Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH Open Access

On the multiparameterized fractional multiplicative integral inequalities

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Abstract

We introduce a novel multiparameterized fractional multiplicative integral identity and utilize it to derive a range of inequalities for multiplicatively s-convex mappings in connection with different quadrature rules involving one, two, and three points. Our results cover both new findings and established ones, offering a holistic framework for comprehending these inequalities. To validate our outcomes, we provide an illustrative example with visual aids. Furthermore, we highlight the practical significance of our discoveries by applying them to special means of real numbers within the realm of multiplicative calculus.

Mathematics Subject Classification: 26A33; 26A51; 26D10; 26D15

Keywords: Multiparameterized identity; Integral inequalities; Fractional multiplicative integral; Multiplicative s-convexity

1 Introduction

Multiplicative calculus, initially introduced by Grosman and Katz [1] in 1967, was developed as an alternative to traditional calculus to address issues related to rates of change and multiplicative processes. This type of calculus, which covers only positive functions, gained its mathematical formalization in Bashirov et al.'s comprehensive work [2] in 2008. Its importance lies in its ability to handle phenomena involving growth, decay, and proportional relationships more effectively. Over time, it has gained relevance in fields like finance [3], biology [4], and physics [5], offering a fresh perspective on modeling and analysis for scenarios where traditional calculus may fall short.

On the other hand, convexity is a fundamental mathematical concept that plays a pivotal role in various scientific disciplines. Its significance lies in its ability to capture the essential characteristics of many real-world phenomena, making it a powerful tool for modeling and analysis. Convex functions, in particular, exhibit remarkable properties that simplify optimization, economics, and even the understanding of physical systems. In mathematical terms, a function $\mathcal{R}$ is convex on the interval $[a,b]$ if for all $u_1, u_2 \in [a,b]$, we have the inequality

$$\mathcal{R}\big(\gamma u_1 + (1-\gamma)u_2\big) \leq \gamma \mathcal{R}(u_1) + (1-\gamma)\mathcal{R}(u_2)$$

for all $\gamma \in [0,1]$.

One of the most important results associated with convexity is the Hermite–Hadamard (H-H) inequality, which states that for a convex function $\mathcal{R}$ on an interval $[a,b]$, we have

$$\mathcal{R}\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b \mathcal{R}(u)\,du \leq \frac{\mathcal{R}(a)+\mathcal{R}(b)}{2}.$$

Various extensions and versions of the convexity concept have emerged and found application in estimating the accuracy of certain quadrature formulas; see [6–14] and the references tehrein. However, when considering multiplicative calculus, the most appropriate variant is the logarithmic convexity, also termed as multiplicative convexity. It can be defined as follows.

Definition 1.1 ([15]) A positive function $\mathcal{R}$ is said to be multiplicatively convex over the interval I if for all $u_1, u_2 \in I$, we have

$$\mathcal{R}\big(\gamma u_1 + (1-\gamma)u_2\big) \leq \big[\mathcal{R}(u_1)\big]^{\gamma}\big[f(u_2)\big]^{1-\gamma}$$

for all $\gamma \in [0,1]$.

In a recent development, Ali et al. [16] provided the H-H inequality for multiplicative integrals.

Theorem 1.2 *Let $\mathcal{R}$ be a positive multiplicatively convex function on the interval $[a,b]$. Then we have the following inequalities*:

$$\mathcal{R}\left(\frac{a+b}{2}\right) \leq \left(\int_a^b \mathcal{R}(u)^{du}\right)^{\frac{1}{b-a}} \leq \sqrt{\mathcal{R}(a)\mathcal{R}(b)}. \tag{1}$$

In the realm of multiplicative integral inequalities, significant research has been conducted. The midpoint- and trapezoid-type inequalities were established in [17] via multiplicative convexity. Ali et al. [18] scrutinized the Ostrowski and Simpson-type inequalities within the context of multiplicatively convex functions. Additionally, a distinct study detailed Maclaurin inequalities in [19], whereas Meftah and Lakhdari [20] explored dual Simpson-type inequalities for the same class of functions. These contributions collectively enhance our understanding of inequalities in the realm of multiplicative calculus. For more papers dealing with multiplicative integral inequalities, we refer the readers to [21–26].

In 2016, Abdeljawad and Grossman introduced the multiplicative Riemann–Liouville fractional integrals as follows.

Definition 1.3 [27] The left- and right-sided multiplicative Riemann–Liouville fractional integral operators of order $\delta \in \mathbb{C}$, $Re(\delta) > 0$, are defined respectively by

$${}_aI_*^{\delta}\mathcal{R}(u) = \exp\big\{\big(J_{a^+}^{\delta}(\ln\circ\mathcal{R})\big)(u)\big\}, \quad a < u, \tag{2}$$

and

$${}_*I_b^{\delta}\mathcal{R}(u) = \exp\big\{\big(J_{b^-}^{\delta}(\ln\circ\mathcal{R})\big)(u)\big\}, \quad u < b, \tag{3}$$

where $J^{\delta}_{\sigma^+}$ and $J^{\delta}_{\sigma^-}$ denote the left- and right-sided Riemann–Liouville operators defined respectively by

$$J^{\delta}_{a^+}\mathcal{T}(v) = \frac{1}{\Gamma(\delta)}\int_a^v (v-u)^{\delta-1}\mathcal{T}(u)\,du$$

and

$$J^{\delta}_{b^-}\mathcal{T}(v) = \frac{1}{\Gamma(\delta)}\int_v^b (u-v)^{\delta-1}\mathcal{T}(u)\,du.$$

Budak and Özçelik [28] established the H-H inequalities pertaining to multiplicative Riemann–Liouville fractional integrals.

Theorem 1.4 *Let $\mathcal{R}$ be a positive multiplicatively convex function on the interval $[a,b]$. For $\delta > 0$, we have the following inequalities*:

$$\mathcal{R}\left(\frac{a+b}{2}\right) \le \left[{}_*I^{\delta}_{\frac{a+b}{2}}\mathcal{R}(a)\,{}_{\frac{a+b}{2}}I^{\delta}_*\mathcal{R}(b)\right]^{\frac{2^{\delta-1}\Gamma(\delta+1)}{(b-a)^{\delta}}} \le \sqrt{\mathcal{R}(a)\mathcal{R}(b)}. \tag{4}$$

Fu et al. [29] conducted an investigation into multiplicative tempered fractional integrals, offering a generalization of the findings presented by Ali et al. [16] and Budak et al. [28]. Moreover, within the scope of fractional multiplicative integral inequalities, Boulares et al. [30] demonstrated Bullen-type inequalities, whereas the establishment of Simpson inequalities was attributed to Moumen et al. [31]. Furthermore, Peng et al. [32] contributed to the field with their work on fractional multiplicative Maclaurin-type inequalities. Additional pertinent results can be found in [33–36] and the references therein.

An interesting study was conducted in [37], where the authors introduced a one-parameter identity from which they were able to establish some fractional multiplicative midpoint-, trapezium-, Simpson-, and Bullen-type inequalities. Taking into account the insights from the previously mentioned works, this study commences with the introduction of a biparameterized multiplicative integral identity tailored for multiplicative differentiable functions. Expanding upon this identity, we proceed to establish a series of three-point Newton–Cotes-type inequalities, purposefully designed for multiplicatively s-convex functions. To conclude our investigation, we offer an illustrative example accompanied by graphical representations to validate the accuracy of the results obtained. Additionally, we present practical applications that exemplify the utility and relevance of our derived outcomes.

2 Preliminaries

In this section, we commence by revisiting select definitions, properties, and concepts related to differentiation and multiplicative integration.

Definition 2.1 ([2]) The multiplicative derivative of a positive function $\mathcal{R}$, denoted as $\mathcal{R}^*$, is defined as follows:

$$\frac{d^*\mathcal{R}}{du} = \mathcal{R}^*(u) = \lim_{h\to 0}\left(\frac{\mathcal{R}(u+h)}{\mathcal{R}(u)}\right)^{\frac{1}{h}}.$$

Remark 2.2 Each positive differentiable function $\mathcal{R}$ inherently exhibits multiplicative differentiability, with the interconnection between $\mathcal{R}'$ and $\mathcal{R}^*$ governed by the relationship

$$\mathcal{R}^*(u) = \exp\{(\ln\mathcal{R}(u))'\} = \exp\left\{\frac{\mathcal{R}'(u)}{\mathcal{R}(u)}\right\}.$$

Proposition 2.3 ([2]) *Let $\mathcal{R}$ and $\mathcal{S}$ be two multiplicatively differentiable functions, let $\mathcal{T}$ be a differentiable function, and let c be an arbitrary positive constant. Then the functions $c\mathcal{R}$, $\mathcal{R}\mathcal{S}$, $\mathcal{R}+\mathcal{S}$, $\mathcal{R}/\mathcal{S}$, and $\mathcal{R}^{\mathcal{T}}$ are multiplicatively differentiable, and*

- $(c\mathcal{R})^*(u) = \mathcal{R}^*(u)$,
- $(\mathcal{R}+\mathcal{S})^*(u) = \mathcal{R}^*(u)^{\frac{\mathcal{R}(u)}{\mathcal{R}(u)+\mathcal{S}(u)}}\mathcal{S}^*(u)^{\frac{\mathcal{S}(u)}{\mathcal{R}(u)+\mathcal{S}(u)}}$,
- $(\mathcal{R}\mathcal{S})^*(u) = \mathcal{R}^*(u)\mathcal{S}^*(u)$,
- $(\frac{\mathcal{R}}{\mathcal{S}})^*(u) = \frac{\mathcal{R}^*(u)}{\mathcal{S}^*(u)}$,
- $(\mathcal{R}^{\mathcal{T}})^*(u) = \mathcal{R}^*(u)^{\mathcal{T}(u)}\mathcal{R}(u)^{\mathcal{T}'(u)}$,
- $(\mathcal{R}\circ\mathcal{T})^*(u) = \mathcal{R}^*(\mathcal{T}(u))^{\mathcal{T}'(u)}$.

Definition 2.4 ([2]) The multiplicative integral of a positive function $\mathcal{R}$ is defined as follows:

$$\int_a^b (\mathcal{R}(u))^{du} = \exp\left\{\int_a^b \ln(\mathcal{R}(u))\,du\right\}.$$

Proposition 2.5 ([2]) *Let $\mathcal{R}$ and $\mathcal{S}$ be positive Riemann-integrable functions. Then $\mathcal{R}$ and $\mathcal{S}$ are multiplicatively integrable, and the following properties hold*:

- $\int_a^b((\mathcal{R}(u))^p)^{du} = (\int_a^b(\mathcal{R}(u))^{du})^p$, $p\in\mathbb{R}$,
- $\int_a^b(\mathcal{R}(u))^{du} = \int_a^c(\mathcal{R}(u))^{du}\int_c^b(\mathcal{R}(u))^{du}$, $a\le c\le b$,
- $\int_a^a(\mathcal{R}(u))^{du} = 1$ *and* $\int_a^b(\mathcal{R}(u))^{du} = (\int_b^a(\mathcal{R}(u))^{du})^{-1}$,
- $\int_a^b(\mathcal{R}(u)\mathcal{S}(u))^{du} = \int_a^b(\mathcal{R}(u))^{du}\int_a^b(\mathcal{S}(u))^{du}$,
- $\int_a^b(\frac{\mathcal{R}(u)}{\mathcal{S}(u)})^{du} = \frac{\int_a^b(\mathcal{R}(u))^{du}}{\int_a^b(\mathcal{S}(u))^{du}}$.

Theorem 2.6 ([2]) *Let $\mathcal{R}$ be a positive multiplicatively differentiable function on the interval $[a,b]$, and let $\mathcal{S}:[a,b]\to\mathbb{R}$ be differentiable. Then the function $(\mathcal{R}^*)^{\mathcal{S}}$ is multiplicatively integrable, and*

$$\int_a^b\left(\mathcal{R}^*(u)^{\mathcal{S}(u)}\right)^{du} = \frac{\mathcal{R}(b)^{\mathcal{S}(b)}}{\mathcal{R}(a)^{\mathcal{S}(a)}}\times\frac{1}{\int_a^b(\mathcal{R}(u)^{\mathcal{S}'(u)})^{du}}.$$

Lemma 2.7 ([2]) *Let $\mathcal{R}$ be a positive multiplicatively differentiable function on $[a,b]$, and let $\mathcal{T}:J\subset\mathbb{R}\to\mathbb{R}$ and $\mathcal{S}:[a,b]\to\mathbb{R}$ be two distinct differentiable functions. Then we have*

$$\int_a^b\left(\mathcal{R}^*(\mathcal{T}(u))^{\mathcal{T}'(u)\mathcal{S}(u)}\right)^{du} = \frac{\mathcal{R}(\mathcal{T}(b))^{\mathcal{S}(b)}}{\mathcal{R}(\mathcal{T}(a))^{\mathcal{S}(a)}}\times\frac{1}{\int_a^b(\mathcal{R}(\mathcal{T}(u))^{\mathcal{S}'(u)})^{du}}.$$

Now let us review certain specific functions that will be utilized later in our investigation.

Definition 2.8 ([38]) The beta function is defined for any complex numbers u_1, u_2 such that $Re(u_1)>0$ and $Re(u_2)>0$ by

$$B(u_1,u_2) = \int_0^1 v^{u_1-1}(1-v)^{u_2-1}\,dv = \frac{\Gamma(u_1)\Gamma(u_2)}{\Gamma(u_1+u_2)},$$

where Γ is the Euler gamma function.

Definition 2.9 ([38]) The incomplete beta function is defined for any complex numbers u_1, u_2 such that $Re(u_1) > 0$ and $Re(u_2) > 0$ by

$$B_a(u_1,u_2) = \int_0^a v^{u_1-1}(1-v)^{u_2-1}\,dv, \quad 0 \le a < 1.$$

Definition 2.10 ([38]) For any complex numbers u_1, u_2, u_3, and y such that $Re(u_3) > Re(u_2) > 0$ and $|y| < 1$, the hypergeometric function is defined as

$${}_2\mathcal{F}_1(u_1,u_2,u_3;y) = \frac{1}{B(u_2,u_3-u_2)}\int_0^1 v^{u_2-1}(1-v)^{u_3-u_2-1}(1-yv)^{-u_1}\,dv,$$

where $B(\cdot,\cdot)$ is the beta function.

Lastly, we recall the definition of multiplicatively s-convex functions.

Definition 2.11 ([39]) A function $\mathcal{R} : I \subset \mathbb{R} \to \mathbb{R}^+$ is said to be multiplicatively s-convex for some fixed $s \in (0,1]$ if for all $u_1, u_2 \in I$, we have the inequality

$$\mathcal{R}\big(\gamma u_1 + (1-\gamma)u_2\big) \le \big[\mathcal{R}(u_1)\big]^{\gamma^s}\big[\mathcal{R}(u_2)\big]^{(1-\gamma)^s}$$

for all $\gamma \in [0,1]$.

