Kuwait Journal of Science 51 (2024) 100130

Contents lists available at ScienceDirect

Kuwait Journal of Science

journal homepage: www.sciencedirect.com/journal/kuwait-journal-of-science

Full Length Article

On the complementary nabla Pachpatte type dynamic inequalities via convexity

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ARTICLE INFO

Keywords:
Time scale calculus
Hardy's inequality
Copson's inequality
Pachpatte's inequality
Convexity

ABSTRACT

Pachpatte type inequalities are convex generalizations of the well-known Hardy-Copson type inequalities. As Hardy-Copson type inequalities and convexity have numerous applications in pure and applied mathematics, combining these concepts will lead to more significant applications that can be used to develop certain branches of mathematics such as fuctional analysis, operator theory, optimization and ordinary/partial differential equations. We extend classical nabla Pachpatte type dynamic inequalities by changing the interval of the exponent δ from $\delta > 1$ to $\delta < 0$. Our results not only complement the classical nabla Pachpatte type inequalities but also generalize complementary nabla Hardy-Copson type inequalities. As the case of $\delta < 0$ has not been previously examined, these complementary inequalities represent a novelty in the nabla time scale calculus, specialized cases in continuous and discrete scenarios, and in the dual outcomes derived in the delta time scale calculus.

Introduction

Since Hardy's inequality is one of those inequalities which turns information about derivatives of functions into information about the size of the function, it is an essential part of all areas of mathematics and useful in various applications.

In this paper, we obtain new nabla Pachpatte type inequalities, which are convex generalizations of Hardy-Copson type dynamic inequalities, by changing the interval of the exponent δ from $\delta > 1$ to $\delta < 0$. This new interval will provide new inequalities which are complementary to the previous ones obtained for $\delta > 1$.

The classical discrete Hardy inequality was established by Hardy (1920) as

$$\sum_{k=1}^{\infty}\left(\frac{1}{k}\sum_{j=1}^{k}d(j)\right)^{\delta} \leq \left(\frac{\delta}{\delta-1}\right)^{\delta}\sum_{k=1}^{\infty}d^{\delta}(k), \tag{1}$$

where d(k)≥0, δ>1 and the continuous versions were derived by (Hardy et al. (1934, Theorem 330) as

$$\int_0^{\infty}\frac{P^{\delta}(t)}{t^{\xi}}dt \leq \begin{cases}\left(\dfrac{\delta}{\xi-1}\right)^{\delta}\displaystyle\int_0^{\infty}\frac{p^{\delta}(t)}{t^{\xi-\delta}}dt, & \text{if } \xi>1 \text{ and } P(t)=\displaystyle\int_0^{t}p(s)ds,\\ \left(\dfrac{\delta}{1-\xi}\right)^{\delta}\displaystyle\int_0^{\infty}\frac{p^{\delta}(t)}{t^{\xi-\delta}}dt, & \text{if } \xi<1 \text{ and } P(t)=\displaystyle\int_t^{\infty}p(s)ds,\end{cases} \tag{2}$$

where $p(t) \geq 0$, $\delta > 1$.

Since various generalizations and numerous variants of the discrete and continuous Hardy inequalities (1)–(2) exist in the literature, all of which can not be mentioned here, we only focus on the extensions which have been established in Copson (1928, Theorem 1.1, Theorem 2.1) and in Copson (1976 Theorem 1, Theorem 3). As we follow the way that Hardy and Copson lead us, we call these inequalities as Hardy-Copson type inequalities.

Hardy-Copson type inequalities have attracted many mathematicians for almost a century and many refinements and new proofs for the discrete and continuous cases have appeared in the books (Hardy et al., 1934; Masmoudi, 2011; Balinsky et al., 2015; Kufner et al., 2007, 2017) and in the articles (Bennett, 1987; Leindler, 1993; Chu et al., 2014; Liao,

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https://doi.org/10.1016/j.kjs.2023.09.004
Received 16 January 2023; Received in revised form 18 September 2023; Accepted 21 September 2023

2015; Gao and Zhao, 2020; Lefèvre, 2020; Beesack, 1961; Pachpatte, 1999; Iddrisu et al., 2014; Nikolidakis, 2014), respectively.

The convex generalizations of Hardy-Copson type inequalities have appeared in the literature after the celebrated papers of Pachpatte. The discrete Pachpatte type inequalities have been established by Pachpatte (1990a) and by Hwang and Yang (1990) for a real-valued positive convex function $H(u)$ defined for $u > 0$ and for the sequences $d(k) > 0$, $e(k) > 0$ as

$$\sum_{m=1}^{\infty} d(m)H^{\delta}\left(\frac{E(m)}{D(m)}\right) \leq \left(\frac{\delta}{\delta-1}\right)^{\delta} \sum_{m=1}^{\infty} d(m)H^{\delta}(e(m)),\ \delta > 1, \tag{3}$$

where $D(m) = \sum_{j=1}^{m} d(j)$ and $E(m) = \sum_{j=1}^{m} d(j)e(j)$. Note that choosing $H(u) = u$ and $d(m) = 1$ for all $m \geq 1$ in inequality (3) yields Hardy's inequality (1).

The first continuous convex extension of Hardy inequality (2) has been established in Levinson (1964) by assuming a condition on the convex function in this manner: Let $H(u)$ be a real-valued nonnegative convex function defined for $u > 0$ satisfying

$$HH'' \geq \left(\frac{\delta-1}{\delta}\right)(H')^2, \quad 1 < \delta \in \mathbb{R}. \tag{4}$$

If $p(t) > 0$ is nondecreasing function and $r(t) \geq 0$, then Levinson (1964) obtained following inequality

$$\int_0^{\infty} H\left(\frac{R(t)}{P(t)}\right)dt \leq \left(\frac{\delta}{\delta-1}\right)^{\delta} \int_0^{\infty} H(r(t))dt, \quad \delta > 1, \tag{5}$$

where $P(t) = \int_0^t p(s)ds$ and $R(t) = \int_0^t p(s)r(s)ds$.

Pachpatte (1990b) improved and generalized Levinson's inequality (5) by removing condition (4) on a nonnegative convex function H and by taking into account the constant $\xi > 1$ for real-valued integrable functions $p(t) > 0$, $r(t) \geq 0$, and for the constants $\delta \geq 1$, $\xi > 1$ as

$$\int_0^{\infty} \frac{p(s)}{P^{\xi-\delta}(t)} H^{\delta}\left(\frac{R(t)}{P(t)}\right)dt \leq \left(\frac{\delta}{\xi-1}\right)^{\delta} \int_0^{\infty} p(t)\frac{H^{\delta}(r(t))}{P^{\xi-\delta}(t)}dt, \tag{6}$$

where the functions P and R are defined as above.

Another result obtained by Pachpatte (1994) for a real valued nonnegative convex function $H(u)$ defined for $u > 0$, for real-valued integrable functions $p(t) > 0$, $r(t) \geq 0$ and for the constants $\delta \geq 1$, $\kappa \geq 0$ is as folows.

$$\int_0^{\infty} p(s)H^{\delta+\kappa}\left(\frac{R(t)}{P(t)}\right)dt \leq \left(\frac{\delta+\kappa}{\delta+\kappa-1}\right)^{\delta} \int_0^{\infty} \frac{p(t)H^{\delta}(r(t))Q^{\kappa}(t)}{P^{\kappa}(t)}dt, \tag{7}$$

where the functions P and R are defined as above and $Q(t) = \int_0^t p(s)H(r(s))ds$.

