Research Article

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Pathological study on uncertain numbers and proposed solutions for discrete fuzzy fractional order calculus

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Abstract: A pathological study in the definition of uncertain numbers is carried out, and some solutions are proposed. Fundamental theorems for uncertain discrete fractional and integer order calculus are established. The concept of maximal solution for obtaining a unique uncertain solution is introduced. The solutions of uncertain discrete relaxation equations for the integer and the fractional order are obtained. Various numerical examples are accompanied to clarify the theoretical results and study of uncertain system behavior.

Keywords: fuzzy number, r-cuts, uncertain numbers, Hukuhara difference, fractional nabla difference

1 Introduction

Physics is full of phenomena that a perturbed system tends to return to its equilibrium point. Because the equilibrium is a state of the lowest energy, systems tend to reach it. Such a phenomenon is called relaxation and the relaxation equations describe it. Examples of such phenomena are stress relaxation in response to strain for materials and radioactive decay [1,2]. Our motivation in this paper is to build a comprehensive theory concerning uncertainty to discrete relaxation equations and related linear equations.

The complexity of natural processes increases the dimensions of modeled dynamical systems. It is known that high-dimensional systems can be reduced to a low-dimensional system with some appropriate integral operator involving fading memory [3]. Therefore, fractional operators with memory are significantly crucial for modeling such complex systems [4,5]. Solving systems of fractional order differential equations generally is not easy. The analytic solutions are in the form of series with infinite terms and a slow rate of convergence. Thus, a numerical method is used to solve them. Many of them result in discrete difference equations. Therefore, it is tempting to use systems of discrete difference equations for modeling from the very beginning. On the other hand, discrete equations directly use data. Consequently, it is no surprise to see recent attention for discrete calculus as well as discrete fractional calculus [6,7].

The recent progress on discrete fractional operators, especially discrete fractional sum and discrete fractional difference, has been reviewed in the study by Wang *et al.* [8]. The core of these definitions is based on the nabla difference operator $\nabla p(t) = p(t) - p(t-1)$ and the delta difference operators $\Delta p(t) = p(t+1) - p(t)$, for a given function f and the time t [6]. Each has its own advantages [9]. In this article, we use the nabla-based definitions, motivated by the discussion of the study by Hein *et al.* [9]. The properties of nabla calculus are established in several studies [10–13]. Also, there are two fractional nabla difference equations. We will distinguish which one is preferable by the similarity of their behavior to nabla's difference when the order approaches the integer order.

Then, we studied the systems of fractional and integer order discrete difference equations. Such systems have interesting applications in neural networks and epidemiological modeling [14–19]. For fractional order systems with the discrete nabla fractional derivative, one can consult previous studies by Wei *et al.* [20,21].

Uncertainty means a lack of exact information in modeling, measurement parameters, forces, and unpredictability of future events. Fuzzy theory, interval analysis,

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and stochastic analysis can be helpful to analyze uncertainty. Fuzzy sets can help us to show vagueness. This is carried out by introducing membership functions to illustrate the uncertainty of the belongingness of an element to a set [22]. There are many attempts to define uncertain numbers as a fuzzy set. On the other hand, engineers often used intervals to show the uncertainty of a measurement. Fortunately, the fuzzy number (FN) has parametric representation that relates to interval analysis [23,24]. Therefore, a unified uncertainty concept that unifies interval analysis and fuzzy theory with diverse scalar multiplication is introduced in the study by Shiri [25]. However, the definition of what uncertain/FN is still vague.

There exist various definitions for FNs by imposing some restriction on membership function [26–28]. As far as we know, Dubois and Prade [26], introduced the first definition of an FN. Interestingly, this definition is the mostly followed definition by researchers though it has some drawbacks. Based on this definition, an FN is a fuzzy set of real numbers with continuous membership function and compact support of interval type, say it $[c, d]$, $c, d \in \mathbb{R}$, whereas it should gain the maximum value of membership function (i.e., one) on an interval (or possibly a point) $[a, b]$, $a, b \in \mathbb{R}$, i.e., strictly increasing on $[c, a]$ and then strictly decreasing on $[b, d]$.

Obviously, with this definition, a deterministic number is not an FN. This is one of the pitfalls if we consider FNs as an extension of real numbers. Goetschel and Voxman [28] replaced continuity with the upper semi-continuous condition to include crisp/deterministic numbers as FNs and to define a metric. In an interesting work, Dijkman *et al.* [27] categorized the FN with diverse conditions and investigated them. For further works, we would like to take the attention of the readers to the other available literature [29,30].

Later developments of fuzzy equations to solve fuzzy equations enjoy a redefinition of an FN with parametric forms. This redefinition helps to transform a fuzzy equation into a deterministic equation and then solve it. In this respect, we will find many drawbacks that we should consider it in such definitions. Our discussion in this article highlights such drawbacks and we will try to solve this drawback without ruining previous trends of definitions that may include large amounts of studies. For example, we propose to keep continuity as Dubois's definition and add a real number to the set of FNs as an exception.

Briefly, we try to highlight the cons and pros of additional properties on membership function in relation to standard and parametric definitions. These properties include convexity, the existence of a deterministic part, compactness, continuity, and strictly increasing and decreasing conditions.

Based on our discussion of drawbacks and possible solutions to properties of the membership function, we modify the unified definition mentioned in the study by Shiri [25]. The application of such concrete definition for the study of measurement error and other uncertainty sources in modeling with systems of equations is studied in the work by Shiri [25].

The aims of this study are twofold. As we mentioned previously, the first aim is about the drawbacks and proposed solutions to the definition of uncertain numbers. The next aim of this study is related to constructing the calculus of uncertain discrete numbers for fractional and integer order operators. To do this, we follow the new scalar multiplication [25] for the definition of generalized Hukuhara difference. Interestingly, a complex analysis will show that the Hukuhara difference plays an important role in well defined systems rather than the generalized Hukuhara difference.

Our special focus on discrete operators will be fuzzy nabla differences with the generalized Hukuhara difference, $\nabla_{\ominus_g}$, and its fractional generalization $\nabla^{\nu}_{\ominus_g}$, $\nu \in (0, 1)$.

By introducing uncertain discrete sums and differences for integer and fractional order, we provide their classical relationship under the fundamental theorems for both fractional and integer order cases. According to the fundamental theorem, the maximal uncertain anti-nabla function is an uncertain discrete sum of that function. In a parallel discussion, we obtain Leibniz's rule for uncertain difference. Similarly, we will build a fundamental theorem for an uncertain fractional difference operator.

The study of linear uncertain fraction difference equation of the form

$$\nabla^{\nu}_{\ominus_g} P(t) = \lambda P(t) + f(t), \quad t \in N_{a+1}, \ \nu \in (0, 1), \tag{1}$$

is our main contribution, whereas P is an unknown function on N_{a+1} and the source function f is a given function on $N_a \coloneqq \{a, a + 1, \ldots\}$. Usually, some information on initial time $t = a$ is provided. Unlike the deterministic case, initial value problems for uncertain cases are usually ill-posed. Thus, we introduce the concept of maximal solution, and we observe that the maximal solution is unique and the corresponding problem becomes well posed with this reforming concept.

Although Eq. (1) without concerning uncertainty and fuzziness has received some attention these years [31,32], as far as we know, the effect of uncertainty on these equations is not studied yet. But continuous fractional differential equations have received extensive attention concerning fuzziness [33–36].

A subclass of Eq. (1) is relaxation equations. The fractional relaxation equation plays an important role in the modeling of perturbed systems in martial science [4,37]. In connection with interval analysis, such equations have been investigated in the study by Huang *et al.* [38]. We investigated uncertainty for discrete relaxation equations in separate sections.

In Section 2, we survey and review the concept of FNs and related operations. In Sections 3 and 6, fundamental theorems for uncertain fractional derivatives of integer order and fractional are constructed, respectively. In Section 3, the uncertain relaxation equation is studied, and in Section 6, the uncertain relaxation equation with fractional order is investigated. Section 5 is devoted to introducing uncertain fraction sum and difference. Finally, in Section 8, we provide illustrative examples with discussions, comparisons, and clarifications of the previously stated theoretic results.

2 Uncertain number and FNs

An FN u is a fuzzy set with a membership function μ from $\mathbb{R}$ to $[0,1]$, with some extra condition imposed on μ. Its r-cut $r \in [0,1]$ is defined by:

$$u^r = \{x \in \mathbb{R} : \mu(x) \geq r\}. \tag{2}$$

The boundaries of r-cut usually are used for transferring an FN into an interval and investigating them with interval analysis. This plays an important role in solving uncertain systems of equations by transforming them into deterministic systems [39]. Therefore, we review the imposed properties for the definition of FNs related to r-cats.

- **Convexity:** If μ is a convex function, then r-cut is an interval. Figure 1 shows that the 0.6-cuts of a non-convex set are a union of two disjoint intervals U_1 and U_2. The advantage of this assumption is that we can connect fuzzy theory to interval analysis. The disadvantage is that we could not use it for quantum mechanics, for example, for describing double-slit experiments. Thus, FNs could not be used for describing quantum mechanics. Nevertheless, fuzzy sets can be used.
- **Deterministic part:** The existence of a deterministic means that there exists an interval $[a,b]$ (probably, a can be b) such that $\mu(x) = 1$ for each $x \in [a,b]$. The advantage of this definition is that r-cuts are nonempty sets for all $r \in [0,1]$. Therefore, the boundaries of an r-cut set for an FN μ are well defined by:

$$\underline{u}(r) = \inf \mu^r \in [-\infty, a] \tag{3}$$

and

$$\overline{u}(r) = \sup \mu^r \in [b, \infty]. \tag{4}$$

Usually, these boundaries are used to describe FNs by the parametric representation $[\underline{u}(r), \overline{u}(r)]$, $r \in [0,1]$.
- **Compactness:** Emphasizing on compactness of μ^r, forces $\underline{u}(0)$ and $\overline{u}(0)$ become finite numbers. Thus, this condition including the existence of a deterministic part altogether implies that

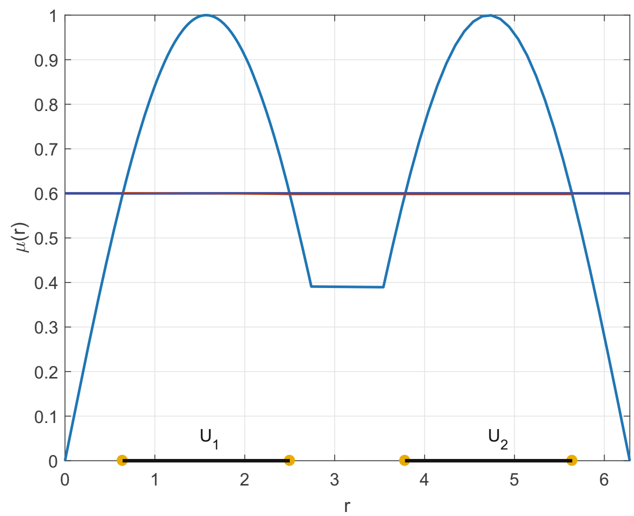

Figure 1: A non-convex membership function has disjoint r-cuts. For example, for $r = 0.6$, $\mu^r = U_1 \cup U_2$. This means that the boundaries of r-cuts are more than two points. Furthermore, the two picks imply the uncertainty of two numbers simultaneously (the pikes), which is intuitively inconsistent. Thus, this membership function can be a fuzzy set instead of an FN. Oddly, we have such fuzzy sets in quantum physics.