3 Main results

We denote

$$\begin{aligned}&\mathcal{Q}(a,b,x,\varrho;\mathcal{R})\\ &\quad= \big(\mathcal{R}(x)\big)^{\frac{2(x-a)+\varrho(a+b-2x)}{2(b-a)}}\left(\mathcal{R}\left(\frac{a+b}{2}\right)\right)^{\frac{(1-\varrho)(a+b-2x)}{b-a}}\big(\mathcal{R}(a+b-x)\big)^{\frac{2(x-a)+\varrho(a+b-2x)}{2(b-a)}},\end{aligned} \tag{5}$$

$$\mathcal{I}(a,b,x;\mathcal{R}) = \begin{cases} \left(({}_aI_*^\delta\mathcal{R})(\frac{a+b}{2})({}_*I_b^\delta\mathcal{R})(\frac{a+b}{2})\right)^{\frac{2^{\delta-1}}{(b-a)^{\delta-1}}} & \text{for } x = a,\\ \left(({}_*I_x^\delta\mathcal{R})(a)({}_{(a+b-x)}I_*^\delta\mathcal{R})(b)\right)^{\frac{1}{(x-a)^{\delta-1}}} & \\ \quad\times\left(({}_xI_*^\delta\mathcal{R})(\frac{a+b}{2})({}_*I_{(a+b-x)}^\delta\mathcal{R})(\frac{a+b}{2})\right)^{\frac{2^{\delta-1}}{(a+b-2x)^{\delta-1}}} & \text{for } a < x < \frac{a+b}{2},\\ \left(({}_*I_{\frac{a+b}{2}}^\delta\mathcal{R})(a)({}_{\frac{a+b}{2}}I_*^\delta\mathcal{R})(b)\right)^{\frac{2^{\delta-1}}{(b-a)^{\delta-1}}} & \text{for } x = \frac{a+b}{2},\end{cases} \tag{6}$$

$$\Theta(\varrho,\delta,s) = \begin{cases} \frac{2\delta(1-\varrho)^{\frac{\delta+s+1}{\delta}}-\delta+(\delta+s+1)\varrho}{(\delta+s+1)(s+1)} & \text{for } 0 \le \varrho \le 1,\\ \frac{\varrho(\delta+s+1)-\delta}{(\delta+s+1)(s+1)} & \text{for } \varrho > 1,\end{cases} \tag{7}$$

and

$$\mathcal{N}(\varrho,\delta,s) = \begin{cases} \frac{1-\varrho}{s+1}\left(1-2(1-(1-\varrho)^{\frac{1}{\delta}})^{s+1}\right) - \mathcal{G}_{(1-\varrho)^{\frac{1}{\delta}}}(\delta+1,s+1) & \text{for } 0 \le \varrho \le 1,\\ \frac{\varrho-1}{s+1} + B(s+1,\delta+1) & \text{for } \varrho > 1,\end{cases} \tag{8}$$

with

$$\mathcal{G}_\rho(u,v) = B_\rho(u,v) - B_{1-\rho}(v,u),$$

where B and B_ρ are beta and incomplete beta functions, respectively.

To substantiate our findings, it is imperative to invoke the following lemma.

Lemma 3.1 *Assuming that $\mathcal{R}$ is a positive multiplicatively differentiable function on $[a,b]$ and $\mathcal{R}^*$ is multiplicatively integrable over $[a,b]$, then for $\delta > 0$, $\varrho \geq 0$, and $x \in [a, \frac{a+b}{2}]$, we have the following fractional multiplicative integral identity*:

$$\begin{aligned}
&\mathcal{Q}(a,b,x,\varrho;\mathcal{R})\big(\mathcal{I}(a,b,x;\mathcal{R})\big)^{-\frac{\Gamma(\delta+1)}{b-a}} \\
&\quad = \left(\int_0^1 \big(\big(\mathcal{R}^*\big((1-\gamma)a+\gamma x\big)\big)^{\gamma^\delta}\big)^{d\gamma}\right)^{\frac{(x-a)^2}{b-a}} \\
&\qquad \times \left(\int_0^1 \left(\left(\mathcal{R}^*\left((1-\gamma)x+\gamma\frac{a+b}{2}\right)\right)^{1-\varrho-(1-\gamma)^\delta}\right)^{d\gamma}\right)^{\frac{(a+b-2x)^2}{4(b-a)}} \\
&\qquad \times \left(\int_0^1 \left(\left(\mathcal{R}^*\left((1-\gamma)\frac{a+b}{2}+\gamma(a+b-x)\right)\right)^{\gamma^\delta-(1-\varrho)}\right)^{d\gamma}\right)^{\frac{(a+b-2x)^2}{4(b-a)}} \\
&\qquad \times \left(\int_0^1 \big(\big(\mathcal{R}^*\big((1-\gamma)(a+b-x)+\gamma b\big)\big)^{-(1-\gamma)^\delta}\big)^{d\gamma}\right)^{\frac{(x-a)^2}{b-a}},
\end{aligned} \tag{9}$$

where $\mathcal{Q}$ and $\mathcal{I}$ are defined in (5) and (6), respectively.

Proof Let

$$I_1 = \left(\int_0^1 \big(\big(\mathcal{R}^*\big((1-\gamma)a+\gamma x\big)\big)^{\gamma^\delta}\big)^{d\gamma}\right)^{\frac{(x-a)^2}{b-a}},$$

$$I_2 = \left(\int_0^1 \left(\left(\mathcal{R}^*\left((1-\gamma)x+\gamma\frac{a+b}{2}\right)\right)^{1-\varrho-(1-\gamma)^\delta}\right)^{d\gamma}\right)^{\frac{(a+b-2x)^2}{4(b-a)}},$$

$$I_3 = \left(\int_0^1 \left(\left(\mathcal{R}^*\left((1-\gamma)\frac{a+b}{2}+\gamma(a+b-x)\right)\right)^{\gamma^\delta-(1-\varrho)}\right)^{d\gamma}\right)^{\frac{(a+b-2x)^2}{4(b-a)}},$$

and

$$I_4 = \left(\int_0^1 \big(\big(\mathcal{R}^*\big((1-\gamma)(a+b-x)+\gamma b\big)\big)^{-(1-\gamma)^\delta}\big)^{d\gamma}\right)^{\frac{(x-a)^2}{b-a}}.$$

First, let us consider the case $x \in (a, \frac{a+b}{2})$.

Using Lemma 2.6, for I_1, we have

$$\begin{aligned}
I_1 &= \left(\int_0^1 \big(\big(\mathcal{R}^*\big((1-\gamma)a+\gamma x\big)\big)^{\gamma^\delta}\big)^{d\gamma}\right)^{\frac{(x-a)^2}{b-a}} \\
&= \int_0^1 \big(\big(\mathcal{R}^*\big((1-\gamma)a+\gamma x\big)\big)^{\frac{(x-a)^2}{b-a}(\gamma^\delta)}\big)^{d\gamma} \\
&= \frac{(\mathcal{R}(x))^{\frac{x-a}{b-a}}}{1} \frac{1}{\int_0^1 ((\mathcal{R}((1-\gamma)a+\gamma x))^{\delta\frac{x-a}{b-a}\gamma^{\delta-1}})^{d\gamma}} \\
&= \big(\mathcal{R}(x)\big)^{\frac{x-a}{b-a}} \frac{1}{\exp\{\delta\frac{x-a}{b-a}\int_0^1(\gamma^{\delta-1}(\ln\mathcal{R}((1-\gamma)a+\gamma x)))\,d\gamma\}}
\end{aligned}$$

$$
\begin{aligned}
&= \big(\mathcal{R}(x)\big)^{\frac{x-a}{b-a}} \frac{1}{\exp\{\delta \frac{(x-a)^{1-\delta}}{b-a} \int_a^x ((u-a)^{\delta-1}(\ln \mathcal{R}(u)))\, du\}} \\
&= \big(\mathcal{R}(x)\big)^{\frac{x-a}{b-a}} \frac{1}{(\exp\{(\frac{1}{\Gamma(\delta)} \int_a^x ((u-a)^{\delta-1}(\ln \mathcal{R}(u)))\, du)\})^{(x-a)^{1-\delta} \frac{\Gamma(\delta+1)}{b-a}}} \\
&= \big(\mathcal{R}(x)\big)^{\frac{x-a}{b-a}} \big(({}_*I_x^\delta \mathcal{R})(a)\big)^{-\frac{(x-a)^{1-\delta}\Gamma(\delta+1)}{b-a}}.
\end{aligned}
\tag{10}
$$

In the same vein,

$$
\begin{aligned}
I_2 &= \left(\int_0^1 \left(\left(\mathcal{R}^*\left((1-\gamma)x + \gamma \frac{a+b}{2}\right)\right)^{1-\varrho-(1-\gamma)^\delta}\right)^{d\gamma}\right)^{\frac{(a+b-2x)^2}{4(b-a)}} \\
&= \left(\int_0^1 \left(\left(\mathcal{R}^*\left((1-\gamma)x + \gamma \frac{a+b}{2}\right)\right)^{\frac{(a+b-2x)^2}{4(b-a)}(1-\varrho-(1-\gamma)^\delta)}\right)^{d\gamma}\right) \\
&= \frac{(\mathcal{R}(\frac{a+b}{2}))^{\frac{(1-\varrho)(a+b-2x)}{2(b-a)}}}{(\mathcal{R}(x))^{-\frac{\varrho(a+b-2x)}{2(b-a)}}} \frac{1}{\int_0^1 ((\mathcal{R}((1-\gamma)x + \gamma \frac{a+b}{2}))^{\delta \frac{(a+b-2x)}{2(b-a)}(1-\gamma)^{\delta-1}})^{d\gamma}} \\
&= \frac{(\mathcal{R}(x))^{\frac{\varrho(a+b-2x)}{2(b-a)}} (\mathcal{R}(\frac{a+b}{2}))^{\frac{(1-\varrho)(a+b-2x)}{2(b-a)}}}{\exp\{\delta \frac{(a+b-2x)}{2(b-a)} \int_0^1 (1-\gamma)^{\delta-1} \ln(\mathcal{R}((1-\gamma)x + \gamma \frac{a+b}{2}))\, d\gamma\}} \\
&= \frac{(\mathcal{R}(x))^{\frac{\varrho(a+b-2x)}{2(b-a)}} (\mathcal{R}(\frac{a+b}{2}))^{\frac{(1-\varrho)(a+b-2x)}{2(b-a)}}}{\exp\{\frac{2^{\delta-1}\Gamma(\delta+1)}{(a+b-2x)^{\delta-1}(b-a)} (\frac{1}{\Gamma(\delta)} \int_x^{\frac{a+b}{2}} (\frac{a+b}{2} - u)^{\delta-1} \ln(\mathcal{R}(u))\, du)\}} \\
&= \big(\mathcal{R}(x)\big)^{\frac{\varrho(a+b-2x)}{2(b-a)}} \left(\mathcal{R}\left(\frac{a+b}{2}\right)\right)^{\frac{(1-\varrho)(a+b-2x)}{2(b-a)}} \left(\big({}_xI_*^\delta \mathcal{R}\big)\left(\frac{a+b}{2}\right)\right)^{-\frac{2^{\delta-1}\Gamma(\delta+1)}{(a+b-2x)^{\delta-1}(b-a)}},
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
I_3 &= \left(\int_0^1 \left(\left(\mathcal{R}^*\left((1-\gamma)\frac{a+b}{2} + \gamma(a+b-x)\right)\right)^{\gamma^\delta-(1-\varrho)}\right)^{d\gamma}\right)^{\frac{(a+b-2x)^2}{4(b-a)}} \\
&= \int_0^1 \left(\left(\mathcal{R}^*\left((1-\gamma)\frac{a+b}{2} + \gamma(a+b-x)\right)\right)^{\frac{(a+b-2x)^2}{4(b-a)}(\gamma^\delta-(1-\varrho))}\right)^{d\gamma} \\
&= \frac{(\mathcal{R}(a+b-x))^{\frac{\varrho(a+b-2x)}{2(b-a)}}}{(\mathcal{R}(\frac{a+b}{2}))^{-\frac{(1-\varrho)(a+b-2x)}{2(b-a)}}} \frac{1}{\int_0^1 ((\mathcal{R}((1-\gamma)\frac{a+b}{2} + \gamma(a+b-x)))^{\delta \frac{(a+b-2x)}{2(b-a)}\gamma^{\delta-1}})^{d\gamma}} \\
&= \frac{(\mathcal{R}(\frac{a+b}{2}))^{\frac{(1-\varrho)(a+b-2x)}{2(b-a)}} (\mathcal{R}(a+b-x))^{\frac{\varrho(a+b-2x)}{2(b-a)}}}{\exp\{\delta \frac{(a+b-2x)}{2(b-a)} \int_0^1 \gamma^{\delta-1} \ln(\mathcal{R}((1-\gamma)\frac{a+b}{2} + \gamma(a+b-x)))\, d\gamma\}} \\
&= \frac{(\mathcal{R}(\frac{a+b}{2}))^{\frac{(1-\varrho)(a+b-2x)}{2(b-a)}} (\mathcal{R}(a+b-x))^{\frac{\varrho(a+b-2x)}{2(b-a)}}}{(\exp\{\frac{1}{\Gamma(\delta)} \int_{\frac{a+b}{2}}^{a+b-x} (u - \frac{a+b}{2})^{\delta-1} \ln(\mathcal{R}(u))\, du\})^{\frac{2^{\delta-1}\Gamma(\delta+1)}{(b-a)(a+b-2x)^{\delta-1}}}} \\
&= \left(\mathcal{R}\left(\frac{a+b}{2}\right)\right)^{\frac{(1-\varrho)(a+b-2x)}{2(b-a)}} \big(\mathcal{R}(a+b-x)\big)^{\frac{\varrho(a+b-2x)}{2(b-a)}} \\
&\quad \times \left(\big({}_*I_{(a+b-x)}^\delta \mathcal{R}\big)\left(\frac{a+b}{2}\right)\right)^{-\frac{2^{\delta-1}\Gamma(\delta+1)}{(b-a)(a+b-2x)^{\delta-1}}},
\end{aligned}
\tag{12}
$$

and

$$\begin{aligned}
I_4 &= \left(\int_0^1 \left(\left(\mathcal{R}^*\big((1-\gamma)(a+b-x)+\gamma b\big)\right)^{-(1-\gamma)^\delta}\right)^{d\gamma}\right)^{\frac{(x-a)^2}{b-a}}\\
&= \left(\int_0^1 \left(\left(\mathcal{R}^*\big((1-\gamma)(a+b-x)+\gamma b\big)\right)^{-\frac{(x-a)^2}{b-a}(1-\gamma)^\delta}\right)^{d\gamma}\right)\\
&= \frac{1}{(\mathcal{R}(a+b-x))^{-\frac{x-a}{b-a}}}\frac{1}{\int_0^1((\mathcal{R}((1-\gamma)(a+b-x)+\gamma b))^{\frac{\delta(x-a)}{b-a}(1-\gamma)^{\delta-1}})^{d\gamma}}\\
&= \big(\mathcal{R}(a+b-x)\big)^{\frac{x-a}{b-a}}\frac{1}{\exp\{\delta\frac{x-a}{b-a}\int_0^1(1-\gamma)^{\delta-1}\ln(\mathcal{R}((1-\gamma)(a+b-x)+\gamma b))\,d\gamma\}}\\
&= \big(\mathcal{R}(a+b-x)\big)^{\frac{x-a}{b-a}}\frac{1}{\exp\{\frac{(x-a)^{1-\delta}\Gamma(\delta+1)}{b-a}(\frac{1}{\Gamma(\delta)}\int_{a+b-x}^b(b-u)^{\delta-1}\ln(\mathcal{R}(u))\,du)\}}\\
&= \big(\mathcal{R}(a+b-x)\big)^{\frac{x-a}{b-a}}\left(\left({}_{(a+b-x)}I_*^\delta\mathcal{R}\right)(b)\right)^{-\frac{(x-a)^{1-\delta}\Gamma(\delta+1)}{b-a}}. \qquad (13)
\end{aligned}$$

Multiplying equalities (10)–(13), we get the desired result.

