{H}}(\mho, \bar{\mathcal{W}}_1)$ and $\mathbf{f}_2 \in \mathbb{C}^{2,2}_{\mathfrak{g}\mathbf{H}}(\mho, \bar{\mathcal{W}}_2)$ fulfill (1.3) having ICs, then one acknowledges that the resulting adequacy verifiable evidence procedure is identical to the evidence of Lemma 4.1 in [38], and it is excluded.

When $\Bbbk = 1$ and consider there is a solution $\big(\mathbf{f}_1(\varphi,\eta), \mathbf{f}_2(\varphi,\eta)\big)^{\upsilon_1}$ of (2.19) and take $z_1(\varphi,\eta) = \mathbf{h}_1(\varphi,\eta,\mathbf{f}_2(\varphi,\eta))$. It continues to follow from (2.19) that after implementing the C-K fractional differential operator ${}^{c}_{\mathfrak{g}\mathbf{H}}\mathcal{D}_1^{\delta;\rho}$ to both sides of the first equation of (2.19), as a result of (2.16) that

$${}^{c}_{\mathfrak{g}\mathbf{H}}\mathcal{D}_1^{\delta;\rho}\mathbf{f}_1(\varphi,\eta) = z_1(\varphi,\eta),$$

which means

$${}^{c}_{\mathfrak{g}\mathbf{H}}\mathcal{D}_1^{\delta;\rho}\mathbf{f}_1(\varphi,\eta) = \mathbf{h}_1(\varphi,\eta,\mathbf{f}_2(\varphi,\eta)).$$

Besides that, $\mathbf{f}_1(\varphi,0) = \omega_1(\varphi)$, $\mathbf{f}_1(0,\eta) = \omega_2(0,\eta) = \omega_2(\eta)$ according to the first equation of (2.19). We acquire, accordingly to the following formula of (2.19),

$${}^{c}_{\mathfrak{g}\mathbf{H}}\mathcal{D}_1^{\gamma;\rho}\mathbf{f}_2(\varphi,\eta) = \mathbf{h}_2(\varphi,\eta,\mathbf{f}_1(\varphi,\eta)), \quad \mathbf{f}_2(\varphi,0) = \chi_1(\varphi), \quad \mathbf{f}_2(0,\eta) = \chi_2(\eta).$$

Therefore, $\big(\mathbf{f}_1(\varphi,\eta), \mathbf{f}_2(\varphi,\eta)\big)^{\upsilon_1}$ is the solution to (1.3) having ICs.

For $\Bbbk = 2$, we apply C-K fractional differential operator ${}^{c}_{\mathfrak{g}\mathbf{H}}\mathcal{D}_2^{\delta;\rho}$ to both sides of the first formula in (2.20). Then, from (2.20), one obtains

$${}^{c}_{\mathfrak{g}\mathbf{H}}\mathcal{D}_2^{\delta;\rho}\mathbf{f}_1(\varphi,\eta) = z_1(\varphi,\eta).$$

That is., ${}^{c}_{\mathfrak{g}\mathbf{H}}\mathcal{D}_2^{\gamma;\rho}\mathbf{f}_2(\varphi,\eta) = \mathbf{h}_1(\varphi,\eta,\mathbf{f}_2(\varphi,\eta))$. Moreover, using (2.20) we have that $\mathbf{f}_1(\varphi,0) = \omega_1(\varphi)$, $\mathbf{f}_1(0,\eta) = \omega_2(\eta)$. Furthermore, we uniformly obtained the second equation of (2.20),

$${}^{c}_{\mathfrak{g}\mathbf{H}}\mathcal{D}_2^{\gamma;\rho}\mathbf{f}_2(\varphi,\eta) = \mathbf{h}_2(\varphi,\eta,\mathbf{f}_1(\varphi,\eta)), \quad \mathbf{f}_2(\varphi,0) = \chi_1(\varphi), \quad \mathbf{f}_2(0,\eta) = \chi_2(\eta),$$

which completes the proof. □

Remark 2.4. In view of Long et al. [38] only provided adequacy, we extend the established task by proposing superiority and stipulation of analogy to (1.3) having ICs in Lemma 2.3.

For every $\tau_1, \upsilon_1 \in [0,1]$, and for each vector $\omega_1 = \begin{pmatrix} \nu \\ \lambda \end{pmatrix}$, $\omega_2 = \begin{pmatrix} \tilde{\nu} \\ \tilde{\lambda} \end{pmatrix} \in \mathbb{C}(\mho, \bar{\mathcal{W}}_1) \times (\mho, \bar{\mathcal{W}}_2)$, suppose that

$$\begin{aligned} \big\|\xi_1\big\| :&= \max\big\{\|\nu\|, \|\lambda\|\big\} \\ &= \max\big\{\theta(\mathbf{f}_2, \hat{0}), \theta(\lambda, \hat{0})\big\} \\ &= \max\Big\{\sup_{(\varphi,\eta)\in\mho,\upsilon_1\in[0,1]} \max\big\{|\nu^-_{\upsilon_1}|, |\nu^+_{\upsilon_1}|\big\}, \sup_{(\varphi,\eta)\in\mho,\upsilon_1\in[0,1]} \max\big\{|\lambda^-_{\upsilon_1}|, |\lambda^+_{\upsilon_1}|\big\}\Big\}, \end{aligned}$$

where $\hat{0}(\varphi,\eta) = \begin{cases} 1 & if \quad \varphi = \eta = 0, \\ 0 & otherwise. \end{cases}$

Therefore, utilizing the notions of [38] and [42], we observed that $\mathbb{C}(\mho, \bar{\mathcal{W}}_1) \times (\mho, \bar{\mathcal{W}}_2)$ is a Banach space. Considering

$$\mathcal{P} = \Big\{ \begin{pmatrix} \nu \\ \lambda \end{pmatrix} \in \mathbb{C}(\mho, \bar{\mathcal{W}}_1) \times (\mho, \bar{\mathcal{W}}_2) | \nu(\varphi,\eta), \lambda(\varphi,\eta) \geq 0,\ \forall (\varphi,\eta) \in \mho \Big\},$$

then $\mathcal{P}$ is normal and regenerating cone of $\mathbb{C}(\mho, \bar{\mathcal{W}}_1) \times (\mho, \bar{\mathcal{W}}_2)$. The semi-order "$\leq$" in $\mathbb{C}(\mho, \bar{\mathcal{W}}_1) \times (\mho, \bar{\mathcal{W}}_2)$ is obtained from cone $\mathcal{P}$, i.e., $\xi_1 \leq \xi_2 \iff \xi_2 - \xi_1 \in \mathcal{P}$ for $\xi_1 = \begin{pmatrix} \nu \\ \lambda \end{pmatrix}$, $\xi_2 = \begin{pmatrix} \tilde{\nu} \\ \tilde{\lambda} \end{pmatrix} \in \mathbb{C}(\mho, \bar{\mathcal{W}}_1) \times (\mho, \bar{\mathcal{W}}_2)$.

In [42,51], authors only provided the Gronwall variant of the version for a univariate mapping. We demonstrate the succeeding form of Gronwall's variant in the vector type of a bivariate mapping by Theorem 3.2 of [52] or Lemma 2.3 of [42], which is essential for acquiring our major findings.

Lemma 2.4. *Suppose that* $\mathbf{h}_1 \in \mathbb{C}_{\mathbb{J}}(\mho, \bar{\mathcal{W}}_2, \bar{\mathcal{W}}_1)$ *and* $\mathbf{h}_2 \in \mathbb{C}_{\mathbb{J}}(\mho, \bar{\mathcal{W}}_1, \bar{\mathcal{W}}_2)$ *holds Lipschitz assumptions having constants* $\mathcal{L}_1$ *and* $\mathcal{L}_2$*. That is., there exist positive real constants* $\mathcal{L}_1$ *and* $\mathcal{L}_2$ *such that,* $\forall\ \wp_1, \wp_2 \in \mathbb{C}(\mho, \bar{\mathcal{W}}_1)$ *and* $\vartheta_1, \vartheta_2 \in \mathbb{C}(\mho, \bar{\mathcal{W}}_2)$*,*

$$\begin{cases} \bar{d}_\infty\left(\mathbf{h}_1(\varphi,\eta,\vartheta_1), \mathbf{h}_1(\varphi,\eta,\vartheta_2)\right) \le \mathcal{L}_1 \bar{d}_\infty(\vartheta_1,\vartheta_2), \\ \bar{d}_\infty\left(\mathbf{h}_2(\varphi,\eta,\theta_1), \mathbf{h}_2(\varphi,\eta,\theta_2)\right) \le \mathcal{L}_2 \bar{d}_\infty(\theta_1,\theta_2). \end{cases}$$

Surmise that Gronwall variant having vector formulation

$$\mathcal{V}(\varphi,\eta) \le \mathbf{A}\mathcal{V}(\varphi,\eta) + \mathcal{G}$$

exists, where $\mathcal{V}(\varphi,\eta) = \begin{pmatrix} \mathbf{f}_{11}(\varphi,\eta) \\ \mathbf{f}_{21}(\varphi,\eta) \end{pmatrix}$*,* $\mathcal{G}(\varphi,\eta) = \begin{pmatrix} \hbar_1(\varphi,\eta) \\ \hbar_2(\varphi,\eta) \end{pmatrix} \in \mathbb{C}(\mho, \bar{\mathcal{W}}_1) \times \mathbb{C}(\mho, \bar{\mathcal{W}}_2)$*,* $\mathbf{A} = \begin{pmatrix} 0 & \mathcal{L}_1 \, {}^{RL}_{F}\mathcal{I}^{\gamma,\rho}_{0^+} \\ \mathcal{L}_1 \, {}^{RL}_{F}\mathcal{I}^{\delta,\rho}_{0^+} & 0 \end{pmatrix}$ *and* $\mathcal{L}_1 \, {}^{RL}_{F}\mathcal{I}^{\delta,\rho}_{0^+}$ *and* $\mathcal{L}_1 \, {}^{RL}_{F}\mathcal{I}^{\gamma,\rho}_{0^+}$ *denotes the fractional integrals of Caputo-Katugampola type. Furthermore, if the aforementioned requirements are true:*

(B$_1$) $\forall\ a_1, b_1, \delta, \gamma, \mathcal{L}_1, \mathcal{L}_2 \in (0,1)$,

(B$_2$) $\max\{\mathcal{L}_1, \mathcal{L}_2\} < \mathcal{M}^{-1}$, *where*

$$\mathcal{M} = \max\left\{ \frac{\rho^{-(\gamma_1+\gamma_2)}(\mathbf{t}^\rho - \mathbf{a}^\rho)^{\gamma_1}(\mathbf{s}^\rho - \mathbf{b}^\rho)^{\gamma_2}}{\Gamma(\gamma_1+1)\Gamma(\gamma_2+1)}, \frac{\rho^{-(\delta_1+\delta_2)}(\mathbf{t}^\rho - \mathbf{a}^\rho)^{\delta_1}(\mathbf{s}^\rho - \mathbf{b}^\rho)^{\delta_2}}{\Gamma(\delta_1+1)\Gamma(\delta_2+1)} \right\},$$

then $\mathcal{V}(\varphi,\eta) \le \sum_{\mathbb{k}=0}^{\infty} \mathbf{A}^{\mathbb{k}}\mathcal{G}$*, where* $\mathbf{A}^{\mathfrak{n}+1} = \mathbf{A}(\mathbf{A}^{\mathfrak{n}})$ *and* $\mathbf{A}^0 = I$*, the identity matrix.*

Proof. Introducing the function $\mathbb{T} : \mathbb{C}(\mho, \bar{\mathcal{W}}_1) \times \mathbb{C}(\mho, \bar{\mathcal{W}}_2) \mapsto \mathbb{C}(\mho, \bar{\mathcal{W}}_1) \times \mathbb{C}(\mho, \bar{\mathcal{W}}_2)$ as

$$(\mathbb{T}\mathcal{V})(\varphi,\eta) = \mathbf{A}\mathcal{V}(\varphi,\eta) + \mathcal{G}.$$

Initially, we demonstrate that $\mathbb{T}$ is a nondecreasing operator. In reality, allowing $\xi_1 = \begin{pmatrix} \mu \\ \lambda \end{pmatrix} \le \xi_2 = \begin{pmatrix} \tilde{\mu} \\ \tilde{\lambda} \end{pmatrix}$, i.e.,

$$\begin{cases} \mu(\varphi,\eta) \le \tilde{\mu}(\varphi,\eta) \\ \\ \lambda(\varphi,\eta) \le \tilde{\lambda}(\varphi,\eta) \end{cases} \quad \forall\ (\varphi,\eta) \in [0,\mathbf{c}] \times [0,\mathbf{d}],$$

then

$$\begin{aligned} \mathbb{T}\xi_2 - \mathbb{T}\xi_1 &= \mathbf{A}\xi_2 - \mathbf{A}\xi_1 \\ &= \begin{pmatrix} \mathcal{L}_1 \, {}^{RL}_{F}\mathcal{I}^{\gamma,\rho}_{0^+}(\tilde{\lambda} - \lambda) \\ \mathcal{L}_2 \, {}^{RL}_{F}\mathcal{I}^{\delta,\rho}_{0^+}(\tilde{\mu} - \mu) \end{pmatrix} \ge \begin{pmatrix} \hat{0} \\ \hat{0} \end{pmatrix}. \end{aligned}$$

Therefore, $\mathbb{T}$ is a non-decreasing operator. Further, we have to show that $\|\mathbf{A}\| < 1$. In fact, since

$$\|\xi_1\| = 1 \iff \max\left\{ \sup_{(\varphi,\eta)\in\mho, \upsilon_1\in[0,1]} \max\left\{|\mu^-_{\upsilon_1}|, |\mu^+_{\upsilon_1}|\right\}, \sup_{(\varphi,\eta)\in\mho, \upsilon_1\in[0,1]} \max\left\{|\lambda^-_{\upsilon_1}|, |\lambda^+_{\upsilon_1}|\right\} \right\} = 1,$$

Definition 2.2 is strung around it, so

$$\begin{aligned} \left\|\mathbf{A}\right\| &= \sup_{\|\xi_1\|=1} \left\|\mathbf{A}\xi_1\right\| \\ &\le \sup_{\|\xi_1\|=1} \max\left\{ \mathcal{L}_1 \, {}^{RL}_{F}\mathcal{I}^{\gamma;\rho}_{0^+}, \mathcal{L}_2 \, {}^{RL}_{F}\mathcal{I}^{\delta;\rho}_{0^+} \right\} \\ &\quad \times \left\{ \sup_{(\varphi,\eta)\in\mho, \upsilon_1\in[0,1]} \max\left\{|\mu^-_{\upsilon_1}|, |\mu^+_{\upsilon_1}|\right\}, \sup_{(\varphi,\eta)\in\mho, \upsilon_1\in[0,1]} \max\left\{|\lambda^-_{\upsilon_1}|, |\lambda^+_{\upsilon_1}|\right\} \right\} \\ &\le \max\{\mathcal{L}_1, \mathcal{L}_2\} \sup_{(\varphi,\eta)\in\mho} \max\left\{ \frac{\rho^{-(\gamma_1+\gamma_2)}(\mathbf{t}^\rho - \mathbf{a}^\rho)^{\gamma_1}(\mathbf{s}^\rho - \mathbf{b}^\rho)^{\gamma_2}}{\Gamma(\gamma_1+1)\Gamma(\gamma_2+1)}, \frac{\rho^{-(\delta_1+\delta_2)}(\mathbf{t}^\rho - \mathbf{a}^\rho)^{\delta_1}(\mathbf{s}^\rho - \mathbf{b}^\rho)^{\delta_2}}{\Gamma(\delta_1+1)\Gamma(\delta_2+1)} \right\} \\ &< 1, \end{aligned}$$

and in light of Theorem 3.2 in [52], which shows that $\mathbb{T}$ has a unique f_p as $\mathcal{V}^*$ and $\lim_{\mathfrak{n}\mapsto\infty} \mathbb{T}^{\mathfrak{n}}\mathcal{Q} = \mathcal{Q}^*$.