A generalization of Pachpatte's inequalities (6) and (7) was established by Pečarić and Hanjš (1999) for a real valued nonnegative convex function $H(u)$ defined for $u > 0$, for real-valued integrable functions $p(t) > 0$, $r(t) \geq 0$ and for the constants $\delta \geq 1$, $\kappa \geq 0$, $\kappa + \xi > 1$ as

$$\int_0^{\infty} \frac{p(s)}{P^{\xi-\delta}(t)} H^{\delta+\kappa}\left(\frac{R(t)}{P(t)}\right)dt \leq \left(\frac{\delta+\kappa}{\xi+\kappa-1}\right)^{\delta} \int_0^{\infty} p(t)\frac{Q^{\kappa}(t)}{P^{\kappa+\xi-\delta}(t)}H^{\delta}(r(t))dt, \tag{8}$$

where the functions P, R and Q are defined as above. Note that choosing $\kappa = 0$ and $\xi = \delta$ in inequality (8) yields inequalities (6) and (7), respectively.

After the invention of the calculus on time scales (Bohner and Peterson, 2001, 2003; Guseinov and Kaymakçalan, 2002; Atici & Guseinov, 2002; Gürses et al., 2005), many well-known inequalities have been expanded to an arbitrary time scale. These inequalities have been obtained in delta time scale calculus (Agarwal et al., 2001, 2014; Saker, 2012) as well as in the nabla case (Anderson, 2005; Özkan et al., 2008; Güvenilir et al., 2015; Bohner et al., 2015; Pelen, 2019; Kayar et al., 2021; Kayar and Kaymakçalan, 2022b).

Establishing dynamic Hardy-Copson type inequalities has been started by the delta approach and these unifications can be found in the book (Agarwal et al., 2016) and in the articles (Řehák, 2005; Saker et al., 2014a,b, 2017, 2018a,b, Saker and Mahmoud, 2019; Agarwal et al., 2017; El-Deeb et al., 2020). For the nabla Hardy-Copson type inequalities, we refer Kayar and Kaymakçalan (2021, 2022a).

Although some Pachpatte type dynamic inequalities, which are convex generalizations of Hardy-Copson type inequalities, were obtained in Saker et al. (2018a, 2019) via delta time scales calculus, there does not exist any result in the nabla case.

Combining Hardy-Copson-type inequalities with convexity, both of which have many applications in many fields in pure and applied mathematics such as functional analysis, optimization theory, control, spectral theory, Fourier analysis, interpolation theory, operator theory, geometry, and ordinary/partial differential equations, leads to Pachpatte-type inequalities and provides more applications in the above fields.

Contrary to Hardy-Copson-type inequalities, there exists few results, especially in the time scale calculus, for Pachpatte-type inequalities, which are convex generalizations of Hardy-Copson-type inequalities. The main aim in this paper is to find $\delta < 0$ versions of the classical nabla Pachpatte type dynamic inequalities ($\delta > 1$) by preserving the directions of the classical inequalities. Since Pachpatte type inequalities are convex generalizations of Hardy-Copson inequalities, our results not only complement the classical nabla Pachpatte type inequalities but also generalize complementary nabla Hardy-Copson type inequalities. Another novelty of this manuscript is to obtain $\delta < 0$ versions of the discrete, continuous and delta Pachpatte type dynamic inequalities, which have not been considered yet.

It is assumed that the readers know the fundamental theory of the time scale calculus. If not, the books (Bohner and Peterson, 2001, 2003) or the articles (Guseinov and Kaymakçalan, 2002; Atici & Guseinov, 2002; Gürses et al., 2005; Saker et al., 2018a; Kayar and Kaymakçalan, 2022a) can be helpful to understand the theory.

The nabla Jensen's inequality will be used in the sequel.

Lemma 1.1. (*Jensen's inequality*). *(Özkan et al., 2008) Let $f \in C_{ld}([t_1, t_2], [t_3, t_4])$ and $g \in C_{ld}([t_1, t_2], \mathbb{R})$ with $\int_{t_1}^{t_2} |g(s)|\nabla s > 0$, where $t_1, t_2 \in \mathbb{T}$ and $t_3, t_4 \in \mathbb{R}$. If $H \in C((t_3, t_4), \mathbb{R})$ is convex, then*

$$H\left(\frac{\int_{t_1}^{t_2} \left|g(s)\right| f(s)\nabla s}{\int_{t_1}^{t_2} \left|g(s)\right|\nabla s}\right) \leq \frac{\int_{t_1}^{t_2} \left|g(s)\right| \left|H(f(s))\right|\nabla s}{\int_{t_1}^{t_2} \left|g(s)\right|\nabla s}. \tag{9}$$

Complementary Pachpatte type inequalities

We start this section by defining the following functions

$$P(t) = \int_t^{\infty} p(s)\nabla s, \quad R(t) = \int_c^t p(s)r(s)\nabla s, \quad Q(t) = \int_c^t p(s)H(r(s))\nabla s,$$
$$\overline{P}(t) = \int_c^t p(s)\nabla s, \quad \overline{R}(t) = \int_t^{\infty} p(s)r(s)\nabla s, \quad \overline{Q}(t) = \int_t^{\infty} p(s)H(r(s))\nabla s, \tag{10}$$

where $p,\ r \geq 0$ are ld-continuous, ∇-differentiable and locally nabla integrable functions, $H(u)$ is a real-valued nonnegative convex function defined for $u \geq 0$ and $0 < c \in \mathbb{T}$.

The following results, which are obtained for $\delta < 0$, $\kappa \geq 0$ and $\kappa + \xi \leq 0$, yield complements of Pachpatte type dynamic inequalities presented for $\delta > 1$, $\kappa \geq 0$ and $\kappa + \xi \leq 0$.

Theorem 2.1. *Let $\frac{P^{\rho}(t)}{P(t)} \leq D_1$, $t \in (c, \infty)_{\mathbb{T}}$ be fulfilled for some $D_1 > 0$ and for the functions P and R defined as in (10). Assume that $\delta < 0$, $\kappa \geq 0$ and $\kappa +$*

$\xi \le 0$ *are real numbers* and H(u) is a real-valued nonnegative convex function defined for $u \ge 0$.

(1) *When* $\kappa+\delta>1$, *one can show*

$$\int_c^\infty \frac{p(t)H^{\kappa+\delta}(R^\rho(t))}{[P(t)]^{\kappa+\xi}}\nabla t \le \left[\frac{D_1^{\kappa+\xi}(\kappa+\delta)}{1-\kappa-\xi}\right]^\delta \int_c^\infty \frac{p(t)H^\delta(r(t))[Q^\rho(t)]^\kappa}{[P(t)]^{\kappa+\xi-\delta}}\nabla t, \quad (11)$$

$$\int_c^\infty \frac{p(t)H^{\kappa+\delta}(R^\rho(t))}{[P^\rho(t)]^{\kappa+\xi}}\nabla t \le \left[\frac{D_1^{\kappa+\xi-1}(\kappa+\delta)}{1-\kappa-\xi}\right]^\delta \int_c^\infty \frac{p(t)H^\delta(r(t))[Q^\rho(t)]^\kappa}{[P^\rho(t)]^{\kappa+\xi-\delta}}\nabla t. \quad (12)$$