Now, for the cases where $x=a$ and $x=\frac{a+b}{2}$, it suffices to follow the same demonstration principle as above, taking into account that when $x=a$, we have $I_1=I_4=1$, and when $x=\frac{a+b}{2}$, $I_2=I_3=1$, and equation (9) will be reduced:

(1) For $x=a$,

$$\begin{aligned}
&(\mathcal{R}(a))^{\frac{\varrho}{2}}\left(\mathcal{R}\left(\frac{a+b}{2}\right)\right)^{1-\varrho}(\mathcal{R}(b))^{\frac{\varrho}{2}}\left(\left({}_aI_*^\delta\mathcal{R}\right)\left(\frac{a+b}{2}\right)\left({}_*I_b^\delta\mathcal{R}\right)\left(\frac{a+b}{2}\right)\right)^{-\frac{2^{\delta-1}\Gamma(\delta+1)}{(b-a)^\delta}}\\
&= \Bigg[\left(\int_0^1\left(\left(\mathcal{R}^*\left((1-\gamma)a+\gamma\frac{a+b}{2}\right)\right)^{1-\varrho-(1-\gamma)^\delta}\right)^{d\gamma}\right)\\
&\quad\times\left(\int_0^1\left(\left(\mathcal{R}^*\left((1-\gamma)\frac{a+b}{2}+\gamma b\right)\right)^{\gamma^\delta-(1-\varrho)}\right)^{d\gamma}\right)\Bigg]^{\frac{b-a}{4}};
\end{aligned}$$

(2) For $x=\frac{a+b}{2}$,

$$\begin{aligned}
&\mathcal{R}\left(\frac{a+b}{2}\right)\left(\left({}_*I^\delta_{\frac{a+b}{2}}\mathcal{R}\right)(a)\left({}_{\frac{a+b}{2}}I_*^\delta\mathcal{R}\right)(b)\right)^{-\frac{2^{\delta-1}\Gamma(\delta+1)}{(b-a)^\delta}}\\
&= \Bigg[\left(\int_0^1\left(\left(\mathcal{R}^*\left((1-\gamma)a+\gamma\frac{a+b}{2}\right)\right)^{\gamma^\delta}\right)^{d\gamma}\right)\\
&\quad\times\left(\int_0^1\left(\left(\mathcal{R}^*\left((1-\gamma)\frac{a+b}{2}+\gamma b\right)\right)^{-(1-\gamma)^\delta}\right)^{d\gamma}\right)\Bigg]^{\frac{b-a}{4}}.
\end{aligned}$$

The proof is completed. □

Theorem 3.2 *Let $\mathcal{R}:[a,b]\to\mathbb{R}^+$ be an increasing multiplicatively differentiable function on $[a,b]$. If $\mathcal{R}^*$ exhibits multiplicative s-convexity on this interval, then for any $x\in[a,\frac{a+b}{2}]$, $\varrho\geq 0$, and $\delta>0$, we have*

$$\left|\mathcal{Q}(a,b,x,\varrho;\mathcal{R})\big(\mathcal{I}(a,b,x;\mathcal{R})\big)^{-\frac{\Gamma(\delta+1)}{b-a}}\right|$$

$$\begin{aligned}
&\leq \left(\mathcal{R}^*(a)\mathcal{R}^*(b)\right)^{\frac{(x-a)^2}{b-a}B(\delta+1,s+1)}\mathcal{R}^*\left(\frac{a+b}{2}\right)^{\frac{(a+b-2x)^2}{2(b-a)}\mathcal{N}(\varrho,\delta,s)} \\
&\quad\times \left(\mathcal{R}^*(x)\mathcal{R}^*(a+b-x)\right)^{\frac{(a+b-2x)^2}{4(b-a)}\Theta(\varrho,\delta,s)+\frac{(x-a)^2}{(\delta+s+1)(b-a)}},
\end{aligned}$$

where $\mathcal{Q}$, $\mathcal{I}$, Θ, and $\mathcal{N}$ are defined in (5)–(8), *and B is the beta function.*

Proof From Lemma 3.1 by the properties of multiplicative integrals we have

$$\begin{aligned}
&\left|\mathcal{Q}(a,b,x,\varrho;\mathcal{R})\left(\mathcal{I}(a,b,x;\mathcal{R})\right)^{-\frac{\Gamma(\delta+1)}{b-a}}\right| \\
&\quad\leq \exp\left(\frac{(x-a)^2}{b-a}\int_0^1 \gamma^\delta \ln\left(\mathcal{R}^*\left((1-\gamma)a+\gamma x\right)\right)d\gamma\right) \\
&\qquad\times \exp\left(\frac{(a+b-2x)^2}{4(b-a)}\int_0^1 \left|1-\varrho-(1-\gamma)^\delta\right| \ln\mathcal{R}^*\left((1-\gamma)x+\gamma\frac{a+b}{2}\right)d\gamma\right) \\
&\qquad\times \exp\left(\frac{(a+b-2x)^2}{4(b-a)}\int_0^1 \left|\gamma^\delta-(1-\varrho)\right| \ln\mathcal{R}^*\left((1-\gamma)\frac{a+b}{2}+\gamma(a+b-x)\right)d\gamma\right) \\
&\qquad\times \exp\left(\frac{(x-a)^2}{b-a}\int_0^1 (1-\gamma)^\delta \ln\mathcal{R}^*\left((1-\gamma)(a+b-x)+\gamma b\right)d\gamma\right).
\end{aligned}$$

From the multiplicative s-convexity of $\mathcal{R}^*$ we have

$$\begin{aligned}
&\left|\mathcal{Q}(a,b,x,\varrho;\mathcal{R})\left(\mathcal{I}(a,b,x;\mathcal{R})\right)^{-\frac{\Gamma(\delta+1)}{b-a}}\right| \\
&\quad\leq \exp\left(\frac{(x-a)^2}{b-a}\int_0^1 \gamma^\delta\left((1-\gamma)^s\ln\mathcal{R}^*(a)+\gamma^s\ln\mathcal{R}^*(x)\right)d\gamma\right) \\
&\qquad\times \exp\left(\frac{(a+b-2x)^2}{4(b-a)}\right. \\
&\qquad\times \left.\int_0^1 \left|1-\varrho-(1-\gamma)^\delta\right|\left((1-\gamma)^s\ln\mathcal{R}^*(x)+\gamma^s\ln\mathcal{R}^*\left(\frac{a+b}{2}\right)\right)d\gamma\right) \\
&\qquad\times \exp\left(\frac{(a+b-2x)^2}{4(b-a)}\right. \\
&\qquad\times \left.\int_0^1 \left|\gamma^\delta-(1-\varrho)\right|\left((1-\gamma)^s\ln\mathcal{R}^*\left(\frac{a+b}{2}\right)+\gamma^s\ln\mathcal{R}^*(a+b-x)\right)d\gamma\right) \\
&\qquad\times \exp\left(\frac{(x-a)^2}{b-a}\int_0^1 (1-\gamma)^\delta\left((1-\gamma)^s\ln\mathcal{R}^*(a+b-x)+\gamma^s\ln\mathcal{R}^*(b)\right)d\gamma\right) \\
&\quad= \exp\left(\frac{(x-a)^2}{b-a}\left(\int_0^1 \gamma^\delta(1-\gamma)^s d\gamma\right)\ln\mathcal{R}^*(a)+\frac{(x-a)^2}{b-a}\left(\int_0^1 \gamma^{\delta+s}d\gamma\right)\ln\mathcal{R}^*(x)\right) \\
&\qquad\times \exp\left(\frac{(a+b-2x)^2}{4(b-a)}\left(\int_0^1 \left|1-\varrho-(1-\gamma)^\delta\right|(1-\gamma)^s d\gamma\right)\ln\mathcal{R}^*(x)\right. \\
&\qquad+ \left.\frac{(a+b-2x)^2}{4(b-a)}\left(\int_0^1 \left|1-\varrho-(1-\gamma)^\delta\right|\gamma^s d\gamma\right)\ln\mathcal{R}^*\left(\frac{a+b}{2}\right)\right) \\
&\qquad\times \exp\left(\frac{(a+b-2x)^2}{4(b-a)}\left(\int_0^1 \left|\gamma^\delta-(1-\varrho)\right|(1-\gamma)^s d\gamma\right)\ln\mathcal{R}^*\left(\frac{a+b}{2}\right)\right. \\
&\qquad+ \left.\frac{(a+b-2x)^2}{4(b-a)}\left(\int_0^1 \left|\gamma^\delta-(1-\varrho)\right|\gamma^s d\gamma\right)\ln\mathcal{R}^*(a+b-x)\,d\gamma\right)
\end{aligned}$$

$$
\begin{aligned}
&\times \exp\left(\frac{(x-a)^2}{b-a}\left(\int_0^1 (1-\gamma)^{\delta+s}\,d\gamma\right)\ln\mathcal{R}^*(a+b-x)\right.\\
&\left.+\frac{(x-a)^2}{b-a}\left(\int_0^1 (1-\gamma)^{\delta}\gamma^s\,d\gamma\right)\ln\mathcal{R}^*(b)\right)\\
&=\left(\mathcal{R}^*(a)\mathcal{R}^*(b)\right)^{\frac{(x-a)^2}{b-a}B(\delta+1,s+1)}\mathcal{R}^*\left(\frac{a+b}{2}\right)^{\frac{(a+b-2x)^2}{2(b-a)}\mathcal{N}(\varrho,\delta,s)}\\
&\times\left(\mathcal{R}^*(x)\mathcal{R}^*(a+b-x)\right)^{\frac{(a+b-2x)^2}{4(b-a)}\Theta(\varrho,\delta,s)+\frac{(x-a)^2}{(\delta+s+1)(b-a)}},
\end{aligned}
$$

where we have used (7), (8), and the equalities

$$
\int_0^1 \gamma^\delta(1-\gamma)^s\,d\gamma=\int_0^1(1-\gamma)^\delta\gamma^s\,d\gamma=B(\delta+1,s+1), \tag{14}
$$

$$
\int_0^1 \gamma^{\delta+s}\,d\gamma=\int_0^1(1-\gamma)^{\delta+s}\,d\gamma=\frac{1}{\delta+s+1}, \tag{15}
$$

$$
\int_0^1 \left|1-\varrho-(1-\gamma)^\delta\right|(1-\gamma)^s\,d\gamma=\int_0^1\left|1-\varrho-\gamma^\delta\right|\gamma^s\,d\gamma=\Theta(\varrho,\delta,s), \tag{16}
$$

and

$$
\int_0^1 \left|\gamma^\delta-(1-\varrho)\right|(1-\gamma)^s\,d\gamma=\int_0^1\left|1-\varrho-(1-\gamma)^\delta\right|\gamma^s\,d\gamma=\mathcal{N}(\varrho,\delta,s). \tag{17}
$$

The proof is completed. □

Corollary 3.3 *Taking $x=a$ and $\varrho\in[0,1]$ in Theorem 3.2, we get*

$$
\begin{aligned}
&\left|(\mathcal{R}(a))^{\frac{\varrho}{2}}\left(\mathcal{R}\left(\frac{a+b}{2}\right)\right)^{1-\varrho}(\mathcal{R}(b))^{\frac{\varrho}{2}}\left(\left({}_aI_*^\delta\mathcal{R}\right)\left(\frac{a+b}{2}\right)\left({}_*I_b^\delta\mathcal{R}\right)\left(\frac{a+b}{2}\right)\right)^{-\frac{2^{\delta-1}\Gamma(\delta+1)}{(b-a)^\delta}}\right|\\
&\leq\left(\left(\mathcal{R}^*(a)\mathcal{R}^*(b)\right)^{\frac{(\delta+s+1)\varrho-\delta+2\delta(1-\varrho)^{\frac{\delta+s+1}{\delta}}}{(s+1)(\delta+s+1)}}\mathcal{R}^*\left(\frac{a+b}{2}\right)^{2\mathcal{T}(\varrho,\delta,s)}\right)^{\frac{b-a}{4}}
\end{aligned}
$$

with

$$
\begin{aligned}
\mathcal{T}(\varrho,\delta,s)&=\frac{1-\varrho}{s+1}\left(1-2\left(1-(1-\varrho)^{\frac{1}{\delta}}\right)^{s+1}\right)\\
&\quad-B_{(1-\varrho)^{\frac{1}{\delta}}}(\delta+1,s+1)+B_{1-(1-\varrho)^{\frac{1}{\delta}}}(s+1,\delta+1).
\end{aligned} \tag{18}
$$

Corollary 3.4 *By setting $\varrho=1$ in Corollary 3.3 we obtain the following trapezium inequality*:

$$
\begin{aligned}
&\left|\sqrt{\mathcal{R}(a)\mathcal{R}(b)}\left(\left({}_aI_*^\delta\mathcal{R}\right)\left(\frac{a+b}{2}\right)\left({}_*I_b^\delta\mathcal{R}\right)\left(\frac{a+b}{2}\right)\right)^{-\frac{2^{\delta-1}\Gamma(\delta+1)}{(b-a)^\delta}}\right|\\
&\leq\left(\left(\mathcal{R}^*(a)\mathcal{R}^*(b)\right)^{\frac{1}{\delta+s+1}}\mathcal{R}^*\left(\frac{a+b}{2}\right)^{2B(s+1,\delta+1)}\right)^{\frac{b-a}{4}}.
\end{aligned}
$$

Corollary 3.5 *Taking $\varrho = \frac{1}{2}$ in Corollary 3.3, we obtain the following Bullen inequality*:

$$\left|\left(\mathcal{R}(a)\left(\mathcal{R}\left(\frac{a+b}{2}\right)\right)^{2}\mathcal{R}(b)\right)^{\frac{1}{4}}\left(\left({}_{a}I_{*}^{\delta}\mathcal{R}\right)\left(\frac{a+b}{2}\right)\left({}_{*}I_{b}^{\delta}\mathcal{R}\right)\left(\frac{a+b}{2}\right)\right)^{-\frac{2^{\delta-1}\Gamma(\delta+1)}{(b-a)^{\delta}}}\right|$$
$$\leq\left(\left(\mathcal{R}^{*}(a)\mathcal{R}^{*}(b)\right)^{\frac{s+1-\delta+2\delta(\frac{1}{2})^{\frac{s+1}{\delta}}}{2(s+1)(\delta+s+1)}}\mathcal{R}^{*}\left(\frac{a+b}{2}\right)^{2\mathcal{T}_{1}(\delta,s)}\right)^{\frac{b-a}{4}}$$

with

$$\mathcal{T}_{1}(\delta,s)=\frac{1}{2(s+1)}\left(1-2\left(1-\left(\frac{1}{2}\right)^{\frac{1}{\delta}}\right)^{s+1}\right)$$
$$-B_{(\frac{1}{2})^{\frac{1}{\delta}}}(\delta+1,s+1)+B_{1-(\frac{1}{2})^{\frac{1}{\delta}}}(s+1,\delta+1).$$

Corollary 3.6 *Taking $\varrho = \frac{1}{3}$ in Corollary 3.3, we obtain the following Simpson inequality*:

$$\left|\left(\mathcal{R}(a)\left(\mathcal{R}\left(\frac{a+b}{2}\right)\right)^{4}\mathcal{R}(b)\right)^{\frac{1}{6}}\left(\left({}_{a}I_{*}^{\delta}\mathcal{R}\right)\left(\frac{a+b}{2}\right)\left({}_{*}I_{b}^{\delta}\mathcal{R}\right)\left(\frac{a+b}{2}\right)\right)^{-\frac{2^{\delta-1}\Gamma(\delta+1)}{(b-a)^{\delta}}}\right|$$
$$\leq\left(\left(\mathcal{R}^{*}(a)\mathcal{R}^{*}(b)\right)^{\frac{s+1-2\delta+4\delta(\frac{2}{3})^{\frac{s+1}{\delta}}}{3(s+1)(\delta+s+1)}}\mathcal{R}^{*}\left(\frac{a+b}{2}\right)^{2\mathcal{T}_{2}(\delta,s)}\right)^{\frac{b-a}{4}}$$

with

$$\mathcal{T}_{2}(\delta,s)=\frac{2}{3(s+1)}\left(1-2\left(1-\left(\frac{2}{3}\right)^{\frac{1}{\delta}}\right)^{s+1}\right)$$
$$-B_{(\frac{2}{3})^{\frac{1}{\delta}}}(\delta+1,s+1)+B_{1-(\frac{2}{3})^{\frac{1}{\delta}}}(s+1,\delta+1).$$