$$\underline{u}(r) \in (-\infty, a]$$

and

$$\overline{u}(r) \in [b, \infty)$$

are well defined. Furthermore, we can substitute max and min with sup and inf in the definition of r-cut boundaries. A consequence of this assumption is that $\mu(x) = 0$ outside of the interval $[c,d]$, where $c = \underline{u}(0)$ and $d = \overline{u}(0)$. This property is usually used for the definition of an FN to emphasize compactness.
- In my opinion, compactness has some useful properties, but it conceals some nice membership functions such as Gaussian functions, which is a probability density function of normal distribution. It is permitted in fuzzy set theory but according to the definition of Dubois et al., it is not permitted to be an FN. Figure 2 (a) and (b) shows a fuzzy set with the Gaussian function $\mu(x) = \exp(-(x-1)^2)$ and their r-cuts representation, respectively. The boundaries of r-cuts representation are infinite in zero. One advantage of such restriction is that we can define at zero the related r-cuts. So the definition of FN is fated to be closer to interval analysis than to statistical analysis.
- **Continuity:** Continuity can have many advantages. However, the main problem is that we could not define a crisp number as an FN. Our expectation is that a number $d \in \mathbb{R}$ with membership function

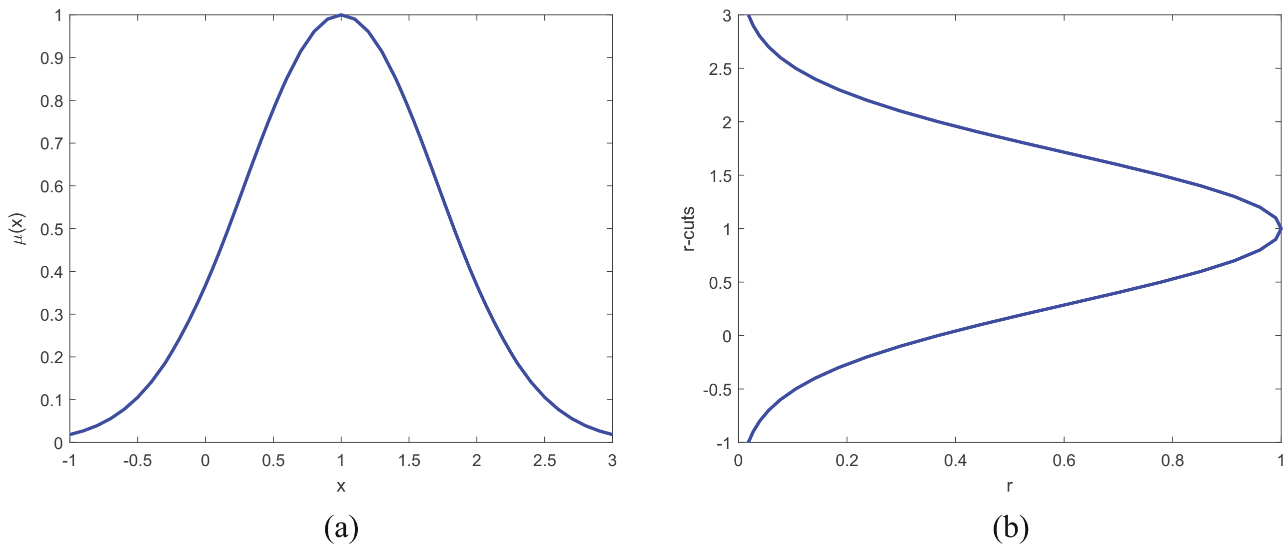

Figure 2: (a) A membership function with Gaussian function and (b) its r-cuts boundaries. It is not an FN.

$$\mu(x) = \begin{cases} 1, & x = d, \\ 0, & \text{otherwise}, \end{cases} \quad (5)$$

can be defined as an FN. It is evident that this number is not continuous. Thus, Goetschel and Voxman [28] proposed replacing semi-continuity conditions instead of continuity. In this case, μ is right continuous in the left-hand side (i.e., (∞, d)) and left continuous in the right-hand side (i.e., (d, ∞)). The last updated definition enjoys imposing such left and right continuity outside of the deterministic part.

- On the other hand, the discontinuity of boundaries of r-cuts can lead to some problems in directly recovering FNs from their parametric representation. Let $[\underline{u}, \overline{u}]$ be a parametric representation and $a = \underline{u}(1)$, $b = \overline{u}(1)$, $c = \underline{u}(0)$, and $d = \overline{u}(0)$. Then, a membership function can be recovered by

$$\mu(x) = \begin{cases} \underline{u}^{-1}(x), & x = [c, a], \\ 1, & x = [a, b], \\ \overline{u}^{-1}(x), & x = [b, d], \\ 0, & \text{otherwise}. \end{cases} \quad (6)$$

But, $\overline{u}^{-1}(x)$ and $\underline{u}^{-1}(x)$ are not defined on discontinuous point (Figure 3). However, this problem can be cured by the recovery function

$$\mu(x) = \begin{cases} \inf\{r : \underline{u}(r) \in [x, a]\}, & t = [c, a], \\ 0, & x = [a, b], \\ \inf\{r : \overline{u}(r) \in [b, x]\}, & x = [b, d], \\ 0, & \text{outside of [c,d]}. \end{cases} \quad (7)$$

Therefore, with this definition, the corresponding FN of Figure 3(a) has representation deputed in Figure 3(b).

- Now, let us recover parameter representation by r-cut definition and in Figure 3(b). By Definition (2), $u^{0.5} = [-1, 4]$. Thus, $\overline{u}(0.5) = 4$. This is in contraction with our original Figure 3(a), which $\overline{u}(0.5) = 3$. This problem can be solved by pressuring left contentious continuous condition on $(0, 1]$ for $\overline{u}$.
- **Strictly decreasing or increasing condition:** Dubious and Prade [26] imposed the strictly decreasing or increasing conditions on FNs. However, to define a topology, Goetsche and Voxman [28] just imposed the decreasing or increasing condition. The membership function depicted in Figure 3(b) is not an FN by the definition of the study by Dubois *et al.* [26]. But it is an FN by definition [28]. We note that omitting the strictness leads to non-continuity in the parametric form, and *vice versa* (Figure 3 and related discussion).

Remark 2.1. Another solution to the paradox of strictly monotonousness conditions is keeping continuity, strictly monotonousness behavior, and adding parametric functions as constant $\underline{u}(r) = \overline{v}(r) = \text{const.}$ to fuzzy parametric numbers. This means, on the other hand, keeping the Dubious condition and adding the FNs of Form (5) to the set of FNs.

Continuing the discussion of the study by Shiri [25], we have the following definition:

$$\mathcal{K} = \{[a, b] : a \le b\},$$

$$\mathbb{U}^M[a, b] = \{u : [a, b] \to \mathbb{R} \\ : u \text{ is strictly decreasing, continuous on } [a, b] \text{ or } u \equiv 0\}, \quad (8)$$

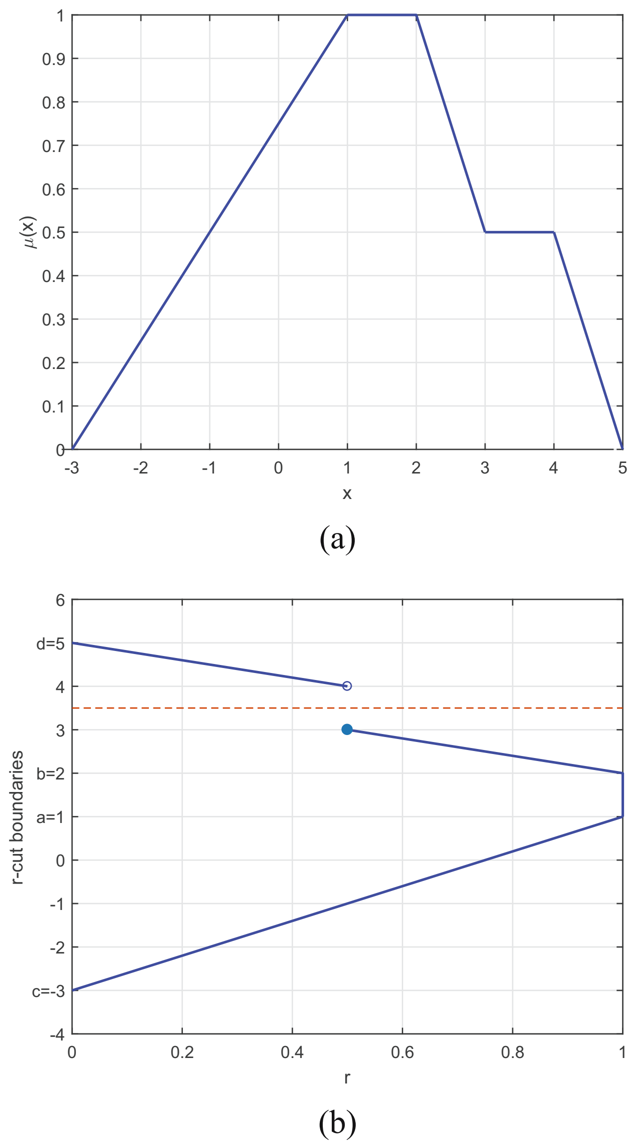

Figure 3: (a) A fuzzy set with membership function μ, without strictly monotonous condition and (b) r-cut representation of the FN depicted in (a). It is evident from (a) that $u^{0.5} = [1, 4]$, $u^{0.5\varepsilon} \subset [1, 4]$, and $u^{0.5+\varepsilon} \supset [1, 3]$. Therefore, $\bar{u}(r)$ has discontinuity at $r = 0.5$. Thus, we could not define $r = \bar{u}^1(3.5)$. In this case, we may use (7) to define $\mu(3.5)$, which are not recommended.

and

$$\mathbb{U}^{M+}[a, b] = \{u : [a, b] \to \mathbb{R}^+ : u \in C^M\}.$$

Definition 2.2. [25] A map $f_v : [0, 1] \to \mathcal{K}$ is called entirely uncertain number (EUN) if

$$f_v(r) = [f - v_1(r), f + v_2(r)], \quad r \in [0, 1], \tag{9}$$

where $f \in \mathbb{R}$ is the deterministic part and $v_1, v_2 \in \mathbb{U}^{M+}[0, 1]$ are uncertain parts. Furthermore, if the condition

$$v_1(1) = v_2(1) = 0 \tag{10}$$

holds, we say that d_v is an uncertain number. If $v_1, v_2 \in C^m[0, 1]$ ($m \in \mathbb{N} \cup \{0\}$), we say that d_v is a C^m-smooth (entirely) uncertain number.