(2) *When* $0<\kappa+\delta<1$, *one can show*

$$\int_c^\infty \frac{p(t)H^{\kappa+\delta}(R(t))}{[P(t)]^{\kappa+\xi}}\nabla t \le \left[\frac{D_1^{\kappa+\xi}(\kappa+\delta)}{1-\kappa-\xi}\right]^\delta \int_c^\infty \frac{p(t)H^\delta(r(t))[Q(t)]^\kappa}{[P(t)]^{\kappa+\xi-\delta}}\nabla t, \quad (13)$$

$$\int_c^\infty \frac{p(t)H^{\kappa+\delta}(R(t))}{[P^\rho(t)]^{\kappa+\xi}}\nabla t \le \left[\frac{\kappa+\delta}{1-\kappa-\xi}\right]^\delta \int_c^\infty \frac{p(t)H^\delta(r(t))[Q(t)]^\kappa}{[P^\rho(t)]^{\kappa+\xi-\delta}}\nabla t. \quad (14)$$

Proof. We first prove inequality (11). Convexity of the function H implies

$$H(R^\rho(t)) = H\left(\int_c^{\rho(t)} p(s)r(s)\nabla s\right) \le \int_c^{\rho(t)} p(s)H(r(s))\nabla s = Q^\rho(t). \quad (15)$$

Then the left hand side of inequality (11) becomes

$$\int_c^\infty \frac{p(t)H^{\kappa+\delta}(R^\rho(t))}{[P(t)]^{\kappa+\xi}}\nabla t \le \int_c^\infty \frac{p(t)[Q^\rho(t)]^{\kappa+\delta}}{[P(t)]^{\kappa+\xi}}\nabla t. \quad (16)$$

Now we will estimate the right hand side of inequality (16) to get nabla Hardy-Copson type inequalities.

We start with the same method of the proof of Kayar et al. (2021, Theorem 3.1), which shows reverse nabla Hardy-Copson type inequalities, even though our aim is to prove nabla Hardy-Copson type dynamic inequalities.

Employing integration by parts formula to the right hand side of inequality (16), we can get

$$\int_c^\infty \frac{p(t)Q^\rho(t)}{[P(t)]^{\kappa+\xi}}\nabla t = u(t)Q(t)|_c^\infty - \int_c^\infty u(t)[Q^{\kappa+\delta}(t)]^\nabla \nabla t,$$

where $u(t) = -\int_t^\infty \frac{p(s)}{[P(s)]^{\kappa+\xi}}\nabla s$. Using $Q(c)=0$ and $u(\infty)=0$ yield

$$\int_c^\infty \frac{p(t)Q^\rho(t)}{[P(t)]^{\kappa+\xi}}\nabla t = -\int_c^\infty u(t)[Q^{\kappa+\delta}(t)]^\nabla \nabla t. \quad (17)$$

For $Q^\nabla(t) = p(t)H(r(t)) \ge 0$, using the chain rule for the nabla derivative provides

$$[Q^{\kappa+\delta}(t)]^\nabla = \int_0^1 \frac{(\kappa+\delta)Q^\nabla(t)dh}{[hQ(t)+(1-h)Q^\rho(t)]^{1-\kappa-\delta}} \ge (\kappa+\delta)p(t)H(r(t))[Q^\rho(t)]^{\kappa+\delta-1},$$

where we have used $Q^\nabla(t)\ge 0$ and $Q^\rho(t)\le Q(t)$ for $\kappa+\delta>1$.

Moreover for $P^\nabla(t) = -p(t) \le 0$, using the chain rule for the nabla derivative provides

$$[P^{1-\kappa-\xi}(t)]^\nabla = \int_0^1 \frac{(1-\kappa-\xi)P^\nabla(t)dh}{[hP(t)+(1-h)P^\rho(t)]^{\kappa+\xi}} \ge \frac{-(1-\kappa-\xi)p(t)}{[P^\rho(t)]^{\kappa+\xi}} \ge \frac{-(1-\kappa-\xi)p(t)}{[P(t)]^{\kappa+\xi}D_1^{\kappa+\xi}}, \quad (18)$$

where we have used $P^\nabla(t)\le 0$ and $P^\rho(t)\ge P(t)$ for $\kappa+\xi\le 0$. Then one can deduce that

$$\frac{p(t)}{[P(t)]^{\kappa+\xi}} \ge \frac{-D_1^{\kappa+\xi}[P^{1-\kappa-\xi}(t)]^\nabla}{(1-\kappa-\xi)}.$$

Then one can obtain that

$$-u(t) = \int_t^\infty \frac{p(s)}{[P(s)]^{\kappa+\xi}}\nabla s \ge \frac{D_1^{\kappa+\xi}}{(1-\kappa-\xi)}\int_t^\infty -[P^{1-\kappa-\xi}(s)]^\nabla \nabla s = \frac{D_1^{\kappa+\xi}(1-\kappa-\xi)}{[P(t)]^{\kappa+\xi-1}}.$$

Then inequality (17) reduces to

$$\int_c^\infty \frac{p(t)[Q^\rho(t)]^{\kappa+\delta}}{[P(t)]^{\kappa+\xi}}\nabla t \ge \frac{D_1^{\kappa+\xi}(\kappa+\delta)}{1-\kappa-\xi}\int_c^\infty \frac{p(t)H(r(t))[Q^\rho(t)]^{\kappa+\delta-1}}{[P(t)]^{\kappa+\xi-1}}\nabla t.$$

The reverse Hölder inequality (Özkan et al., 2008) for the constants $\delta<0$ and $0<\frac{\delta}{\delta-1}<1$, implies

$$\left[\int_c^\infty \frac{p(t)[Q^\rho(t)]^{\kappa+\delta}}{[P(t)]^{\kappa+\xi}}\nabla t\right]^{1/\delta} \ge \frac{D_1^{\kappa+\xi}(\kappa+\delta)}{1-\kappa-\xi}\left[\int_c^\infty \frac{p(t)H^\delta(r(t))[Q^\rho(t)]^\kappa}{[P(t)]^{\kappa+\xi-\delta}}\nabla t\right]^{1/\delta}.$$

For $\delta<0$, taking δ-th power of the both sides of the above inequality yields inequality (11) after taking into account inequality (15).

The proof of inequality (12) can be established by following the above steps for

$$\frac{p(t)}{[P^\rho(t)]^{\kappa+\xi}} \ge \frac{-[P^{1-\kappa-\xi}(t)]^\nabla}{(1-\kappa-\xi)}$$

and

$$-u(t) = \int_t^\infty \frac{p(s)}{[P^\rho(s)]^{\kappa+\xi}}\nabla s \ge \frac{1}{(1-\kappa-\xi)}\int_t^\infty -[P^{1-\kappa-\xi}(s)]^\nabla \nabla s = \frac{1}{1-\kappa-\xi}[P(t)]^{\kappa+\xi-1} \ge \frac{D_1^{\kappa+\xi-1}}{(1-\kappa-\xi)}[P^\rho(t)]^{\kappa+\xi-1}.$$

For inequalities (13)–(14), after employing convexity of the function H, we use integration by parts formula in the following ways.

$$\int_c^\infty \frac{p(t)[Q(t)]^{\kappa+\delta}}{[P(t)]^{\kappa+\xi}}\nabla t = u(t)Q(t)|_c^\infty - \int_c^\infty u^\rho(t)[Q^{\kappa+\delta}(t)]^\nabla \nabla t, \quad (19)$$

or

$$\int_c^\infty \frac{p(t)[Q(t)]^{\kappa+\delta}}{[P^\rho(t)]^{\kappa+\xi}}\nabla t = u(t)Q(t)|_c^\infty - \int_c^\infty u^\rho(t)[Q^{\kappa+\delta}(t)]^\nabla \nabla t, \quad (20)$$

where $u(t) = -\int_t^\infty \frac{p(s)}{[P(s)]^{\kappa+\xi}}\nabla s$ and $u(t) = -\int_t^\infty \frac{p(s)}{[P(s)]^{\kappa+\xi}}\nabla s$, respectively and $Q(c)=0$ and $u(\infty)=0$.