Corollary 3.7 *Taking $\varrho = \frac{7}{15}$ in Corollary 3.3, we obtain the following corrected Simpson inequality*:

$$\left|\left(\left(\mathcal{R}(a)\right)^{7}\left(\mathcal{R}\left(\frac{a+b}{2}\right)\right)^{16}\left(\mathcal{R}(b)\right)^{7}\right)^{\frac{1}{30}}\left(\left({}_{a}I_{*}^{\delta}\mathcal{R}\right)\left(\frac{a+b}{2}\right)\left({}_{*}I_{b}^{\delta}\mathcal{R}\right)\left(\frac{a+b}{2}\right)\right)^{-\frac{2^{\delta-1}\Gamma(\delta+1)}{(b-a)^{\delta}}}\right|$$
$$\leq\left(\left(\mathcal{R}^{*}(a)\mathcal{R}^{*}(b)\right)^{\frac{7s+7-8\delta+16\delta(\frac{8}{15})^{\frac{s+1}{\delta}}}{15(s+1)(\delta+s+1)}}\mathcal{R}^{*}\left(\frac{a+b}{2}\right)^{2\mathcal{T}_{3}(\delta,s)}\right)^{\frac{b-a}{4}}$$

with

$$\mathcal{T}_{3}(\delta,s)=\frac{8}{15(s+1)}\left(1-2\left(1-\left(\frac{8}{15}\right)^{\frac{1}{\delta}}\right)^{s+1}\right)$$
$$-B_{(\frac{8}{15})^{\frac{1}{\delta}}}(\delta+1,s+1)+B_{1-(\frac{8}{15})^{\frac{1}{\delta}}}(s+1,\delta+1).$$

Corollary 3.8 *Taking $\varrho = 0$ in Corollary 3.3, we obtain the following midpoint inequality*:

$$\left|\mathcal{R}\left(\frac{a+b}{2}\right)\left(\left({}_{a}I_{*}^{\delta}\mathcal{R}\right)\left(\frac{a+b}{2}\right)\left({}_{*}I_{b}^{\delta}\mathcal{R}\right)\left(\frac{a+b}{2}\right)\right)^{-\frac{2^{\delta-1}\Gamma(\delta+1)}{(b-a)^{\delta}}}\right|$$
$$\leq\left(\left(\mathcal{R}^{*}(a)\mathcal{R}^{*}(b)\right)^{\frac{\delta}{(s+1)(\delta+s+1)}}\mathcal{R}^{*}\left(\frac{a+b}{2}\right)^{2(\frac{1}{s+1}-B(\delta+1,s+1))}\right)^{\frac{b-a}{4}}.$$

Corollary 3.9 *Taking $x = a$ and $\varrho > 1$ in Theorem 3.2, we get*

$$\left|(\mathcal{R}(a))^{\frac{\varrho}{2}}\left(\mathcal{R}\left(\frac{a+b}{2}\right)\right)^{1-\varrho}(\mathcal{R}(b))^{\frac{\varrho}{2}}\left(\left({}_aI_*^\delta\mathcal{R}\right)\left(\frac{a+b}{2}\right)\left({}_*I_b^\delta\mathcal{R}\right)\left(\frac{a+b}{2}\right)\right)^{-\frac{2^{\delta-1}\Gamma(\delta+1)}{(b-a)^\delta}}\right|$$
$$\leq\left(\left(\mathcal{R}^*(a)\mathcal{R}^*(b)\right)^{\frac{\varrho(\delta+s+1)-\delta}{(s+1)(\delta+s+1)}}\mathcal{R}^*\left(\frac{a+b}{2}\right)^{2(\frac{\varrho-1}{s+1}+B(s+1,\delta+1))}\right)^{\frac{b-a}{4}}.$$

Corollary 3.10 *Taking $\varrho = \frac{4}{3}$ in Corollary 3.9, we obtain the following Milne inequality:*

$$\left|\left((\mathcal{R}(a))^2\left(\mathcal{R}\left(\frac{a+b}{2}\right)\right)^{-1}(\mathcal{R}(b))^2\right)^{\frac{1}{3}}\left(\left({}_aI_*^\delta\mathcal{R}\right)\left(\frac{a+b}{2}\right)\left({}_*I_b^\delta\mathcal{R}\right)\left(\frac{a+b}{2}\right)\right)^{-\frac{2^{\delta-1}\Gamma(\delta+1)}{(b-a)^\delta}}\right|$$
$$\leq\left(\left(\mathcal{R}^*(a)\mathcal{R}^*(b)\right)^{\frac{\delta+4s+4}{3(s+1)(\delta+s+1)}}\mathcal{R}^*\left(\frac{a+b}{2}\right)^{2(\frac{1}{3(s+1)}+B(s+1,\delta+1))}\right)^{\frac{b-a}{4}}.$$

Corollary 3.11 *Taking $x = \frac{3a+b}{4}$ and $\varrho \in [0,1]$ in Theorem 3.2, we get*

$$\left|\left(\mathcal{R}\left(\frac{3a+b}{4}\right)\right)^{\frac{1+\varrho}{4}}\left(\mathcal{R}\left(\frac{a+b}{2}\right)\right)^{\frac{1-\varrho}{2}}\left(\mathcal{R}\left(\frac{a+3b}{4}\right)\right)^{\frac{1+\varrho}{4}}(\mathcal{D}(\mathcal{R}))^{-\frac{4^{\delta-1}\Gamma(\delta+1)}{(b-a)^\delta}}\right|$$
$$\leq\left(\left(\mathcal{R}^*(a)\mathcal{R}^*(b)\right)^{B(\delta+1,s+1)}\mathcal{R}^*\left(\frac{a+b}{2}\right)^{2\mathcal{T}(\varrho,\delta,s)}\right.$$
$$\left.\times\left(\mathcal{R}^*\left(\frac{3a+b}{4}\right)\mathcal{R}^*\left(\frac{a+3b}{4}\right)\right)^{\frac{(\delta+s+1)\varrho-\delta+2\delta(1-\varrho)^{\frac{\delta+s+1}{\delta}}}{(s+1)(\delta+s+1)}+\frac{1}{\delta+s+1}}\right)^{\frac{b-a}{16}},$$

where $\mathcal{T}$ is defined in (18), and

$$\mathcal{D}(\mathcal{R}) = \left(\left({}_*I^\delta_{\frac{3a+b}{4}}\mathcal{R}\right)(a)\left({}_{\frac{a+3b}{4}}I_*^\delta\mathcal{R}\right)(b)\left({}_{\frac{3a+b}{4}}I_*^\delta\mathcal{R}\right)\left(\frac{a+b}{2}\right)\left({}_*I^\delta_{\frac{a+3b}{4}}\mathcal{R}\right)\left(\frac{a+b}{2}\right)\right). \tag{19}$$

Corollary 3.12 *Taking $x = \frac{3a+b}{4}$ and $\varrho > 1$ in Theorem 3.2, we get*

$$\left|\left(\mathcal{R}\left(\frac{3a+b}{4}\right)\right)^{\frac{1+\varrho}{4}}\left(\mathcal{R}\left(\frac{a+b}{2}\right)\right)^{\frac{1-\varrho}{2}}\left(\mathcal{R}\left(\frac{a+3b}{4}\right)\right)^{\frac{1+\varrho}{4}}(\mathcal{D}(\mathcal{R}))^{-\frac{4^{\delta-1}\Gamma(\delta+1)}{(b-a)^\delta}}\right|$$
$$\leq\left(\left(\mathcal{R}^*(a)\mathcal{R}^*(b)\right)^{B(\delta+1,s+1)}\mathcal{R}^*\left(\frac{a+b}{2}\right)^{2(\frac{\varrho-1}{s+1}+B(s+1,\delta+1))}\right.$$
$$\left.\times\left(\mathcal{R}^*\left(\frac{3a+b}{4}\right)\mathcal{R}^*\left(\frac{a+3b}{4}\right)\right)^{\frac{\varrho(\delta+s+1)-\delta}{(s+1)(\delta+s+1)}+\frac{1}{\delta+s+1}}\right)^{\frac{b-a}{16}},$$

where $\mathcal{D}$ is given by (19).

Corollary 3.13 *Taking $\varrho = \frac{5}{3}$ in Corollary 3.12, we obtain the following dual-Simpson inequality:*

$$\left|\left(\left(\mathcal{R}\left(\frac{3a+b}{4}\right)\right)^2\left(\mathcal{R}\left(\frac{a+b}{2}\right)\right)^{-1}\left(\mathcal{R}\left(\frac{a+3b}{4}\right)\right)^2\right)^{\frac{1}{3}}(\mathcal{D}(\mathcal{R}))^{-\frac{4^{\delta-1}\Gamma(\delta+1)}{(b-a)^\delta}}\right|$$

$$\leq \left(\left(\mathcal{R}^*(a)\mathcal{R}^*(b)\right)^{B(\delta+1,s+1)} \mathcal{R}^*\left(\frac{a+b}{2}\right)^{2(\frac{2}{3(s+1)}+B(s+1,\delta+1))} \times \left(\mathcal{R}^*\left(\frac{3a+b}{4}\right)\mathcal{R}^*\left(\frac{a+3b}{4}\right)\right)^{\frac{2\delta+8s+8}{3(s+1)(\delta+s+1)}} \right)^{\frac{b-a}{16}},$$

where $\mathcal{D}$ *is given by* (19).

Remark 3.14 By setting $\varrho = \frac{17}{15}$ in Corollary 3.12 we recover the result established in Theorem 3.2 from [33] concerning the corrected dual-Simpson formula.

Corollary 3.15 *Taking* $x = \frac{5a+b}{6}$ *and* $\varrho \in [0,1]$ *in Theorem* 3.2, *we get*

$$\left| \left(\mathcal{R}\left(\frac{5a+b}{6}\right)\right)^{\frac{1+2\varrho}{6}} \left(\mathcal{R}\left(\frac{a+b}{2}\right)\right)^{\frac{2(1-\varrho)}{3}} \left(\mathcal{R}\left(\frac{a+5b}{6}\right)\right)^{\frac{1+2\varrho}{6}} (\mathcal{M}(\mathcal{R}))^{-\frac{3^{\delta-1}\Gamma(\delta+1)}{(b-a)^{\delta}}} \right|$$
$$\leq \left(\left(\mathcal{R}^*(a)\mathcal{R}^*(b)\right)^{B(\delta+1,s+1)} \mathcal{R}^*\left(\frac{a+b}{2}\right)^{8\mathcal{T}(\varrho,\delta,s)} \times \left(\mathcal{R}^*\left(\frac{5a+b}{6}\right)\mathcal{R}^*\left(\frac{a+5b}{6}\right)\right)^{4\frac{(\delta+s+1)\varrho-\delta+2\delta(1-\varrho)^{\frac{\delta+s+1}{\delta}}}{(s+1)(\delta+s+1)}+\frac{1}{\delta+s+1}} \right)^{\frac{b-a}{36}},$$

where $\mathcal{T}$ *is defined by* (18), *and*

$$\mathcal{M}(\mathcal{R}) = \left(\left(\left({}_*I^{\delta}_{\frac{5a+b}{6}}\mathcal{R}\right)(a)\left({}_{\frac{a+5b}{6}}I^{\delta}_*\mathcal{R}\right)(b)\right)^{2^{\delta-1}} \times \left(\left({}_{\frac{5a+b}{6}}I^{\delta}_*\mathcal{R}\right)\left(\frac{a+b}{2}\right)\left({}_*I^{\delta}_{\frac{a+5b}{6}}\mathcal{R}\right)\left(\frac{a+b}{2}\right)\right)\right). \tag{20}$$

Corollary 3.16 *Taking* $\varrho = \frac{5}{8}$ *in Corollary* 3.15, *we obtain the following Maclaurin inequality*:

$$\left| \left(\left(\mathcal{R}\left(\frac{5a+b}{6}\right)\right)^{3} \left(\mathcal{R}\left(\frac{a+b}{2}\right)\right)^{2} \left(\mathcal{R}\left(\frac{a+5b}{6}\right)\right)^{3} \right)^{\frac{1}{8}} (\mathcal{M}(\mathcal{R}))^{-\frac{3^{\delta-1}\Gamma(\delta+1)}{(b-a)^{\delta}}} \right|$$
$$\leq \left(\left(\mathcal{R}^*(a)\mathcal{R}^*(b)\right)^{B(\delta+1,s+1)} \mathcal{R}^*\left(\frac{a+b}{2}\right)^{8\mathcal{T}_4(\delta,s)} \times \left(\mathcal{R}^*\left(\frac{5a+b}{6}\right)\mathcal{R}^*\left(\frac{a+5b}{6}\right)\right)^{\frac{5s+5-3\delta+6\delta(\frac{3}{8})^{\frac{s+1}{\delta}}}{2(s+1)(\delta+s+1)}+\frac{1}{\delta+s+1}} \right)^{\frac{b-a}{36}},$$

where $\mathcal{M}$ *is given by* (20), *and*

$$\mathcal{T}_4(\delta,s) = \frac{3}{(s+1)}\left(1-2\left(1-\left(\frac{3}{8}\right)^{\frac{1}{\delta}}\right)^{s+1}\right) - B_{(\frac{3}{8})^{\frac{1}{\delta}}}(\delta+1,s+1) + B_{1-(\frac{3}{8})^{\frac{1}{\delta}}}(s+1,\delta+1).$$

Corollary 3.17 *Taking $\varrho = \frac{41}{80}$ in Corollary 3.15, we obtain the following corrected Maclaurin inequality*:

$$\begin{aligned}&\left|\left(\left(\mathcal{R}\left(\frac{5a+b}{6}\right)\right)^{27}\left(\mathcal{R}\left(\frac{a+b}{2}\right)\right)^{26}\left(\mathcal{R}\left(\frac{a+5b}{6}\right)\right)^{27}\right)^{\frac{1}{80}}(\mathcal{M}(f))^{-\frac{3^{\delta-1}\Gamma(\delta+1)}{(b-a)^{\delta}}}\right|\\&\leq\left(\left(\mathcal{R}^*(a)\mathcal{R}^*(b)\right)^{B(\delta+1,s+1)}\mathcal{R}^*\left(\frac{a+b}{2}\right)^{8\mathcal{T}_5(\delta,s)}\right.\\&\quad\times\left.\left(\mathcal{R}^*\left(\frac{5a+b}{6}\right)\mathcal{R}^*\left(\frac{a+5b}{6}\right)\right)^{\frac{41s+41-39\delta+78\delta(\frac{39}{80})^{\frac{s+1}{\delta}}}{20(s+1)(\delta+s+1)}+\frac{1}{\delta+s+1}}\right)^{\frac{b-a}{36}},\end{aligned}$$

where $\mathcal{M}$ is given by (20), and

$$\begin{aligned}\mathcal{T}_5(\delta,s)&=\frac{39}{80(s+1)}\left(1-2\left(1-\left(\frac{39}{80}\right)^{\frac{1}{\delta}}\right)^{s+1}\right)\\&\quad-B_{(\frac{39}{80})^{\frac{1}{\delta}}}(\delta+1,s+1)+B_{1-(\frac{39}{80})^{\frac{1}{\delta}}}(s+1,\delta+1).\end{aligned}$$