Every EUN can be characterized by the triple

$$[d, u_1, u_2] \in \mathbb{R} \times \mathbb{U}^{M+}[0, 1] \times \mathbb{U}^{M+}[0, 1].$$

We denote the set of EUNs by $\mathbb{R}_{\text{EU}}$ and the set of uncertain numbers by $\mathbb{R}_U$. The important characterization of $\mathbb{R}_U$ is that they have unique representation in $\mathbb{R} \times \mathbb{U}^{M+}[0, 1] \times \mathbb{U}^{M+}[0, 1]$ [25].

Pathological remark 2.3. The presentation of an EUN may not be unique. Suppose $d_u = \bar{d}_{\bar{u}}$. Then,

$$[d - u_1(r), d + u_2(r)] = [\bar{d} - \bar{u}_1(r), \bar{d} + \bar{u}_2(r)], \quad r \in [0, 1].$$

Thus,

$$d - u_i(r) = \bar{d} - \bar{u}_i(r).$$

This means that $\bar{u}_i(r) - u_i(r) = \bar{d} - d$ is a crisp number and let us name it c. To make the definition of EUN well defined, we can use the equivalent classes as separate elements, or choose one particular element of a class by imposing more conditions. We follow the latter. One such element is the centralized EUN (CEUN). A CEUN is an EUN with the extra condition:

$$u_2(1) = u_1(1). \tag{11}$$

We denote the set of CEUNs by $\mathbb{R}_{\text{CEU}}$.

Theorem 2.4. *Every EUN has an equivalent CEUN.*

Proof. Let u_d be a given EUN. Set

$$c = \frac{u_2(1) - u_1(1)}{2},$$

and let $\bar{d} = d + c$ and $\bar{u}_i(r) = u_i(r) - (-1)^i c$. It is straightforward to check that $\bar{u}_i \in \mathbb{U}^{M+}[0, 1]$. Therefore, $\bar{d}_{\bar{u}}$ is an EUN, while

$$\bar{u}_1(1) = u_1(1) + \frac{u_2(1) - u_1(1)}{2} = \bar{u}_2(1).$$

Thus, $\bar{d}_{\bar{u}}$ is a CEUN. □

Theorem 2.5. *Every CEUN has a unique presentation.*

Proof. Let $\bar{d}_{\bar{u}} = d_u$ be two representations of CEUNs. Then,

$$d + (-1)^i u_i(r) = \bar{d} + (-1)^i \bar{u}_1(r), \quad i = 1, 2. \tag{12}$$

Therefore,

$$2d + u_2(r) - u_1(r) = 2\bar{d} + \bar{u}_2(r) - \bar{u}_2(r). \tag{13}$$

Substituting $r = 1$ into (13), we have

$$2d + u_2(1) - u_1(1) = 2\bar{d} + \bar{u}_2(1) - \bar{u}_2(1). \tag{14}$$

Since $\bar{d}_{\bar{u}}$ and d_u are CEUN, $u_2(1) - u_1(1) = 0$ as well as $\bar{u}_2(1) - \bar{u}_2(1) = 0$. Thus, it follows from (14) that

$$d = \bar{d}, \tag{15}$$

and immediately by substituting f from (15) into (12), we obtain $u_i(r) = \bar{u}_i(r)$. This completes the proof. □

A parametric representation of an FN $p = (p_1, p_2)$ has a CEUN presentation [25] as:

$$d(p)_u = [d(p) - u_1(p)(r), d(p) + u_2(p)(r)], \tag{16}$$

where

$$d(p) = \frac{p_1(1) + p_2(1)}{2}, \tag{17}$$

and

$$u_i(p)(r) \coloneqq (-1)^{\{i+1\}} d(p) - p_i(r), \quad i = 1, 2. \tag{18}$$

We can note that

$$\begin{aligned} u_1(p)(1) &= \frac{p_1(1) + p_2(1)}{2} - p_1(1) = \frac{p_2(1) - p_1(1)}{2} \\ &= p_2(1) - \frac{p_1(1) + p_2(1)}{2} = u_2(p)(1). \end{aligned} \tag{19}$$

Let p and q be two CEUNs. Then, the addition is defined by:

$$\begin{aligned} d(p + q) &= d(p) + d(q) \\ u_i(p + q) &= u_i(p)(r) + u_i(q)(r), \quad i = 1, 2. \end{aligned} \tag{20}$$

The new scalar multiplication has recently redefined in the study by Shiri [25]:

$$cf = [cd(p) - |c|u_1(p)(r), cd(p) + |c|u_2(p)(r)], c \in \mathbb{R}. \tag{21}$$

Therefore, the Hukuhara difference [40] and its generalization [24] are defined by:

$$p \ominus q = h \Leftrightarrow p = q + h \tag{22}$$

and

$$p \ominus_g q = h \Leftrightarrow \begin{cases} p = q + h, \\ \text{or } q = p + (-1)h. \end{cases} \tag{23}$$

Remark 2.6. Hukuhara difference is independent of the scalar product. Thus, with both definitions of scalar product, we have unique results. However, the generalized Hukuhara definition depends on scalar multiplication. This definition varies with different types of scalar products.

Theorem 2.7. [25] *Suppose p and q are two FNs. The differences (I) $p \ominus q$ and (II) $p \ominus_g q$ exist if*

(I) $u_i(p) - u_i(q) \in \mathbb{U}^{M+}[0, 1]$ for $i = 1, 2$,

(II) $|u_i(p) - u_i(q)| \in C^M[0, 1]$ for $i = 1, 2$,

respectively. Furthermore, if (I) and (II) hold, then

$$\begin{aligned} p \ominus q = [&d(p) - d(q) - (u_1(p)(r) - u_1(g)(r)), \\ &d(p) - d(q) + (u_2(p)(r) - u_2(q)(r))] \end{aligned} \tag{24}$$

and

$$\begin{aligned} p \ominus_g q = [&d(p) - d(q) - |u_1(p)(r) - u_1(q)(r)|, \\ &d(p) - d(q) + |u_2(p)(r) - u_2(q)(r)|]. \end{aligned} \tag{25}$$

3 Fundamental theorem of uncertain discrete calculus

3.1 Nabla calculus

Let $p : \mathbb{N}_a \to \mathbb{R}_{\text{CEU}}$ be a fuzzy-valued discrete function. Then,

$$\nabla_{\ominus_g} p(t) = p(t) \ominus_g p(t-1), \quad t \in \mathbb{N}_{a+1}, \tag{26}$$

is the nabla difference operator. From Theorem 2.7, this definition is well defined if

$$|u_i(p)(t) - u_i(p)(t-1)| \in \mathbb{U}^M[0, 1], \quad i = 1, 2.$$

The continuity conditions automatically hold. However, the monotonic conditions should be checked.

We can extent the definition for $n \in \mathbb{N}$ by the recursive formula:

$$\nabla^n_{\ominus_g} p(t) = \nabla_{\ominus_g}(\nabla^{n-1}_{\ominus_g} p)(t),$$

for $t \in \mathbb{N}_{a+n}$.

Let $c \le d$ and $c, d \in \mathbb{N}_a$. Then, the nabla integral is defined by:

$$\int_c^d p(t)\nabla t = \sum_{i=c+1}^{d} p(i),$$

and for $d < c$,

$$\int_d^c p(t)\nabla t = (-1) \sum_{i=c+1}^{d} p(i),$$

where the especial product by (-1) is defined by (21). For definitions of nabla integral in crisp values, see the study by Kelley and Peterson [7]. The fundamental theorem for uncertain fractional nabla integral no longer holds.

For a given $f : \mathbb{N}_{a+1} \to \mathbb{R}_{CEU}$, the simplest fuzzy discrete difference equation is

$$\nabla_{\ominus_g} P(t) = f(t), \quad t \in N_{a+1}. \tag{27}$$

The solution of Eq. (27) is a fuzzy-valued function $P : \mathbb{N}_a \to \mathbb{R}_{CEU}$ such that satisfies (27). It is called the anti-nabla function in calculus. To have a unique solution, we can add some more information such as boundary conditions. We will use the initial condition

$$P(a) = P_a, \tag{28}$$

for a given $P_a \in \mathbb{R}_{CEU}$.

Remark 3.1. For a crisp function by the fundamental theorem of the nabla calculus [9], we have

$$P(t) = P_a + \int_a^t f(s)\nabla s. \tag{29}$$

However, this is no longer true for fuzzy-valued equations.

Example 3.2. Let us substitute $t = a + 1$ into Eq. (27). Then,

$$d(P(a+1)) - d(P_a) = d(f(a+1))$$

and

$$|u_i(P(a+1)) - u_i(P_a)| = u_i(f(a+1)), \quad i = 1, 2.$$

Therefore, the deterministic part has a unique solution:

$$d(P(a+1)) = d(P_a) + d(f(a+1)). \tag{30}$$

But, the uncertain part has two solutions. To distinguish these two solutions, we use extra subscriptions:

$$u_{i,j}(P(a+1)) = u_i(P_a) - (-1)^j u_i(f(a+1)), \quad i, j = 1, 2. \tag{31}$$

Trivially, $u_{i,1}(P(a+1)) \in \mathbb{U}^{M+}$, while we should check that if $u_{i,2}(P(a+1)) \in \mathbb{U}^{M+}$. In this case, we have four solutions of the form:

$$[d(P(a+1)), u_{1,k}(P(a+1)), u_{2,j}(P(a+1))], \tag{32}$$

$k, j = 1, 2$. Which solution do we choose?

The best choice is related to the biggest uncertainties. But, we know that

$$u_{i,1}(P(a+1)) > u_{i,2}(P(a+1)). \tag{33}$$

Therefore, Case (a) is well defined and includes other cases. It can prevent the bifurcation problem and provides a unique solution. We name such a solution a maximal solution.

Definition 3.3. Let $\{[d, u_{1,i}, u_{2,i}]\}_{i \in I}$ be all solutions of an uncertain problem. Then, the solution defined by $[d, \sup_{i\in I} u_{1,i}, \sup_{i\in I} u_{2,i}]$ is called a maximal solution. Here, the sup is pointwise with respect to r, i.e., $(\sup_{i\in I} u_{1,i})(r) = \sup_{i\in I}(u_{1,i}(r))$.

Remark 3.4. The compensation for introducing a unique solution (maximal solution) is losing information.

We now turn to state the fundamental theorem of discrete uncertain calculus with CEUNs.