Now, $\mathcal{G}$ is assumed to be the initial iterative process influence, that can be acquired by performing the relevant computation:

$$\begin{aligned} &\mathcal{V}_0 = \mathcal{G}, \\ &\mathcal{V}_1 = \mathbb{T}\mathcal{V}_0 = \mathbf{A}\mathcal{G} + \mathcal{G}, \\ &\mathcal{V}_2 = \mathbb{T}\mathcal{V}_1 = \mathbf{A}^2\mathcal{G} + \mathbf{A}\mathcal{G} + \mathcal{G}, \\ &\vdots \\ &\mathcal{V}_{\mathfrak{n}} = \mathbb{T}\mathcal{V}_{\mathfrak{n}-1} = \mathbf{A}^{\mathfrak{n}}\mathcal{G} + \cdots + \mathbf{A}\mathcal{G} + \mathcal{G} = \sum_{\mathbb{k}=0}^{\mathfrak{n}} \mathbf{A}^{\mathbb{k}}\mathcal{G}, \end{aligned}$$

$$\mathcal{V}^* = \lim_{\mathfrak{n}\mapsto\infty} \mathcal{V}_{\mathfrak{n}} = \sum_{\Bbbk=0}^{\infty} \mathbf{A}^{\Bbbk}\mathcal{G}.$$

As a result, Lemma 2.3 of [42] shows that $\mathcal{V}(\varphi,\eta) \le \sum_{\Bbbk=0}^{\infty} \mathbf{A}^{\Bbbk}\mathcal{G}$. This concludes the proof. □

3. Existence and uniqueness results

Employing computational induction technique and the Banach f_p hypothesis, we show the E-U of two types of $\mathfrak{g}\mathbf{H}$-weak findings, known as **(a)**-weak result and **(b)**-weak finding, for (1.3) with ICs. An illustrative example is also provided to validate the information described in this part.

Theorem 3.1. *Suppose that $\mathbf{h}_1 \in \complement_{\mathbb{J}}(\mho, \bar{\mathcal{W}}_2, \bar{\mathcal{W}}_1)$ and $\mathbf{h}_2 \in \complement_{\mathbb{J}}(\mho, \bar{\mathcal{W}}_1, \bar{\mathcal{W}}_2)$ fulfills the Lipschitz assumptions; then (1.3) having ICs has a unique **(a)**-weak solution described on $\mho$.*

Proof. The implementation of Picard's iterative process is applied to prove the Theorem 3.1. For this purpose, we identify two functional $\mathbb{T}_1 : \complement(\mho, \bar{\mathcal{W}}_1) \mapsto \complement(\mho, \bar{\mathcal{W}}_1)$ and $\mathbb{T}_2 : \complement(\mho, \bar{\mathcal{W}}_2) \mapsto \complement(\mho, \bar{\mathcal{W}}_2)$ as follows

$$\mathbb{T}_1(\mathbf{f}_1(\varphi,\eta)) := \Psi(\varphi,\eta) + {}^{RL}_{F}\mathcal{I}^{\delta;\rho}_{0+}\mathbf{h}_1(\varphi,\eta,\Phi(\varphi,\eta)) + {}^{RL}_{F}\mathcal{I}^{\gamma;\rho}_{0+}\mathbf{h}_2(\varphi,\eta,\mathbf{f}_1(\varphi,\eta)),$$
$$\mathbb{T}_2(\mathbf{f}_2(\varphi,\eta)) := \Phi(\varphi,\eta) + {}^{RL}_{F}\mathcal{I}^{\gamma;\rho}_{0+}\mathbf{h}_2(\varphi,\eta,\Psi(\varphi,\eta)) + {}^{RL}_{F}\mathcal{I}^{\gamma;\rho}_{0+}\mathbf{h}_1(\varphi,\eta,\mathbf{f}_2(\varphi,\eta)).$$

So, $\mathbb{T}_1$ and $\mathbb{T}_2$ are involved with $\psi(\varphi,\eta)$ and $\phi(\varphi,\eta)$, respectively. We now recognize from Lemma 2.1 **(i)** that

$$\begin{aligned}
&\bar{d}_\infty\big(\mathbb{T}_1(u_1), \mathbb{T}_1(u_2)\big)\\
&\le \bar{d}_\infty\big(\Psi(\varphi,\eta),\Psi(\varphi,\eta)\big) + \bar{d}_\infty \left\{ \begin{array}{l} \big({}^{RL}_{F}\mathcal{I}^{\delta}_{0+}\mathbf{h}_1(\varphi,\eta,\Phi(\varphi,\eta)) + {}^{RL}_{F}\mathcal{I}^{\gamma}_{0+}\mathbf{h}_2(\varphi,\eta,u_1(\varphi,\eta)),\\ {}^{RL}_{F}\mathcal{I}^{\delta}_{0+}\mathbf{h}_1(\varphi,\eta,\Phi(\varphi,\eta)) + {}^{RL}_{F}\mathcal{I}^{\gamma}_{0+}\mathbf{h}_2(\varphi,\eta,u_2(\varphi,\eta))\big) \end{array}\right.\\
&\le \frac{\rho^{2-(\delta_1+\delta_2)}\mathcal{L}_1}{\Gamma(\delta_1)\Gamma(\delta_2)} \int_0^{\varphi}\int_0^{\eta} \mathbf{s}^{\rho-1}(x^\rho - \mathbf{s}^\rho)^{\delta_1-1}\mathbf{t}^{\rho-1}(y^\rho - \mathbf{t}^\rho)^{\delta_2-1}\\
&\quad\times \bar{d}_\infty\Big({}^{RL}_{F}\mathcal{I}^{\gamma}_{0+}\mathbf{h}_2(\mathbf{s},\mathbf{t},u_1(\mathbf{s},\mathbf{t})), {}^{RL}_{F}\mathcal{I}^{\gamma}_{0+}\mathbf{h}_2(\varphi,\eta,u_2(\varphi,\eta))\Big) d\mathbf{t}d\mathbf{s}
\end{aligned}$$

and according to

$$\begin{aligned}
&\bar{d}_\infty\Big({}^{RL}_{F}\mathcal{I}^{\gamma}_{0+}\mathbf{h}_2(\mathbf{s},\mathbf{t},u_1(\mathbf{s},\mathbf{t})), {}^{RL}_{F}\mathcal{I}^{\gamma}_{0+}\mathbf{h}_2(\varphi,\eta,u_2(\varphi,\eta))\Big)\\
&\le \frac{\rho^{2-(\gamma_1+\gamma_2)}\mathcal{L}_2}{\Gamma(\gamma_1)\Gamma(\gamma_2)} \int_0^{\varphi}\int_0^{\eta} \mathbf{s}^{\rho-1}(x^\rho - \mathbf{s}^\rho)^{\gamma_1-1}\mathbf{t}^{\rho-1}(y^\rho - \mathbf{t}^\rho)^{\gamma_2-1}\bar{d}_\infty\big(u_1(\psi,\phi),u_2(\psi,\phi)\big)d\phi d\psi.
\end{aligned}$$

Utilizing the fact of (2.11) that

$$\begin{aligned}
&\bar{d}_\infty(\mathbb{T}_1(u_1), \mathbb{T}_1(u_2))\\
&\le \frac{\rho^{2-(\delta_1+\delta_2)}\mathcal{L}_1}{\Gamma(\delta_1)\Gamma(\delta_2)} \frac{\rho^{2-(\gamma_1+\gamma_2)}\mathcal{L}_2}{\Gamma(\gamma_1)\Gamma(\gamma_2)} \int_0^{\varphi}\int_0^{\eta} \mathbf{s}^{\rho-1}(x^\rho - \mathbf{s}^\rho)^{\delta_1-1}\mathbf{t}^{\rho-1}(y^\rho - \mathbf{t}^\rho)^{\delta_2-1}\\
&\quad\times\bigg(\int_0^{\mathbf{s}}\int_0^{\mathbf{t}} \psi^{\rho-1}(s^\rho - \psi^\rho)^{\gamma_1-1}\phi^{\rho-1}(\mathbf{t}^\rho - \phi^\rho)^{\gamma_2-1}\bar{d}_\infty(u_1(\psi,\phi),u_2(\psi,\phi))d\phi d\psi\bigg) d\mathbf{t}d\mathbf{s}\\
&\le \frac{\rho^{4-(\delta_1+\delta_2+\gamma_1+\gamma_2)}\mathcal{L}_1\mathcal{L}_2\Gamma(\gamma_1)\Gamma(\gamma_2)}{\Gamma(2\gamma_1+\delta_1)\Gamma(2\gamma_2+\delta_2)} \varphi^{2\gamma_1+\delta_1-1}\eta^{2\gamma_2+\delta_2-1}\bar{d}_{1-\gamma}(u_1,u_2),
\end{aligned}$$

that is identical to

$$\varphi^{1-\gamma_1}\eta^{1-\gamma_2}\bar{d}_\infty(\mathbb{T}_1(u_1),\mathbb{T}_1(u_2)) \le \frac{\rho^{4-(\delta_1+\delta_2+\gamma_1+\gamma_2)}\mathcal{L}_1\mathcal{L}_2\Gamma(\gamma_1)\Gamma(\gamma_2)}{\Gamma(2\gamma_1+\delta_1)\Gamma(2\gamma_2+\delta_2)} \varphi^{\gamma_1+\delta_1}\eta^{\gamma_2+\delta_2}\bar{d}_{1-\gamma}(u_1,u_2). \tag{3.1}$$

Then, for every $\mathfrak{n} \in \mathbb{N}$, we established the functionals:

$$\mathbb{T}_1^{\mathfrak{n}}(\mathbf{f}_1(\varphi,\eta)) = \mathbb{T}_1\big(\mathbb{T}_1^{\mathfrak{n}-1}(\mathbf{f}_1(\varphi,\eta))\big), \quad \mathbb{T}_2^{\mathfrak{n}}(\mathbf{f}_2(\varphi,\eta)) = \mathbb{T}_2\big(\mathbb{T}_2^{\mathfrak{n}-1}(\mathbf{f}_2(\varphi,\eta))\big),$$

and demonstrate that the respective variant exists, employing mathematical induction:

$$\begin{aligned}
&\bar{d}_\infty(\mathbb{T}_1^{\mathfrak{n}}u_1(\varphi,\eta), \mathbb{T}_1^{\mathfrak{n}}u_2(\varphi,\eta))\\
&\le \frac{\rho^{2(2\mathfrak{n})-\mathfrak{n}(\delta_1+\delta_2+\gamma_1+\gamma_2)}\mathcal{L}_1^{\mathfrak{n}}\mathcal{L}_2^{\mathfrak{n}}\Gamma(\gamma_1)\Gamma(\gamma_2)}{\Gamma((\mathfrak{n}+1)\gamma_1+\mathfrak{n}\delta_1)\Gamma((\mathfrak{n}+1)\gamma_2+\mathfrak{n}\delta_2)} \varphi^{(\mathfrak{n}+1)\gamma_1+\mathfrak{n}\delta_1-1}\eta^{(\mathfrak{n}+1)\gamma_2+\mathfrak{n}\delta_2-1}\bar{d}_{1-\gamma}(u_1,u_2),
\end{aligned} \tag{3.2}$$

that also indicates that $\mathbb{T}_{\mathfrak{n}}$ is a contraction mapping (CM) if $\mathfrak{n}$ is large enough.

For $\mathfrak{n} = 1$, we get (3.2) from (3.1).

When $\mathfrak{n}=\Bbbk$, suppose (3.2) also satisfies, accordingly,

$$\begin{aligned}&\bar{d}_{\infty}(\mathbb{T}_1^{\Bbbk}u_1(\varphi,\eta),\mathbb{T}_1^{\Bbbk}u_2(\varphi,\eta))\\&\le\frac{\rho^{2(2\Bbbk)-\Bbbk(\delta_1+\delta_2+\gamma_1+\gamma_2)}\mathcal{L}_1^{\Bbbk}\mathcal{L}_2^{\Bbbk}\Gamma(\gamma_1)\Gamma(\gamma_2)}{\Gamma((\Bbbk+1)\gamma_1+\Bbbk\delta_1)\Gamma((\Bbbk+1)\gamma_2+\Bbbk\delta_2)}\varphi^{(\Bbbk+1)\gamma_1+\Bbbk\delta_1-1}\eta^{(\Bbbk+1)\gamma_2+\Bbbk\delta_2-1}\bar{d}_{1-\gamma}(u_1,u_2).\end{aligned}$$

Then we find when $\mathfrak{n}=\Bbbk+1$,

$$\begin{aligned}&\bar{d}_{\infty}(\mathbb{T}_1^{\Bbbk+1}u_1(\varphi,\eta),\mathbb{T}_1^{\Bbbk+1}u_2(\varphi,\eta))\\&\le\bar{d}_{\infty}\big(\Psi(\varphi,\eta),\Psi(\varphi,\eta)\big)+\bar{d}_{\infty}\begin{cases}{}^{RL}_{F}\mathcal{I}^{\delta;\rho}_{0+}\mathbf{h}_1(\varphi,\eta,\Phi(\varphi,\eta))+{}^{RL}_{F}\mathcal{I}^{\gamma;\rho}_{0+}\mathbf{h}_2\big(\varphi,\eta,\mathbb{T}_1^{\Bbbk}u_1(\varphi,\eta)\big)\\+{}^{RL}_{F}\mathcal{I}^{\delta;\rho}_{0+}\mathbf{h}_1(\varphi,\eta,\Phi(\varphi,\eta))+{}^{RL}_{F}\mathcal{I}^{\gamma;\rho}_{0+}\mathbf{h}_2\big(\varphi,\eta,\mathbb{T}_1^{\Bbbk}u_2(\varphi,\eta)\big)\end{cases}\\&\le\frac{\rho^{2-(\delta_1+\delta_2)}\mathcal{L}_1}{\Gamma(\delta_1)\Gamma(\delta_2)}\int_0^{\varphi}\int_0^{\eta}\mathbf{s}^{\rho-1}(x^{\rho}-\mathbf{s}^{\rho})^{\delta_1-1}\mathbf{t}^{\rho-1}(y^{\rho}-\mathbf{t}^{\rho})^{\delta_2-1}\\&\quad\times\bar{d}_{\infty}\Big({}^{RL}_{F}\mathcal{I}^{\gamma}_{0+}\mathbf{h}_2(\mathbf{s},\mathbf{t},\mathbb{T}_1^{\Bbbk}u_1(\mathbf{s},\mathbf{t})),\,{}^{RL}_{F}\mathcal{I}^{\gamma}_{0+}\mathbf{h}_2(\varphi,\eta,\mathbb{T}_2^{\Bbbk}u_2(\mathbf{s},\mathbf{t}))\Big)d\mathbf{t}d\mathbf{s}\end{aligned}$$

and since

$$\begin{aligned}&\bar{d}_{\infty}\Big({}^{RL}_{F}\mathcal{I}^{\gamma}_{0+}\mathbf{h}_2(\mathbf{s},\mathbf{t},\mathbb{T}_1^{\Bbbk}u_1(\mathbf{s},\mathbf{t})),\,{}^{RL}_{F}\mathcal{I}^{\gamma}_{0+}\mathbf{h}_2(\varphi,\eta,\mathbb{T}_2^{\Bbbk}u_2(\mathbf{s},\mathbf{t}))\Big)\\&\le\frac{\rho^{4\frac{\ell_\iota+1-\gamma_\iota}{\delta_\iota+\gamma_\iota}-\frac{\ell_\iota+1-\gamma_\iota}{\delta_\iota+\gamma_\iota}(\delta_1+\delta_2+\gamma_1+\gamma_2)}\mathcal{L}_1^{\Bbbk}\mathcal{L}_2^{\Bbbk}\bar{d}_{1-\gamma}(u_1,u_2)}{\Gamma(\ell_1+1)\Gamma(\ell_2+1)}\int_0^{\mathbf{s}}\int_0^{\mathbf{t}}\psi^{\ell_1(\rho-1)}(\mathbf{s}^{\rho}-\psi_1^{\rho})^{\gamma_1-1}\phi_1^{\ell_2(\rho-1)}(\mathbf{t}^{\rho}-\phi^{\rho})^{\gamma_2-1}d\phi d\psi.\end{aligned}$$