Corollary 3.18 *Taking $x=\frac{a+b}{2}$ in Theorem 3.2, we get*

$$\begin{aligned}&\left|\mathcal{R}\left(\frac{a+b}{2}\right)\left(\left({}_*I^{\delta}_{\frac{a+b}{2}}\mathcal{R}\right)(a)\left({}_{\frac{a+b}{2}}I^{\delta}_*\mathcal{R}\right)(b)\right)^{-\frac{2^{\delta-1}\Gamma(\delta+1)}{(b-a)^{\delta}}}\right|\\&\leq\left(\left(\mathcal{R}^*(a)\mathcal{R}^*(b)\right)^{B(\delta+1,s+1)}\left(\mathcal{R}^*\left(\frac{a+b}{2}\right)\right)^{\frac{2}{\delta+s+1}}\right)^{\frac{b-a}{4}}.\end{aligned}$$

Corollary 3.19 *Taking $s=1$ and $\varrho\in[0,1]$ in Theorem 3.2, we get*

$$\begin{aligned}&\left|\mathcal{Q}(a,b,x,\varrho;\mathcal{R})\left(\mathcal{I}(a,b,x;\mathcal{R})\right)^{-\frac{\Gamma(\delta+1)}{b-a}}\right|\\&\leq\left(\mathcal{R}^*(a)\mathcal{R}^*(b)\right)^{\frac{(x-a)^2}{(\delta+1)(\delta+2)(b-a)}}\\&\quad\times\mathcal{R}^*\left(\frac{a+b}{2}\right)^{\frac{(a+b-2x)^2}{2(b-a)}\left(\frac{2-(1-\varrho)(\delta+1)(\delta+2)}{2(\delta+1)(\delta+2)}+\frac{2\delta}{\delta+1}(1-\varrho)^{\frac{\delta+1}{\delta}}-\frac{\delta}{\delta+2}(1-\varrho)^{\frac{\delta+2}{\delta}}\right)}\\&\quad\times\left(\mathcal{R}^*(x)\mathcal{R}^*(a+b-x)\right)^{\frac{(a+b-2x)^2}{4(b-a)}\frac{(\delta+2)\varrho-\delta+2\delta(1-\varrho)^{\frac{\delta+2}{\delta}}}{2(\delta+2)}+\frac{(x-a)^2}{(\delta+2)(b-a)}}.\end{aligned}$$

Corollary 3.20 *Taking $s=1$ in Theorem 3.2, we get, for $\varrho>1$,*

$$\begin{aligned}&\left|\mathcal{Q}(a,b,x,\varrho;\mathcal{R})\left(\mathcal{I}(a,b,x;\mathcal{R})\right)^{-\frac{\Gamma(\delta+1)}{b-a}}\right|\\&\leq\left(\mathcal{R}^*(a)\mathcal{R}^*(b)\right)^{\frac{(x-a)^2}{(\delta+1)(\delta+2)(b-a)}B(\delta+1,2)}\mathcal{R}^*\left(\frac{a+b}{2}\right)^{\frac{(a+b-2x)^2}{2(b-a)}\left(\frac{\varrho-1}{2}+\frac{1}{(\delta+1)(\delta+2)}\right)}\\&\quad\times\left(\mathcal{R}^*(x)\mathcal{R}^*(a+b-x)\right)^{\frac{(\varrho(\delta+2)-\delta)(a+b-2x)^2}{8(\delta+2)(b-a)}+\frac{(x-a)^2}{(\delta+2)(b-a)}}.\end{aligned}$$

Corollary 3.21 *Taking $x = a$ in Corollary 3.19, we obtain*

$$\begin{aligned}&\left|(\mathcal{R}(a))^{\frac{\varrho}{2}}\left(\mathcal{R}\left(\frac{a+b}{2}\right)\right)^{1-\varrho}(\mathcal{R}(b))^{\frac{\varrho}{2}}\left(\left({}_aI_*^\delta\mathcal{R}\right)\left(\frac{a+b}{2}\right)\left({}_*I_b^\delta\mathcal{R}\right)\left(\frac{a+b}{2}\right)\right)^{-\frac{2^{\delta-1}\Gamma(\delta+1)}{(b-a)^\delta}}\right|\\&\le\Bigg(\left(\mathcal{R}^*(a)\mathcal{R}^*(b)\right)^{\frac{(\delta+2)\varrho-\delta+2\delta(1-\varrho)^{\frac{\delta+2}{\delta}}}{2(\delta+2)}}\\&\quad\times\mathcal{R}^*\left(\frac{a+b}{2}\right)^{\frac{2-(1-\varrho)(\delta+1)(\delta+2)}{(\delta+1)(\delta+2)}+\frac{4\delta}{\delta+1}(1-\varrho)^{\frac{\delta+1}{\delta}}-\frac{2\delta}{\delta+2}(1-\varrho)^{\frac{\delta+2}{\delta}}}\Bigg)^{\frac{b-a}{4}}.\end{aligned}$$

Remark 3.22 In Corollary 3.21, if we take

1) $x = a$ and $\varrho = \frac{1}{3}$, then we obtain Theorem 10 from [31],

2) $x = a$ and $\varrho = \frac{1}{2}$, then we obtain Theorem 5 from [30].

Corollary 3.23 *By setting $\delta = 1$ in Theorem 3.2 we get*

$$\begin{aligned}&\left|\left(\mathcal{R}(x)\mathcal{R}(a+b-x)\right)^{\frac{2(x-a)+\varrho(a+b-2x)}{2(b-a)}}\left(\mathcal{R}\left(\frac{a+b}{2}\right)\right)^{\frac{(1-\varrho)(a+b-2x)}{b-a}}\left(\int_a^b(\mathcal{R}(u))^{du}\right)^{\frac{1}{a-b}}\right|\\&\le\left(\mathcal{R}^*(a)\mathcal{R}^*(b)\right)^{\frac{(x-a)^2}{(s+1)(s+2)(b-a)}}\mathcal{R}^*\left(\frac{a+b}{2}\right)^{\frac{(a+b-2x)^2}{2(b-a)}\widetilde{\mathcal{N}}(\varrho,s)}\\&\quad\times\left(\mathcal{R}^*(x)\mathcal{R}^*(a+b-x)\right)^{\frac{(a+b-2x)^2}{4(b-a)}\widetilde{\Theta}(\varrho,s)+\frac{(x-a)^2}{(s+2)(b-a)}},\end{aligned}$$

where

$$\widetilde{\Theta}(\varrho,s)=\begin{cases}\frac{(s+2)\varrho-1+2(1-\varrho)^{s+2}}{(s+1)(s+2)} & \textit{for } 0\le\varrho\le 1,\\ \frac{\varrho(s+2)-1}{(s+1)(s+2)} & \textit{for } \varrho>1,\end{cases}$$

and

$$\widetilde{\mathcal{N}}(\varrho,s)=\begin{cases}\frac{s+1-\varrho(s+2)+2\varrho^{s+2}}{(s+1)(s+2)} & \textit{for } 0\le\varrho\le 1,\\ \frac{(s+2)(\varrho-1)+1}{(s+1)(s+2)} & \textit{for } \varrho>1.\end{cases}$$

Corollary 3.24 *Taking $\delta = s = 1$ in Theorem 3.2, we get*

$$\begin{aligned}&\left|\left(\mathcal{R}(x)\mathcal{R}(a+b-x)\right)^{\frac{2(x-a)+\varrho(a+b-2x)}{2(b-a)}}\left(\mathcal{R}\left(\frac{a+b}{2}\right)\right)^{\frac{(1-\varrho)(a+b-2x)}{b-a}}\left(\int_a^b(\mathcal{R}(u))^{du}\right)^{\frac{1}{a-b}}\right|\\&\le\left(\mathcal{R}^*(a)\mathcal{R}^*(b)\right)^{\frac{(x-a)^2}{6(b-a)}}\mathcal{R}^*\left(\frac{a+b}{2}\right)^{\frac{(a+b-2x)^2}{2(b-a)}\widehat{\mathcal{N}}(\varrho)}\\&\quad\times\left(\mathcal{R}^*(x)\mathcal{R}^*(a+b-x)\right)^{\frac{(a+b-2x)^2}{4(b-a)}\widehat{\Theta}(\varrho)+\frac{(x-a)^2}{3(b-a)}},\end{aligned}$$

where

$$\widehat{\Theta}(\varrho)=\begin{cases}\frac{1-3\varrho+6\varrho^2-2\varrho^3}{6} & \textit{for } 0\le\varrho\le 1,\\ \frac{3\varrho-1}{6} & \textit{for } \varrho>1,\end{cases}$$

and

$$\widehat{\mathcal{N}}(\varrho)=\begin{cases}\frac{2-3\varrho+2\varrho^3}{6} & \textit{for } 0\le\varrho\le 1,\\ \frac{3\varrho-2}{6} & \textit{for } \varrho>1.\end{cases}$$

Remark 3.25 In Corollary 3.24, if we take

1) $x=\frac{a+b}{2}$, then we obtain Theorem 3.3 from [17];

2) $x=a$ and $\varrho=1$, then using the multiplicative convexity of $\mathcal{R}^*$, we get Theorem 3.6 from [17];

3) $x=a$ and $\varrho=\frac{1}{3}$, then we obtain Corollary 3 from [31];

4) $x=a$ and $\varrho=\frac{1}{2}$, then we obtain Corollary 3 from [30];

5) $x=\frac{5a+b}{6}$ and $\varrho=\frac{5}{8}$, then we obtain Theorem 3.2 from [19];

6) $x=\frac{3a+b}{4}$ and $\varrho=\frac{5}{3}$, then we obtain Theorem 3.2 from [20].

Example 3.26 Consider the function $\mathcal{R}(u)=2^{u^{s+1}}$ for $s\in(0,1]$ with $a=0$ and $b=1$, the multiplicative derivative of this function is $\mathcal{R}^*(u)=2^{(s+1)u^s}$, which exhibits multiplicative s-convexity on the interval $[0,1]$. Thus by Theorem 3.2, for $0<\delta\le 1$,

$$\frac{2^{[x^{s+1}(\frac{(2x+\varrho(1-2x))}{2})+2^{-(s+1)}(1-\varrho)(1-2x)+(1-x)^{s+1}\frac{(2x+\varrho(1-2x))}{2}]}}{e^{\delta\ln 2[(\int_0^x u^{\delta+s}\,du+\int_{1-x}^1 u^{s+1}(1-u)^{\delta-1}\,du)x^{1-\delta}+(\int_x^{\frac{1}{2}} u^{s+1}(\frac{1}{2}-u)^{\delta-1}\,du+\int_{\frac{1}{2}}^{1-x} u^{s+1}(u-\frac{1}{2})^{\delta-1}\,du)(\frac{1-2x}{2})^{1-\delta}]}}$$
$$\le 2^{(s+1)[x^2B(\delta+1,s+1)+\frac{(1-2x)^2}{2^{s+1}}\mathcal{N}(\varrho,\delta,s)+(x^s+(1-x)^s)(\frac{(1-2x)^2}{4}\Theta(\varrho,\delta,s)+\frac{x^2}{(\delta+s+1)})]},$$

where Θ and $\mathcal{N}$ are defined by (7) and (8), and B is the beta function.

Since this result depends on four parameters, we will set $\delta=1$ and $x=a=0$ and then represent the result with respect to the remaining two. The outcome is illustrated in Fig. 1.

$$2^{[2^{-(s+1)}(1-\varrho)+\frac{\varrho}{2}-\frac{1}{s+2}]}\le\begin{cases}2^{(s+1)[\frac{s+1-\varrho(s+2)+2\varrho^{s+2}}{2^{s+1}(s+1)(s+2)}+\frac{(s+2)\varrho-1+2(1-\varrho)^{s+2}}{4(s+1)(s+2)}]} & \text{for } \varrho\in[0,1],\\ 2^{(s+1)[\frac{(\varrho-1)(s+2)+1}{2^{s+1}(s+1)(s+2)}+\frac{\varrho(s+2)-1}{4(s+1)(s+2)}]} & \text{for } \varrho>1.\end{cases}$$

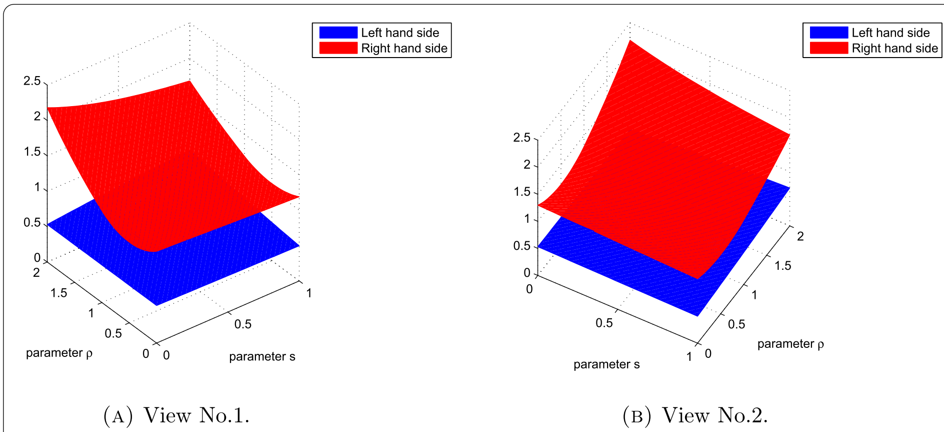

Figure 1 $x=0$, $\delta=1$, $\varrho\in[0,2]$, and $s\in(0,1]$

Based on the observations from Fig. 1, it is evident that the right-hand term consistently surpasses the left-hand term. This holds for all $\varrho \in [0,2]$ and $s \in (0,1]$, providing substantiation for the validity of our findings.

Theorem 3.27 *Let $\mathcal{R} : [a,b] \to \mathbb{R}^+$ be an increasing multiplicatively differentiable function on $[a,b]$. If for $q > 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, $(\ln \mathcal{R}^*)^q$ is s-convex on this interval, then for all $x \in [a, \frac{a+b}{2}]$, $\varrho \geq 0$, and $\delta > 0$, we have*

$$\begin{aligned} &\left|\mathcal{Q}(a,b,x,\varrho;\mathcal{R})\big(\mathcal{I}(a,b,x;\mathcal{R})\big)^{-\frac{\Gamma(\delta+1)}{b-a}}\right| \\ &\quad\leq \big(\mathcal{R}^*(a)\mathcal{R}^*(x)\mathcal{R}^*(a+b-x)\mathcal{R}^*(b)\big)^{\frac{(x-a)^2}{b-a}(\frac{1}{\delta p+1})^{\frac{1}{p}}(\frac{1}{s+1})^{\frac{1}{q}}} \\ &\quad\quad\times \left(\mathcal{R}^*(x)\left(\mathcal{R}^*\left(\frac{a+b}{2}\right)\right)^2\mathcal{R}^*(a+b-x)\right)^{\frac{(a+b-2x)^2}{4(b-a)}(\frac{1}{s+1})^{\frac{1}{q}}(\Lambda(\delta,p,\varrho))^{\frac{1}{p}}}, \end{aligned}$$

where $\mathcal{Q}$ and $\mathcal{I}$ are defined by (5) and (6), respectively, and

$$\Lambda(\delta,p,\varrho) = \begin{cases} \frac{(1-\varrho)^{p+\frac{1}{\delta}}}{\delta}B(\frac{1}{\delta},p+1) + \frac{\varrho^{p+\frac{1}{\delta}}}{\delta(p+1)}\,{}_2\mathcal{F}_1(1-\frac{1}{\delta},1,p+2;\varrho) & \text{for } \varrho \in [0,1], \\ \varrho^p\,{}_2\mathcal{F}_1(-p,1,\frac{1}{\delta}+1;\frac{1}{\varrho}) & \text{for } \varrho \in [1,2], \\ \frac{(\varrho-1)^p}{\delta}\,{}_2\mathcal{F}_1(-p,\frac{1}{\delta},\frac{1}{\delta}+1;\frac{1}{1-\varrho}) & \text{for } \varrho > 2, \end{cases} \tag{21}$$

with B and ${}_2\mathcal{F}_1$ the beta and hypergeometric functions, respectively.