Theorem 3.5. *The initial value problem for Eq.* (27) *has a unique maximal solution* $P : \mathbb{N}_{a+1} \to \mathbb{N}_{CEU}$ *and*

$$P(t) = P_a + \int_a^t f(s)\nabla(s). \tag{34}$$

Proof. By Example 3.2, the maximal solution of (27) on $t = a + 2$ is given by (30) and (31). Substituting $t = a + 1$ into Eq. (27), we obtain

$$P(a+2) \ominus_g P(a+1) = f(a+2). \tag{35}$$

This means that the maximal solution in this point is given by:

$$d(P(a+2)) = d(P(a+1)) + d(f(a+2)) \tag{36}$$

and

$$u_i(P(a+2)) = u_i(P(a+1)) + u_i(f(a+2)), \quad i = 1, 2. \tag{37}$$

For $t \in \mathbb{N}_{a+1}$, with a similar argument, we obtain

$$d(P(t)) = d(P(t-1)) + d(f(t)), \tag{38}$$

and

$$u_i(P(t)) = u_i(P(t-1)) + u_i(f(t)), \quad i = 1, 2. \tag{39}$$

The solution of the recursive Formulas (38) and (39) is

$$\begin{aligned} d(P(t)) &= d(P_a) + \sum_{i=a+1}^{t} d(f(i)) \\ &= d(P_a) + \int_a^t d(f(s))\nabla(s) \end{aligned} \tag{40}$$

and

$$\begin{aligned} u_i(P(t)) &= u_i(P_a) + \sum_{s=a+1}^{t} u_i(f(s)) \\ &= u_i(P_a) + \int_a^t u_i(f(s))\nabla(s), \end{aligned} \tag{41}$$

for $i = 1, 2$, respectively. Finally, the summation property of CEUNs results in Eq. (34). □

Similarly, Leibniz's rule for fuzzy difference can be stated as follows.

Theorem 3.6. *Suppose* $q(t,s):\mathbb{N}_a\times\mathbb{N}_a\to\mathbb{R}_{\mathrm{CEU}}$ *and* $u_i(q(t,s))$ *with respect to t is an increasing function, i.e.,*

$$(i)\quad u_i(q(s,t)) - u_i(q(s,t-1)) \in \mathbb{U}^{M+},\quad t\in\mathbb{N}_a.$$

Then,

$$\nabla_{\ominus_{g,t}}\int_b^t q(s,t)\nabla s = \int_b^t \nabla_{\ominus_{g,t}} q(s,t)\nabla s + q(s,t-1). \tag{42}$$

Proof. For $t\in\mathbb{N}_{b+1}$ and $b\in\mathbb{N}_a$

$$\nabla_{\ominus_{g,t}}\int_b^t q(s,t)\nabla s = \sum_{s=b+1}^{t} q(s,t)\ominus_g \sum_{s=b+1}^{t} q(s,t-1), \tag{43}$$

and hence,

$$\begin{aligned} u_i\left(\nabla_{\ominus_{g,t}}\int_b^t q(s,t)\nabla s\right) &= \left|\sum_{s=b+1}^{t} u_i(q(s,t)) - \sum_{s=b+1}^{t-1} u_i(q(s,t-1))\right| \\ &\le \sum_{s=b+1}^{t} |u_i(q(s,t)) - u_i(q(s,t-1))| \\ &\quad + |u_i(q(s,t-1))| \end{aligned} \tag{44}$$

Equality holds if Condition (i) of the theorem holds. In this case, we have

$$\begin{aligned} u_i\left(\nabla_{\ominus_{g,t}}\int_b^t q(s,t)\nabla s\right) &= \int_b^t (u_i\nabla_{\ominus_{g,t}} q(s,t))\nabla s + |u_i(q(s,t-1))| \\ &= \int_b^t u_i(\nabla_{\ominus_{g,t}} q(s,t))\nabla s + u_i(q(s,t-1)). \end{aligned} \tag{45}$$

Similarly,

$$\begin{aligned} d\left(\nabla_{\ominus_{g',t}}\int_b^t q(s,t)\nabla s\right) &= \int_b^t d(\nabla_{\ominus_{g,t}} q(s,t))\nabla s + d(q(s,t-1)). \end{aligned} \tag{46}$$

Remark 3.7. Since Condition (i) of Theorem 3.6 automatically holds for the existence of nabla difference for Hukuhara difference, Theorem 3.6 is also valid for nabla difference with the Hukuhara definition.

4 Uncertain discrete relaxation equation

Relaxation means the return of a perturbed system to its equilibrium. The fuzzy/uncertain discrete relaxation equation with a rate of relaxation $\lambda\in\mathbb{R}$ can be described by:

$$\nabla_{\ominus_g} P(t) = \lambda P(t),\quad t\in N_{a+1}, \tag{47}$$

with a fuzzy initial condition:

$$P(a) = P_a \in \mathbb{R}_{\mathrm{CEU}}.$$

Let $t=a+1$. Then,

$$P(t)\ominus_g P(t-1) = \lambda P(t). \tag{48}$$

Immediately, it follows from (48) that

$$d(P(a+1)) = \frac{1}{1-\lambda} d(P(a)). \tag{49}$$

The possible solutions for uncertain parts are

$$u_i(P(a+1)) = u_i(P(a)) \pm |\lambda| u_i(P(a+1)). \tag{50}$$

Thus,

$$u_i(P(a+1)) = \frac{1}{1-|\lambda|} u_i(P(a)) \tag{51}$$

and

$$u_i(P(a+1)) = \frac{1}{1+|\lambda|} u_i(P(a)). \tag{52}$$

The choice of the solution depends on λ. If $|\lambda|<1$, Eq. (51) is maximal solution since

$$\frac{1}{1-|\lambda|} > \frac{1}{1+|\lambda|}.$$

However, for $|\lambda|>1$, the right-hand side of Eq. (51) does not belong to $\mathbb{U}^{M+}[0,1]$, since it is negative. Thus, in this case, only Eq. (52) is an acceptable solution. Exactly, with similar arguments for $n\in\mathbb{N}$

$$d(P(a+n)) = \left(\frac{1}{1-\lambda}\right)^n d(P_a),\quad n\in\mathbb{N}, \tag{53}$$

and

$$u_i(P(a+n)) = \left(\frac{1}{1-|\lambda|}\right)^n u_i(P_a),\quad n\in\mathbb{N}, \tag{54}$$

for $|\lambda|<1$, and

$$u_i(P(a+n)) = \left(\frac{1}{1+|\lambda|}\right)^n u_i(P_a),\quad n\in\mathbb{N}, \tag{55}$$

for $|\lambda|>1$ are the maximal solutions.

5 Uncertain fractional discrete calculus

To define the nabla fractional difference, we need to define the nabla fractional sum. For crisp function, there exist two definitions [9,41]. For uncertain valued functions, we generalize those definitions.

Definition 5.1. Let $f: \mathbb{N}_a \to \mathbb{R}_{\text{CEU}}$ and $\nu \in (0,1)$. Then, the Nubla fractional sum for $t \in \mathbb{N}_a$ is defined by:

$$\nabla_a^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \int_a^t (t - \rho(s))^{\overline{\nu - 1}} f(s) \nabla(s) = \frac{1}{\Gamma(\nu)} \sum_{s=a+1}^{t} \frac{\Gamma(t - s + \nu)}{\Gamma(t - s + 1)} f(s) \tag{56}$$

where $\rho(s) = s - 1$ and

$$k^{\bar{\nu}} \coloneqq \begin{cases} \frac{\Gamma(k+\nu)}{\Gamma(k)}, & k \in \mathbb{R} \backslash \mathbb{Z}^-, \quad \nu \in \mathbb{R}, \\ 0, & k \in \mathbb{Z}^-, \quad k + \nu \in \mathbb{R} \backslash \mathbb{Z}^-, \\ \text{undefined}, & \text{otherwise}, \end{cases}$$

and $\mathbb{Z}^- = \{0, -1, \ldots\}$. However, to include the effect of $s = a$, one may consider the second definition as:

$$\nabla_a^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \int_{a-1}^t (t - \rho(s))^{\overline{\nu - 1}} f(s) \nabla(s) = \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t} \frac{\Gamma(t - s + \nu)}{\Gamma(t - s + 1)} f(s). \tag{57}$$

Now, we can define the nabla fractional difference in the following.

Definition 5.2. [9] Let $f: \mathbb{N}_a \to \mathbb{R}_{\text{CEU}}$ and $N - 1 < \nu \le N$, $N \in \mathbb{N}$. Then,

$$\nabla_{a,\ominus_g}^{\nu} f(t) \coloneqq \nabla_{\ominus_g}^{N} \nabla_a^{-(N-\nu)} f(t), \quad t \in \mathbb{N}_{a+N}. \tag{58}$$

We compare the definition of nabla difference for both cases for a crisp function. Particularly, for $\nu \in (0,1]$ from (56), we have

$$\nabla_a^{\nu} f(t) \coloneqq \nabla_a^{-(N-\nu)} f(t) - \nabla_a^{-(N-\nu)} f(t-1) = \frac{1}{\Gamma(1-\nu)} \sum_{s=a+1}^{t-1} \frac{\Gamma(t-s-\nu)}{\Gamma(t-s)} (f(s+1) - f(s)) + \frac{1}{\Gamma(1-\nu)} \frac{\Gamma(t-a-\nu)}{\Gamma(t-a)} f(a). \tag{59}$$

Thus,

$$\lim_{\nu \to 1} \nabla_a^{\nu} f(a+1) = f(a).$$

But from (57), we have

$$\nabla_a^{\nu} f(t) \coloneqq \nabla_a^{-(N-\nu)} f(t) - \nabla_a^{-(N-\nu)} f(t-1) = \frac{1}{\Gamma(1-\nu)} \sum_{s=a+1}^{t} \frac{\Gamma(t-s+1-\nu)}{\Gamma(t-s+1)} (f(s) - f(s-1)) + \frac{1}{\Gamma(1-\nu)} \frac{\Gamma(t-a+1-\nu)}{\Gamma(t-a+1)} f(a). \tag{60}$$

Thus,

$$\lim_{\nu \to 1} \nabla_a^{\nu} f(a+1) = \nabla f(a+1).$$

It can easily be checked that for both definitions,

$$\lim_{\nu \to 0} \nabla_a^{\nu} f(a+1) = f(a+1).$$

Conclusively, the nabla fractional difference definition by (57) is preferable since it acts like an integer order nabla difference when $\nu \to 1$ and $\nu \to 0$.