Here $\ell_\iota=(\Bbbk+1)\gamma_\iota+\Bbbk\delta_\iota-1,\ (\iota=1,2)$, it is clear that

$$\begin{aligned}&\bar{d}_{\infty}(\mathbb{T}_1^{\Bbbk+1}u_1(\varphi,\eta),\mathbb{T}_1^{\Bbbk+1}u_2(\varphi,\eta))\\&\le\Bbbk\int_0^{\varphi}\int_0^{\eta}\mathbf{s}^{\rho-1}(\varphi^{\rho}-\mathbf{s}^{\rho})^{\delta_1-1}\mathbf{t}^{\rho-1}(\eta^{\rho}-\mathbf{t}^{\rho})^{\delta_2-1}\\&\quad\times\bigg(\int_0^{\mathbf{s}}\psi^{\ell_2(\rho-1)}(\mathbf{s}^{\rho}-\psi_1^{\rho})^{\gamma_1-1}d\psi\int_0^{\mathbf{t}}\phi^{\ell_2(\rho-1)}(\mathbf{t}^{\rho}-\phi_1^{\rho})^{\gamma_2-1}d\phi\bigg)d\mathbf{t}d\mathbf{s}\\&=\frac{\rho^{2(2(\Bbbk+1))-(\Bbbk+1)(\delta_1+\delta_2+\gamma_1+\gamma_2)}\mathcal{L}_1^{\Bbbk+1}\mathcal{L}_2^{\Bbbk+1}\Gamma(\gamma_1)\Gamma(\gamma_2)}{\Gamma((\Bbbk+2)\gamma_1+(\Bbbk+1)\delta_1)\Gamma((\Bbbk+2)\gamma_2+(\Bbbk+1)\delta_2)}\varphi^{(\Bbbk+2)\gamma_1+(\Bbbk+1)\delta_1-1}\eta^{(\Bbbk+2)\gamma_2+(\Bbbk+1)\delta_2-1}\bar{d}_{1-\gamma}(u_1,u_2),\end{aligned}$$

where $\Bbbk=\frac{\mathcal{L}_1^{\Bbbk+1}\mathcal{L}_2^{\Bbbk+1}\bar{d}_{1-\gamma}(u_1,u_2)}{\Gamma(\delta_1)\Gamma(\delta_2)\Gamma(\ell_1+1)\Gamma(\ell_2+1)}$, which indicates that (3.2) is true for $\mathfrak{n}=\Bbbk+1$ and we have

$$\bar{d}_{1-\gamma}(\mathbb{T}_1^{\mathfrak{n}}u_1,\mathbb{T}_1^{\mathfrak{n}}u_2)\le\frac{\rho^{4\mathfrak{n}-\mathfrak{n}(\delta_1+\delta_2+\gamma_1+\gamma_2)}\mathcal{L}_1^{\mathfrak{n}}\mathcal{L}_2^{\mathfrak{n}}\Gamma(\gamma_1)\Gamma(\gamma_2)}{\Gamma((\mathfrak{n}+1)\gamma_1+\mathfrak{n}\delta_1)\Gamma((\mathfrak{n}+1)\gamma_2+\mathfrak{n}\delta_2)}\mathbf{a}^{\mathfrak{n}\gamma_1+\mathfrak{n}\delta_1}\mathbf{b}^{(\mathfrak{n}+1)\gamma_2+n_2\delta_2}\bar{d}_{1-\gamma}(u_1,u_2)$$

for every $\mathfrak{n}\in\mathbb{N}$. This yields that

$$\lim_{\mathfrak{n}\mapsto\infty}\frac{(\rho^{4-(\delta_1+\delta_2+\gamma_1+\gamma_2)}\mathcal{L}_1\mathcal{L}_2\mathbf{a}^{\gamma_1+\delta_1}\mathbf{b}^{\gamma_2+\delta_2})^{\mathfrak{n}}}{\Gamma((\mathfrak{n}+1)\gamma_1+\mathfrak{n}\delta_1)\Gamma((\mathfrak{n}+1)\gamma_2+\mathfrak{n}\delta_2)}=0,$$

indicate that $\mathbb{T}_1^{\mathfrak{n}}$ is a CM when $\mathfrak{n}$ is sufficiently large. With the analogous argument, we can deduce that $\mathbb{T}_2^{\mathfrak{n}}$ is a CM when $\mathfrak{n}$ is sufficiently large. Therefore, $\exists$ a one and only one $(\bar{u}_1,\bar{v}_1)\in\bar{\mathcal{W}}_1\times\bar{\mathcal{W}}_2$ such that the aforementioned assumptions hold:

$$\begin{aligned}\mathbf{f}_1(\varphi,\eta)&=\Psi(\varphi,\eta)+{}^{RL}_{F}\mathbb{I}^{\delta;\rho}_{0+}\mathbf{h}_1(\varphi,\eta,\Phi(\varphi,\eta))+{}^{RL}_{F}\mathbb{I}^{\gamma;\rho}_{0+}\mathbf{h}_2(\varphi,\eta,\mathbf{f}_1(\varphi,\eta)),\\\mathbf{f}_2(\varphi,\eta)&=\Phi(\varphi,\eta)+{}^{RL}_{F}\mathbb{I}^{\delta;\rho}_{0+}\mathbf{h}_2(\varphi,\eta,\Psi(\varphi,\eta))+{}^{RL}_{F}\mathbb{I}^{\delta;\rho}_{0+}\mathbf{h}_1(\varphi,\eta,\mathbf{f}_2(\varphi,\eta)),\end{aligned}$$

which is the **(a)**-weak result of (1.3) having ICs. □

Furthermore, we will demonstrate the E-U of the **(b)**-weak solution for (1.3) with ICs in the upcoming sections by making the appropriate suppositions for $\hat{\mathfrak{C}}^{\mathbf{h}_1}_{\Psi}(\mho,\bar{\mathcal{W}}_2)$ described by (2.8) and $\hat{\mathfrak{C}}^{\mathbf{h}_2}_{\Phi}(\mho,\bar{\mathcal{W}}_1)$ calculated by (2.9):

($\mathbf{q}_1$) $\hat{\mathfrak{C}}^{\mathbf{h}_1}_{\Psi}(\mho,\bar{\mathcal{W}}_2)\neq\emptyset$, $\hat{\mathfrak{C}}^{\mathbf{h}_2}_{\Phi}(\mho,\bar{\mathcal{W}}_1)\neq\emptyset$.

($\mathbf{q}_2$) If $\mathbf{f}_2(.,.)\in\hat{\mathfrak{C}}^{\mathbf{h}_1}_{\Psi}(\mho,\bar{\mathcal{W}}_2)$, then $\mathcal{Q}(.,.)\in\hat{\mathfrak{C}}^{\mathbf{h}_1}_{\Psi}(\mho,\bar{\mathcal{W}}_2)$, where

$$\mathcal{Q}(\varphi,\eta)=\Psi(\varphi,\eta)\ominus(-1)\,{}^{RL}_{F}\mathcal{I}^{\delta;\rho}_{0+}\mathbf{h}_1(\varphi,\eta,\mathbf{f}_2(\varphi,\eta)),\quad\forall\,(\varphi,\eta)\in\mho.$$

When $\mathbf{f}_1(.,.)\in\hat{\mathfrak{C}}^{\mathbf{h}_2}_{\Phi}(\mho,\bar{\mathcal{W}}_1)$, here

$$\mathcal{V}(\varphi,\eta)=\Phi(\varphi,\eta)\ominus(-1)\,{}^{RL}_{F}\mathcal{I}^{\gamma;\rho}_{0+}\mathbf{h}_2(\varphi,\eta,\mathbf{f}_1(\varphi,\eta)),\quad\forall\,(\varphi,\eta)\in\mho.$$

Theorem 3.2. *Suppose the assumption stated above* **(q**$_1$**)** *satisfies. Also, suppose that* $\mathbf{h}_1 \in \mathbb{C}_{\mathbb{J}}(\mho, \bar{\mathcal{W}}_2, \bar{\mathcal{W}}_1)$ *and* $\mathbf{h}_2 \in \mathbb{C}_{\mathbb{J}}(\mho, \bar{\mathcal{W}}_1, \bar{\mathcal{W}}_2)$ *fulfills the Lipschitz assumptions. Then* (1.3) *with ICs has unique* ***(b)****-weak solution.*

Proof. Utilizing the supposition (q_1), we are considerate that two $\mathbf{H}$-differences

$$\Psi(\varphi,\eta) \ominus (-1)\, {}^{RL}_{F}\mathcal{I}^{\delta;\rho}_{0^+}\mathbf{h}_1(\varphi,\eta,\mathbf{f}_2(\varphi,\eta))$$

and

$$\Phi(\varphi,\eta) \ominus (-1)\, {}^{RL}_{F}\mathcal{I}^{\gamma;\rho}_{0^+}\mathbf{h}_2(\varphi,\eta,\mathbf{f}_1(\varphi,\eta))$$

exist $\forall\ (\varphi,\eta) \in \mho$.

According to the supposition $(\mathbf{q}_2)$, it is appropriate to describe the functionals $\tilde{\mathbb{T}}_1 : \hat{\mathbb{C}}^{\mathbf{h}_1}_{\Psi}(\mho, \bar{\mathcal{W}}_2) \mapsto \hat{\mathbb{C}}^{\mathbf{h}_2}_{\Phi}(\mho, \bar{\mathcal{W}}_2)$ and $\tilde{\mathbb{T}}_2 : \hat{\mathbb{C}}^{\mathbf{h}_2}_{\Phi}(\mho, \bar{\mathcal{W}}_1) \mapsto \hat{\mathbb{C}}^{\mathbf{h}_1}_{\Psi}(\mho, \bar{\mathcal{W}}_1)$ as follows

$$\begin{aligned}
\tilde{\mathbb{T}}_2(\mathbf{f}_1(\varphi,\eta)) &:= \Psi(\varphi,\eta) \ominus (-1)\, {}^{RL}_{F}\mathcal{I}^{\delta;\rho}_{0^+}\mathbf{h}_1(\varphi,\eta,\Phi(\varphi,\eta)) \ominus {}^{RL}_{F}\mathcal{I}^{\gamma;\rho}_{0^+}\mathbf{h}_2(\varphi,\eta,\mathbf{f}_1(\varphi,\eta)),\\
\tilde{\mathbb{T}}_1(\mathbf{f}_2(\varphi,\eta)) &:= \Phi(\varphi,\eta) \ominus (-1)\, {}^{RL}_{F}\mathcal{I}^{\gamma;\rho}_{0^+}\mathbf{h}_2(\varphi,\eta,\Psi(\varphi,\eta)) \ominus {}^{RL}_{F}\mathcal{I}^{\delta;\rho}_{0^+}\mathbf{h}_1(\varphi,\eta,\mathbf{f}_2(\varphi,\eta)),
\end{aligned}$$

that further shows that $\tilde{\mathbb{T}}_1$ and $\tilde{\mathbb{T}}_2$ are related to $\mathbf{f}_2(\varphi,\eta)$ and $\mathbf{f}_1(\varphi,\eta)$, respectively. As a result of Lemma 2.1 **(ii)**, we have

$$\begin{aligned}
&\bar{d}_{\infty}\big(\tilde{\mathbb{T}}_2(u_1), \tilde{\mathbb{T}}_2(u_2)\big)\\
&\le \frac{\rho^{4-(\delta_1+\delta_2+\gamma_1+\gamma_2)}\mathcal{L}_1\mathcal{L}_2\Gamma(\gamma_1)\Gamma(\gamma_2)}{\Gamma(2\gamma_1+\delta_1)\Gamma(2\gamma_2+\delta_2)}\varphi^{2\gamma_1+\delta_1-1}\eta^{2\gamma_2+\delta_2-1}\bar{d}_{1-\gamma}(u_1,u_2),
\end{aligned}$$

which leads to

$$\begin{aligned}
&\varphi^{1-\gamma_1}\eta^{1-\gamma_1}\bar{d}_{\infty}\big(\tilde{\mathbb{T}}_2(u_1), \tilde{\mathbb{T}}_2(u_2)\big)\\
&\le \frac{\rho^{4-(\delta_1+\delta_2+\gamma_1+\gamma_2)}\mathcal{L}_1\mathcal{L}_2\Gamma(\gamma_1)\Gamma(\gamma_2)}{\Gamma(2\gamma_1+\delta_1)\Gamma(2\gamma_2+\delta_2)}\varphi^{\gamma_1+\delta_1}\eta^{\gamma_2+\delta_2}\bar{d}_{1-\gamma}(u_1,u_2).
\end{aligned}$$

As a verification of Theorem 3.1, we employ the inductive technique to acquire the functional sequence $\{\tilde{\mathbb{T}}^{\mathfrak{n}}_2\}_{\mathfrak{n}\ge 1}$ formed by

$$\tilde{\mathbb{T}}^{\mathfrak{n}}_2\big(\mathbf{f}_1(\varphi,\eta)\big) = \tilde{\mathbb{T}}_2\big(\tilde{\mathbb{T}}^{\mathfrak{n}-1}_2\mathbf{f}_1(\varphi,\eta)\big)$$

and

$$\bar{d}_{1-\gamma}(\tilde{\mathbb{T}}^{\mathfrak{n}}_2 u_1, \tilde{\mathbb{T}}^{\mathfrak{n}}_2 u_2) \le \frac{\rho^{4\mathfrak{n}-\mathfrak{n}(\delta_1+\delta_2+\gamma_1+\gamma_2)}\mathcal{L}^{\mathfrak{n}}_1\mathcal{L}^{\mathfrak{n}}_2\mathbf{a}^{\mathfrak{n}\gamma_1+\mathfrak{n}\delta_1}\mathbf{b}^{\mathfrak{n}\gamma_2+\mathfrak{n}\delta_2}\Gamma(\gamma_1)\Gamma(\gamma_2)}{\Gamma((\mathfrak{n}+1)\gamma_1+\mathfrak{n}\delta_1)\Gamma((\mathfrak{n}+1)\gamma_2+\mathfrak{n}\delta_2)}\bar{d}_{1-\gamma}(u_1,u_2).$$

Such as

$$\lim_{\mathfrak{n}\mapsto\infty}\frac{\rho^{4\mathfrak{n}-\mathfrak{n}(\delta_1+\delta_2+\gamma_1+\gamma_2)}\mathcal{L}^{\mathfrak{n}}_1\mathcal{L}^{\mathfrak{n}}_2\mathbf{a}^{\mathfrak{n}\gamma_1+\mathfrak{n}\delta_1}\mathbf{b}^{\mathfrak{n}\gamma_2+\mathfrak{n}\delta_2}\Gamma(\gamma_1)\Gamma(\gamma_2)}{\Gamma((\mathfrak{n}+1)\gamma_1+\mathfrak{n}\delta_1)\Gamma((\mathfrak{n}+1)\gamma_2+\mathfrak{n}\delta_2)} = 0.$$

If $\mathfrak{n}$ is sufficiently large, it describes a contraction mapping as $\tilde{\mathbb{T}}^{\mathfrak{n}}_2$. Analogously, it is recognized that $\tilde{\mathbb{T}}^{\mathfrak{n}}_1$ is also a CM when $\mathfrak{n}$ is sufficiently large. As a result, $\exists$ a unique $(\mathbf{f}_1,\mathbf{f}_2) \in \bar{\mathcal{W}}_1 \times \bar{\mathcal{W}}_2$ for which the respective expressions represent:

$$\begin{aligned}
\mathbf{f}_1(\varphi,\eta) &:= \Psi(\varphi,\eta) \ominus (-1)\, {}^{RL}_{F}\mathcal{I}^{\delta;\rho}_{0^+}\mathbf{h}_1(\varphi,\eta,\Phi(\varphi,\eta)) \ominus {}^{RL}_{F}\mathcal{I}^{\gamma;\rho}_{0^+}\mathbf{h}_2(\varphi,\eta,\mathbf{f}_1(\varphi,\eta)),\\
\mathbf{f}_2(\varphi,\eta) &:= \Phi(\varphi,\eta) \ominus (-1)\, {}^{RL}_{F}\mathcal{I}^{\gamma;\rho}_{0^+}\mathbf{h}_2(\varphi,\eta,\Psi(\varphi,\eta)) \ominus {}^{RL}_{F}\mathcal{I}^{\delta;\rho}_{0^+}\mathbf{h}_1(\varphi,\eta,\mathbf{f}_2(\varphi,\eta)). \quad \square
\end{aligned}$$

3.1. Continuous dependence and ε-approximation

By re-configuring the initial settings and applying Lemmas 2.3 and 2.4, we demonstrate the continuous reliance of two types of $\mathfrak{g}\mathbf{H}$-weak results on beginning data and ε-approximation findings of the dynamic model for (1.3) with ICs. Furthermore, we will demonstrate that the aforementioned is a subcase of the former.