For $Q^\nabla(t) = p(t)H(r(t)) \ge 0$, using the chain rule for the nabla derivative provides

$$[Q^{\kappa+\delta}(t)]^{\nabla}=\int_0^1\frac{(\kappa+\delta)Q^{\nabla}(t)dh}{[hQ(t)+(1-h)Q^{\rho}(t)]^{1-\kappa-\delta}}\geq(\kappa+\delta)p(t)H(r(t))[Q(t)]^{\kappa+\delta-1},$$

where we have used $Q^{\nabla}(t)\geq 0$ and $Q^{\rho}(t)\leq Q(t)$ for $0\leq\kappa+\delta<1$.

Then one can obtain from inequality (18) that

$$-u^{\rho}(t)=\int_{\rho(t)}^{\infty}\frac{p(s)}{[P(s)]^{\kappa+\xi}}\nabla s\geq\frac{D_1^{\kappa+\xi}}{(1-\kappa-\xi)}\int_{\rho(t)}^{\infty}-\left[P^{1-\kappa-\xi}(s)\right]^{\nabla}\nabla s$$
$$=\frac{D_1^{\kappa+\xi}}{(1-\kappa-\xi)[P^{\rho}(t)]^{\kappa+\xi-1}}\geq\frac{D_1^{\kappa+\xi}}{(1-\kappa-\xi)[P(t)]^{\kappa+\xi-1}}.$$

or

$$-u^{\rho}(t)=\int_{\rho(t)}^{\infty}\frac{p(s)}{[P^{\rho}(s)]^{\kappa+\xi}}\nabla s\geq\frac{1}{(1-\kappa-\xi)}\int_{\rho(t)}^{\infty}-[P^{1-\kappa-\xi}(s)]^{\nabla}\nabla s$$
$$=\frac{1}{(1-\kappa-\xi)[P^{\rho}(t)]^{\kappa+\xi-1}}.$$

The rests of the proofs depend on substituting the necessary terms to the inequalities (19) and (20) and applying reverse Hölder's inequality to the resulting inequalities.

Remark 2.2. Since $P^{\rho}(t)\geq P(t)$ and $\kappa+\xi\leq 0$, inequalities (12) and (14) are valid if the terms $[P^{\rho}(t)]^{\kappa+\xi}$ on the left hand sides of these inequalities are changed by $[P(t)]^{\kappa+\xi}$.

Since $Q^{\rho}(t)\leq Q(t)$ and $\kappa\geq 0$, inequalities (11) and (12) are valid if the terms $[Q^{\rho}(t)]^{\kappa}$ on the right hand sides of these inequalities are changed by $[Q(t)]^{\kappa}$.

Remark 2.3. Since the condition $\delta<0$ has not been considered so far, nabla Pachpatte type dynamic inequalities (11)-(14) have been obtained for the first time. Moreover the condition $0<\kappa+\delta<1$ in Theorem 2.1 first ever appears in the literature. Hence the gap in the literature may be fulfilled by these inequalities, which are obtained for $\delta<0,\kappa\geq 0,\kappa+\xi\leq 0$ via this theorem.

The nabla Pachpatte type dynamic inequalities (11)-(14) obtained for $\delta<0$, $\kappa\geq 0$ and $\kappa+\xi\leq 0$ are complements of the nabla Pachpatte type dynamic inequalities established for $\delta>1$, $\kappa\geq 0$ and $\kappa+\xi\leq 0$.

The nabla Pachpatte type inequalities (11)-(14) obtained for $\delta<0$, $\kappa\geq 0$ and $\kappa+\xi\leq 0$ are convex generalizations of the nabla Hardy-Copson type inequalities obtained for $\delta<0$, $\kappa\geq 0$ and $\kappa+\xi\leq 0$ where $H(u)=u$.

Remark 2.4. Delta versions of the nabla inequalities (11)-(14) can be obtained by replacing P^{ρ}, P, R^{ρ}, R, Q^{ρ}, Q presented in (10) by P, P^{σ}, R, R^{σ}, Q, Q^{σ} defined as

$$P(t)=\int_t^{\infty}p(s)\Delta s,\quad R(t)=\int_c^t p(s)r(s)\Delta s,\quad Q(t)=\int_c^t p(s)H(r(s))\Delta s,$$
$$\overline{P}(t)=\int_c^t p(s)\Delta s,\quad \overline{R}(t)=\int_t^{\infty}p(s)r(s)\Delta s,\quad \overline{Q}(t)=\int_t^{\infty}p(s)H(r(s))\Delta s. \tag{21}$$

Let $\frac{P(t)}{P^{\sigma}(t)}\leq E_1,\ t\in(c,\infty)^{\mathbb{T}}$ be fulfilled for some $E_1>0$ and for the functions P and R defined as in (21). Suppose that $\delta<0$, $\kappa\geq 0$ and $\kappa+\xi\leq 0$ are real constants. Then nabla Pachpatte type dynamic inequalities (11)-(14) turn into new delta Pachpatte type dynamic inequalities. For instance, inequalities (12) and (14) become the following forms, respectively. When $\kappa+\delta>1$, we get

$$\int_c^{\infty}\frac{p(t)H^{\kappa+\delta}(R(t))}{[P(t)]^{\kappa+\xi}}\Delta t\leq\left[\frac{E_1^{\kappa+\xi-1}(\kappa+\delta)}{1-\kappa-\xi}\right]^{\delta}\int_c^{\infty}\frac{p(t)H^{\delta}(r(t))[Q(t)]^{\kappa}}{[P(t)]^{\kappa+\xi-\delta}}\Delta t,$$

and when $0<\kappa+\delta<1$, we get

$$\int_c^{\infty}\frac{p(t)H^{\kappa+\delta}(R^{\sigma}(t))}{[P(t)]^{\kappa+\xi}}\Delta t\leq\left[\frac{\kappa+\delta}{1-\kappa-\xi}\right]^{\delta}\int_c^{\infty}\frac{p(t)H^{\delta}(r(t))[Q^{\sigma}(t)]^{\kappa}}{[P(t)]^{\kappa+\xi-\delta}}\Delta t. \tag{22}$$

Since the condition $\delta<0$ has not been considered so far, delta variants of nabla Pachpatte type dynamic inequalities (11)-(14) have appeared in the literature for the first time. Moreover the condition $0<\kappa+\delta<1$ in Theorem 2.1 first ever appears in the literature. Hence the gap in the literature may be fulfilled by these inequalities, which are obtained for $\delta<0$, $\kappa\geq 0$, $\kappa+\xi\leq 0$ via this remark.

The delta counterparts of the nabla Pachpatte type dynamic inequalities (11)-(14) derived for $\delta<0$, $\kappa\geq 0$ and $\kappa+\xi\leq 0$ complement the delta Pachpatte type dynamic inequalities established for $\delta>1$, $\kappa\geq 0$ and $\kappa+\xi\leq 0$.

The delta Hardy-Copson type inequalities established for $\delta<0$, $\kappa\geq 0$ and $\kappa+\xi\leq 0$ can be obtained in the special case where $H(u)=u$ from the delta analogues of the inequalities (11)-(14).