For uncertain functions with $\nu \in (0,1)$, we have

$$\begin{aligned} \nabla_{a,\ominus_g}^{\nu} f(t) &= \nabla_{\ominus_g} \nabla_a^{-(N-\nu)} f(t), \quad t \in \mathbb{N}_{a+1} \\ &= \nabla_a^{-(N-\nu)} f(t) \ominus_g \nabla_a^{-(N-\nu)} f(t-1) \\ &= \frac{1}{\Gamma(1-\nu)} \sum_{s=a}^{t} \frac{\Gamma(t-s+1-\nu)}{\Gamma(t-s+1)} f(s) \\ &\ominus_g \frac{1}{\Gamma(1-\nu)} \sum_{s=a}^{t-1} \frac{\Gamma(t-s-\nu)}{\Gamma(t-s)} f(s). \end{aligned} \tag{61}$$

Thus,

$$d(\nabla_a^{\nu} f(t)) = \frac{1}{\Gamma(1-\nu)} \sum_{s=a+1}^{t} \frac{\Gamma(t-s+1-\nu)}{\Gamma(t-s+1)} \nabla d(f(s)) + \frac{1}{\Gamma(1-\nu)} \frac{\Gamma(t-a+1-\nu)}{\Gamma(t-a+1)} d(f(a)) \tag{62}$$

and

$$u_i(\nabla_a^{\nu} f(t)) = \left| \frac{1}{\Gamma(1-\nu)} \sum_{s=a+1}^{t} \frac{\Gamma(t-s+1-\nu)}{\Gamma(t-s+1)} \nabla u_i(f(s)) + \frac{1}{\Gamma(1-\nu)} \frac{\Gamma(t-a+1-\nu)}{\Gamma(t-a+1)} u_i(f(a)) \right|. \tag{63}$$

Supposing $u_i f(s)$ is an increasing function (then, $f(s) \ominus f(s-1)$ exists), we can omit the absolute value function and we will have

$$u_i(\nabla_a^{\nu} f(t)) = \frac{1}{\Gamma(1-\nu)} \sum_{s=a+1}^{t} \frac{\Gamma(t-s+1-\nu)}{\Gamma(t-s+1)} \nabla u_i(f(s)) + \frac{1}{\Gamma(1-\nu)} \frac{\Gamma(t-a+1-\nu)}{\Gamma(t-a+1)} u_i(f(a)). \tag{64}$$

For solving the related equation, we may use the following formulas:

$$d(\nabla_a^\nu f(t)) = d(f(t)) - \frac{\nu}{\Gamma(1-\nu)} \sum_{s=a}^{t-1} \frac{\Gamma(t-s-\nu)}{\Gamma(t-s+1)} d(f(s)) \tag{65}$$

and

$$u_i(\nabla_a^\nu f(t)) = |u_i(f(t)) - \frac{\nu}{\Gamma(1-\nu)} \sum_{s=a}^{t-1} \frac{\Gamma(t-s-\nu)}{\Gamma(t-s+1)} u_i(f(s))|. \tag{66}$$

Remark 5.3. Equations (65) and (66) are important. They show the effect of memory and locality for discrete fuzzy derivatives. Let us denote the coefficients by η_i, i.e.,

$$\eta_k(\nu) \coloneqq -\frac{\nu}{\Gamma(1-\nu)} \frac{\Gamma(k-\nu)}{\Gamma(k+1)}, \tag{67}$$

As depicted in Figure 4, the effect of the memory due to these coefficients achieves its highest amount not on zero or one but on the other points.

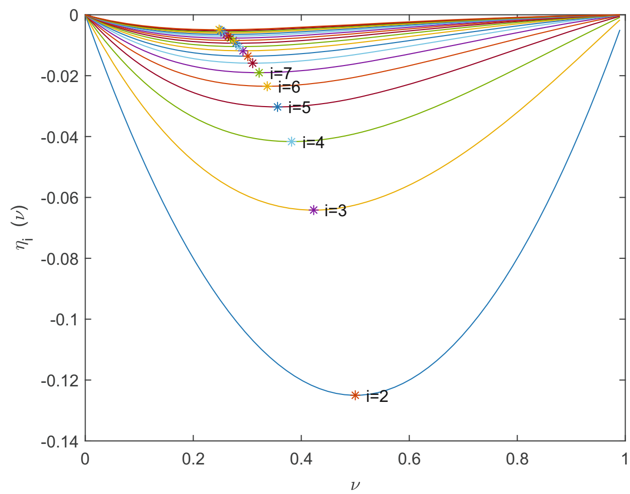

Figure 4: Coefficient of fractional nabla difference (65) for $i \in \mathbb{N}_2$. The maximum effect of the memory for that coefficient achieves at the minimum values of each curve. Such points are not zero or one. The important implication of this figure is against the belief that in related equations by increasing or decreasing fractional order toward integer order, we may obtain higher memory. Also, the curves are not symmetric, which means that $0+\varepsilon$ and $1\ \ \varepsilon$ do not receive similar memories. For $\varepsilon = 0.25$, the coefficients of $1\ \ \varepsilon = 0.75$ vanish faster than the coefficients of $0+\varepsilon = 0.25$.

6 Fundamental theorem of uncertain fractional discrete calculus

In this section, we solve an inverse problem to find a possible inverse of an uncertain fractional discrete difference. Indeed, we seek the solution of the following equation:

$$\nabla^\nu_{a,\ominus_g} P(t) = f(t), \quad t \in N_{a+1}, \quad \nu \in (0,1), \tag{68}$$

with initial condition $P(a) = P_a$, where $f : \mathbb{N}_{a+1} \to \mathbb{R}_{\text{CEU}}$.

We solve it by recursively substituting $t \in N_{a+1}$ into Eq. (68). Setting $t = a+1$, we obtain

$$d(P(a+1)) = d(f(a+1)) + \nu d(P(a)) \tag{69}$$

and

$$u_i(P(a+1)) = \nu u_i(P(a)) \pm u_i(f(a+1)). \tag{70}$$

Thus, the maximal solution is

$$P(a+1) = \nu P(a) + f(a+1). \tag{71}$$

Now, putting $t = a+2$, we obtain

$$\begin{aligned} d(\nabla_a^\nu P(a+2)) &= d(P(a+2)) - \nu d(P(a+1)) - \frac{\nu(1-\nu)}{2!} d(P(a)) \\ &= d(f(a+2)). \end{aligned} \tag{72}$$

Thus, the maximal solution is

$$\begin{aligned} P(a+2) &= \nu P(a+1) + \frac{\nu(1-\nu)}{2!} P(a) + f(a+2) \\ &= \frac{\nu(1+\nu)}{2!} P(a) + \nu f(a+1) + f(a+2). \end{aligned} \tag{73}$$

Similarly, the maximum solution satisfies

$$P(t) = \frac{\nu}{\Gamma(1-\nu)} \sum_{s=a}^{t-1} \frac{\Gamma(t-s-\nu)}{\Gamma(t-s+1)} P(s) + f(t), \tag{74}$$

for $t \in \mathbb{N}_{a+1}$. By defining $f(a) = P(a) = P_a$, we can suppose

$$P(t) = \sum_{i=0}^{t-a} c_i(\nu) f(t-i). \tag{75}$$

Substituting (75) into (74), we obtain

$$\begin{aligned} \sum_{i=0}^{t-a} c_i(\nu) f(t-i) &= \sum_{s=a}^{t-1} \sum_{i=0}^{s-a} \frac{\nu\Gamma(t-s-\nu)}{\Gamma(1-\nu)\Gamma(t-s+1)} c_i(\nu) f(s-i) + f(t) \\ &= \sum_{i=0}^{t-1-a} c_i(\nu) \sum_{s=a+i}^{t-1} \frac{\nu\Gamma(t-s-\nu)}{\Gamma(1-\nu)\Gamma(t-s+1)} f(s-i) + f(t). \end{aligned} \tag{76}$$

Equating coefficient of $f(a)$, we obtain

$$c_{t-a}(\nu) = \sum_{i=0}^{t-1-a} c_i(\nu) \frac{\nu}{\Gamma(1-\nu)} \frac{\Gamma(t-a-i-\nu)}{\Gamma(t-a-i+1)}, \tag{77}$$

or equivalently,

$$c_m(\nu) = \sum_{i=0}^{m-1} c_i(\nu)\frac{\nu}{\Gamma(1-\nu)}\frac{\Gamma(m-i-\nu)}{\Gamma(m-i+1)}, \quad m \geq 1. \tag{78}$$

Thus, by this recursive formula, we can obtain easily beautiful fading coefficients

$$c_0(\nu) = 1 = \frac{\Gamma(\nu+0)}{\Gamma(\nu)\Gamma(1+0)} = \frac{\nu^{(0)}}{\Gamma(1+0)} \tag{79}$$

and

$$c_n(\nu) = \frac{\nu\cdots(n-1+\nu)}{n!} = \frac{\Gamma(\nu+n)}{\Gamma(\nu)\Gamma(1+n)} = \frac{\nu^{(n)}}{\Gamma(1+n)}, \tag{80}$$

for $n \in \mathbb{N}_0$. They are related to the Pochhammer symbol, which is defined as follows:

$$\nu^{(n)} = \frac{\Gamma(\nu+n)}{\Gamma(\nu)}.$$

Therefore, the maximal exact solution is

$$\begin{aligned} P(t) &= \sum_{i=0}^{t-a}\frac{\nu^{(i)}}{i!}f(t-i) \\ &= \sum_{s=a}^{t}\frac{\nu^{(s-a)}}{(s-a)!}f(t+a-s) \\ &= \frac{\nu^{(t-a)}}{(t-a)!}P_a + \sum_{s=a}^{t-1}\frac{\nu^{(s-a)}}{(s-a)!}f(t+a-s) \\ &= \frac{\nu^{(t-a)}}{(t-a)!}P_a + \sum_{z=a+1}^{t}\frac{\nu^{(t-z)}}{(t-z)!}f(z) \\ &= \frac{\nu^{(t-a)}}{(t-a)!}P_a + \int_a^t \frac{\nu^{(t-z)}}{(t-z)!}f(z)\nabla z \\ &= \frac{\nu^{(t-a)}}{(t-a)!}P_a + \frac{1}{\Gamma(\nu)}\int_a^t \frac{\Gamma(\nu+t-z)}{\Gamma(1+t-z)!}f(z)\nabla z \\ &= \frac{\nu^{(t-a)}}{(t-a)!}P_a + \nabla_a^{-\nu}f(t). \end{aligned} \tag{81}$$

As we expected, we can consider $\nabla_a^{-\nu}$ as an inverse of ∇_a^{ν}. In this context, we should call it the maximal inverse of $\nabla_a^{-\nu}$, because it may have other inverses, which hold more information. The maximal inverse has the highest uncertainty among the solution. We summarize our discussion in the fundamental theorem for fractional discrete calculus.

Theorem 6.1. *The initial value problem for Eq.* (68) *has a unique maximal solution $P : \mathbb{N}_{a+1} \to \mathbb{N}_{\text{CEU}}$ and*

$$P(t) = \frac{\Gamma(\nu+t-a)}{\Gamma(\nu)(t-a)!}P_a + \nabla_a^{-\nu}f(t). \tag{82}$$

6.1 Uncertain discrete fractional relaxation equation

An uncertain discrete fractional relaxation equation with a rate of relaxation $\lambda \in \mathbb{R}$ can be described by:

$$\nabla_{\ominus_g}^{\nu}P(t) = \lambda P(t), \quad t \in N_{a+1},\ \nu \in (0,1), \tag{83}$$

with a fuzzy initial condition $\nabla_{\ominus_g}P(a) = P_a \in \mathbb{R}_{\text{CEU}}$. We use primary recursive methods to solve this equation by substituting $t \in N_{a+1}$. First, we put $t = a+1$, and we obtain

$$d(P(a+1)) = \frac{\nu}{(1-\lambda)}d(P(a)), \tag{84}$$

and the uncertain part accepts two solutions

$$u_i(P(a+1)) = \frac{\nu}{(1-|\lambda|)}u_i(P(a)) \tag{85}$$

and

$$u_i(P(a+1)) = \frac{\nu}{(1+|\lambda|)}u_i(P(a)). \tag{86}$$

If $|\lambda| < 1$, we can chose both solutions (85) and (86). Trivially, in this case, the maximal solution is (85). If $|\lambda| > 1$, we have only one option, Eq. (86). Unfortunately, for $\lambda = 1$, the problem is ill-posed and we have no solution.