To begin, take into account that findings are continuous based on the starting documentation.

By reconfiguring ICs of (1.3), we obtain a novel unified framework for fuzzified fractional PDEs:

$$\begin{cases}
{}^{c}_{\mathfrak{g}\mathbf{H}}\mathcal{D}^{\delta;\rho}_{\varrho}\mathbf{f}_1(\varphi,\eta) = \mathbf{h}_1(\varphi,\eta,\mathbf{f}_2(\varphi,\eta)),\\
{}^{c}_{\mathfrak{g}\mathbf{H}}\mathcal{D}^{\gamma;\rho}_{\varrho}\mathbf{f}_2(\varphi,\eta) = \mathbf{h}_2(\varphi,\eta,\mathbf{f}_1(\varphi,\eta)),\\
\mathbf{f}_1(\varphi,0) = \omega_{11}(\varphi), \quad \mathbf{f}_2(\varphi,0) = \chi_{11}(\varphi),\\
\mathbf{f}_1(0,\eta) = \omega_{21}(\eta), \quad \mathbf{f}_2(0,\eta) = \chi_{21}(\eta),
\end{cases} \tag{3.3}$$

or

$$\begin{cases}
{}^{c}_{\mathfrak{g}\mathbf{H}}\mathcal{D}^{\delta;\rho}_{\varrho}\mathbf{f}_1(\varphi,\eta) = \mathbf{h}_1(\varphi,\eta,\mathbf{f}_2(\varphi,\eta)),\\
{}^{c}_{\mathfrak{g}\mathbf{H}}\mathcal{D}^{\gamma;\rho}_{\varrho}\mathbf{f}_2(\varphi,\eta) = \mathbf{h}_2(\varphi,\eta,\mathbf{f}_1(\varphi,\eta)),\\
\mathbf{f}_1(\varphi,0) = \omega_{12}(\varphi), \quad \mathbf{f}_2(\varphi,0) = \chi_{12}(\varphi),\\
\mathbf{f}_1(0,\eta) = \omega_{22}(\eta), \quad \mathbf{f}_2(0,\eta) = \chi_{22}(\eta),
\end{cases} \tag{3.4}$$

and choosing $(u_1(.,.), v_1(.,.))^{\mathbf{T}}$ and $(u_2(.,.), v_2(.,.))^{\mathbf{T}}$ be p_1-weak solutions of (3.3) and (3.4) for $p_1 = \mathbf{(a)}, \mathbf{(b)}$, respectively, therefore, we have

$$\begin{cases} u_1(\varphi,\eta) = \Psi_1(\varphi,\eta) + {}^{RL}_{F}\mathcal{I}^{\delta;\rho}_{0^+}\mathbf{h}_1(\varphi,\eta,v_1(\varphi,\eta)), \\ v_1(\varphi,\eta) = \Phi_1(\varphi,\eta) + {}^{RL}_{F}\mathcal{I}^{\gamma;\rho}_{0^+}\mathbf{h}_2(\varphi,\eta,u_1(\varphi,\eta)), \\ u_2(\varphi,\eta) = \Psi_2(\varphi,\eta) + {}^{RL}_{F}\mathcal{I}^{\delta;\rho}_{0^+}\mathbf{h}_1(\varphi,\eta,v_2(\varphi,\eta)), \\ v_2(\varphi,\eta) = \Phi_1(\varphi,\eta) + {}^{RL}_{F}\mathcal{I}^{\gamma;\rho}_{0^+}\mathbf{h}_2(\varphi,\eta,u_2(\varphi,\eta)), \end{cases} \tag{3.5}$$

or

$$\begin{cases} u_1(\varphi,\eta) = \Psi_1(\varphi,\eta) \ominus (-1)\,{}^{RL}_{F}\mathcal{I}^{\delta;\rho}_{0^+}\mathbf{h}_1(\varphi,\eta,v_1(\varphi,\eta)), \\ v_1(\varphi,\eta) = \Phi_1(\varphi,\eta) \ominus (-1)\,{}^{RL}_{F}\mathcal{I}^{\gamma;\rho}_{0^+}\mathbf{h}_2(\varphi,\eta,u_1(\varphi,\eta)), \\ u_2(\varphi,\eta) = \Psi_2(\varphi,\eta) \ominus (-1)\,{}^{RL}_{F}\mathcal{I}^{\delta;\rho}_{0^+}\mathbf{h}_1(\varphi,\eta,v_2(\varphi,\eta)), \\ v_2(\varphi,\eta) = \Phi_1(\varphi,\eta) \ominus (-1)\,{}^{RL}_{F}\mathcal{I}^{\gamma;\rho}_{0^+}\mathbf{h}_2(\varphi,\eta,u_2(\varphi,\eta)), \end{cases} \tag{3.6}$$

where

$$\Psi_1(\varphi,\eta) = \xi_{11}(\varphi) + \xi_{21}(\eta) \ominus \xi_{11}(0), \quad \Phi_1(\varphi,\eta) = \chi_{11}(\varphi) + \chi_{21}(\eta) \ominus \chi_{11}(0),$$

$$\Psi_2(\varphi,\eta) = \xi_{12}(\varphi) + \xi_{22}(\eta) \ominus \xi_{12}(0), \quad \Phi_2(\varphi,\eta) = \chi_{12}(\varphi) + \chi_{22}(\eta) \ominus \chi_{12}(0).$$

Theorem 3.3. *Under the assumption of Lemma 2.4 and* $\mathbf{h}_1 \in \mathbb{C}_{\mathbb{J}}(\mho, \bar{\mathcal{W}}_2, \bar{\mathcal{W}}_1)$ *and* $\mathbf{h}_2 \in \mathbb{C}_{\mathbb{J}}(\mho, \bar{\mathcal{W}}_1, \bar{\mathcal{W}}_2)$ *fulfills the Lipschitz assumptions. If* $(u_1(.,.), v_1(.,.))^{\mathbf{T}}$ *and* $(u_2(.,.), v_2(.,.))^{\mathbf{T}}$ *are* p_1*-weak solutions of* (3.3) *and* (3.4) *for* $p_1 = \mathbf{(a)}, \mathbf{(b)}$*, respectively, with the respective starting values are* $(\Psi_1, \Phi_1)^{\mathbf{T}}$ *and* $(\Psi_2, \Phi_2)^{\mathbf{T}}$*, several times; then the succeeding variant stands true:*

$$\begin{pmatrix} \Upsilon(u_1,u_2) \\ \Upsilon(v_1,v_2) \end{pmatrix} \le \begin{pmatrix} \Upsilon(\Psi_1,\Psi_2) \\ \Upsilon(\Phi_1,\Phi_2) \end{pmatrix} + \begin{pmatrix} \frac{\rho^{-(\delta_1+\delta_2)}\mathbf{a}^{\delta_1}\mathbf{b}^{\delta_2}}{\Gamma(\delta_1+1)\Gamma(\delta_2+1)} & 0 \\ 0 & \frac{\rho^{-(\gamma_1+\gamma_2)}\mathbf{a}^{\gamma_1}\mathbf{b}^{\gamma_2}}{\Gamma(\gamma_1+1)\Gamma(\gamma_2+1)} \end{pmatrix} \sum_{\Bbbk=0}^{\infty} \mathbf{A}^{\Bbbk} \begin{pmatrix} \mathcal{L}_1\Upsilon(\Phi_1,\Phi_2) \\ \mathcal{L}_2\Upsilon(\Psi_1,\Psi_2) \end{pmatrix}, \tag{3.7}$$

where $\mathbf{A}$ *corresponds to Lemma 2.4.*

Proof. Choosing $z_1(\varphi,\eta) = {}^{c}_{\mathrm{gH}}\mathcal{D}^{\delta;\rho}_{1}u_1(\varphi,\eta)$, $z_2(\varphi,\eta) = {}^{c}_{\mathrm{gH}}\mathcal{D}^{\delta;\rho}_{1}u_2(\varphi,\eta)$, $w_1(\varphi,\eta) = {}^{c}_{\mathrm{gH}}\mathcal{D}^{\gamma;\rho}_{1}v_1(\varphi,\eta)$ and $w_2(\varphi,\eta) = {}^{c}_{\mathrm{gH}}\mathcal{D}^{\gamma;\rho}_{2}v_2(\varphi,\eta)$, then, without loss of generality, we take into account (3.5) as describes:

$$u_1(\varphi,\eta) = \Psi_1(\varphi,\eta) + {}^{RL}_{F}\mathcal{I}^{\delta;\rho}_{0^+}z_1(\varphi,\eta), \quad u_2(\varphi,\eta) = \Psi_2(\varphi,\eta) + {}^{RL}_{F}\mathcal{I}^{\delta;\rho}_{0^+}z_2(\varphi,\eta),$$

$$v_1(\varphi,\eta) = \Phi_1(\varphi,\eta) + {}^{RL}_{F}\mathcal{I}^{\gamma;\rho}_{0^+}w_1(\varphi,\eta), \quad v_2(\varphi,\eta) = \Phi_2(\varphi,\eta) + {}^{RL}_{F}\mathcal{I}^{\gamma;\rho}_{0^+}w_2(\varphi,\eta), \quad \forall\, (\varphi,\eta) \in \mho.$$

Therefore, (1.3) having ICs can be written as

$$\begin{cases} z_1(\varphi,\eta) = \mathbf{h}_1(\varphi,\eta,\Phi_1(\varphi,\eta)) + {}^{RL}_{F}\mathcal{I}^{\gamma;\rho}_{0^+}w_1(\varphi,\eta), \\ w_1(\varphi,\eta) = \mathbf{h}_2(\varphi,\eta,\Psi_1(\varphi,\eta)) + {}^{RL}_{F}\mathcal{I}^{\delta;\rho}_{0^+}z_1(\varphi,\eta), \end{cases}$$

and

$$\begin{cases} z_2(\varphi,\eta) = \mathbf{h}_1(\varphi,\eta,\Phi_2(\varphi,\eta)) + {}^{RL}_{F}\mathcal{I}^{\gamma;\rho}_{0^+}w_2(\varphi,\eta), \\ w_2(\varphi,\eta) = \mathbf{h}_2(\varphi,\eta,\Psi_2(\varphi,\eta)) + {}^{RL}_{F}\mathcal{I}^{\delta;\rho}_{0^+}z_2(\varphi,\eta). \end{cases}$$

In view of Lemma 2.1 **(i)**, we have

$$\begin{aligned} &\bar{d}_\infty\big(z_1(\varphi,\eta), z_2(\varphi,\eta)\big) \\ &\le \bar{d}_\infty\Big(\mathbf{h}_1\big(\varphi,\eta,\Phi_1(\varphi,\eta) + {}^{RL}_{F}\mathcal{I}^{\gamma;\rho}_{0^+}w_1(\varphi,\eta)\big) + \mathbf{h}_1\big(\varphi,\eta,\Phi_2(\varphi,\eta) + {}^{RL}_{F}\mathcal{I}^{\gamma;\rho}_{0^+}w_1(\varphi,\eta)\big)\Big) \\ &\quad + \bar{d}_\infty\Big(\mathbf{h}_1\big(\varphi,\eta,\Phi_2(\varphi,\eta) + {}^{RL}_{F}\mathcal{I}^{\gamma;\rho}_{0^+}w_1(\varphi,\eta)\big) + \mathbf{h}_1\big(\varphi,\eta,\Phi_2(\varphi,\eta) + {}^{RL}_{F}\mathcal{I}^{\gamma;\rho}_{0^+}w_2(\varphi,\eta)\big)\Big) \\ &\le \mathcal{L}_1\bar{d}_\infty\big(\Phi_1(\varphi,\eta),\Phi_2(\varphi,\eta)\big) + \mathcal{L}_1\,{}^{RL}_{F}\mathcal{I}^{\gamma;\rho}_{0^+}\bar{d}_\infty\big(w_1(\varphi,\eta), w_2(\varphi,\eta)\big). \end{aligned}$$

Analogously, we have

$$\bar{d}_\infty\big(w_1(\varphi,\eta), w_2(\varphi,\eta)\big) \le \mathcal{L}_2\bar{d}_\infty\big(\Psi_1(\varphi,\eta),\Psi_2(\varphi,\eta)\big) + \mathcal{L}_2\,{}^{RL}_{F}\mathcal{I}^{\delta;\rho}_{0^+}\bar{d}_\infty\big(z_1(\varphi,\eta), z_2(\varphi,\eta)\big).$$

Taking into account Lemma 2.4, we have

$$\begin{aligned} \begin{pmatrix} \bar{d}_\infty(z_1(\varphi,\eta), z_2(\varphi,\eta)) \\ \bar{d}_\infty\big(w_1(\varphi,\eta), w_2(\varphi,\eta)\big) \end{pmatrix} &\le \begin{pmatrix} 0 & \mathcal{L}_1\,{}^{RL}_{F}\mathcal{I}^{\gamma;\rho}_{0^+} \\ \mathcal{L}_2\,{}^{RL}_{F}\mathcal{I}^{\delta;\rho}_{0^+} & 0 \end{pmatrix} \begin{pmatrix} \bar{d}_\infty(z_1(\varphi,\eta), z_2(\varphi,\eta)) \\ \bar{d}_\infty\big(w_1(\varphi,\eta), w_2(\varphi,\eta)\big) \end{pmatrix} \\ &\quad + \begin{pmatrix} \mathcal{L}_1\bar{d}_\infty(\Phi_1(\varphi,\eta),\Phi_2(\varphi,\eta)) \\ \mathcal{L}_2\bar{d}_\infty\big(\Psi_1(\varphi,\eta),\Psi_2(\varphi,\eta)\big) \end{pmatrix} \\ &\le \sum_{\Bbbk=0}^{\infty} \begin{pmatrix} 0 & \mathcal{L}_1\,{}^{RL}_{F}\mathcal{I}^{\gamma;\rho}_{0^+} \\ \mathcal{L}_2\,{}^{RL}_{F}\mathcal{I}^{\delta;\rho}_{0^+} & 0 \end{pmatrix}^{\Bbbk} \begin{pmatrix} \mathcal{L}_1\bar{d}_\infty(\Phi_1(\varphi,\eta),\Phi_2(\varphi,\eta)) \\ \mathcal{L}_2\bar{d}_\infty\big(\Psi_1(\varphi,\eta),\Psi_2(\varphi,\eta)\big) \end{pmatrix}. \end{aligned}$$

Therefore, utilizing Lemma 2.1 **(i)** and Lemma 2.3, we can write

$$\begin{pmatrix} \bar{d}_\infty(u_1(\varphi,\eta),u_2(\varphi,\eta)) \\ \bar{d}_\infty\big(v_1(\varphi,\eta),v_2(\varphi,\eta)\big) \end{pmatrix} \le \begin{pmatrix} \bar{d}_\infty(\Psi_1(\varphi,\eta),\Psi_2(\varphi,\eta)) \\ \bar{d}_\infty\big(\Phi_1(\varphi,\eta),\Phi_2(\varphi,\eta)\big) \end{pmatrix} + \begin{pmatrix} {}^{RL}_{F}\mathcal{I}^{\delta;\rho}_{0+} & 0 \\ 0 & {}^{RL}_{F}\mathcal{I}^{\gamma;\rho}_{0+} \end{pmatrix} \sum_{\Bbbk=0}^{\infty}\mathbf{A}^{\Bbbk}\begin{pmatrix} \mathcal{L}_1\bar{d}_\infty(\Phi_1(\varphi,\eta),\Phi_2(\varphi,\eta)) \\ \mathcal{L}_2\bar{d}_\infty\big(\Psi_1(\varphi,\eta),\Psi_2(\varphi,\eta)\big) \end{pmatrix}. \tag{3.8}$$