The delta Pachpatte type inequality (22) derived for $\delta<0$, $\kappa\geq 0$, $\kappa+\xi\leq 0$ is a convex extension and a complementary inequality of the delta Hardy-Copson type inequality presented in Saker et al. (2018a, Theorem 2.2) where $H(u)=u$ and $\delta>1$, $\kappa\geq 0$, $0\leq\kappa+\xi<1$.

Let $\kappa=0$ and $H(u)=u$. Then inequality (22) is a complementary inequality of the delta Hardy-Copson inequality (2.36) in Saker et al. (2014b, Theorem 2.9) established for $\delta>1$, $0\leq\kappa+\xi<1$.

The following results, which are obtained for $\delta<0$, $\kappa\geq 0$ and $0\leq\kappa+\xi<1$, yield complements of Pachpatte type dynamic inequalities presented for $\delta>1$, $\kappa\geq 0$ and $0\leq\kappa+\xi<1$.

Theorem 2.5. *Let $D_2\leq\frac{P(t)}{P^{\rho}(t)}\leq 1,\ t\in(c,\infty)_{\mathbb{T}}$ be fulfilled for some $D_2>0$ and for the functions P, R and Q defined as in (10). Assume that $\delta<0$, $\kappa\geq 0$ and $0\leq\kappa+\xi<1$ are real numbers* and H(u) is a real-valued nonnegative convex function defined for u ≥ 0.

(1) *When $\kappa+\delta>1$, we get*

$$\int_c^{\infty}\frac{p(t)H^{\kappa+\delta}(R^{\rho}(t))}{[P^{\rho}(t)]^{\kappa+\xi}}\nabla t\leq\left[\frac{D_2(\kappa+\delta)}{1-\kappa-\xi}\right]^{\delta}\int_c^{\infty}\frac{p(t)H^{\delta}(r(t))[Q^{\rho}(t)]^{\kappa}}{[P^{\rho}(t)]^{\kappa+\xi-\delta}}\nabla t, \tag{23}$$

$$\int_c^{\infty}\frac{p(t)H^{\kappa+\delta}(R^{\rho}(t))}{[P(t)]^{\kappa+\xi}}\nabla t\leq\left[\frac{\kappa+\delta}{1-\kappa-\xi}\right]^{\delta}\int_c^{\infty}\frac{p(t)H^{\delta}(r(t))[Q^{\rho}(t)]^{\kappa}}{[P(t)]^{\kappa+\xi-\delta}}\nabla t. \tag{24}$$

(2) *When $0<\kappa+\delta<1$, one can show*

$$\int_c^{\infty}\frac{p(t)H^{\kappa+\delta}(R(t))}{[P^{\rho}(t)]^{\kappa+\xi}}\nabla t\leq\left[\frac{D_2^{\kappa+\xi}(\kappa+\delta)}{1-\kappa-\xi}\right]^{\delta}\int_c^{\infty}\frac{p(t)H^{\delta}(r(t))[Q(t)]^{\kappa}}{[P^{\rho}(t)]^{\kappa+\xi-\delta}}\nabla t, \tag{25}$$

$$\int_c^{\infty}\frac{p(t)H^{\kappa+\delta}(R(t))}{[P(t)]^{\kappa+\xi}}\nabla t\leq\left[\frac{\kappa+\delta}{1-\kappa-\xi}\right]^{\delta}\int_c^{\infty}\frac{p(t)H^{\delta}(r(t))[Q(t)]^{\kappa}}{[P(t)]^{\kappa+\xi-\delta}}\nabla t. \tag{26}$$

Proof. We first show the proof of inequality (23). Convexity of the function H implies

$$H(R^{\rho}(t))=H\left(\int_c^{\rho(t)}p(s)r(s)\nabla s\right)\leq\int_c^{\rho(t)}p(s)H(r(s))\nabla s=Q^{\rho}(t). \tag{27}$$

We use exactly the same method as the proof of Theorem 2.1.

By using $[Q^{\kappa+\delta}(t)]^{\nabla}\geq(\kappa+\delta)p(t)H(r(t))[Q^{\rho}(t)]^{\kappa+\delta-1}$ for $\kappa+\delta>1$ and

$$[P^{1-\kappa-\xi}(t)]^{\nabla}\geq\frac{-(1-\kappa-\xi)p(t)}{[P(t)]^{\kappa+\xi}},\quad 0\leq\kappa+\xi<1,$$

and by following the same steps of the proof of Theorem 2.1 for the functions P and Q, we can show that

$$\int_c^{\infty}\frac{p(t)[Q^{\rho}(t)]^{\kappa+\delta}}{[P^{\rho}(t)]^{\kappa+\xi}}\nabla t \geq \frac{D_2^{\kappa+\xi}(\kappa+\delta)}{1-\kappa-\xi}\int_c^{\infty}\frac{p(t)H(r(t))[Q^{\rho}(t)]^{\kappa+\delta-1}}{[P(t)]^{\kappa+\xi-1}}\nabla t$$
$$\geq \frac{D_2(\kappa+\delta)}{1-\kappa-\xi}\int_c^{\infty}\frac{p(t)H(r(t))[Q^{\rho}(t)]^{\kappa+\delta-1}}{[P^{\rho}(t)]^{\kappa+\xi-1}}\nabla t.$$

The reverse Hölder inequality (Özkan et al., 2008) for the constants $\delta < 0$ and $0 < \frac{\delta}{\delta-1} < 1$, implies

$$\left[\int_c^{\infty}\frac{p(t)[Q^{\rho}(t)]^{\kappa+\delta}}{[P^{\rho}(t)]^{\kappa+\xi}}\nabla t\right]^{1/\delta} \geq \frac{D_2(\kappa+\delta)}{1-\kappa-\xi}\left[\int_c^{\infty}\frac{p(t)H^{\delta}(r(t))[Q^{\rho}(t)]^{\kappa}}{[P^{\rho}(t)]^{\kappa+\xi-\delta}}\nabla t\right]^{1/\delta}.$$

For $\delta < 0$, taking δ-th power of the both sides of the above inequality yields the reverse inequality (23) after taking into account inequality (27). The proofs of inequalities (24)–(26) can be obtained by following the same method as above. □

Remark 2.6. Since the condition $\delta < 0$ has not been considered so far, nabla Pachpatte type dynamic inequalities (23)-(26) have been obtained for the first time. Moreover the condition $0 < \kappa + \delta < 1$ in Theorem 2.5 first ever appears in the literature. Hence the gap in the literature may be fulfilled by these inequalities, which are obtained for $\delta < 0, \kappa \geq 0, 0 \leq \kappa + \xi < 1$ via this theorem.

The nabla Pachpatte type dynamic inequalities (23)-(26) obtained for $\delta < 0$, $\kappa \geq 0$ and $0 \leq \kappa + \xi < 1$ are complements of the nabla Pachpatte type dynamic inequalities established for $\delta > 1$, $\kappa \geq 0$ and $0 \leq \kappa + \xi < 1$.

The nabla Pachpatte type inequalities (23)-(26) obtained for $\delta < 0$, $\kappa \geq 0$ and $0 \leq \kappa + \xi < 1$ are convex generalizations of the nabla Hardy-Copson type inequalities presented for $\delta < 0$, $\kappa \geq 0$ and $0 \leq \kappa + \xi < 1$ where $H(u) = u$.