Now, put $t = a+2$ for the deterministic part, we have

$$\begin{aligned} d(\nabla_a^{\nu}P(a+2)) &= d(P(a+2)) - \nu d(P(a+1)) \\ &\quad - \frac{\nu(1-\nu)}{2!}d(P(a)) \\ &= \lambda d(P(a+2)), \end{aligned} \tag{87}$$

and hence,

$$\begin{aligned} d(P(a+2)) &= \frac{1}{(1-\lambda)}\left(\nu d(P(a+1)) + \frac{\nu(1-\nu)}{2!}d(P(a))\right) \\ &= \left[\left(\frac{\nu}{(1-\lambda)}\right)^2 + \left(\frac{\nu(1-\nu)}{2!(1-\lambda)}\right)\right]d(P(a)), \end{aligned} \tag{88}$$

and for the uncertain part, we have

$$\begin{aligned} u_i(\nabla_a^{\nu}P(a+2)) &= |u_i(P(a+2)) - \nu u_i(P(a+1)) \\ &\quad - \frac{\nu(1-\nu)}{2!}u_i(P(a))| \\ &= |\lambda| u_i(P(a+2)). \end{aligned} \tag{89}$$

Again, we will obtain two solutions. The maximal solution depends on the behavior of λ. If $|\lambda| < 1$, the maximal solution is

$$u_i(P(a+2)) = \left[\left(\frac{\nu}{(1-|\lambda|)}\right)^2 + \left(\frac{\nu(1-\nu)}{2!(1-|\lambda|)}\right)\right] u_i(P(a)), \quad (90)$$

and if $|\lambda| > 1$, the maximal solution is

$$u_i(P(a+2)) = \left[\left(\frac{\nu}{(1+|\lambda|)}\right)^2 + \left(\frac{\nu(1-\nu)}{2!(1+|\lambda|)}\right)\right] u_i(P(a)). \quad (91)$$

Finally, it follows from (65)

$$d(P(t)) = \frac{\nu}{\Gamma(1-\nu)(1-\lambda)} \times \sum_{s=a}^{t-1} \frac{\Gamma(t-s-\nu)}{\Gamma(t-s+1)} d(P(s)), \quad (92)$$

or equivalently,

$$d(P(a+n)) = \frac{\nu}{\Gamma(1-\nu)(1-\lambda)} \sum_{i=0}^{n-1} \frac{\Gamma(n-i-\nu)}{\Gamma(n-i+1)} d(P(a+i)). \quad (93)$$

Supposing

$$d(P(a+i)) = \sum_{j=1}^{i} \left(\frac{1}{1-\lambda}\right)^j c_{i,j}(\nu) d(P_a),\ i \geq 1, \quad (94)$$

Eq. (93) leads to:

$$\begin{aligned} \sum_{j=1}^{n} \left(\frac{1}{1-\lambda}\right)^j c_{n,j}(\nu) &= \frac{\nu}{\Gamma(1-\nu)} \sum_{i=1}^{n-1} \frac{\Gamma(n-i-\nu)}{\Gamma(n-i+1)} \sum_{j=1}^{i} \left(\frac{1}{1-\lambda}\right)^{j+1} c_{i,j}(\nu) \\ &\quad + \frac{\nu}{\Gamma(1-\nu)(1-\lambda)} \frac{\Gamma(n-\nu)}{\Gamma(n+1)} \\ &= \sum_{j=1}^{n-1}\sum_{i=j}^{n-1} \frac{\nu}{\Gamma(1-\nu)} \frac{\Gamma(n-i-\nu)}{\Gamma(n-i+1)} \left(\frac{1}{1-\lambda}\right)^{j+1} c_{i,j}(\nu) \\ &\quad + \frac{\nu}{\Gamma(1-\nu)(1-\lambda)} \frac{\Gamma(n-\nu)}{\Gamma(n+1)}. \end{aligned} \quad (95)$$

Equating the coefficient of $\left(\frac{1}{1-\lambda}\right)^j$, we obtain

$$c_{n,1} = \frac{\nu}{\Gamma(1-\nu)} \frac{\Gamma(n-\nu)}{\Gamma(n+1)}$$

and

$$c_{n,j+1} = \sum_{k=j}^{n-1} \frac{\nu}{\Gamma(1-\nu)} \frac{\Gamma(n-k-\nu)}{\Gamma(n-k+1)} c_{k,j}(\nu), \quad (96)$$

for $n \in \mathbb{N}_{j+1}$. we can obtain $c_{n,j+1}$ recursively. For example, by substituting $j = 1$, we obtain

$$c_{n,2} = \sum_{i=1}^{n-1} \frac{\nu}{\Gamma(1-\nu)} \frac{\Gamma(n-i-\nu)}{\Gamma(n-i+1)} \frac{\nu}{\Gamma(1-\nu)} \frac{\Gamma(i-\nu)}{\Gamma(i+1)}.$$

Similarly, for uncertain part, if $|\lambda| < 1$, the maximal solution is

$$u_k(P(a+i)) = \sum_{j=1}^{i} \left(\frac{1}{1-|\lambda|}\right)^j c_{i,j}(\nu) u_k(P_a), \quad i \geq 1, \quad (97)$$

and if $|\lambda| > 1$, the maximal solution is

$$u_k(P(a+i)) = \sum_{j=1}^{i} \left(\frac{1}{1+|\lambda|}\right)^j c_{i,j}(\nu) u_k(P_a), \quad i \geq 1, \quad (98)$$

where $k = 1, 2$.

7 General linear case

In this section, we focus on Eq. (1), which is a general form of the linear uncertain discrete equation. Substituting $t = a + 1$ into (1), we obtain

$$d(P(a+1)) = \frac{1}{(1-\lambda)}(\nu d(P(a)) + d(f(a+1))), \quad (99)$$

and the uncertain part accepts two solutions

$$u_i(P(a+1)) = \frac{1}{(1\pm|\lambda|)}(\nu u_i(P(a)) + u_i(f(a+1))), \quad (100)$$

which the choice ± for maximal solution depends on λ. For other points, assume

$$d(P(a+i)) = \sum_{j=1}^{i} \left(\frac{1}{1-\lambda}\right)^j c_{i,j}(\nu)(d(P_a) + d(f(a+j))) \quad (101)$$

for $i \geq 1$. From (65), we know that

$$\nabla^{\nu}_{\ominus_g} d(P(a+n)) = d(P(a+n)) - \frac{\nu}{\Gamma(1-\nu)} \sum_{i=0}^{n-1} \times \frac{\Gamma(n-i-\nu)}{\Gamma(n-i+1)} d(P(a+i)), \quad (102)$$

and hence, by substituting (102) into Eq. (1), we obtain

$$d(P(a+n)) = \frac{1}{1-\lambda}\left(f(a+n) + \frac{\nu}{\Gamma(1-\nu)} \sum_{i=0}^{n-1} \times \frac{\Gamma(n-i-\nu)}{\Gamma(n-i+1)} d(P(a+i))\right). \quad (103)$$

Thus, from (101), we have

$$\begin{aligned}
&\sum_{j=1}^{n}\left(\frac{1}{1-\lambda}\right)^{j} c_{n,j}(\nu)(d(P_a)+f(a+j))\\
&= \frac{1}{1-\lambda}\Bigg(d(f(a+n)) + \frac{\nu}{\Gamma(1-\nu)}\frac{\Gamma(n-\nu)}{\Gamma(n+1)}d(P(a))\\
&\quad + \frac{\nu}{\Gamma(1-\nu)}\sum_{i=1}^{n-1}\frac{\Gamma(n-i-\nu)}{\Gamma(n-i+1)}\\
&\quad \times \sum_{j=1}^{i}\left(\frac{1}{1-\lambda}\right)^{j} c_{i,j}(\nu)(d(P_a)+f(a+j))\Bigg)\\
&= \frac{1}{1-\lambda}\Bigg(d(f(a+n)) + \frac{\nu}{\Gamma(1-\nu)}\frac{\Gamma(n-\nu)}{\Gamma(n+1)}d(P(a))\\
&\quad + \frac{\nu}{\Gamma(1-\nu)}\sum_{j=1}^{n-1}\sum_{i=j}^{n-1}\frac{\Gamma(n-i-\nu)}{\Gamma(n-i+1)}\\
&\quad \times \left(\frac{1}{1-\lambda}\right)^{j} c_{i,j}(\nu)(d(P_a)+f(a+j))\Bigg).
\end{aligned} \tag{104}$$

Thus,

$$c_{n,1}(\nu) = \frac{\frac{\nu}{\Gamma(1-\nu)}\frac{\Gamma(n-\nu)}{\Gamma(n+1)}d(P_a) + f(a+n)}{d(P_a)+f(a+1)}$$

and

$$c_{n,j+1}(\nu) = \frac{\nu}{\Gamma(1-\nu)}\sum_{i=j}^{n-1}\frac{\Gamma(n-i-\nu)}{\Gamma(n-i+1)}c_{i,j}(\nu), \tag{105}$$

for $n \in \mathbb{N}_{j+1}$. Interestingly, Formulas (105) and (96) are the same. Conclusively, we obtain the following result.

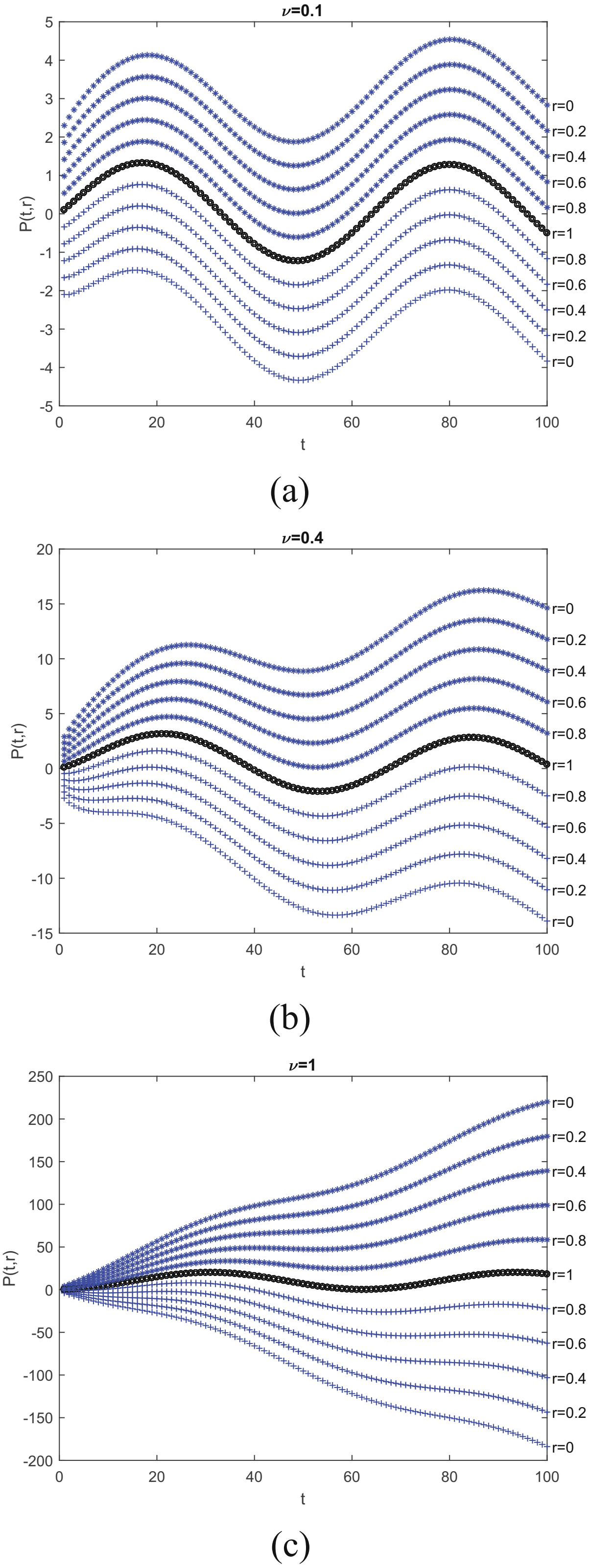

Figure 5: Solution of Example 8 for (a) $\nu = 0.1$, (b) $\nu = 0.4$, and (c) $\nu = 1$ and various r on [1, 100]. The markers o, $*$, and $+$ are used for the deterministic part, left part, and right part of uncertain numbers.