Considering (2.10) on domain $\mho$, it is observed from (3.8) that

$$\begin{aligned} \begin{pmatrix} \Upsilon(u_1,u_2) \\ \Upsilon(v_1,v_2) \end{pmatrix} &\le \begin{pmatrix} \Upsilon(\Psi_1,\Psi_2) \\ \Upsilon(\Phi_1,\Phi_2) \end{pmatrix} + \begin{pmatrix} {}^{RL}_{F}\mathcal{I}^{\delta;\rho}_{0+} & 0 \\ 0 & {}^{RL}_{F}\mathcal{I}^{\gamma;\rho}_{0+} \end{pmatrix} \sum_{\Bbbk=0}^{\infty}\mathbf{A}^{\Bbbk}\begin{pmatrix} \mathcal{L}_1\Upsilon(\Phi_1,\Phi_2) \\ \mathcal{L}_2\Upsilon(\Psi_1,\Psi_2) \end{pmatrix} \\ &\le \begin{pmatrix} \Upsilon(\Psi_1,\Psi_2) \\ \Upsilon(\Phi_1,\Phi_2) \end{pmatrix} + \begin{pmatrix} \frac{\rho^{-(\delta_1+\delta_2)}\mathbf{a}^{\delta_1}\mathbf{b}^{\delta_2}}{\Gamma(\delta_1+1)\Gamma(\delta_2+1)} & 0 \\ 0 & \frac{\rho^{-(\gamma_1+\gamma_2)}\mathbf{a}^{\gamma_1}\mathbf{b}^{\gamma_2}}{\Gamma(\gamma_1+1)\Gamma(\gamma_2+1)} \end{pmatrix} \sum_{\Bbbk=0}^{\infty}\mathbf{A}^{\Bbbk}\begin{pmatrix} \mathcal{L}_1\Upsilon(\Phi_1,\Phi_2) \\ \mathcal{L}_2\Upsilon(\Psi_1,\Psi_2) \end{pmatrix}. \end{aligned} \tag{3.9}$$

This is the desired result (3.7), which indicates that the finding is still reliant on the initial data for (1.3) with fuzzified coupled equations that can be derived in the domain $\mho$. In a similar way, we have an analogous solution for (3.6). This has an immediate consequence. □

In what follow, we suggest the ε-approximate solution of (3.3) or (3.4).

Definition 3.1. Suppose there is a mapping $\big(\mathbf{f}_1(\varphi,\eta),\mathbf{f}_2(\varphi,\eta)\big)^{\mathbf{T}}$ termed as the ε-approximate solution of (3.3) and (3.4); here $\varepsilon=(\hat{\varepsilon},\bar{\varepsilon})$ if $\big(\mathbf{f}_1(\varphi,\eta),\mathbf{f}_2(\varphi,\eta)\big)^{\mathbf{T}}$ fulfills a coupled system of fuzzified FPDEs as follows

$$\begin{cases} \bar{d}_\infty\Big({}^{c}_{\mathbf{gH}}\mathcal{D}^{\delta;\rho}_{1}\mathbf{f}_1(\varphi,\eta),\mathbf{h}_1(\varphi,\eta,\mathbf{f}_2(\varphi,\eta))\Big) \le \hat{\varepsilon}, \\ \bar{d}_\infty\Big({}^{c}_{\mathbf{gH}}\mathcal{D}^{\gamma;\rho}_{1}\mathbf{f}_2(\varphi,\eta),\mathbf{h}_2(\varphi,\eta,\mathbf{f}_1(\varphi,\eta))\Big) \le \bar{\varepsilon}. \end{cases}$$

Theorem 3.4. *Under the assumption of Lemma 2.4 and* $\mathbf{h}_1 \in \mathbb{C}_{\mathbb{J}}(\mho,\bar{\mathcal{W}}_2,\bar{\mathcal{W}}_1)$ *and* $\mathbf{h}_2 \in \mathbb{C}_{\mathbb{J}}(\mho,\bar{\mathcal{W}}_1,\bar{\mathcal{W}}_2)$ *fulfills the Lipschtiz assumptions, for* $\iota=1,2$, $(u_\iota(.,.),v_\iota(.,.))^{\mathbf{T}}$ *is independently the approximate* ε_ι*-solutions of* (3.3) *and* (3.4)*, where* $\varepsilon=(\hat{\varepsilon},\bar{\varepsilon})$ *with the respective starting values are* $(\Psi_\iota,\Phi_\iota)^{\mathbf{T}}$*. Then the succeeding variant stands true:*

$$\begin{pmatrix} \Upsilon(u_1,u_2) \\ \Upsilon(v_1,v_2) \end{pmatrix} \le \begin{pmatrix} \Upsilon(\Psi_1,\Psi_2) \\ \Upsilon(\Phi_1,\Phi_2) \end{pmatrix} + \begin{pmatrix} \frac{\rho^{-(\delta_1+\delta_2)}\mathbf{a}^{\delta_1}\mathbf{b}^{\delta_2}}{\Gamma(\delta_1+1)\Gamma(\delta_2+1)} & 0 \\ 0 & \frac{\rho^{-(\gamma_1+\gamma_2)}\mathbf{a}^{\gamma_1}\mathbf{b}^{\gamma_2}}{\Gamma(\gamma_1+1)\Gamma(\gamma_2+1)} \end{pmatrix} \sum_{\Bbbk=0}^{\infty}\mathbf{A}^{\Bbbk}\begin{pmatrix} \mathcal{L}_1\Upsilon(\Phi_1,\Phi_2)+\hat{\varepsilon_1}+\hat{\varepsilon_2} \\ \mathcal{L}_2\Upsilon(\Psi_1,\Psi_2)+\bar{\varepsilon_1}+\bar{\varepsilon_2} \end{pmatrix}, \tag{3.10}$$

where $\mathbf{A}$ *corresponds to Lemma 2.4.*

Proof. By means of Definition 3.1, clearly, we observe that for $\iota=1,2$,

$$\begin{cases} \bar{d}_\infty\Big({}^{c}_{\mathbf{gH}}\mathcal{D}^{\delta;\rho}_{1}\mathbf{f}_{1_\iota}(\varphi,\eta),\mathbf{h}_1(\varphi,\eta,\mathbf{f}_{2_\iota}(\varphi,\eta))\Big) \le \hat{\varepsilon}_\iota, \\ \bar{d}_\infty\Big({}^{c}_{\mathbf{gH}}\mathcal{D}^{\gamma;\rho}_{1}\mathbf{f}_{2_\iota}(\varphi,\eta),\mathbf{h}_2(\varphi,\eta,\mathbf{f}_{1_\iota}(\varphi,\eta))\Big) \le \bar{\varepsilon}_\iota. \end{cases} \tag{3.11}$$

Choosing $z_1(\varphi,\eta)={}^{c}_{\mathbf{gH}}\mathcal{D}^{\delta;\rho}_{1}u_1(\varphi,\eta)$, $z_2(\varphi,\eta)={}^{c}_{\mathbf{gH}}\mathcal{D}^{\delta;\rho}_{1}u_2(\varphi,\eta)$, $w_1(\varphi,\eta)={}^{c}_{\mathbf{gH}}\mathcal{D}^{\gamma;\rho}_{1}v_1(\varphi,\eta)$ and $w_2(\varphi,\eta)={}^{c}_{\mathbf{gH}}\mathcal{D}^{\gamma;\rho}_{2}v_2(\varphi,\eta)$. Generally, concerning to (3.5), one recognizes that (3.11) is similar to

$$\begin{cases} \bar{d}_\infty\Big(z_\iota(\varphi,\eta),\mathbf{h}_1\Big(\varphi,\eta\Phi_\iota(\varphi,\eta)+{}^{RL}_{F}\mathcal{I}^{\gamma;\rho}_{0+}\bar{w}_\iota(\varphi,\eta)\Big)\Big) \le \hat{\varepsilon}_\iota, \\ \bar{d}_\infty\Big(w_\iota(\varphi,\eta),\mathbf{h}_2\Big(\varphi,\eta\Psi_\iota(\varphi,\eta)+{}^{RL}_{F}\mathcal{I}^{\gamma;\rho}_{0+}\bar{z}_\iota(\varphi,\eta)\Big)\Big) \le \bar{\varepsilon}_\iota, \end{cases}$$

for $\iota=1,2$. As $\mathbf{h}_1$ and $\mathbf{h}_2$ fulfills the Lipschitz assumptions, so that, following Lemma's 2.1 **(i)**, 2.3 and 2.4 that

$$\begin{aligned} \begin{pmatrix} \bar{d}_\infty(z_1(\varphi,\eta),z_2(\varphi,\eta)) \\ \bar{d}_\infty\big(w_1(\varphi,\eta),w_2(\varphi,\eta)\big) \end{pmatrix} &\le \begin{pmatrix} 0 & \mathcal{L}_1\,{}^{RL}_{F}\mathcal{I}^{\gamma;\rho}_{0+} \\ \mathcal{L}_2\,{}^{RL}_{F}\mathcal{I}^{\delta;\rho}_{0+} & 0 \end{pmatrix} \begin{pmatrix} \bar{d}_\infty(z_1(\varphi,\eta),z_2(\varphi,\eta)) \\ \bar{d}_\infty\big(w_1(\varphi,\eta),w_2(\varphi,\eta)\big) \end{pmatrix} \\ &\quad + \begin{pmatrix} \mathcal{L}_1\bar{d}_\infty(\Phi_1(\varphi,\eta),\Phi_2(\varphi,\eta))+\hat{\varepsilon}_1+\hat{\varepsilon}_2 \\ \mathcal{L}_2\bar{d}_\infty\big(\Psi_1(\varphi,\eta),\Psi_2(\varphi,\eta)\big)+\bar{\varepsilon}_1+\bar{\varepsilon}_2 \end{pmatrix} \\ &\le \sum_{\Bbbk=0}^{\infty}\mathbf{A}^{\Bbbk}\begin{pmatrix} \mathcal{L}_1\bar{d}_\infty(\Phi_1(\varphi,\eta),\Phi_2(\varphi,\eta))+\hat{\varepsilon}_1+\hat{\varepsilon}_2 \\ \mathcal{L}_2\bar{d}_\infty\big(\Psi_1(\varphi,\eta),\Psi_2(\varphi,\eta)\big)+\bar{\varepsilon}_1+\bar{\varepsilon}_2 \end{pmatrix} \end{aligned}$$

and

$$\begin{pmatrix} \bar{d}_\infty(u_1(\varphi,\eta),u_2(\varphi,\eta)) \\ \bar{d}_\infty\big(v_1(\varphi,\eta),v_2(\varphi,\eta)\big) \end{pmatrix} \le \begin{pmatrix} \bar{d}_\infty(\Psi_1(\varphi,\eta),\Psi_2(\varphi,\eta)) \\ \bar{d}_\infty\big(\Phi_1(\varphi,\eta),\Phi_2(\varphi,\eta)\big) \end{pmatrix} + \begin{pmatrix} {}^{RL}_{F}\mathcal{I}^{\delta;\rho}_{0+} & 0 \\ {}^{RL}_{F}\mathcal{I}^{\gamma;\rho}_{0+} & 0 \end{pmatrix}$$

$$\leq \sum_{\Bbbk=0}^{\infty} \mathbf{A}^{\Bbbk} \begin{pmatrix} \mathcal{L}_1 \bar{d}_\infty(\Phi_1(\varphi,\eta), \Phi_2(\varphi,\eta)) + \hat{\varepsilon}_1 + \hat{\varepsilon}_2 \\ \mathcal{L}_2 \bar{d}_\infty(\Psi_1(\varphi,\eta), \Psi_2(\varphi,\eta)) + \bar{\varepsilon}_1 + \bar{\varepsilon}_2 \end{pmatrix}. \tag{3.12}$$

Letting the upper estimate of aforesaid (3.12) on the domain $\mho$, further, utilizing (2.10), we have

$$\begin{pmatrix} \Upsilon(u_1,u_2) \\ \Upsilon(v_1,v_2) \end{pmatrix} \leq \begin{pmatrix} \Upsilon(\Psi_1,\Psi_2) \\ \Upsilon(\Phi_1,\Phi_2) \end{pmatrix} + \begin{pmatrix} \frac{\rho^{-(\delta_1+\delta_2)}\mathbf{a}^{\delta_1}\mathbf{b}^{\delta_2}}{\Gamma(\delta_1+1)\Gamma(\delta_2+1)} & 0 \\ 0 & \frac{\rho^{-(\gamma_1+\gamma_2)}\mathbf{a}^{\gamma_1}\mathbf{b}^{\gamma_2}}{\Gamma(\gamma_1+1)\Gamma(\gamma_2+1)} \end{pmatrix} \sum_{\Bbbk=0}^{\infty} \mathbf{A}^{\Bbbk} \begin{pmatrix} \mathcal{L}_1 \Upsilon(\Phi_1,\Phi_2) + \hat{\varepsilon_1} + \hat{\varepsilon_2} \\ \mathcal{L}_2 \Upsilon(\Psi_1,\Psi_2) + \bar{\varepsilon_1} + \bar{\varepsilon_2} \end{pmatrix}. \tag{3.13}$$

This indicates that (3.10) satisfies. Analogously, we get a similar consequence relating to (1.3). This completes the proof of Theorem 3.4. □

Remark 3.1. For $\hat{\varepsilon}_\imath = \bar{\varepsilon}_\imath = 0$ ($\imath = 1,2$), then (3.10) yields (3.7) for the continuous reliance of the findings of (1.3) on the initial data.

According to Example 5.1 of [38], which is the illustration subjectively and comprehensively exemplifies the E-U outcomes of Theorems 3.1 and 3.2.

Example 3.5. The subsequent coupled system of fuzzified FPDEs is supposed: for every $(\varphi,\eta) \in \mho = [0,\mathbf{c}] \times [0,\mathbf{d}]$ and $\Bbbk = 1,2$

$$\begin{cases} {}^{c}_{\mathfrak{g}\mathbf{H}}\mathcal{D}^{\delta;\rho}_{\Bbbk} \mathbf{f}_1(\varphi,\eta) = \varsigma_1(\varphi,\eta)\mathbf{f}_2(\varphi,\eta) + \varsigma_2(\varphi,\eta), \\ {}^{c}_{\mathfrak{g}\mathbf{H}}\mathcal{D}^{\gamma;\rho}_{\Bbbk} \mathbf{f}_2(\varphi,\eta) = \sigma_1(\varphi,\eta)\mathbf{f}_1(\varphi,\eta) + \sigma_2(\varphi,\eta), \\ \mathbf{f}_1(\varphi,0) = \mathbf{f}_1(0,\eta) = \mathbf{f}_1(0,0) = -2\Xi, \\ \mathbf{f}_2(\varphi,0) = \mathbf{f}_2(0,\eta) = \mathbf{f}_2(0,0) = 2\Xi, \end{cases} \tag{3.14}$$

where $\delta,\gamma \in [0,1]\times[0,1], \varsigma_1(\varphi,\eta), \varsigma_2(\varphi,\eta), \sigma_1(\varphi,\eta)$ and $\sigma_1(\varphi,\eta)$ are the polynomial mappings and Ξ is a fuzzy number.