The following results, which are obtained for $\delta < 0$, $\kappa \geq 0$ and $\kappa + \xi \leq 0$, yield complements of Pachpatte type dynamic inequalities presented for $\delta > 1$, $\kappa \geq 0$ and $\kappa + \xi \leq 0$.

Theorem 2.7. *Let $\frac{P(t)}{P^{\rho}(t)} \leq D_3$, $t \in (c,\infty)_{\mathbb{T}}$ be fulfilled for some $D_3 > 0$ and for the functions P, R and Q defined as in (10). Assume that $\delta < 0, \kappa \geq 0$ and $\kappa + \xi \leq 0$ are real numbers* and H(u) is a real-valued nonnegative convex function defined for u $\geq$ 0.

(1) *When $\kappa + \delta > 1$, one can show*

$$\int_c^{\infty}\frac{p(t)H^{\kappa+\delta}\big(R(t)\big)}{[P^{\rho}(t)]^{\kappa+\xi}}\nabla t \leq \left[\frac{D_3^{\kappa+\xi}(\kappa+\delta)}{1-\kappa-\xi}\right]^{\delta}\int_c^{\infty}\frac{p(t)H^{\delta}(r(t))[Q(t)]^{\kappa}}{[P^{\rho}(t)]^{\kappa+\xi-\delta}}\nabla t, \quad (28)$$

$$\int_c^{\infty}\frac{p(t)H^{\kappa+\delta}\big(R(t)\big)}{[P(t)]^{\kappa+\xi}}\nabla t \leq \left[\frac{D_3^{\kappa+\xi-1}(\kappa+\delta)}{1-\kappa-\xi}\right]^{\delta}\int_c^{\infty}\frac{p(t)H^{\delta}(r(t))[Q(t)]^{\kappa}}{[P(t)]^{\kappa+\xi-\delta}}\nabla t. \quad (29)$$

(2) *When $0 < \kappa + \delta < 1$, one can show*

$$\int_c^{\infty}\frac{p(t)H^{\kappa+\delta}\big(R^{\rho}(t)\big)}{[P^{\rho}(t)]^{\kappa+\xi}}\nabla t \leq \left[\frac{D_3^{\kappa+\xi}(\kappa+\delta)}{1-\kappa-\xi}\right]^{\delta}\int_c^{\infty}\frac{p(t)H^{\delta}(r(t))[Q^{\rho}(t)]^{\kappa}}{[P^{\rho}(t)]^{\kappa+\xi-\delta}}\nabla t, \quad (30)$$

$$\int_c^{\infty}\frac{p(t)H^{\kappa+\delta}\big(R^{\rho}(t)\big)}{[P(t)]^{\kappa+\xi}}\nabla t \leq \left[\frac{\kappa+\delta}{1-\kappa-\xi}\right]^{\delta}\int_c^{\infty}\frac{p(t)H^{\delta}(r(t))[Q^{\rho}(t)]^{\kappa}}{[P(t)]^{\kappa+\xi-\delta}}\nabla t. \quad (31)$$

Proof. We first show the proof of inequality (29). Convexity of the function H implies

$$H\big(R(t)\big) = H\left(\int_t^{\infty}p(s)r(s)\nabla s\right) \leq \int_t^{\infty}p(s)H(r(s))\nabla s = Q(t). \quad (32)$$

We use exactly the same method as the proof of Theorem 2.1.

By using $-[Q^{\kappa+\delta}(t)]^{\nabla} \geq (\kappa+\delta)p(t)H(r(t))[Q(t)]^{\kappa+\delta-1}$ for $\kappa+\delta > 1$ and

$$[P^{1-\kappa-\xi}(t)]^{\nabla} \leq \frac{(1-\kappa-\xi)p(t)}{[P(t)]^{\kappa+\xi}}, \quad \kappa+\xi \leq 0,$$

and by following the same steps of the proof of Theorem 2.1 for the functions P and Q, we can show that

$$\int_c^{\infty}\frac{p(t)[Q(t)]^{\kappa+\delta}}{[P(t)]^{\kappa+\xi}}\nabla t \geq \frac{D_3^{\kappa+\xi-1}(\kappa+\delta)}{1-\kappa-\xi}\int_c^{\infty}\frac{p(t)H(r(t))[Q(t)]^{\kappa+\delta-1}}{[P(t)]^{\kappa+\xi-1}}\nabla t.$$

The reverse Hölder inequality (Özkan et al., 2008) for the constants $\delta < 0$ and $0 < \frac{\delta}{\delta-1} < 1$, implies

$$\left[\int_c^{\infty}\frac{p(t)[Q(t)]^{\kappa+\delta}}{[P(t)]^{\kappa+\xi}}\nabla t\right]^{1/\delta} \geq \frac{D_3^{\kappa+\xi-1}(\kappa+\delta)}{1-\kappa-\xi}\left[\int_c^{\infty}\frac{p(t)H^{\delta}(r(t))[Q(t)]^{\kappa}}{[P(t)]^{\kappa+\xi-\delta}}\nabla t\right]^{1/\delta}.$$

For $\delta < 0$, taking δ-th power of the both sides of the above inequality yields the reverse inequality (29) after taking into account inequality (32). The proofs of inequalities (28), (30) and (31) can be obtained by following the same method as above. □

Remark 2.8. Since the condition $\delta < 0$ has not been considered so far, nabla Pachpatte type dynamic inequalities (28)-(31) have been obtained for the first time. Moreover the condition $0 < \kappa + \delta < 1$ in Theorem 2.7 first ever appears in the literature. Hence the gap in the literature may be fulfilled by these inequalities, which are obtained for $\delta < 0, \kappa \geq 0, \kappa + \xi \leq 0$ via this theorem.

The nabla Pachpatte type dynamic inequalities (28)-(31) obtained for $\delta < 0$, $\kappa \geq 0$ and $\kappa + \xi \leq 0$ are complements of the nabla Pachpatte type dynamic inequalities established for $\delta > 1$, $\kappa \geq 0$ and $\kappa + \xi \leq 0$.

The nabla Pachpatte type inequalities (28)-(31) obtained for $\delta < 0$, $\kappa \geq 0$ and $\kappa + \xi \leq 0$ are convex generalizations of the nabla Hardy-Copson type inequalities presented for $\delta < 0$, $\kappa \geq 0$ and $\kappa + \xi \leq 0$ where $H(u) = u$.

The following results, which are obtained for $\delta < 0$, $\kappa \geq 0$ and $0 \leq \kappa + \xi < 1$, yield complements of Pachpatte type dynamic inequalities presented for $\delta > 1$, $\kappa \geq 0$ and $0 \leq \kappa + \xi < 1$.

Theorem 2.9. *Let $D_4 \leq \frac{P(t)}{P^{\rho}(t)} \leq 1$, $t \in (c,\infty)_{\mathbb{T}}$ be fulfilled for some $D_4 > 0$ and for the functions P, R and Q defined as in (10). Assume that $\delta < 0$, $\kappa \geq 0$ and $0 \leq \kappa + \xi < 1$ are real numbers* and H(u) is a real-valued nonnegative convex function defined for u $\geq$ 0.