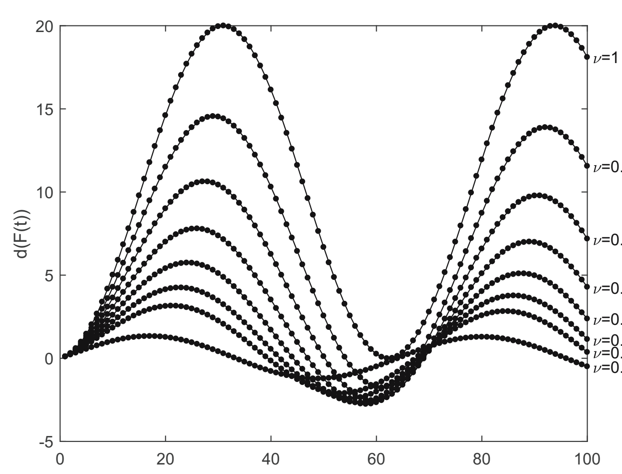

Figure 6: Deterministic part of solution of Example 8 for various ν on [1, 100].

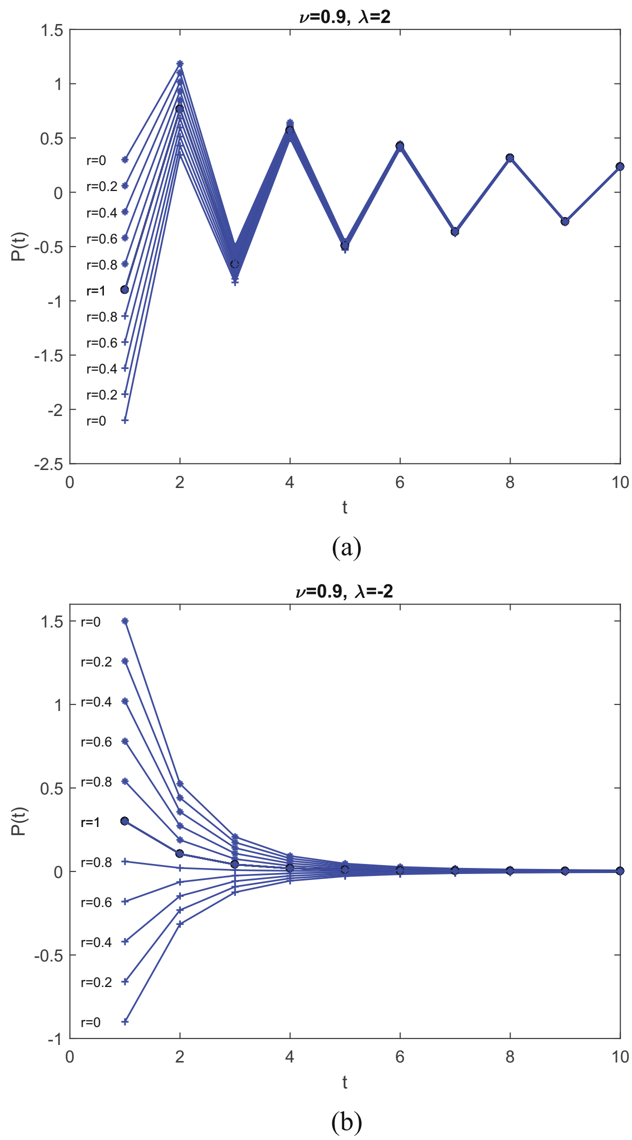

Figure 7: Solution of Example 8.3 for $\nu = 0.9$: (a) $\lambda = 2$ and (b) $\lambda = 2$ for various t on [1, 10].

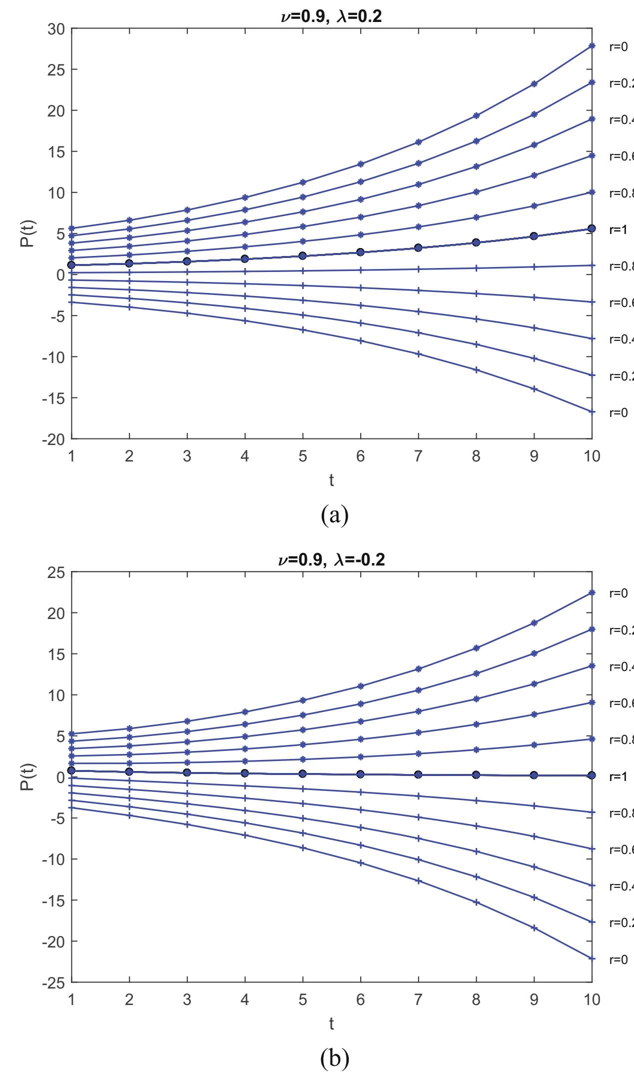

Figure 8: Solution of Example 8.3 for $\nu = 0.9$: (a) $\lambda = 0.2$ and (b) $\lambda = 0.2$ for various t on [1, 10].

Theorem 7.1. *Suppose* $\lambda \in \mathbb{R}$ *and* $f : \mathbb{N}_a \to \mathbb{R}_{\mathrm{CEU}}$. *Then, discrete Eq.* (1) *has a unique maximal solution described by* (101) *for the deterministic part*:

$$d(P(a+i)) = \begin{cases} \sum_{j=1}^{i} \left(\frac{1}{1-|\lambda|}\right)^{j} c_{i,j}(\nu)(d(P_a) & |\lambda| < 1, \\ \quad + d(f(a+j))), & \\ \sum_{j=1}^{i} \left(\frac{1}{1+|\lambda|}\right)^{j} c_{i,j}(\nu)(d(P_a) & |\lambda| > 1. \\ \quad + d(f(a+j))), & \end{cases} \tag{106}$$

8 Examples and discussions

First, we give a comparative example of anti-derivative.

Example 8.1. We consider an uncertain force function with time-independent uncertainty described by:

$$f(t) = [\sin(0.1t) - u_1(r), \sin(0.1t) + u_2(r)],$$

where

$$u_1(r) = u_2(r) = 2 - 2r.$$

The initial condition at $a = 0$ is given by $P_a = [-u_1(r), u_2(r)]$.

We obtain the discrete anti-derivative of F by solving Eq. (27). Of course, the solution by (82) is

$$\begin{aligned} P(t) &= P_a + \int_0^t f(s)\nabla(s) \\ &= [-u_1(r), u_2(r)] + \sum_{s=1}^{t} [\sin(0.1s) - u_1(r), \sin(0.1s) \\ &\quad + u_2(r)], \ \ t \in \mathbb{N}_1. \end{aligned} \tag{107}$$

For fractional order discrete Eq. (68), we will have

$$\begin{aligned} P(t) = c_{t-a}(\nu)[-u_1(r), u_2(r)] + \sum_{i=0}^{t-a-1} c_i(\nu) \\ \times [\sin(0.1(t-i)) - u_1(r), \sin(0.1(t-i)) \\ + u_2(r)]. \end{aligned} \tag{108}$$

Remark 8.2. We note that

$$\lim_{\nu\to 1} c_i(\nu) = c_i(1) = 1, \quad i \in \mathbb{N}_0.$$

Thus, the solution is non-local and when $\nu \to 1$ the solution will approach the integer order case. But,

$$\lim_{\nu\to 0} c_i(\nu) = c_i(0) = 0, \quad i \in \mathbb{N}_1.$$

Therefore, the solution will tend to $f(t)$. This is a local solution. The fadedness increases when $\nu \to 0$. The memory of the solution increases when $\nu \to 1$. This discussion for the discrete fractional difference is not the same (see Remark 5.3). For fractional differences, cases $\nu = 0$ and $\nu = 1$ have local discrete differences.

In Figure 5(a)–(c) we demonstrated the deterministic part and the boundaries of the solutions (right and left boundaries) for various r and ν. These figures illustrate that by increasing ν uncertainty increases. It is trivial since by increasing ν the effects of the memory coefficients increase and thus it accumulates larger uncertainties from previous terms. A similar argument can show why by increasing ν the deterministic part show a higher amplitude of oscillations. Figure 6 shows this phenomenon.

Table 1: Coefficient of the (94) versus fractional order

ν	$c_{6,1}(\nu)$	$c_{6,2}(\nu)$	$c_{6,3}(\nu)$	$c_{6,4}(\nu)$	$c_{6,5}(\nu)$	$c_{6,6}(\nu)$
0.99000	0.0003	0.0010	0.0025	0.0066	0.0238	0.9415
0.99900	0.0000	0.0001	0.0003	0.0007	0.0025	0.9940
0.99990	0.0000	0.0000	0.0000	0.0001	0.0002	0.9994
0.99999	0.0000	0.0000	0.0000	0.0000	0.0000	0.9999

Example 8.3. The final example is devoted to the relaxation equation. For this example, we consider

$$u_1(r) = u_2(r) = 4 - 4r,$$
$$a = 0,$$

and $P_a = [-u_1(r), u_2(r)]$. We compare the result with various values of ν and λ.