Clearly, the mappings $\mathbf{h}_1(\varphi,\eta,\mathbf{f}_2(\varphi,\eta)) := \varsigma_1(\varphi,\eta)\mathbf{f}_2(\varphi,\eta)+\varsigma_2(\varphi,\eta)$ and $\mathbf{h}_2(\varphi,\eta,\mathbf{f}_1(\varphi,\eta)) := \sigma_1(\varphi,\eta)\mathbf{f}_1(\varphi,\eta)+\sigma_2(\varphi,\eta)$ in (3.14) satisfy the Lipschitz assumptions having $\mathcal{L}_1 = \max\limits_{(\varphi,\eta)\in\mho} |\varsigma_1(\varphi,\eta)|$ and $\mathcal{L}_2 = \max\limits_{(\varphi,\eta)\in\mho} |\sigma_1(\varphi,\eta)|$ and hence (3.14) satisfies as a unique **(a)**-weak solution in $\mathbb{C}(\mho,\bar{\mathcal{W}}_1)\times\mathbb{C}(\mho,\bar{\mathcal{W}}_2)$.

Furthermore, let us demonstrate the presence of the **(b)**-weak findings for (3.14). To initiate, select $\delta = \gamma = 0.667$, $a_1 = 1, b_1 = 0.5$, $\rho = 1$, $\varsigma_1(\varphi,\eta) = \frac{9}{2(\Gamma(0.33))^2}\varphi^{1/3}\eta^{1/3}$, $\varsigma_2(\varphi,\eta) = \frac{-9\Xi}{2(\Gamma(0.33))^2}\varphi^{4/3}\eta^{4/3}$, $\sigma_1(\varphi,\eta) = \frac{9}{2(\Gamma(0.33))^2}\varphi^{1/3}\eta^{1/3}$ and $\sigma_2(\varphi,\eta) = \frac{9\Xi}{2(\Gamma(0.33))^2}\varphi^{4/3}\eta^{4/3}$, then (3.14) reduces to the coupled PDE:

$$\begin{cases} {}^{c}_{\mathfrak{g}\mathbf{H}}\mathcal{D}^{2/3;1}_{2} \mathbf{f}_1(\varphi,\eta) = \frac{-9}{2(\Gamma(0.33))^2}\varphi^{1/3}\eta^{1/3}\mathbf{f}_2(\varphi,\eta) - \frac{9\Xi}{2(\Gamma(0.33))^2}\varphi^{4/3}\eta^{4/3}, \\ {}^{c}_{\mathfrak{g}\mathbf{H}}\mathcal{D}^{2/3;1}_{2} \mathbf{f}_2(\varphi,\eta) = \frac{9}{2(\Gamma(0.33))^2}\varphi^{1/3}\eta^{1/3}\mathbf{f}_1(\varphi,\eta) + \frac{9\Xi}{2(\Gamma(0.33))^2}\varphi^{4/3}\eta^{4/3}, \\ \mathbf{f}_1(\varphi,0) = \mathbf{f}_1(0,\eta) = \mathbf{f}_1(0,0) = -2\Xi, \\ \mathbf{f}_2(\varphi,0) = \mathbf{f}_2(0,\eta) = \mathbf{f}_2(0,0) = 2\Xi. \end{cases} \tag{3.15}$$

Thus, the Lipschitz parameters are directly achieved as $\mathcal{L}_1 = \frac{4.5}{\sqrt[3]{2}(\Gamma(0.33))^2}$ and $\mathcal{L}_2 = \frac{4.5}{\sqrt[3]{2}(\Gamma(0.33))^2}$ with $\Psi(\varphi,\eta) = -2\Xi$ and $\Phi(\varphi,\eta) = 2\Xi$.

In the present review, we will fuzzify the predetermined solutions using the Buckley-Feuring approach proposed by Long et al. [38], we obtain $(\Lambda(\varphi,\eta,\Xi), \nabla(\varphi,\eta,\Xi)) = (-2\Xi - \Xi_{\varphi\eta}, 2\Xi - \Xi_{\varphi\eta})$, thus, by the Buckley-Feuring approach [49] for (3.15) to check the situation $(\mathbf{q}_1)$ in Theorem 3.2.

In Example 3.5, employing the Gaussian $\mathbf{F_n}$ Ξ along with the membership mapping $\Xi(\mathbf{t}) = 1/\exp(9(\mathbf{t}-\Omega_1))^2$, having Ω_1 as a crisp number. The τ_1-cuts and υ_1-cuts of Ξ are independently presented as

$$\left[\Omega_1(\tau_1), \Omega_2(\tau_1)\right] = \left(\frac{3\Omega_1 - \sqrt{-\ln\tau_1}}{3}, \frac{3\Omega_1 + \sqrt{-\ln\tau_1}}{3}\right),$$

$$\left[\Omega_1(\upsilon_1), \Omega_2(\upsilon_1)\right] = \left(\frac{3\Omega_1 - \sqrt{-\ln\upsilon_1}}{3}, \frac{3\Omega_1 + \sqrt{-\ln\upsilon_1}}{3}\right)$$

and the extended principle's continuity demonstrates that the fuzzified findings of (3.15) are

$$\left[\Lambda(\varphi,\Omega_1,\Xi)\right]^{\tau_1} = \left(-2\left(\frac{3\Omega_1 - \sqrt{-\ln\tau_1}}{3}\right) - \left(\frac{3\Omega_1 - \sqrt{-\ln\tau_1}}{3}\right)\varphi\eta, -2\left(\frac{3\Omega_1 + \sqrt{-\ln\tau_1}}{3}\right) - \left(\frac{3\Omega_1 + \sqrt{-\ln\tau_1}}{3}\right)\varphi\eta\right),$$

and

$$\left[\nabla(\varphi,\Omega_1,\Xi)\right]^{\upsilon_1} = \left(2\left(\frac{3\Omega_1 - \sqrt{-\ln\upsilon_1}}{3}\right) - \left(\frac{3\Omega_1 - \sqrt{-\ln\upsilon_1}}{3}\right)\varphi\eta, 2\left(\frac{3\Omega_1 + \sqrt{-\ln\upsilon_1}}{3}\right) - \left(\frac{3\Omega_1 + \sqrt{-\ln\upsilon_1}}{3}\right)\varphi\eta\right).$$

Now we demonstrate that the requirement $(\mathbf{q}_1)$ in Theorem 3.2 is satisfied, and we demonstrate the E-U of the **(b)**-weak findings for (3.15). For simplicity, choosing $\mathcal{K} = \frac{9}{(\Gamma(0.3))^2}$, we have

$$\left[(-1)\,{}^{RL}_{F}\mathcal{I}^{2/3;\rho}_{0^+}\mathbf{h}_1(\varphi,\eta,\mathbf{f}_2(\varphi,\eta))\right]^{\tau_1}$$

$$= \frac{\rho^{2/3}\mathcal{K}[\mathbf{f}_2]^{\tau_1}}{2(\Gamma(0.67))^2}\int_0^{\varphi}\int_0^{\eta} \mathbf{s}^{\rho-1}\mathbf{t}^{\rho-1}(\varphi^\rho - \mathbf{s}^\rho)^{-1/3}(\eta^\rho - \mathbf{t}^\rho)^{-1/3}\mathbf{s}^{1/3}\mathbf{t}^{1/3}d\mathbf{s}d\mathbf{t}$$

$$+\frac{2\rho^{2/3}\mathcal{K}[\Xi]^{\tau_1}}{(\Gamma(0.67))^2}\int_0^{\varphi}\int_0^{\eta}\mathbf{s}^{\rho-1}\mathbf{t}^{\rho-1}(\varphi^{\rho}-\mathbf{s}^{\rho})^{-1/3}(\eta^{\rho}-\mathbf{t}^{\rho})^{-1/3}\mathbf{s}^{1/3}\mathbf{t}^{1/3}d\mathbf{s}d\mathbf{t}$$
$$=\frac{[\mathbf{f}_2]^{\tau_1}\rho^{2/3}\varphi\eta}{2}+\frac{2\rho^{2/3}[\Xi]^{\tau_1}\varphi^2\eta^2}{9},$$

which indicates

$$len\big[(-1)\,{}^{RL}_{F}\mathcal{I}^{2/3;\rho}_{0^+}\mathbf{h}_1(\varphi,\eta,\mathbf{f}_2(\varphi,\eta))\big]^{\tau_1}\leq 0.2823\sqrt{-\ln\tau_1},$$

therefore

$$len[\Psi(\varphi,\eta)]^{\tau_1}\leq len\big[(-1)\,{}^{RL}_{F}\mathcal{I}^{2/3;\rho}_{0^+}\mathbf{h}_1(\varphi,\eta,\mathbf{f}_2(\varphi,\eta))\big]^{\tau_1}.$$

According to result of [53], it is noted that the **H**-difference $\Psi(\varphi,\eta)\ominus(-1)\,{}^{RL}_{F}\mathcal{I}^{2/3;\rho}_{0^+}\mathbf{h}_1(\varphi,\eta,\mathbf{f}_2(\varphi,\eta))$ holds.

It describes from the preceding evidence that

$$[\Psi(\varphi,\eta)]^{\tau_1}=\Big(-2\Big(\frac{3\Omega_1-\sqrt{-\ln\tau_1}}{3}\Big),-2\Big(\frac{3\Omega_1+\sqrt{-\ln\tau_1}}{3}\Big)\Big)$$

and

$$\big[(-1)\,{}^{RL}_{F}\mathcal{I}^{2/3;\rho}_{0^+}\mathbf{h}_1(\varphi,\eta,\mathbf{f}_2(\varphi,\eta))\big]^{\tau_1}=\Big(\frac{18\varphi\eta-5\varphi^2\eta^2}{18}\,\frac{3\Omega_1-\sqrt{-\ln\tau_1}}{3},\frac{18\varphi\eta-5\varphi^2\eta^2}{18}\,\frac{3\Omega_1+\sqrt{-\ln\tau_1}}{3}\Big).$$

Letting

$$\mathcal{Q}(\varphi,\eta)=\Psi(\varphi,\eta)\ominus(-1)\,{}^{RL}_{F}\mathcal{I}^{2/3;\rho}_{0^+}\mathbf{h}_1(\varphi,\eta,\mathbf{f}_2(\varphi,\eta)),$$

therefore, in view of Example 5.1 in [38], we have

$$\big[\mathcal{Q}(\varphi,\eta)\big]^{\tau_1}=\Big(\frac{-36-18\varphi\eta+5\varphi^2\eta^2}{18}\,\frac{3\Omega_1-\sqrt{-\ln\tau_1}}{3},\frac{-36-18\varphi\eta+5\varphi^2\eta^2}{18}\,\frac{3\Omega_1+\sqrt{-\ln\tau_1}}{3}\Big)$$

and

$$len\big[\mathcal{Q}(\varphi,\eta)\big]^{\tau_1}=0.667\sqrt{-\ln\tau_1}\,\frac{-36-18\varphi\eta+5\varphi^2\eta^2}{18}.$$

According to the preceding procedure, we have

$$\big[(-1)\,{}^{RL}_{F}\mathcal{I}^{2/3;\rho}_{0^+}\mathbf{h}_1(\varphi,\eta,\mathcal{Q}(\varphi,\eta))\big]^{\tau_1}=\frac{[\mathcal{V}]^{\tau_1}\rho^{2/3}\varphi\eta}{2}+\frac{2\rho^{2/3}[\Xi]^{\tau_1}\varphi^2\eta^2}{9}$$

and

$$\begin{aligned}len\big[(-1)\,{}^{RL}_{F}\mathcal{I}^{2/3;\rho}_{0^+}\mathbf{h}_1(\varphi,\eta,\mathcal{Q}(\varphi,\eta))\big]^{\tau_1}&=0.667\sqrt{-\ln\tau_1}\Big(\frac{-36\varphi\eta-10\varphi^2\eta^2}{36}\Big)\\&\leq 0.2823\sqrt{-\ln\tau_1},\end{aligned}$$

which demonstrates that the **H**-difference

$$\Psi(\varphi,\eta)\ominus(-1)\,{}^{RL}_{F}\mathcal{I}^{2/3;\rho}_{0^+}\mathbf{h}_1(\varphi,\eta,\mathcal{Q}(\varphi,\eta))$$

holds.

Similarly, we have

$$\big[(-1)\,{}^{RL}_{F}\mathcal{I}^{2/3;\rho}_{0^+}\mathbf{h}_2(\varphi,\eta,\psi(\varphi,\eta))\big]^{\upsilon_1}=\frac{[\mathbf{f}_1]^{\upsilon_1}\rho^{2/3}\varphi\eta}{2}-\frac{2\rho^{2/3}[\Xi]^{\upsilon_1}\varphi^2\eta^2}{9}$$

and

$$\begin{aligned}len\big[(-1)\,{}^{RL}_{F}\mathcal{I}^{2/3;\rho}_{0^+}\mathbf{h}_2(\varphi,\eta,\mathbf{f}_1(\varphi,\eta))\big]^{\upsilon_1}&=0.667\sqrt{-\ln\tau_1}\Big(\frac{-36\varphi\eta-10\varphi^2\eta^2}{36}\Big)\\&\leq 0.3373\sqrt{-\ln\upsilon_1}.\end{aligned}$$

It follows that

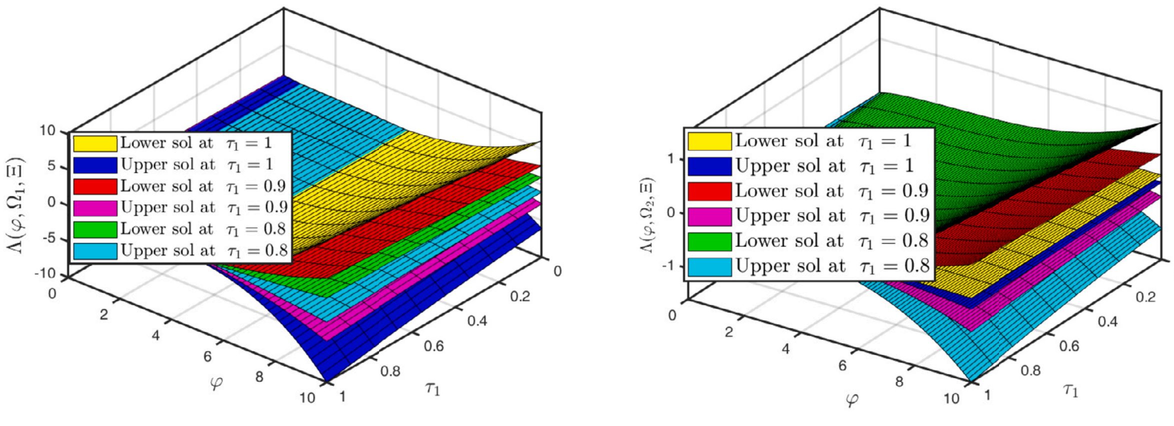
$$len[\Phi(\varphi,\eta)]^{\upsilon_1}\geq len\big[(-1)\,{}^{RL}_{F}\mathcal{I}^{2/3;\rho}_{0^+}\mathbf{h}_2(\varphi,\eta,\mathbf{f}_1(\varphi,\eta))\big]^{\upsilon_1}.$$

It is clear that $\mathcal{V}(\varphi,\eta)=\Phi(\varphi,\eta)\ominus(-1)\mathcal{I}^{2/3;\rho}_{0^+}\mathbf{h}_2(\varphi,\eta,\mathbf{f}_1(\varphi,\eta))$ holds using the results of [53] and

$$len\big[(-1)\,{}^{RL}_{F}\mathcal{I}^{2/3;\rho}_{0^+}\mathbf{h}_2(\varphi,\eta,\mathcal{V}(\varphi,\eta))\big]^{\upsilon_1}\leq 0.3373\sqrt{-\ln\upsilon_1}.$$

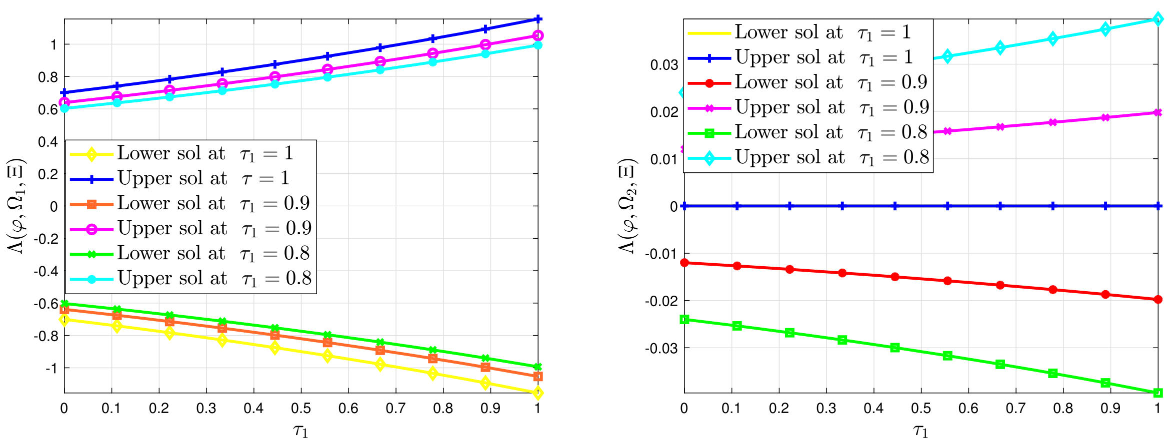
In other words, the **H**-difference $\Phi(\varphi,\eta)\ominus(-1)\mathcal{I}^{2/3;\rho}_{0^+}\mathbf{h}_2(\varphi,\eta,\mathcal{V}(\varphi,\eta))$ exists. Hence, (3.15) has a unique ((**b**))-weak results in $\mathbb{C}(\mho_1,\bar{\mathcal{W}}_1)\times\mathbb{C}(\mho_1,\bar{\mathcal{W}}_2)$.