(1) *When $\kappa + \delta > 1$, we get*

$$\int_c^{\infty}\frac{p(t)H^{\kappa+\delta}(R(t))}{[P(t)]^{\kappa+\xi}}\nabla t \leq \left[\frac{D_4(\kappa+\delta)}{1-\kappa-\xi}\right]^{\delta}\int_c^{\infty}\frac{p(t)H^{\delta}(r(t))[Q(t)]^{\kappa}}{[P(t)]^{\kappa+\xi-\delta}}\nabla t, \quad (33)$$

$$\int_c^{\infty}\frac{p(t)H^{\kappa+\delta}(R(t))}{[P^{\rho}(t)]^{\kappa+\xi}}\nabla t \leq \left[\frac{\kappa+\delta}{1-\kappa-\xi}\right]^{\delta}\int_c^{\infty}\frac{p(t)H^{\delta}(r(t))[Q(t)]^{\kappa}}{[P^{\rho}(t)]^{\kappa+\xi-\delta}}\nabla t. \quad (34)$$

(2) *When $0 < \kappa + \delta < 1$, one can show*

$$\int_c^{\infty}\frac{p(t)H^{\kappa+\delta}(R^{\rho}(t))}{[P(t)]^{\kappa+\xi}}\nabla t \leq \left[\frac{D_4^{\kappa+\xi}(\kappa+\delta)}{1-\kappa-\xi}\right]^{\delta}\int_c^{\infty}\frac{p(t)H^{\delta}(r(t))[Q^{\rho}(t)]^{\kappa}}{[P(t)]^{\kappa+\xi-\delta}}\nabla t, \quad (35)$$

$$\int_c^{\infty}\frac{p(t)H^{\kappa+\delta}(R^{\rho}(t))}{[P^{\rho}(t)]^{\kappa+\xi}}\nabla t \leq \left[\frac{\kappa+\delta}{1-\kappa-\xi}\right]^{\delta}\int_c^{\infty}\frac{p(t)H^{\delta}(r(t))[Q^{\rho}(t)]^{\kappa}}{[P^{\rho}(t)]^{\kappa+\xi-\delta}}\nabla t. \quad (36)$$

Proof. We first show the proof of inequality (35). Convexity of the function H implies

$$H\left(R^{\rho}(t)\right)=H\left(\int_{\rho(t)}^{\infty}p(s)r(s)\nabla s\right)\leq\int_{\rho(t)}^{\infty}p(s)H(r(s))\nabla s=Q^{\rho}(t). \quad (37)$$

We use exactly the same method as the proof of Theorem 2.1.

By using $-[Q^{\kappa+\delta}(t)]^{\nabla}\geq(\kappa+\delta)p(t)H(r(t))[Q^{\rho}(t)]^{\kappa+\delta-1}$ for $0<\kappa+\delta<1$ and

$$[\overline{P}^{1-\kappa-\xi}(t)]^{\nabla}\leq\frac{(1-\kappa-\xi)p(t)}{[\overline{P}^{\rho}(t)]^{\kappa+\xi}}\leq\frac{(1-\kappa-\xi)p(t)}{D_4^{\kappa+\xi}[\overline{P}(t)]^{\kappa+\xi}},\ 0\leq\kappa+\xi<1,$$

and by following the same steps of the proof of Theorem 2.1 for the functions P and Q, we can show that

$$\int_c^{\infty}\frac{p(t)[Q^{\rho}(t)]^{\kappa+\delta}}{[P(t)]^{\kappa+\xi}}\nabla t\geq\frac{D_4^{\kappa+\xi}(\kappa+\delta)}{1-\kappa-\xi}\int_c^{\infty}\frac{p(t)H(r(t))[Q^{\rho}(t)]^{\kappa+\delta-1}}{[P(t)]^{\kappa+\xi-1}}\nabla t.$$

The reverse Hölder inequality (Özkan et al., 2008) for the constants $\delta<0$ and $0<\frac{\delta}{\delta-1}<1$, implies

$$\left[\int_c^{\infty}\frac{p(t)[Q^{\rho}(t)]^{\kappa+\delta}}{[P(t)]^{\kappa+\xi}}\nabla t\right]^{1/\delta}\geq\frac{D_4^{\kappa+\xi}(\kappa+\delta)}{1-\kappa-\xi}\left[\int_c^{\infty}\frac{p(t)H(r(t))[Q^{\rho}(t)]^{\kappa}}{[P(t)]^{\kappa+\xi-\delta}}\nabla t\right]^{1/\delta}.$$

For $\delta<0$, taking δ-th power of the both sides of the above inequality yields the reverse inequality (35) after taking into account inequality (37). The proofs of inequalities (33), (34) and (36) can be obtained by following the same method as above. □

Remark 2.10. Since $P^{\rho}(t)\leq P(t)$ and $0\leq\kappa+\xi<1$, inequalities (34) and (36) are valid if the terms $[P^{\rho}(t)]^{\kappa+\xi}$ on the left hand sides of these inequalities are changed by $[P(t)]^{\kappa+\xi}$.

Since $P^{\rho}(t)\leq P(t)$ and $\kappa+\xi-\delta\geq 0$, inequalities (33) and (35) are valid if the terms $[P(t)]^{\kappa+\xi-\delta}$ on the right hand sides of these inequalities are changed by $[P^{\rho}(t)]^{\kappa+\xi-\delta}$.

Since $Q^{\rho}(t)\geq Q(t)$ and $\kappa\geq 0$, inequalities (33) and (34) are valid if the terms $[Q(t)]^{\kappa}$ on the right hand sides of these inequalities are changed by $[Q^{\rho}(t)]^{\kappa}$.

The following results, which are obtained for $\delta<0$, $\kappa\geq 0$ and $\kappa+\xi>1$, yield complements of the former Pachpatte type dynamic inequalities presented for $\delta>1$, $\kappa\geq 0$ and $0\leq\kappa+\xi>1$.

Theorem 2.11. *Let $D_4>0$ be defined as in Theorem 2.9 and the functions P, R and Q be defined as in (10). Assume that $\delta<0$, $\kappa\geq 0$ and $\kappa+\xi>1$ are real numbers* and H(u) is a real-valued nonnegative convex function defined for u ≥ 0.

(1) *When $\kappa+\delta>1$, we get*

$$\int_c^{\infty}\frac{p(t)H^{\kappa+\delta}(R^{\rho}(t))}{[P(t)]^{\kappa+\xi}}\nabla t\leq\left[\frac{D_4^{\kappa+\xi}(\kappa+\delta)}{\kappa+\xi-1}\right]^{\delta}\int_c^{\infty}\frac{p(t)H^{\delta}(r(t))[Q^{\rho}(t)]^{\kappa}}{[P(t)]^{\kappa+\xi-\delta}}\nabla t, \quad (38)$$

$$\int_c^{\infty}\frac{p(t)}{[\overline{P}^{\rho}(t)]^{\xi-\delta}}H^{\kappa+\delta}\left(\frac{R^{\rho}(t)}{\overline{P}^{\rho}(t)}\right)\nabla t\leq\left[\frac{D_4^{\kappa+\xi-1}(\kappa+\delta-1)}{\kappa+\xi-1}\right]^{\delta}\int_c^{\infty}\frac{p(t)H^{\delta}(r(t))[Q^{\rho}(t)]^{\kappa}}{[\overline{P}^{\rho}(t)]^{\kappa+\xi-\delta}}\nabla t. \quad (39)$$