We know that the solution for $|\lambda| > 1$ and $|\lambda| < 1$ is different. For $|\lambda| > 1$, coefficients of uncertain terms effected by $(1 + |\lambda|)^{-j}$, which tends to zero geometrically as $j \to \infty$. Thus, the uncertainty is reduced. Interestingly, for stable systems, we have deterministic results, even with uncertain parameters. See also the discussion on stability in the study by Alijani *et al.* [35] for the continuous case. In Figure 7(a) and (b), we depicted the numerical results for $\lambda = 2$ and $\lambda = -2$, respectively. Fading the uncertainty as expected is evident. For $|\lambda| < 1$, the coefficients $(1 - |\lambda|)^{-j}$ increase geometrically. A geometric rise in both uncertainty and determinism part can be observed. Figure 8(a) and (b) demonstrates such growth for both $\lambda = 0.2$ and $\lambda = -0.2$.

Remark 8.4. The solution of uncertain discrete relaxation equation is given by Eqs (53)–(55), while the solution of fractional order uncertain discrete relaxation equation is given by more complex Eqs (94), (97), and (98). It is interesting to ask if $\nu \to 1$, does such a complex solution of fractional order tend to integer order one. To answer this question, we first investigate the behavior of the coefficient $c_{i,j}(\nu)$. In Table 1, we reported the numeric computation of $c_{n,j}$ for $n = 6$, for various ν approaching to 1. The result shows the tendency of $c_{n,j}$ toward $\delta_{n,j}$, as ν approach to 1, and hence, Eq. (94) will tend to (53). Such a result directly can be implied by the recursive Formula (96) at the limit.

9 Conclusion

Unlike the fuzzy set, the FN does not have a unique definition and is defined by adding properties such as continuity and uniformity to the membership function. We review the available definitions concerning the advantages and drawbacks of adding, removing, or refining such properties. In conclusion, we keep

1) Convexity: the r-cut become a unique interval.
2) Existence of the deterministic part: the advantage is that r-cuts are nonempty sets.
3) Compactness: imposes the boundaries of r-cuts that become finite.

4) Strict monotonicity and continuity with accepting the exception of deterministic numbers: this leads to a one-to-one relation between parametric representations and membership representations.

Based on this discussion, we modified the concrete definition of the fuzzy/uncertain number [25] that separates the uncertain part and has a unique representation. We used the recent advantages in the definition of uncertain difference operators with new scalar multiplication.

There exist two nabla fractional differences in the literature, and we selected the one that behaves locally similar to integer order in limit. In this respect, we think it is necessary to have a principal base study for the definition of fractional difference in a separate study similar to the study by Shiri and Baleanu [42].

The main aim of this study was to obtain the calculus of uncertain discrete operators. This part covered fundamental theorems that relate uncertain anti-difference operators to uncertain sum operators. We have done it for integer and fractional order in separate sections. The main tool to obtain such beautiful formulas was the concept of the maximal solution. The next part of our calculus covered linear uncertain nabla difference equations for both integer and fractional order. In this respect, the explicit solutions of uncertain discrete relaxation equations, which are an important subclass of such equations, are obtained, separately.

The relaxation equations are used in diverse physical phenomena, particularly the phenomenon with superposition laws. For example, the models of stress-strain for material and radioactive decay use such equations. Depending on the materials the fractional or integer order relaxation equation may be applied. This paper covers both cases and the result of uncertainty analysis will be useful for considering measurement errors in such modeling and their consequence results.

We expect such analysis for more complex equations such as fractional or integer order discrete Hopfield neural networks described in the study by Huang *et al.* [43], possibly extending such investigation for continuous Hopfield neural networks [44]. Indeed, investigating the effect of uncertain data in the learning process and the entropy of an artificial brain can be very important. Partially, it may have some analysis of security issues of learning by knowing the effect of uncertain data on the result.

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References

[1] Radvanyi P, Villain J. The discovery of radioactivity. C R Phys. 2017;18(9–10):544–50.
[2] Cruz-Lopez CA, Espinosa-Paredes G. Fractional radioactive decay law and Bateman equations. Nucl Eng Technol. 2022;54(1):275–82.
[3] Nigmatullin R, Baleanu D, Fernandez A. Balance equations with generalised memory and the emerging fractional kernels. Nonlinear Dyn. 2021;104:4149–61.
[4] Kochubei AN. General fractional calculus, evolution equations, and renewal processes. Integral Equ Oper Theory. 2011;71(4):583–600.
[5] Luchko Y, Yamamoto M. General time-fractional diffusion equation: some unique and existence results for the initial-boundary-value problems. Fract Calc Appl Anal. 2016;19(3):676–95.
[6] Goodrich C, Peterson AC. Discrete fractional calculus. Cham: Springer; 2015.
[7] Kelley W, Peterson A. Difference equations: an introduction with applications. 2nd edition. Harcourt: Academic Press; 2001.
[8] Wang Q, Xu R. A review of definitions of fractional differences and sums. Found Comput Math. 2023;6(2):136–60.
[9] Hein J, McCarthy Z, Gaswick N, McKain B, Speer K. Laplace transforms for the Nabla-difference operator. Panam Math J. 2011;21(3):79–96.
[10] Atici FM, Eloe P. Discrete fractional calculus with the Nabla operator. Electron J Qual Theory Differ Equ. 2009;2009:1–12.
[11] Anastassiou GA. Nabla discrete fractional calculus and nabla inequalities. Math Comput Model. 2010;51(5-6):562–71.
[12] Abdeljawad T, Baleanu D. Monotonicity analysis of a nabla discrete fractional operator with discrete Mittag-Leffler kernel. Chaos Solit Fractals. 2017;102:106–10.
[13] Wei Y, Gao Q, Liu DY, Wang Y. On the series representation of nabla discrete fractional calculus. Commun Nonlinear Sci Numer Simul. 2019;69:198–218.
[14] Wu GC, Abdeljawad T, Liu J, Baleanu D, Wu KT. Mittag-Leffler stability analysis of fractional discrete-time neural networks via fixed point technique. Nonlinear Anal Model Control. 2019;24(6):919–36.
[15] Gu Y, Wang H, Yu Y. Synchronization for fractional-order discrete-time neural networks with time delays. Appl Math Comput. 2020;372:124995.
[16] Abdeljawad T, Banerjee S, Wu GC. Discrete tempered fractional calculus for new chaotic systems with short memory and image encryption. Optik. 2020;218:163698.
[17] Wang ZR, Shiri B, Baleanu D. Discrete fractional watermark technique. Front Inform Technol Electron Eng. 2020;21(6):880–3.

[18] Khan A, Alshehri HM, Abdeljawad T, Al-Mdallal QM, Khan H. Stability analysis of fractional nabla difference COVID-19 model. Results Phys. 2021;22:103888.
[19] Wu GC, Song TT, Wang S. Caputo-Hadamard fractional differential equations on time scales: numerical scheme, asymptotic stability, and chaos. Chaos. 2022;32(9):093143.
[20] Wei Y, Chen Y, Liu T, Wang Y. Lyapunov functions for nabla discrete fractional order systems. ISA Trans. 2019;88:82–90.
[21] Wei Y, Wei Y, Chen Y, Wang Y. Mittag-Leffler stability of nabla discrete fractional-order dynamic systems. Nonlinear Dyn. 2020;101:407–17.
[22] Zadeh LA. Fuzzy sets. Inf Control. 1965;8(3):338–53.
[23] Lodwick WA. Interval and fuzzy analysis: A unified approach. Adv Imaging Electron Phys. 2007;148:75–192.
[24] Stefanini L. A generalization of Hukuhara difference and division for interval and fuzzy arithmetic. Fuzzy Sets and Systems. 2010;161(11):1564–84.
[25] Shiri B. A unified generalization for Hukuhara types differences and derivatives: solid analysis and comparisons. AIMS Math. 2023;8:2168–90.
[26] Dubois D, Prade H. Operations on fuzzy numbers. Int J Syst Sci. 1978;9(6):613–26.
[27] Dijkman JG, van Haeringen H, de Lange SJ. Fuzzy numbers. J Math Anal Appl. 1983;92(2):301–41.
[28] Goetschel JrR, Voxman W. Topological properties of fuzzy numbers. Fuzzy Sets Syst. 1983;10(1–3):87–99.
[29] Brunelli M, Mezei J. An inquiry into approximate operations on fuzzy numbers. Int J Approx Reason. 2017;81:147–59.
[30] Garg H, Ansha. Arithmetic operations on generalized parabolic fuzzy numbers and its application. Proc National Acad Sci India Sec A phys Sci. 2018;88(1):15–26.
[31] Atici F, Eloe P. Initial value problems in discrete fractional calculus. Proc Am Math Soc. 2009;137(3):981–9.
[32] Baleanu D, Wu GC, Bai YR, Chen FL. Stability analysis of Caputo-like discrete fractional systems. Commun Nonlinear Sci Numer Simul. 2017;48:520–30.
[33] Alijani Z, Kangro U. Collocation method for fuzzy Volterra integral equations of the second kind. Math Model Anal. 2020;25(1):146–66.
[34] Shiri B, Alijani Z, Karaca Y. A power series method for the fuzzy fractional logistic differential equation. Fractals. 2023;31(10):2340086.
[35] Alijani Z, Shiri B, Perfilieva I, Baleanu D. Numerical solution of a new mathematical model for intravenous drug administration. Evol Intell. 2023;2023:1–7.
[36] Luo C, Wu GC, Huang LL. Fractional uncertain differential equations with general memory effects: Existences and alpha-path solutions. Nonlinear Anal Model Control. 2023;28:11–28.
[37] Mainardi F. Fractional relaxation-oscillation and fractional diffusion-wave phenomena. Chaos Solit Fractals 1996;7(9):1461–77.
[38] Huang LL, Wu GC, Baleanu D, Wang HY. Discrete fractional calculus for interval-valued systems. Fuzzy Sets Syst. 2021;404:141–58.
[39] Stefanini L, Sorini L, Guerra ML. Parametric representation of fuzzy numbers and application to fuzzy calculus. Fuzzy Sets Syst. 2006;157(18):2423–55.
[40] Hukuhara M. Integration des applications measurables dont la valeur est un compact convexe. Funkcialaj Ekvacioj. 1967;10(3):205–23.
[41] Abdeljawad T, Atici FM. On the definitions of nabla fractional operators. Abstr. Appl. Anal. 2012;2012:1–13.
[42] Shiri B, Baleanu D. All linear fractional derivatives with power functions' convolution kernel and interpolation properties. Chaos Solit Fractals 2023;170:113399.
[43] Huang LL, Park JH, Wu GC, Mo ZW. Variable-order fractional discrete-time recurrent neural networks. J. Comput. Appl. Math. 2020;370:112633.
[44] Kaslik E, Sivasundaram S. Nonlinear dynamics and chaos in fractional-order neural networks. Neural Netw. 2012;32:245–56.