Proof By Lemma 3.1, applying the Hölder inequality, we get

$$\begin{aligned} &\left|\mathcal{Q}(a,b,x,\varrho;\mathcal{R})\big(\mathcal{I}(a,b,x;\mathcal{R})\big)^{-\frac{\Gamma(\delta+1)}{b-a}}\right| \\ &\quad\leq \exp\left(\frac{(x-a)^2}{b-a}\left(\int_0^1 \gamma^{\delta p}\,d\gamma\right)^{\frac{1}{p}}\left(\int_0^1 \left|\ln\big(\mathcal{R}^*\big((1-\gamma)a+\gamma x\big)\big)\right|^q d\gamma\right)^{\frac{1}{q}}\right) \\ &\quad\quad\times \exp\left(\frac{(a+b-2x)^2}{4(b-a)}\left(\int_0^1 \left|1-\varrho-(1-\gamma)^\delta\right|^p d\gamma\right)^{\frac{1}{p}}\right. \\ &\quad\quad\times \left.\left(\int_0^1 \left|\ln\mathcal{R}^*\left((1-\gamma)x+\gamma\frac{a+b}{2}\right)\right|^q d\gamma\right)^{\frac{1}{q}}\right) \\ &\quad\quad\times \exp\left(\frac{(a+b-2x)^2}{4(b-a)}\left(\int_0^1 \left|\gamma^\delta-(1-\varrho)\right|^p d\gamma\right)^{\frac{1}{p}}\right. \\ &\quad\quad\times \left.\left(\int_0^1 \left|\ln\mathcal{R}^*\left((1-\gamma)\frac{a+b}{2}+\gamma(a+b-x)\right)\right|^q d\gamma\right)^{\frac{1}{q}}\right) \\ &\quad\quad\times \exp\left(\frac{(x-a)^2}{b-a}\left(\int_0^1 (1-\gamma)^{\delta p}\,d\gamma\right)^{\frac{1}{p}}\right. \\ &\quad\quad\times \left.\left(\int_0^1 \left|\ln\mathcal{R}^*\big((1-\gamma)(a+b-x)+\gamma b\big)\right|^q d\gamma\right)^{\frac{1}{q}}\right). \end{aligned}$$

From the s-convexity of $(\ln \mathcal{R}^*)^q$, we have

$$\left|\mathcal{Q}(a,b,x,\varrho;\mathcal{R})\big(\mathcal{I}(a,b,x;\mathcal{R})\big)^{-\frac{\Gamma(\delta+1)}{b-a}}\right|$$

$$
\begin{aligned}
&\le \exp\left(\frac{(x-a)^2}{b-a}\left(\frac{1}{\delta p+1}\right)^{\frac{1}{p}}\left(\int_0^1 \left((1-\gamma)^s\left(\ln\left(\mathcal{R}^*(a)\right)\right)^q+\gamma^s\left(\ln\left(\mathcal{R}^*(x)\right)\right)^q\right)d\gamma\right)^{\frac{1}{q}}\right)\\
&\quad\times \exp\left(\frac{(a+b-2x)^2}{4(b-a)}\left(\Lambda(\delta,p,\varrho)\right)^{\frac{1}{p}}\right.\\
&\quad\times\left.\left(\int_0^1\left((1-\gamma)^s\left(\ln\mathcal{R}^*(x)\right)^q+\gamma^s\left(\ln\mathcal{R}^*\left(\frac{a+b}{2}\right)\right)^q\right)d\gamma\right)^{\frac{1}{q}}\right)\\
&\quad\times \exp\left(\frac{(a+b-2x)^2}{4(b-a)}\left(\Lambda(\delta,p,\varrho)\right)^{\frac{1}{p}}\right.\\
&\quad\times\left.\left(\int_0^1\left((1-\gamma)^s\left(\ln\mathcal{R}^*\left(\frac{a+b}{2}\right)\right)^q+\gamma^s\left(\ln\mathcal{R}^*(a+b-x)\right)^q\right)d\gamma\right)^{\frac{1}{q}}\right)\\
&\quad\times \exp\left(\frac{(x-a)^2}{b-a}\left(\frac{1}{\delta p+1}\right)^{\frac{1}{p}}\right.\\
&\quad\times\left.\left(\int_0^1\left((1-\gamma)^s\left(\ln\mathcal{R}^*(a+b-x)\right)^q+\gamma^s\left(\ln\mathcal{R}^*(b)\right)^q\right)d\gamma\right)^{\frac{1}{q}}\right)\\
&=\exp\left(\frac{(x-a)^2}{b-a}\left(\frac{1}{\delta p+1}\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\left(\left(\ln\left(\mathcal{R}^*(a)\right)\right)^q+\left(\ln\left(\mathcal{R}^*(x)\right)\right)^q\right)^{\frac{1}{q}}\right)\\
&\quad\times \exp\left(\frac{(a+b-2x)^2}{4(b-a)}\left(\Lambda(\delta,p,\varrho)\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\right.\\
&\quad\times\left.\left(\left(\ln\mathcal{R}^*(x)\right)^q+\left(\ln\mathcal{R}^*\left(\frac{a+b}{2}\right)\right)^q\right)^{\frac{1}{q}}\right)\\
&\quad\times \exp\left(\frac{(a+b-2x)^2}{4(b-a)}\left(\Lambda(\delta,p,\varrho)\right)^{\frac{1}{p}}\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\right.\\
&\quad\times\left.\left(\left(\ln\mathcal{R}^*\left(\frac{a+b}{2}\right)\right)^q+\left(\ln\mathcal{R}^*(a+b-x)\right)^q\right)^{\frac{1}{q}}\right)\\
&\quad\times \exp\left(\frac{(x-a)^2}{b-a}\left(\frac{1}{\delta p+1}\right)^{\frac{1}{p}}\right.\\
&\quad\times\left.\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\left(\left(\ln\mathcal{R}^*(a+b-x)\right)^q+\left(\ln\mathcal{R}^*(b)\right)^q\right)^{\frac{1}{q}}\right),
\end{aligned}
\tag{22}
$$

where we have used that

$$
\int_0^1\left|\gamma^\delta-(1-\varrho)\right|^p d\gamma=\frac{(1-\varrho)^{p+\frac{1}{\delta}}}{\delta}B\left(\frac{1}{\delta},p+1\right)+\frac{\varrho^{p+\frac{1}{\delta}}}{\delta(p+1)}\ {}_2\mathcal{F}_1\left(1-\frac{1}{\delta},1,p+2;\varrho\right)
$$

for $\varrho\in[0,1]$,

$$
\int_0^1\left|\gamma^\delta-(1-\varrho)\right|^p d\gamma=\varrho^p\ {}_2\mathcal{F}_1\left(-p,1,\frac{1}{\delta}+1;\frac{1}{\varrho}\right)
$$

for $1<\varrho\le 2$, and

$$
\int_0^1\left|\gamma^\delta-(1-\varrho)\right|^p d\gamma=\frac{(\varrho-1)^p}{\delta}\ {}_2\mathcal{F}_1\left(-p,\frac{1}{\delta},\frac{1}{\delta}+1;\frac{1}{1-\varrho}\right)
$$

for $\varrho > 2$. Using the inequality $\mathcal{X}^q + \mathcal{Y}^q \le (\mathcal{X} + \mathcal{Y})^q$ for $\mathcal{X} \ge 0$, $\mathcal{Y} \ge 0$, and $q \ge 1$, (22) gives

$$\begin{aligned}
&\left|\mathcal{Q}(a,b,x,\varrho;\mathcal{R})\bigl(\mathcal{I}(a,b,x;\mathcal{R})\bigr)^{-\frac{\Gamma(\delta+1)}{(b-a)^\delta}}\right| \\
&\quad\le \exp\biggl(\frac{(x-a)^2}{b-a}\biggl(\frac{1}{\delta p+1}\biggr)^{\frac{1}{p}}\biggl(\frac{1}{s+1}\biggr)^{\frac{1}{q}}\bigl(\ln\mathcal{R}^*(a)+\ln\mathcal{R}^*(x)\bigr)\biggr) \\
&\qquad\times \exp\biggl(\frac{(a+b-2x)^2}{4(b-a)}\bigl(\Lambda(\delta,p,\varrho)\bigr)^{\frac{1}{p}}\biggl(\frac{1}{s+1}\biggr)^{\frac{1}{q}}\biggl(\ln\mathcal{R}^*(x)+\ln\mathcal{R}^*\biggl(\frac{a+b}{2}\biggr)\biggr)\biggr) \\
&\qquad\times \exp\biggl(\frac{(a+b-2x)^2}{4(b-a)}\bigl(\Lambda(\delta,p,\varrho)\bigr)^{\frac{1}{p}} \\
&\qquad\times \biggl(\frac{1}{s+1}\biggr)^{\frac{1}{q}}\biggl(\ln\mathcal{R}^*\biggl(\frac{a+b}{2}\biggr)+\ln\mathcal{R}^*(a+b-x)\biggr)\biggr) \\
&\qquad\times \exp\biggl(\frac{(x-a)^2}{b-a}\biggl(\frac{1}{\delta p+1}\biggr)^{\frac{1}{p}}\biggl(\frac{1}{s+1}\biggr)^{\frac{1}{q}}\bigl(\ln\mathcal{R}^*(a+b-x)+\ln\mathcal{R}^*(b)\bigr)\biggr) \\
&\quad= \bigl(\mathcal{R}^*(a)\mathcal{R}^*(x)\mathcal{R}^*(a+b-x)\mathcal{R}^*(b)\bigr)^{\frac{(x-a)^2}{b-a}(\frac{1}{\delta p+1})^{\frac{1}{p}}(\frac{1}{s+1})^{\frac{1}{q}}} \\
&\qquad\times \biggl(\mathcal{R}^*(x)\biggl(\mathcal{R}^*\biggl(\frac{a+b}{2}\biggr)\biggr)^2\mathcal{R}^*(a+b-x)\biggr)^{\frac{(a+b-2x)^2}{4(b-a)}(\frac{1}{s+1})^{\frac{1}{q}}(\Lambda(\delta,p,\varrho))^{\frac{1}{p}}}.
\end{aligned}$$

The proof is completed. □

Corollary 3.28 *By setting $s = 1$ in Theorem 3.27 we get*

$$\begin{aligned}
&\left|\mathcal{Q}(a,b,x,\varrho;\mathcal{R})\bigl(\mathcal{I}(a,b,x;\mathcal{R})\bigr)^{-\frac{\Gamma(\delta+1)}{b-a}}\right| \\
&\quad\le \bigl(\mathcal{R}^*(a)\mathcal{R}^*(x)\mathcal{R}^*(a+b-x)\mathcal{R}^*(b)\bigr)^{\frac{(x-a)^2}{b-a}(\frac{1}{\delta p+1})^{\frac{1}{p}}(\frac{1}{2})^{\frac{1}{q}}} \\
&\qquad\times \biggl(\mathcal{R}^*(x)\biggl(\mathcal{R}^*\biggl(\frac{a+b}{2}\biggr)\biggr)^2\mathcal{R}^*(a+b-x)\biggr)^{\frac{(a+b-2x)^2}{4(b-a)}(\frac{1}{2})^{\frac{1}{q}}(\Lambda(\delta,p,\varrho))^{\frac{1}{p}}},
\end{aligned}$$

where $\mathcal{Q}$, $\mathcal{I}$, and Λ are defined by (5), (6), and (21), respectively.

Corollary 3.29 *By setting $\delta = 1$ in Theorem 3.27 we get, for $\varrho \in [0,1]$,*

$$\begin{aligned}
&\left|\bigl(\mathcal{R}(x)\mathcal{R}(a+b-x)\bigr)^{\frac{2(x-a)+\varrho(a+b-2x)}{2(b-a)}}\biggl(\mathcal{R}\biggl(\frac{a+b}{2}\biggr)\biggr)^{\frac{(1-\varrho)(a+b-2x)}{b-a}}\biggl(\int_a^b\bigl(\mathcal{R}(u)\bigr)^{du}\biggr)^{\frac{1}{a-b}}\right| \\
&\quad\le \biggl(\bigl(\mathcal{R}^*(a)\mathcal{R}^*(x)\mathcal{R}^*(a+b-x)\mathcal{R}^*(b)\bigr)^{\frac{(x-a)^2}{b-a}} \\
&\qquad\times \biggl(\mathcal{R}^*(x)\biggl(\mathcal{R}^*\biggl(\frac{a+b}{2}\biggr)\biggr)^2\mathcal{R}^*(a+b-x)\biggr)^{\frac{(a+b-2x)^2}{4(b-a)}((1-\varrho)^{p+1}+\varrho^{p+1})^{\frac{1}{p}}}\biggr)^{(\frac{1}{p+1})^{\frac{1}{p}}(\frac{1}{s+1})^{\frac{1}{q}}}.
\end{aligned}$$

Corollary 3.30 *Taking $\delta = 1$ and $\varrho > 1$ in Theorem 3.27, we get*

$$\left|\bigl(\mathcal{R}(x)\mathcal{R}(a+b-x)\bigr)^{\frac{2(x-a)+\varrho(a+b-2x)}{2(b-a)}}\biggl(\mathcal{R}\biggl(\frac{a+b}{2}\biggr)\biggr)^{\frac{(1-\varrho)(a+b-2x)}{b-a}}\biggl(\int_a^b\bigl(\mathcal{R}(u)\bigr)^{du}\biggr)^{\frac{1}{a-b}}\right|$$

$$\leq \left(\left(\mathcal{R}^*(a)\mathcal{R}^*(x)\mathcal{R}^*(a+b-x)\mathcal{R}^*(b)\right)^{\frac{(x-a)^2}{b-a}} \right.$$
$$\left. \times \left(\mathcal{R}^*(x)\left(\mathcal{R}^*\left(\frac{a+b}{2}\right)\right)^2\mathcal{R}^*(a+b-x)\right)^{\frac{(a+b-2x)^2}{4(b-a)}(\varrho^{p+1}-(\varrho-1)^{p+1})^{\frac{1}{p}}} \right)^{(\frac{1}{p+1})^{\frac{1}{p}}(\frac{1}{s+1})^{\frac{1}{q}}}.$$