(2) *When $0<\kappa+\delta<1$, one can show*

$$\int_c^{\infty}\frac{p(t)}{[P(t)]^{\xi-\delta}}H^{\kappa+\delta}\left(\frac{R(t)}{P(t)}\right)\nabla t\leq\left[\frac{D_4^{\kappa+\xi}(\kappa+\delta)}{\kappa+\xi-1}\right]^{\delta}\int_c^{\infty}\frac{p(t)H^{\delta}(r(t))[Q(t)]^{\kappa}}{[P(t)]^{\kappa+\xi-\delta}}\nabla t, \quad (40)$$

$$\int_c^{\infty}\frac{p(t)H^{\kappa+\delta}(R(t))}{[P^{\rho}(t)]^{\kappa+\xi}}\nabla t\leq\left[\frac{\kappa+\delta}{\kappa+\xi-1}\right]^{\delta}\int_c^{\infty}\frac{p(t)H^{\delta}(r(t))[Q(t)]^{\kappa}}{[P^{\rho}(t)]^{\kappa+\xi-\delta}}\nabla t. \quad (41)$$

Proof. After employing Jensen's inequality (9) with a convex function H and by using

$$-[P^{1-\kappa-\xi}(t)]^{\nabla}\leq\frac{(\kappa+\xi-1)p(t)}{[P^{\rho}(t)]^{\kappa+\xi}}\leq\frac{(\kappa+\xi-1)p(t)}{D_4^{\kappa+\xi}[P(t)]^{\kappa+\xi}}$$

for $\kappa+\xi>1$, the combination of the techniques used in the proof of Kayar et al. (2021, Theorem 3.12) and in the proofs of Theorem 2.1 and Theorem 2.7 work for the proof of this theorem. □

Remark 2.12. Since the condition $\delta<0$ has not been considered so far, nabla Pachpatte type dynamic inequalities (38)-(41) have been obtained for the first time. Moreover the condition $0<\kappa+\delta<1$ in Theorem 2.11 first ever appears in the literature. Hence the gap in the literature may be fulfilled by these inequalities, which are obtained for $\delta<0$, $\kappa\geq 0$, $\kappa+\xi>1$ via this theorem.

The nabla Pachpatte type dynamic inequalities (38)-(41) obtained for $\delta<0$, $\kappa\geq 0$ and $\kappa+\xi>1$ are complements of the nabla Pachpatte type dynamic inequalities established for $\delta>1$, $\kappa\geq 0$ and $\kappa+\xi>1$.

The nabla Pachpatte type inequalities (38)-(41) obtained for $\delta<0$, $\kappa\geq 0$ and $\kappa+\xi>1$ are convex generalizations of the nabla Hardy-Copson type inequalities presented for $\delta<0$, $\kappa\geq 0$ and $\kappa+\xi>1$ where $H(u)=u$.

The following results, which are obtained for $\delta<0$, $\kappa\geq 0$ and $\kappa+\xi>1$, yield complements of the former Pachpatte type dynamic inequalities presented for $\delta>1$, $\kappa\geq 0$ and $\kappa+\xi>1$.

Theorem 2.13. *Let $D_2>0$ be defined in Theorem 2.5 and the functions P, R and Q be defined as in (10). Assume that $\delta<0$, $\kappa\geq 0$ and $\kappa+\xi>1$ are real numbers* and H(u) is a real-valued nonnegative convex function defined for u ≥ 0.

(1) *When $\kappa+\delta>1$, we get*

$$\int_c^{\infty}\frac{p(t)}{[P(t)]^{\xi-\delta}}H^{\kappa+\delta}\left(\frac{R(t)}{P(t)}\right)\nabla t\leq\left[\frac{D_4^{\kappa+\xi-1}(\kappa+\delta)}{\kappa+\xi-1}\right]^{\delta}\int_c^{\infty}\frac{p(t)H^{\delta}(r(t))[Q(t)]^{\kappa}}{[P(t)]^{\kappa+\xi-\delta}}\nabla t, \quad (42)$$

$$\int_c^{\infty}\frac{p(t)H^{\kappa+\delta}(R(t))}{[P^{\rho}(t)]^{\kappa+\xi}}\nabla t\leq\left[\frac{D_4^{\kappa+\xi}(\kappa+\delta)}{\kappa+\xi-1}\right]^{\delta}\int_c^{\infty}\frac{p(t)H^{\delta}(r(t))[Q(t)]^{\kappa}}{[P^{\rho}(t)]^{\kappa+\xi-\delta}}\nabla t. \quad (43)$$

(2) *When $0<\kappa+\delta<1$, one can show*

$$\int_c^{\infty}\frac{p(t)}{[P^{\rho}(t)]^{\xi-\delta}}H^{\kappa+\delta}\left(\frac{R^{\rho}(t)}{P^{\rho}(t)}\right)\nabla t\leq\left[\frac{D_4^{\kappa+\xi}(\kappa+\delta)}{\kappa+\xi-1}\right]^{\delta}\int_c^{\infty}\frac{p(t)H^{\delta}(r(t))[Q^{\rho}(t)]^{\kappa}}{[P^{\rho}(t)]^{\kappa+\xi-\delta}}\nabla t, \quad (44)$$

$$\int_c^{\infty}\frac{p(t)H^{\kappa+\delta}(R^{\rho}(t))}{[P(t)]^{\kappa+\xi}}\nabla t\leq\left[\frac{\kappa+\delta}{\kappa+\xi-1}\right]^{\delta}\int_c^{\infty}\frac{p(t)H^{\delta}(r(t))[Q^{\rho}(t)]^{\kappa}}{[P(t)]^{\kappa+\xi-\delta}}\nabla t. \quad (45)$$

Proof. After employing Jensen's inequality (9) with a convex function H, the combination of the techniques used in the proof of Kayar et al. (2021, Theorem 3.4) and in the proofs of Theorem 2.1 and Theorem 2.7 work for the proof of this theorem. □

Remark 2.14. Since $P^{\rho}(t)\geq P(t)$ and $\kappa+\xi>1$, inequality (45) is valid if the term $[P(t)]^{\kappa+\xi}$ on the left hand side of this inequality is changed by $[P^{\rho}(t)]^{\kappa+\xi}$.

Since $(t) \geq P(t)$ and $\kappa + \xi - \delta \geq 0$, inequalities (43) and (44) are valid if the terms $[P^{\rho}(t)]^{\kappa+\xi-\delta}$ on the right hand sides of these inequalities are changed by $[P^{\rho}(t)]^{\kappa+\xi-\delta}$.

Since $Q^{\rho}(t) \geq Q(t)$ and $\kappa \geq 0$, inequalities (42) and (43) are valid if the terms $[Q(t)]^{\kappa}$ on the right hand sides of these inequalities are changed by $[Q^{\rho}(t)]^{\kappa}$.

Remark 2.15. Since the condition $\delta < 0$ has not been considered so far, nabla Pachpatte type dynamic inequalities (42)-(45) have been obtained for the first time. Moreover the condition $0 < \kappa + \delta < 1$ in Theorem 2.11 first ever appears in the literature. Hence the gap in the literature may be fulfilled by these inequalities, which are obtained for $\delta < 0, \kappa \geq 0, \kappa + \xi > 1$ via this theorem.

The nabla Pachpatte type dynamic inequalities (42)-(45) obtained for $\delta < 0$, $\kappa \geq 0$ and $\kappa + \xi > 1$ are complements of the nabla Pachpatte type dynamic inequalities given for $\delta > 1$, $\kappa \geq 0$ and $\kappa + \xi > 1$.

The nabla Pachpatte type inequalities (42)-(45) obtained for $\delta < 0$, $\kappa \geq 0$ and $\kappa + \xi > 1$ are convex generalizations of the nabla Hardy-Copson type inequalities presented for $\delta < 0$, $\kappa \geq 0$ and $\kappa + \xi > 1$ where $H(u) = u$.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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