Fig. 1 indicates the surface view of $[\Lambda(\varphi,\Omega_1,\Xi)]^{\tau_1}$ with lower and upper accuracies. With the regulating parameters φ, η and the fluctuating values of $\tau_1\in[0,1]$, the graph anticipates the lower and upper solutions.



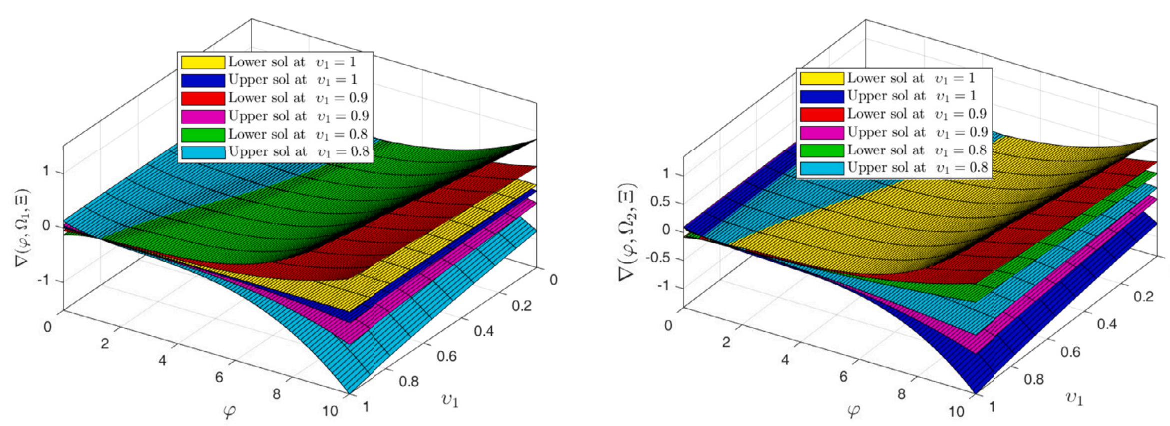
(a) (b)

Fig. 1. Three-dimensional view for fuzzified findings of (3.15) when Gaussian fuzzy numbers are $[\Xi]^{\tau_1} = [\frac{3-\sqrt{-\ln\tau_1}}{3}, \frac{3+\sqrt{-\ln\tau_1}}{3}]$ with $\tau_1 \in [0, 1]$.



(a) (b)

Fig. 2. Two-dimensional view for fuzzified findings of (3.15) when Gaussian $\mathbf{F}_{\mathbf{n}}s$ $[\Xi]^{\tau_1} = [\frac{3-\sqrt{-\ln\tau_1}}{3}, \frac{3+\sqrt{-\ln\tau_1}}{3}]$ with $\tau_1 \in [0, 1]$.



(a) (b)

Fig. 3. Three-dimensional view for fuzzified findings of (3.15) when Gaussian $\mathbf{F}_{\mathbf{n}}s$ $[\Xi]^{\upsilon_1} = [\frac{3-\sqrt{-\ln\upsilon_1}}{3}, \frac{3+\sqrt{-\ln\upsilon_1}}{3}]$ with $\upsilon_1 \in [0, 1]$.

Fig. 2 indicates the two-dimensional view of $[\Lambda(\varphi, \Omega_2, \Xi)]^{\tau_1}$ with lower and upper accuracies. With the regulating parameters φ and fixed $\eta = 0.7$, the fluctuating values of $\tau_1 \in [0, 1]$, the graph anticipates the lower and upper solutions.

Analogously, Fig. 3 indicates the surface view of $[\nabla(\varphi, \Omega_1, \Xi)]^{\upsilon_1}$ with lower and upper accuracies. The graph predicts the lower and upper solutions with the governing parameters φ, η and the varying values of $\upsilon_1 \in [0, 1]$.

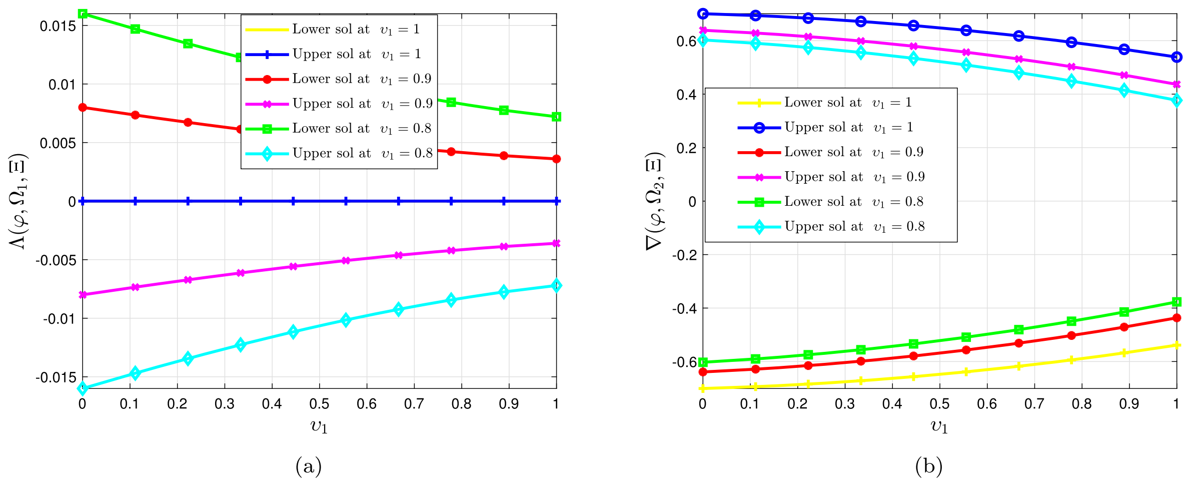


Fig. 4. Two-dimensional view for fuzzified findings of (3.15) when Gaussian fuzzy numbers are $[\Xi]^{\upsilon_1} = [\frac{3-\sqrt{-\ln \upsilon_1}}{3}, \frac{3+\sqrt{-\ln \upsilon_1}}{3}]$ with $\upsilon_1 \in [0,1]$.

Fig. 4 indicates the two-dimensional view of $[\nabla(\varphi, \Omega_2, \Xi)]^{\upsilon_1}$ with lower and upper accuracies. The graph predicts the lower and upper solutions with the governing parameters φ and the varying values of $\upsilon_1 \in [0,1]$ when $\eta = 0.7$.

If the crisp quantity Ω_1 is found in the membership mappings of the $\mathbf{F}_{\mathfrak{n}}$s, the illustration of the recombination solution and its level set is altered with several explanatory factors, as shown in Figs. 1, 2, 3 and 4. Nevertheless, we may further obtain the impression when Ω_1 is constantly changing. This should be accelerated.

4. Application in engineering

In this part, we demonstrate the technical soundness of the suggested approach using several mathematical simulations and a handful of potential applications in engineering. The C-K fuzzy fractional differentiability of order $\delta \in (0,1]$ is the basis for the absolute errors of the issues under multiple scenarios, which show the efficacy of the continuous dependence and ϵ-approximation approach.

Keep in mind that the approaches' effectiveness is evaluated by calculating their absolute errors. For a constant φ and different choices of η, the formulas $\underline{\mathcal{E}}(\varphi,\eta) = \left|\underline{\mathbf{h}}_{\mathbf{e}}(\varphi,\eta) - \underline{\mathbf{H}}_{\mathbf{e}}(\varphi,\eta)\right|$ and $\overline{\mathcal{E}}(\varphi,\eta) = \left|\overline{\mathbf{h}}_{\mathbf{e}}(\varphi,\eta) - \overline{\mathbf{H}}_{\mathbf{e}}(\varphi,\eta)\right|$ yield the error solutions and $\mathbf{h}_{\mathbf{e}}(\varphi,\eta) = \left(\underline{\mathbf{h}}_{\mathbf{e}}(\varphi,\eta), \overline{\mathbf{h}}_{\mathbf{e}}(\varphi,\eta)\right)$ are proven the approximate solutions, while $\mathbf{H}_{\mathbf{e}}(\varphi,\eta) = \left(\underline{\mathbf{H}}_{\mathbf{e}}(\varphi,\eta), \overline{\mathbf{H}}_{\mathbf{e}}(\varphi,\eta)\right)$ is the exact solution. Regarding this, the suggested approach has a comparison in absolute errors with the existing methods [55] and [56], respectively.

This example demonstrates one potential implementation of the FFD model in the viscoelasticity domain. Viscoelasticity is the quality of substances that, when subjected to stretching, display both viscosity and elastic features. As a consequence, a substance may exhibit time-dependent behavior, whereby its reaction to displacement or stress may vary with time. Depending on the speed at which we transmit strain or how long the pressure or displacement continues to exist, basic synthetic substances respond to it in a similar way. When constructing equipment that operates with or interfaces with organic substances or polymer compounds, researchers must have a solid understanding of viscoelasticity in order to prevent incidents like the **Big Dig ceiling collapse**, which occurred on July 10, 2006 [54].

In mathematical terms, we take into account viscoelasticity within fuzzy-valued mappings that indicate unpredictability. Let us begin by investigating the connection involving stress and strain for Newtonian substances (Newton's law) and solids (Hooke's law), respectively:

$$\begin{cases} \zeta(\eta) = \mathfrak{I}\mathbf{e}(\eta), \\ \zeta(\eta) = \Theta \frac{d}{d\eta}\mathbf{e}(\eta). \end{cases} \tag{4.1}$$

In (4.1), $\mathfrak{I}$ and Θ denote the viscosity and spring constant, respectively. However, given that strain is proportional to both the first and zeroth differentiation of strain for fluids and solids, it makes sense to assume that, in the case of "intermediate" materials, stress may be related to the stress derivative of "intermediate" (classical derivative).

$$\zeta(\eta) = \theta\, {}^{c}_{\mathfrak{g}\mathbf{H}}\mathcal{D}^{\delta;\rho}_{\eta}\mathbf{e}(\eta), \quad \delta \in (0,1), \quad \rho > 0. \tag{4.2}$$

Here, (4.1) indicates Hooke's law and (4.2) signifies the Scott Blair law model which contains the C-K $\mathfrak{g}\mathbf{H}$ derivative of stress and strain that can be subsequently expanded by including more terms on each side, this results in a generalized Voigt model:

$$\zeta(\eta) = \mathfrak{I}\mathbf{e}(\eta) + \theta\, {}^{c}_{\mathfrak{g}\mathbf{H}}\mathcal{D}^{\delta;\rho}\mathbf{e}(\eta). \tag{4.3}$$

It explains how an immovable sheet submerged in a Newtonian fluid moves.

Currently, we employ the fuzzy IVP $\mathbf{e}_0$, the fuzzy-valued mapping $\zeta(\eta)$, and the idea of C-K $\mathfrak{g}\mathbf{H}$-differentiability for the first order derivative of $\mathbf{e}(\eta)$, $\mathbf{e}'(\eta)$, to attempt to explore this issue in a real-world scenario.

Let us examine the fuzzy framework that describes how a stiff plate moves when submerged in a Newtonian fluid:

$$\begin{cases} {}^{c}_{\mathfrak{g}\mathbf{H}}\mathcal{D}^{\delta;\rho}_{0^+}\mathbf{e}(\eta) + \mathbf{e}(\eta) = \eta^4 - \frac{\eta^3}{2} - \frac{3\rho^{1-\delta}\eta^{3-\delta}}{\Gamma(4-\delta)} + \frac{24\rho^{2-\delta}\eta^{4-\delta}}{\Gamma(5-\delta)}, \\ \mathbf{e}(0,\mathfrak{r}) = \left[-1+\mathfrak{r}, 1-\mathfrak{r}\right], \quad \delta \in (0,1], \quad \rho > 0, \quad \mathfrak{r} \in [0,1], \quad \eta \in [0,1] \end{cases} \tag{4.4}$$

for which $\mathbf{e} \in \mathbf{f}_1 \in \mathbb{C}(\mho, \bar{\mathcal{W}}) \cap \mathbb{L}^1(\mho, \bar{\mathcal{W}})$ indicates a continuous fuzzy mapping and $\mathfrak{I} = \Theta = 1$.

Table 1
Comparison analysis of the absolute error $\mathcal{E}$ for viscoelasticity when $\rho = 1$ and $\delta = 0.96$.

$\mathfrak{r}$	$\underline{\mathbf{H}_e}(1,\mathfrak{r})$	$\underline{\mathcal{E}}(1,\mathfrak{r})$	$\underline{\mathcal{E}}(1,\mathfrak{r})$ ([55])	$\underline{\mathcal{E}}(1,\mathfrak{r})$ ([56])	$\overline{\mathbf{H}_e}(1,\mathfrak{r})$	$\overline{\mathcal{E}}(1,\mathfrak{r})$	$\overline{\mathcal{E}}(1,\mathfrak{r})$ ([55])	$\overline{\mathcal{E}}(1,\mathfrak{r})$ ([56])
0	0.18732	5.010050E-4	1.010055E-3	3.010053E-1	0.87126	5.542000E-4	1.084000E-3	2.894931E-1
0.1	0.15642	4.906505E-4	9.906755E-4	3.906630E-1	0.83123	4.650750E-4	9.501500E-4	2.968828E-1
0.2	0.18073	4.053467E-4	8.353522E-4	3.453994E-1	0.80562	4.175635E-4	8.155127E-4	2.960875E-1
0.3	0.22869	3.596683E-4	7.596941E-4	3.596812E-1	0.77534	3.686555E-4	7.573110E-4	2.819464E-1
0.4	0.28912	3.535201E-4	6.435693E-4	3.635447E-1	0.73210	3.157595E-4	6.457595E-4	3.075564E-1
0.5	0.31672	2.418016E-4	5.118828E-4	3.418422E-1	0.70012	2.459165E-4	5.118330E-4	3.040526E-1
0.6	0.33657	2.189691E-4	4.189810E-4	3.389751E-1	0.67634	2.037750E-4	4.187550E-4	3.097247E-1
0.7	0.39452	1.663159E-4	3.163298E-4	3.632290E-1	0.59891	1.525400E-4	3.173080E-4	3.109852E-1
0.8	0.39823	1.067701E-4	2.077710E-4	3.777058E-1	0.56891	1.043350E-4	2.048670E-4	3.108110E-1
0.9	0.47231	5.120056E-5	1.020078E-4	3.120067E-1	0.52319	5.165150E-5	1.130300E-4	3.106350E-1
1	0.49001	9.976778E-9	9.4976070E-7	3.197872E-1	0.50000	9.957552E-9	9.415103E-7	3.168396E-1