Corollary 3.31 *Taking $\delta = s = 1$ in Theorem 3.27, we get, for $\varrho \in [0,1]$,*

$$\left| \left(\mathcal{R}(x)\mathcal{R}(a+b-x)\right)^{\frac{2(x-a)+\varrho(a+b-2x)}{2(b-a)}} \left(\mathcal{R}\left(\frac{a+b}{2}\right)\right)^{\frac{(1-\varrho)(a+b-2x)}{b-a}} \left(\int_a^b (\mathcal{R}(u))^{du}\right)^{\frac{1}{a-b}} \right|$$
$$\leq \left(\left(\mathcal{R}^*(a)\mathcal{R}^*(x)\mathcal{R}^*(a+b-x)\mathcal{R}^*(b)\right)^{\frac{(x-a)^2}{b-a}} \right.$$
$$\left. \times \left(\mathcal{R}^*(x)\left(\mathcal{R}^*\left(\frac{a+b}{2}\right)\right)^2\mathcal{R}^*(a+b-x)\right)^{\frac{(a+b-2x)^2}{4(b-a)}((1-\varrho)^{p+1}+\varrho^{p+1})^{\frac{1}{p}}} \right)^{(\frac{1}{p+1})^{\frac{1}{p}}(\frac{1}{2})^{\frac{1}{q}}}.$$

Corollary 3.32 *Taking $\delta = s = 1$ in Theorem 3.27, we get, for $\varrho > 1$,*

$$\left| \left(\mathcal{R}(x)\mathcal{R}(a+b-x)\right)^{\frac{2(x-a)+\varrho(a+b-2x)}{2(b-a)}} \left(\mathcal{R}\left(\frac{a+b}{2}\right)\right)^{\frac{(1-\varrho)(a+b-2x)}{b-a}} \left(\int_a^b (\mathcal{R}(u))^{du}\right)^{\frac{1}{a-b}} \right|$$
$$\leq \left(\left(\mathcal{R}^*(a)\mathcal{R}^*(x)\mathcal{R}^*(a+b-x)\mathcal{R}^*(b)\right)^{\frac{(x-a)^2}{b-a}} \right.$$
$$\left. \times \left(\mathcal{R}^*(x)\left(\mathcal{R}^*\left(\frac{a+b}{2}\right)\right)^2\mathcal{R}^*(a+b-x)\right)^{\frac{(a+b-2x)^2}{4(b-a)}(\varrho^{p+1}-(\varrho-1)^{p+1})^{\frac{1}{p}}} \right)^{(\frac{1}{p+1})^{\frac{1}{p}}(\frac{1}{2})^{\frac{1}{q}}}.$$

Theorem 3.33 *Let $\mathcal{R} : [a,b] \to \mathbb{R}^+$ be an increasing multiplicatively differentiable function on $[a,b]$. If $(\ln \mathcal{R}^*)^q$ is s-convex on this interval for $q > 1$, then for all $x \in [a, \frac{a+b}{2}]$, $\varrho \geq 0$, and $\delta > 0$, we have*

$$\left| \mathcal{Q}(a,b,x,\varrho;\mathcal{R})\left(\mathcal{I}(a,b,x;\mathcal{R})\right)^{-\frac{\Gamma(\delta+1)}{b-a}} \right|$$
$$\leq \left(\mathcal{R}^*(a)\mathcal{R}^*(b)\right)^{\frac{(x-a)^2}{(\delta+1)(b-a)}((\delta+1)B(\delta+1,s+1))^{\frac{1}{q}}}$$
$$\times \left(\mathcal{R}^*(x)\mathcal{R}^*(a+b-x)\right)^{\frac{(x-a)^2}{(\delta+1)(b-a)}(\frac{\delta+1}{\delta+s+1})^{\frac{1}{q}}}$$
$$\times \left(\mathcal{R}^*(x)\mathcal{R}^*(a+b-x)\right)^{\frac{(a+b-2x)^2}{4(b-a)}(\Omega(\delta,\varrho))^{1-\frac{1}{q}}(\Theta(\varrho,\delta,s))^{\frac{1}{q}}}$$
$$\times \left(\mathcal{R}^*\left(\frac{a+b}{2}\right)\right)^{\frac{(a+b-2x)^2}{2(b-a)}(\Omega(\delta,\varrho))^{1-\frac{1}{q}}(\mathcal{N}(\varrho,\delta,s))^{\frac{1}{q}}},$$

where $\mathcal{Q}$, $\mathcal{I}$, Θ, and $\mathcal{N}$ are defined by (5)–(8), respectively, B is the beta function, and

$$\Omega(\delta,\varrho) = \begin{cases} \frac{\varrho-\delta(1-\varrho)}{\delta+1} + \frac{2\delta}{\delta+1}(1-\varrho)^{\frac{\delta+1}{\delta}} & \text{for } \varrho \in [0,1], \\ \frac{(\delta+1)(\varrho-1)+1}{\delta+1} & \text{for } \varrho > 1. \end{cases} \tag{23}$$

Proof By Lemma 3.1, applying the power mean inequality, we deduce

$$\left| \mathcal{Q}(a,b,x,\varrho;\mathcal{R})\left(\mathcal{I}(a,b,x;\mathcal{R})\right)^{-\frac{\Gamma(\delta+1)}{b-a}} \right|$$

$$
\begin{aligned}
&\le \exp\left(\frac{(x-a)^2}{b-a}\left(\int_0^1 \gamma^\delta\,d\gamma\right)^{1-\frac{1}{q}}\left(\int_0^1 \gamma^\delta\left|\ln\left(\mathcal{R}^*\left((1-\gamma)a+\gamma x\right)\right)\right|^q d\gamma\right)^{\frac{1}{q}}\right)\\
&\quad\times \exp\left(\frac{(a+b-2x)^2}{4(b-a)}\left(\int_0^1 \left|1-\varrho-(1-\gamma)^\delta\right| d\gamma\right)^{1-\frac{1}{q}}\right.\\
&\quad\times \left.\left(\int_0^1 \left|1-\varrho-(1-\gamma)^\delta\right|\left|\ln\mathcal{R}^*\left((1-\gamma)x+\gamma\frac{a+b}{2}\right)\right|^q d\gamma\right)^{\frac{1}{q}}\right)\\
&\quad\times \exp\left(\frac{(a+b-2x)^2}{4(b-a)}\left(\int_0^1 \left|\gamma^\delta-(1-\varrho)\right| d\gamma\right)^{1-\frac{1}{q}}\right.\\
&\quad\times \left.\left(\int_0^1 \left|\gamma^\delta-(1-\varrho)\right|\left|\ln\mathcal{R}^*\left((1-\gamma)\frac{a+b}{2}+\gamma(a+b-x)\right)\right|^q d\gamma\right)^{\frac{1}{q}}\right)\\
&\quad\times \exp\left(\frac{(x-a)^2}{b-a}\left(\int_0^1 (1-\gamma)^\delta\,d\gamma\right)^{1-\frac{1}{q}}\right.\\
&\quad\times \left.\left(\int_0^1 (1-\gamma)^\delta\left|\ln\mathcal{R}^*\left((1-\gamma)(a+b-x)+\gamma b\right)\right|^q d\gamma\right)^{\frac{1}{q}}\right).
\end{aligned}
$$

Since $(\ln\mathcal{R}^*)^q$ is s-convex, we obtain

$$
\begin{aligned}
&\left|\mathcal{Q}(a,b,x,\varrho;\mathcal{R})\left(\mathcal{I}(a,b,x;\mathcal{R})\right)^{-\frac{\Gamma(\delta+1)}{b-a}}\right|\\
&\le \exp\left(\frac{(x-a)^2}{b-a}\left(\frac{1}{\delta+1}\right)^{1-\frac{1}{q}}\right.\\
&\quad\times \left.\left(\int_0^1 \gamma^\delta\left((1-\gamma)^s\left(\ln\left(\mathcal{R}^*(a)\right)\right)^q+\gamma^s\left(\ln\left(\mathcal{R}^*(x)\right)\right)^q\right) d\gamma\right)^{\frac{1}{q}}\right)\\
&\quad\times \exp\left(\frac{(a+b-2x)^2}{4(b-a)}\left(\Omega(\delta,\varrho)\right)^{1-\frac{1}{q}}\right.\\
&\quad\times \left.\left(\int_0^1 \left|1-\varrho-(1-\gamma)^\delta\right|\left((1-\gamma)^s\left(\ln\mathcal{R}^*(x)\right)^q+\gamma^s\left(\ln\mathcal{R}^*\left(\frac{a+b}{2}\right)\right)^q\right) d\gamma\right)^{\frac{1}{q}}\right)\\
&\quad\times \exp\left(\frac{(a+b-2x)^2}{4(b-a)}\left(\Omega(\delta,\varrho)\right)^{1-\frac{1}{q}}\right.\\
&\quad\times \left(\int_0^1 \left|\gamma^\delta-(1-\varrho)\right|\left((1-\gamma)^s\left(\ln\mathcal{R}^*\left(\frac{a+b}{2}\right)\right)^q\right.\right.\\
&\quad+ \left.\left.\left.\gamma^s\left(\ln\mathcal{R}^*(a+b-x)\right)^q\right) d\gamma\right)^{\frac{1}{q}}\right)\\
&\quad\times \exp\left(\frac{(x-a)^2}{b-a}\left(\frac{1}{\delta+1}\right)^{1-\frac{1}{q}}\right.\\
&\quad\times \left.\left(\int_0^1 (1-\gamma)^\delta\left((1-\gamma)^s\left(\ln\mathcal{R}^*(a+b-x)\right)^q+\gamma^s\left(\ln\mathcal{R}^*(b)\right)^q\right) d\gamma\right)^{\frac{1}{q}}\right)\\
&= \exp\left(\frac{(x-a)^2}{b-a}\left(\frac{1}{\delta+1}\right)^{1-\frac{1}{q}}\right.\\
&\quad\times \left(\left(\left(B(\delta+1,s+1)\right)^{\frac{1}{q}}\ln\left(\mathcal{R}^*(a)\right)\right)^q+\left(\left(\frac{1}{\delta+s+1}\right)^{\frac{1}{q}}\ln\left(\mathcal{R}^*(x)\right)\right)^q\right)^{\frac{1}{q}}\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&\times \exp\left(\frac{(a+b-2x)^2}{4(b-a)}\big(\Omega(\delta,\varrho)\big)^{1-\frac{1}{q}}\right.\\
&\times \left(\big((\widehat{\Theta}(\varrho,\delta,s))^{\frac{1}{q}}\ln\mathcal{R}^*(x)\big)^q+\left(\big(\widehat{\mathcal{N}}(\varrho,\delta,s)\big)^{\frac{1}{q}}\ln\mathcal{R}^*\left(\frac{a+b}{2}\right)\right)^q\right)^{\frac{1}{q}}\Bigg)\\
&\times \exp\left(\frac{(a+b-2x)^2}{4(b-a)}\big(\Omega(\delta,\varrho)\big)^{1-\frac{1}{q}}\right.\\
&\times \left(\left(\big(\widehat{\mathcal{N}}(\varrho,\delta,s)\big)^{\frac{1}{q}}\ln\mathcal{R}^*\left(\frac{a+b}{2}\right)\right)^q+\big((\widehat{\Theta}(\varrho,\delta,s))^{\frac{1}{q}}\ln\mathcal{R}^*(a+b-x)\big)^q\right)^{\frac{1}{q}}\Bigg)\\
&\times \exp\left(\frac{(x-a)^2}{b-a}\left(\frac{1}{\delta+1}\right)^{1-\frac{1}{q}}\right.\\
&\times \left(\left(\left(\frac{1}{\delta+s+1}\right)^{\frac{1}{q}}\ln\mathcal{R}^*(a+b-x)\right)^q+\big((B(\delta+1,s+1))^{\frac{1}{q}}\ln\mathcal{R}^*(b)\big)^q\right)^{\frac{1}{q}}\Bigg), \qquad (24)
\end{aligned}
$$

where we have used the equalities

$$
\int_0^1 \left|1-\varrho-(1-\gamma)^\delta\right| d\gamma = \int_0^1 \left|1-\varrho-\gamma^\delta\right| d\gamma = \Omega(\delta,\varrho).
$$

Substituting (14)–(17) and using the inequality $\mathcal{X}^q+\mathcal{Y}^q\le(\mathcal{X}+\mathcal{Y})^q$ for $\mathcal{X}\ge 0$, $\mathcal{Y}\ge 0$, and $q\ge 1$, (24) gives

$$
\begin{aligned}
&\left|\mathcal{Q}(a,b,x,\varrho;\mathcal{R})\big(\mathcal{I}(a,b,x;\mathcal{R})\big)^{-\frac{\Gamma(\delta+1)}{(b-a)^\delta}}\right|\\
&\le \exp\left(\frac{(x-a)^2}{b-a}\left(\frac{1}{\delta+1}\right)^{1-\frac{1}{q}}\right.\\
&\quad\times \left(\big(B(\delta+1,s+1)\big)^{\frac{1}{q}}\ln\big(\mathcal{R}^*(a)\big)+\left(\frac{1}{\delta+s+1}\right)^{\frac{1}{q}}\ln\big(\mathcal{R}^*(x)\big)\right)\Bigg)\\
&\quad\times \exp\left(\frac{(a+b-2x)^2}{4(b-a)}\big(\Omega(\delta,\varrho)\big)^{1-\frac{1}{q}}\right.\\
&\quad\times \left(\big(\Theta(\varrho,\delta,s)\big)^{\frac{1}{q}}\ln\mathcal{R}^*(x)+\big(\mathcal{N}(\varrho,\delta,s)\big)^{\frac{1}{q}}\ln\mathcal{R}^*\left(\frac{a+b}{2}\right)\right)\Bigg)\\
&\quad\times \exp\left(\frac{(a+b-2x)^2}{4(b-a)}\big(\Omega(\delta,\varrho)\big)^{1-\frac{1}{q}}\right.\\
&\quad\times \left(\big(\mathcal{N}(\varrho,\delta,s)\big)^{\frac{1}{q}}\ln\mathcal{R}^*\left(\frac{a+b}{2}\right)+\big(\Theta(\varrho,\delta,s)\big)^{\frac{1}{q}}\ln\mathcal{R}^*(a+b-x)\right)\Bigg)\\
&\quad\times \exp\left(\frac{(x-a)^2}{b-a}\left(\frac{1}{\delta+1}\right)^{1-\frac{1}{q}}\right.\\
&\quad\times \left(\left(\frac{1}{\delta+s+1}\right)^{\frac{1}{q}}\ln\mathcal{R}^*(a+b-x)+\big(B(\delta+1,s+1)\big)^{\frac{1}{q}}\ln\mathcal{R}^*(b)\right)\Bigg)\\
&= \big(\mathcal{R}^*(a)\mathcal{R}^*(b)\big)^{\frac{(x-a)^2}{(\delta+1)(b-a)}((\delta+1)B(\delta+1,s+1))^{\frac{1}{q}}}\\
&\quad\times \big(\mathcal{R}^*(x)\mathcal{R}^*(a+b-x)\big)^{\frac{(x-a)^2}{(\delta+1)(b-a)}(\frac{\delta+1}{\delta+s+1})^{\frac{1}{q}}}
\end{aligned}
$$

$$\times \left(\mathcal{R}^*(x)\mathcal{R}^*(a+b-x)\right)^{\frac{(a+b-2x)^2}{4(b-a)}(\Omega(\delta,\varrho))^{1-\frac{1}{q}}(\Theta(\varrho,\delta,s))^{\frac{1}{q}}}$$
$$\times \left(\mathcal{R}^*\left(\frac{a+b}{2}\right)\right)^{\frac{(a+b-2x)^2}{2(b-a)}(\Omega(\delta,\varrho))^{1-\frac{1}{q}}(\mathcal{N}(\varrho,\delta,s))^{\frac{1}{q}}}.$$

The proof is completed. □

Corollary 3.34 *Taking $s=1$ in Theorem 3.33, we get, for $\varrho \in [0,1]$,*

$$\left|\mathcal{Q}(a,b,x,\varrho;\mathcal{R})\left(\mathcal{I}(a,b,x;\mathcal{R})\right)^{-\frac{\Gamma(\delta+1)}{b-a}}\right|$$
$$\leq \left(\mathcal{R}^*(a)\mathcal{R}^*(b)\right)^{\frac{(x-a)^2}{(\delta+1)(b-a)}(\frac{1}{(\delta+2)})^{\frac{1}{q}}}$$
$$\times \left(\mathcal{R}^*(x)\mathcal{R}^*(a+b-x)\right)^{\frac{(x-a)^2}{(\delta+1)(b-a)}(\frac{\delta+1}{\delta+2})^{\frac{1}{q}}+\frac{(a+b-2x)^2}{4(b-a)}\mathcal{V}(\varrho,\delta)}$$
$$\times \left(\mathcal{R}^*\left(\frac{a+b}{2}\right)\right)^{\frac{(a+b-2x)^2}{2(b-a)}(\frac{\varrho-\delta(1-\varrho)+2\delta(1-\varrho)^{\frac{\delta+1}{\delta}}}{\delta+1})^{1-\frac{1}{q}}(\mathcal{W}(\varrho,\delta))^{\frac{1}{q}}}$$

with

$$\mathcal{V}(\varrho,\delta)=\left(\frac{\varrho-\delta(1-\varrho)+2\delta(1-\varrho)^{\frac{\delta+1}{\delta}}}{\delta+1}\right)^{1-\frac{1}{q}}\left(\frac{(\delta+2)\varrho-\delta+2\delta(1-\varrho)^{\frac{\delta+2}{\delta}}}{2(\delta+2)}\right)^{\frac{1}{q}}$$

and

$$\mathcal{W}(\varrho,\delta)=\frac{2-(1-\varrho)(\delta+1)(\delta+2)}{2(\delta+1)(\delta+2)}+\frac{2\delta}{\delta+1}(1-\varrho)^{\frac{\delta+1}{\delta}}-\frac{\delta}{\delta+2}(1-\varrho)^{\frac{\delta+2}{\delta}},$$

where $\mathcal{Q}$ and $\mathcal{I}$ are defined by (5) and (6), respectively.

Corollary 3.35 *Taking $s=1$ in Theorem 3.33, we get, for $\varrho > 1$,*

$$\left|\mathcal{Q}(a,b,x,\varrho;\mathcal{R})\left(\mathcal{I}(a,b,x;\mathcal{R})\right)^{-\frac{\Gamma(\delta+1)}{b-a}}\right|$$
$$\leq \left(\mathcal{R}^*(a)\mathcal{R}^*(b)\right)^{\frac{(x-a)^2}{(\delta+1)(b-a)}(\frac{1}{(\delta+2)})^{\frac{1}{q}}}$$
$$\times \left(\mathcal{R}^*(x)\mathcal{R}^*(a+b-x)\right)^{\frac{(x-a)^2}{(\delta+1)(b-a)}(\frac{\delta+1}{\delta+2})^{\frac{1}{q}}+\frac{(a+b-2x)^2}{4(b-a)}(\frac{(\delta+1)(\varrho-1)+1}{\delta+1})^{1-\frac{1}{q}}(\frac{(\varrho(\delta+2)-\delta)}{8(\delta+2)})^{\frac{1}{q}}}$$
$$\times \left(\mathcal{R}^*\left(\frac{a+b}{2}\right)\right)^{\frac{(a+b-2x)^2}{2(b-a)}(\frac{(\delta+1)(\varrho-1)+1}{\delta+1})^{1-\frac{1}{q}}(\frac{\varrho-1}{2}+\frac{1}{(\delta+1)(\delta+2)})^{\frac{1}{q}}},$$

where $\mathcal{Q}$ and $\mathcal{I}$ are defined by (5) and (6), respectively.

Corollary 3.36 *Taking $\delta=1$ in Theorem 3.33, we get, for $\varrho \in [0,1]$,*

$$\left|\left(\mathcal{R}(x)\mathcal{R}(a+b-x)\right)^{\frac{2(x-a)+\varrho(a+b-2x)}{2(b-a)}}\left(\mathcal{R}\left(\frac{a+b}{2}\right)\right)^{\frac{(1-\varrho)(a+b-2x)}{b-a}}\left(\int_a^b(\mathcal{R}(u))^{du}\right)^{\frac{1}{a-b}}\right|$$
$$\leq \left(\mathcal{R}^*(a)\mathcal{R}^*(b)\right)^{\frac{(x-a)^2}{2(b-a)}(\frac{2}{(s+1)(s+2)})^{\frac{1}{q}}}\left(\mathcal{R}^*(x)\mathcal{R}^*(a+b-x)\right)^{\frac{(x-a)^2}{2(b-a)}(\frac{2}{s+2})^{\frac{1}{q}}}$$

$$\times \left(\mathcal{R}^*(x)\mathcal{R}^*(a+b-x)\right)^{\frac{(a+b-2x)^2}{4(b-a)}(\frac{1-2\varrho+2\varrho^2}{2})^{1-\frac{1}{q}}(\frac{(s+2)\varrho-1+2(1-\varrho)^{s+2}}{(s+1)(s+2)})^{\frac{1}{q}}}$$
$$\times \left(\mathcal{R}^*\left(\frac{a+b}{2}\right)\right)^{\frac{(a+b-2x)^2}{2(b-a)}(\frac{1-2\varrho+2\varrho^2}{2})^{1-\frac{1}{q}}(\frac{s+1-(s+2)\varrho+2\varrho^{s+2}}{(s+1)(s+2)})^{\frac{1}{q}}}.$$

Corollary 3.37 *By setting $\delta = 1$ in Theorem 3.33 we get, for $\varrho > 1$,*

$$\left|\left(\mathcal{R}(x)\mathcal{R}(a+b-x)\right)^{\frac{2(x-a)+\varrho(a+b-2x)}{2(b-a)}}\left(\mathcal{R}\left(\frac{a+b}{2}\right)\right)^{\frac{(1-\varrho)(a+b-2x)}{b-a}}\left(\int_a^b(\mathcal{R}(u))^{du}\right)^{\frac{1}{a-b}}\right|$$
$$\leq \left(\mathcal{R}^*(a)\mathcal{R}^*(b)\right)^{\frac{(x-a)^2}{2(b-a)}(\frac{2}{(s+1)(s+2)})^{\frac{1}{q}}}\left(\mathcal{R}^*(x)\mathcal{R}^*(a+b-x)\right)^{\frac{(x-a)^2}{2(b-a)}(\frac{2}{s+2})^{\frac{1}{q}}}$$
$$\times \left(\mathcal{R}^*(x)\mathcal{R}^*(a+b-x)\right)^{\frac{(a+b-2x)^2}{4(b-a)}(\varrho-\frac{1}{2})^{1-\frac{1}{q}}(\frac{\varrho(s+2)-1}{(s+1)(s+2)})^{\frac{1}{q}}}$$
$$\times \left(\mathcal{R}^*\left(\frac{a+b}{2}\right)\right)^{\frac{(a+b-2x)^2}{2(b-a)}(\varrho-\frac{1}{2})^{1-\frac{1}{q}}(\frac{(s+2)(\varrho-1)+1}{(s+1)(s+2)})^{\frac{1}{q}}}.$$

Corollary 3.38 *Taking $\delta = s = 1$ and $\varrho \in [0,1]$ in Theorem 3.33, we get*

$$\left|\left(\mathcal{R}(x)\mathcal{R}(a+b-x)\right)^{\frac{2(x-a)+\varrho(a+b-2x)}{2(b-a)}}\left(\mathcal{R}\left(\frac{a+b}{2}\right)\right)^{\frac{(1-\varrho)(a+b-2x)}{b-a}}\left(\int_a^b(\mathcal{R}(u))^{du}\right)^{\frac{1}{a-b}}\right|$$
$$\leq \left(\mathcal{R}^*(a)\mathcal{R}^*(b)\right)^{\frac{(x-a)^2}{2(b-a)}(\frac{1}{3})^{\frac{1}{q}}}\left(\mathcal{R}^*(x)\mathcal{R}^*(a+b-x)\right)^{\frac{(x-a)^2}{2(b-a)}(\frac{2}{3})^{\frac{1}{q}}}$$
$$\times \left(\mathcal{R}^*(x)\mathcal{R}^*(a+b-x)\right)^{\frac{(a+b-2x)^2}{4(b-a)}(\frac{1-2\varrho+2\varrho^2}{2})^{1-\frac{1}{q}}(\frac{1-3\varrho+6\varrho^2-2\varrho^3}{6})^{\frac{1}{q}}}$$
$$\times \left(\mathcal{R}^*\left(\frac{a+b}{2}\right)\right)^{\frac{(a+b-2x)^2}{2(b-a)}(\frac{1-2\varrho+2\varrho^2}{2})^{1-\frac{1}{q}}(\frac{2-3\varrho+2\varrho^3}{6})^{\frac{1}{q}}}.$$

Corollary 3.39 *Taking $\delta = s = 1$ and $\varrho > 1$ in Theorem 3.33, we get*

$$\left|\left(\mathcal{R}(x)\mathcal{R}(a+b-x)\right)^{\frac{2(x-a)+\varrho(a+b-2x)}{2(b-a)}}\left(\mathcal{R}\left(\frac{a+b}{2}\right)\right)^{\frac{(1-\varrho)(a+b-2x)}{b-a}}\left(\int_a^b(\mathcal{R}(u))^{du}\right)^{\frac{1}{a-b}}\right|$$
$$\leq \left(\mathcal{R}^*(a)\mathcal{R}^*(b)\right)^{\frac{(x-a)^2}{2(b-a)}(\frac{1}{3})^{\frac{1}{q}}}\left(\mathcal{R}^*(x)\mathcal{R}^*(a+b-x)\right)^{\frac{(x-a)^2}{2(b-a)}(\frac{2}{3})^{\frac{1}{q}}}$$
$$\times \left(\mathcal{R}^*(x)\mathcal{R}^*(a+b-x)\right)^{\frac{(a+b-2x)^2}{4(b-a)}(\varrho-\frac{1}{2})^{1-\frac{1}{q}}(\frac{3\varrho-1}{6})^{\frac{1}{q}}}$$
$$\times \left(\mathcal{R}^*\left(\frac{a+b}{2}\right)\right)^{\frac{(a+b-2x)^2}{2(b-a)}(\varrho-\frac{1}{2})^{1-\frac{1}{q}}(\frac{3\varrho-2}{6})^{\frac{1}{q}}}.$$

4 Applications to special means

In this section, we provide several applications of the obtained results to special means.

Proposition 4.1 *For real numbers a, b with $0 < a < b$, we have*

$$\exp\left\{H(a,a,a,b)+2H(a,b)+H(a,b,b,b)-4G^2(a,b)L^{-1}(a,b)\right\}$$
$$\leq \exp\left\{-\left(b^2+a^2+4\left(\frac{2ab}{a+b}\right)^2+3\left(\frac{4ab}{3a+b}\right)^2+3\left(\frac{4ab}{a+3b}\right)^2\right)\frac{b-a}{24ab}\right\},$$

where $H(\cdot,\cdot)$, $G(\cdot,\cdot)$, and $L(\cdot,\cdot)$ are the harmonic, geometric, and logarithmic means given by $H(a,b)=\frac{2ab}{a+b}$, $G(a,b)=\sqrt{ab}$, and $L(a,b)=\frac{b-a}{\ln b-\ln a}$.

Proof The statement follows from Corollary 3.24 by taking $x=\frac{3a+b}{4ab}$ and $\varrho=0$ for the function $\mathcal{R}(u)=\exp\{\frac{1}{u}\}$ on the interval $[\frac{1}{b},\frac{1}{a}]$. The multiplicative derivative is given by $\mathcal{R}^*(u)=\exp\{-\frac{1}{u^2}\}$, and the integral is given by $(\int_{\frac{1}{b}}^{\frac{1}{a}}\mathcal{R}(u)^{du})^{\frac{ab}{a-b}}=\exp\{(-G^2(a,b)L^{-1}(a,b))\}$. □

Proposition 4.2 *For real numbers a, b with $0<a<b$, we have*

$$\begin{aligned}&\exp\left\{\frac{1}{s+2}\left(A^{\frac{s+2}{2}}(a,a,a,a,a,b)+A^{\frac{s+2}{2}}(a,b,b,b,b,b)-2L_{\frac{s+2}{2}}^{\frac{s+2}{2}}(a,b)\right)\right\}\\&\quad\leq\left(\exp\left\{a^{\frac{s}{2}}+5\left(\frac{5a+b}{6}\right)^{\frac{s}{2}}+8\left(\frac{a+b}{2}\right)^{\frac{s}{2}}+5\left(\frac{a+5b}{6}\right)^{\frac{s}{2}}+b^{\frac{s}{2}}\right\}\right)^{\frac{b-a}{108}(\frac{3}{s+1})^{\frac{1}{2}}},\end{aligned}$$

where $A(\cdot,\dots,\cdot)$ is the arithmetic mean, and $L_p(\cdot,\cdot)$ is the p-logarithmic mean given by $A(a_1,a_2,\dots,a_n)=\frac{a_1+a_2+\cdots+a_n}{n}$ and $L_p(a,b)=(\frac{b^{p+1}-a^{p+1}}{(p+1)(b-a)})^{\frac{1}{p}}$.

Proof The statement follows from Corollary 3.29 by taking $x=\frac{5a+b}{6}$ and $\varrho=1$ for the function $\mathcal{R}(u)=\exp\{\frac{2}{s+2}u^{\frac{s+2}{2}}\}$, $s\in(0,1]$. The multiplicative derivative is given by $\mathcal{R}^*(u)=\exp\{u^{\frac{s}{2}}\}$ with $(\ln\mathcal{R}^*(u))^2=u^s$, which is multiplicative s-convex, and $(\int_a^b\mathcal{R}(u)^{du})^{\frac{1}{a-b}}=\exp\{-\frac{2}{s+2}L_{\frac{s+2}{2}}^{\frac{s+2}{2}}(a,b)\}$. □

5 Conclusions

The impetus for this study stemmed from a quest to deepen our understanding and propel advancements in the domain of fractional multiplicative integral inequalities. Through the introduction of a pioneering parameterized identity, we have not only unveiled a spectrum of inequalities tailored for multiplicatively s-convex mappings but have also established their connections to a variety of quadrature rules involving one, two, and three points. Our findings amalgamate both innovative discoveries and previously established results, underscoring the adaptability and efficacy of the proposed integral identity. In summary, this endeavor constitutes a significant contribution to the field, presenting a rich tapestry of new insights and generalizations poised to enrich further inquiry in this area.

Author contributions
Conceptualization: M.B.A., W.S. and A.L. Methodology: M.B.A., W.S. and A.L. Validation: M.B.A., F.J., W.S. and B.M. Investigation: M.B.A. and W.S. Writing-original draft preparation: M.B.A. and A.L. Writing-review and editing: M.B.A., F.J. and B.M. Visualization: W.S., A.L. and F.J. Supervision: F.J. and B.M. Project administration: M.B.A. and W.S. All authors have read and agreed to the fnal version of the manuscript.

Funding
The authors extend their appreciation to the Deputyship for Research & Innovation, Ministry of Education in Saudi Arabia for funding this research work through the project number 445-9-839.

Data availability
Data sharing is not relevant to this paper, as there was no generation or analysis of new data during the course of this study.

Declarations

Ethics approval and consent to participate
The authors state that they do not have any conflicts of interest.

Competing interests
The authors declare no competing interests.

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Received: 25 September 2023 Accepted: 21 March 2024 Published online: 08 April 2024

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