By means of Definition 2.4, we have

$$\begin{cases} {}^{c}_{\mathfrak{g}\mathbf{H}}\mathcal{D}^{\delta;\rho}_{0^+}\mathbf{e}_-(\eta,\mathfrak{r}) + \mathbf{e}_-(\eta,\mathfrak{r}) = \eta^4 - \frac{\eta^3}{2} - \frac{3\rho^{1-\delta}\eta^{3-\delta}}{\Gamma(4-\delta)} + \frac{24\rho^{2-\delta}\eta^{4-\delta}}{\Gamma(5-\delta)}, \\ \mathbf{e}_-(0,\mathfrak{r}) = -1+\mathfrak{r}, \quad \delta\in(0,1], \ \rho>0, \quad \mathfrak{r}\in[0,1], \ \eta\in[0,1] \end{cases} \tag{4.5}$$

and

$$\begin{cases} {}^{c}_{\mathfrak{g}\mathbf{H}}\mathcal{D}^{\delta;\rho}_{0^+}\mathbf{e}_+(\eta,\mathfrak{r}) + \mathbf{e}_+(\eta,\mathfrak{r}) = \eta^4 - \frac{\eta^3}{2} - \frac{3\rho^{1-\delta}\eta^{3-\delta}}{\Gamma(4-\delta)} + \frac{24\rho^{2-\delta}\eta^{4-\delta}}{\Gamma(5-\delta)}, \\ \mathbf{e}_+(0,\mathfrak{r}) = 1-\mathfrak{r}, \quad \delta\in(0,1], \ \rho>0, \quad \mathfrak{r}\in[0,1], \ \eta\in[0,1] \end{cases} \tag{4.6}$$

have the integer-order solution as:

$$\begin{cases} \mathbf{e}_-(\eta,\mathfrak{r}) = (\mathfrak{r}-1)\mathbf{E}_{\delta,1}\Big[-\frac{\eta^\rho}{\rho}\Big] + \int\limits_0^\eta \varphi^{\rho-1}(\eta^\rho-\varphi^\rho)^{\delta-1}\mathbf{E}_{\delta,\delta}\Big[-\Big(\frac{\eta^\rho-\varphi^\rho}{\rho}\Big)^\delta\Big]\zeta(\varphi)d\varphi, \quad \mathfrak{r}\in[0,1], \\ \mathbf{e}_+(\eta,\mathfrak{r}) = (1-\mathfrak{r})\mathbf{E}_{\delta,1}\Big[-\frac{\eta^\rho}{\rho}\Big] + \int\limits_0^\eta \varphi^{\rho-1}(\eta^\rho-\varphi^\rho)^{\delta-1}\mathbf{E}_{\delta,\delta}\Big[-\Big(\frac{\eta^\rho-\varphi^\rho}{\rho}\Big)^\delta\Big]\zeta(\varphi)d\varphi, \quad \mathfrak{r}\in[0,1] \end{cases} \tag{4.7}$$

for which $\zeta(\eta) = \eta^4 - \frac{\eta^3}{2} - \frac{3\rho^{1-\delta}\eta^{3-\delta}}{\Gamma(4-\delta)} + \frac{24\rho^{2-\delta}\eta^{4-\delta}}{\Gamma(5-\delta)}$. In view of (4.5)-(4.7) and Theorem 3.2, choosing $\mathcal{N}=2$, we can express the lower and upper estimates of the fuzzy solution scheme as:

$$\mathbf{e}_-(\eta,0.1) = (-0.4543 \quad -0.0269 \quad 0.0598)\begin{pmatrix} 1 \\ 2\eta-1 \\ 8\eta^2-8\eta+1 \end{pmatrix} \tag{4.8}$$

and

$$\mathbf{e}_+(\eta,0.1) = (0.6831 \quad 0.5221 \quad 0.1892)\begin{pmatrix} 1 \\ 2\eta-1 \\ 8\eta^2-8\eta+1 \end{pmatrix}. \tag{4.9}$$

It is actually possible to find a novel approximate solution by varying the higher and lower numbers. On the other hand, this strategy will result in a greater number of inaccuracies; this is one of the primary distinctions between our suggested perspective and classical techniques that examined the outcome by merely utilizing the lower and upper approximations without taking the fuzzy condition into account. Thus, we simply verify the fact that $\mathbf{e}_-(\eta,0.1)$ is less than $\mathbf{e}_+(\eta,0.1)$ at the conclusion of the estimate. Actually, we execute it to get a more accurate outcome that has fewer errors.

Table 1 presents the relationship between the absolute errors of the suggested procedure and utilizing the technique of Mazandarani and Kamyad [55] and the scheme applied by Bhrawy et al. [56] at $\varphi = 1$. It is evident that the suggested approach uses just a handful of values for [55] to obtain an accurate estimate alongside the integer-order solution. Table 1 shows that the fractional Euler technique [55] does not constitute an excellent choice for complex FFDE structures due to its inadequate precision. Additionally, the suggested approach is contrasted with the laguerre mapping [56], which has an identical number of functionalities. However, our technique performs more efficiently at the culminating points of $\mathfrak{r}$-cuts in addition to having improved precision.

Absolute error is shown for multiple values of the $C-K$ $\mathfrak{g}\mathbf{H}$-differentiability in Fig. 5(**a**). It should be noted that the numerical estimate of the classical FDE comes together to the analytical solution as δ,ρ gets closer to 1 (i.e., the error diminishes progressively). Absolute error is shown for multiple values of $\mathfrak{r}$ in Fig. 5(**b**). It is evident that the absolute inaccuracy is reducing for the fractional order $\delta = 0.96$ (see; Fig. 5(**c**)). It is crucial to remember that this trend fails to appear as the number of weak solutions increases because the suggested approach consistently corresponds to the result.

Fig. 6(**a**) shows the closed form solution, which is specified on the domains $\eta\in[0,1]$ and $\mathfrak{r}\in[0,1]$. The weak fuzzified solution for (4.4) is found and shown in Fig. 6(**b**). Clearly, the weak fuzzified solution is exceptionally precise throughout the interval points, especially close to the starting and end points, as Figs. 5(**c**)) and 6 demonstrate. The profiles of Figs. 6(**a**) and **(b)** are nearly identical, notwithstanding this.

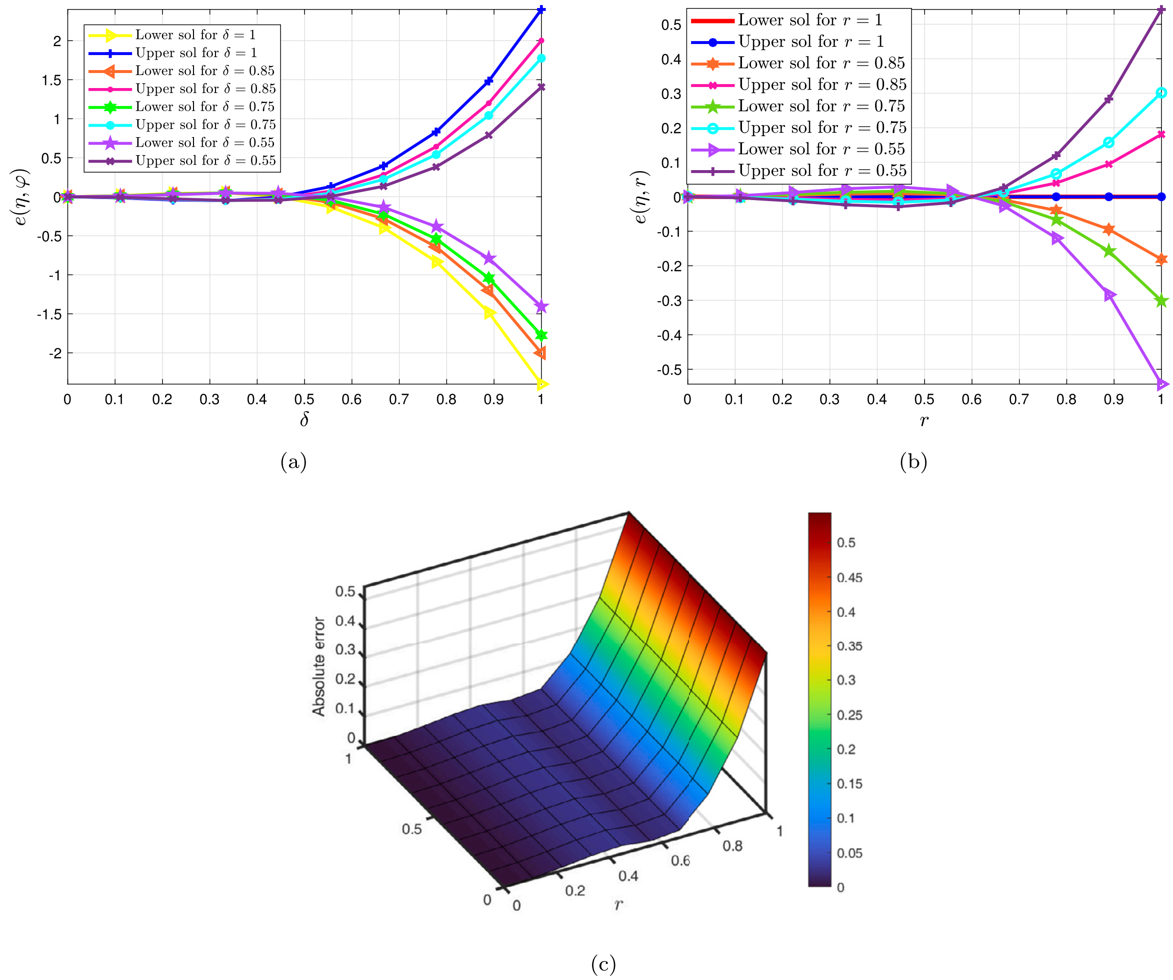


Fig. 5. Viscoelasticity: **(a)** Absolute errors $\underline{\mathcal{E}}$ and $\overline{\mathcal{E}}$ of the suggested technique for various values of fractional order using C-K $\mathfrak{g}\mathbf{H}$ differentiability **(b)** for various values $\mathfrak{r}$ when $\delta = 0.96$ **(c)** Absolute error $\underline{\mathcal{E}}$ for $\mathfrak{r} \in [0,1]$, $\eta \in [0,1]$ $\rho = 1$ and $\delta = 0.96$.

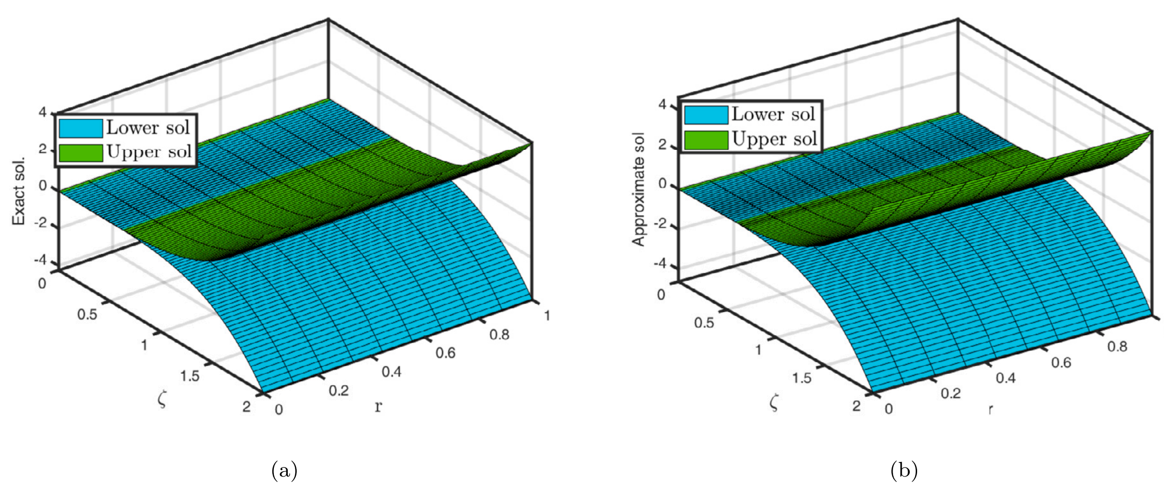


Fig. 6. Three-dimensional view for fuzzified findings **(a)** the exact solution and **(b)** the fuzzy solution of the motion model of a rigid plate immersed in a Newtonian fluid when $\delta = 1$.

It seems that the fractional Euler method is not an appropriate option for certain types of FDE, particularly if we demand better precision, even if the [55] execution is much more straightforward than our suggested solution. Table 2 demonstrates that the result may be obtained for iterating up to 10, that the maximum absolute error is $3.096967E{-}4$ and that the computational cost on an Intel (Core i7) $3.40GHz$ processor is 6.9832 seconds. Moreover, authors [56] might only obtain a maximum absolute error of $8.657634E{-}3$, and the computation cost is 12.59 seconds, assuming the same number of iterations. The last approach obviously has a greater computation cost than what we suggested, but it additionally features a much smaller maximum absolute error. This is for two very significant considerations.

Table 2
Comparison analysis of the maximum absolute error $\mathcal{E}$ for viscoelasticity when $\rho = 1$ and $\delta = 0.96$.

Technique	max($\mathcal{E}$)	Processing time (seconds)
Suggested approach	3.09697E-4	6.9832
[55]	3.5672E-1	0.00989
[56]	8,65763E-3	12.5900

5. Conclusion

The primary goal of this research is to demonstrate several requisite comparison formalisms that will be used to achieve our core concept. We take into account an IVP for the C-K fuzzy coupled system of FPDEs and employ the mathematical inductive procedure via the generalized Lipschitz assumptions to verify the E-U of the solution. The results of fuzzy FPDEs must be applied to governing equations in the context of the C-K $\mathfrak{g}\mathbf{H}$-type derivative. We demonstrated the E-U of two sorts of $\mathfrak{g}\mathbf{H}$-weak solutions utilizing the Banach f_p hypothesis and the mathematical inductive technique. Furthermore, we illustrated the E-U theorems graphically and suggested analytical computations of the $\mathfrak{g}\mathbf{H}$-weak findings for the proposed framework. The transformation of (1.3) to a collection of dynamic fractional coupled Volterra integro-differential frameworks was established and the vector type's Gronwall variant has been acquired.

Besides that, after modifying the ICs, the continuous dependence on the provided information and ϵ-approximate results of (1.3) were formed creatively on the justification of the advanced Gronwall variant of the matrix pattern, which is due to the interacting component in (1.3). Application from an engineering perspective has been provided in the graphical and simulation contexts. In comparison with the previous findings, the approach provided by [55] and [56] is the simplest to put into practice, but it has restrictions and an inadequate degree of correctness when it comes to offering a numerical solution for a variety of fuzzy fractional situations. Indeed, our system has several features over previous FFDE procedures, including **(i)** excellent preciseness, **(ii)** inexpensive computation cost and **(iii)** straightforward execution.

While this research currently only addresses FPDEs, we plan to use the similar methodology presented in this paper to expand our research to encompass additional forms of mechanisms, including fuzzy random FDE, fuzzy functional FDE, nonlinear systems, time-delay, neural networks, signal processing, associative memory, pattern recognition and other mathematical and engineering problems. Both variational inequalities and optimization challenges can be successfully resolved using it. How can the existence and uniqueness of fuzzy optimum control solutions involving parameters be established by employing the output of fuzzy fractional systems of PDEs employing projection operators? On the other hand, the proposed methodology can be extended to time delays, stochastic disturbances, and fractional derivatives of the Atanagana-Baleanu $\mathfrak{g}\mathbf{H}$ type. These are the kinds of concerns that should be addressed during the subsequent stage of this project. This is a significant piece of research that merits more investigation.

Nomenclature

PDEs	Partial differential equations;
FPDEs	Fractional partial differential equations;
DEs	Differential equations;
FDEs	Fractional Differential equations;
$\mathfrak{g}\mathbf{H}$	Generalized Hukuhara;
IVP	Initial value problems;
C-K	Caputo-Katugampola;
ICs	Initial Conditions;
f_p	fixed point;
CM	Contraction mapping;
E-U	Existence-Uniqueness.

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All authors have equal contribution.

Declaration of competing interest

Authors declare that there are no conflicts of interest.

Data availability

All the data used in this work is included within the paper.

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