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# Inference in multivariate linear regression models with elliptically distributed errors

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In this study we investigate the problem of estimation and testing of hypotheses in multivariate linear regression models when the errors involved are assumed to be non-normally distributed. We consider the class of heavy-tailed distributions for this purpose. Although our method is applicable for any distribution in this class, we take the multivariate $t$-distribution for illustration. This distribution has applications in many fields of applied research such as Economics, Business, and Finance. For estimation purpose, we use the modified maximum likelihood method in order to get the so-called modified maximum likelihood estimates that are obtained in a closed form. We show that these estimates are substantially more efficient than least-square estimates. They are also found to be robust to reasonable deviations from the assumed distribution and also many data anomalies such as the presence of outliers in the sample, etc. We further provide test statistics for testing the relevant hypothesis regarding the regression coefficients.

**Keywords:** least-squares estimates; maximum likelihood estimates; modified maximum likelihood estimates; multivariate distributions; multivariate $t$-distribution; robust estimates

*AMS Subject Classification*: 62J05; 62F35; 62H12

## 1. Introduction

A simple linear regression model is considered that relates a multivariate-dependent variable ($y$) with an independent variable ($x$) through the following model:

$$y_{ji} = \gamma_j + \delta_j x_i + e_{ji}, \quad 1 \leq i \leq n, \quad 1 \leq j \leq q. \tag{1}$$

The error terms $e_{ji}(E(e_{ji}) = 0)$ are having the following variance and covariance:

$$V(e_{ji}) = \sigma_j^2, \quad \mathrm{Cov}(e_{ji}, e_{j^*i^*}) = \begin{cases} \sigma_{jj^*}, & i = i^*, \ j \neq j^*, \\ 0 & \text{otherwise.} \end{cases}$$

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Let a distinct pair $(e_j, e_k)(2 \le k \le q,\ 1 \le j \le k-1)$ of random errors $e_j$ and $e_k$ have a bivariate distribution involving scale parameters $\sigma_j$ and $\sigma_k$ and covariances $\sigma_{kj} = \rho_{kj}\sigma_k\sigma_j(\sigma_{jk} = \sigma_{kj})$, where $\rho_{kj}(= \rho_{jk})$ are the Pearson correlation coefficients. Let

$$z_{ji} = \frac{e_{ji}}{\sigma_j} = \frac{y_{ji} - \gamma_j - \delta_j x_i}{\sigma_j} \tag{2}$$

and

$$z_{ki} = \sqrt{\nu_j^*}\sqrt{w_{ji}}\frac{e_{ki} - \theta_{kj}e_{ji}}{\sigma_{k\cdot j}} = \sqrt{\nu_j^*}\sqrt{w_{ji}}\frac{(y_{ki} - \gamma_k - \delta_k x_i) - \theta_{kj}\sigma_j z_{ji}}{\sigma_{k\cdot j}}, \tag{3}$$

where

$$\nu_j^* = \frac{\nu_j + 1}{\nu_j}, \quad w_{ji} = \frac{1}{1 + z_{ji}^2/\nu_j}, \quad \theta_{kj} = \frac{\sigma_k}{\sigma_j}\rho_{kj}, \quad \sigma_{k\cdot j} = \sigma_k\sqrt{1 - \rho_{kj}^2},$$

be independently distributed as Student's $t$ with $\nu_j$ and $\nu_k$ degrees of freedom (DF), respectively.

Although, for estimation purpose, the method of maximum likelihood is applicable here, the maximum likelihood estimates (MLE) are elusive because the likelihood equations are nonlinear and have no explicit solution. Solving them by iteration is problematic [7] in various respects. For example, if the sample contains outliers, the maximum likelihood equations have convergence problems; they may not converge or may converge to a wrong value [6]. Therefore, we employ the method of modified maximum likelihood estimation [9,10,12,14,15,17,18] in which the so-called modified likelihood estimates (MMLE) are derived by expressing the intractable nonlinear likelihood equations in terms of standardized ordered variates and then the intractable terms are replaced by linear approximations in order to provide the estimates. Interestingly these estimates are obtained in a closed form and, hence, can be manipulated analytically. In fact, they are asymptotically equivalent to MLEs [1,19]. Furthermore, in small samples the MMLE are found to be approximately the same as MLE that are obtained iteratively [2,3,5,8,9,15–17].

## 2. Estimation

### 2.1 *Modified MLEs*

Consider the random sample $(e_{ji}, e_{ki})(1 \le i \le n)$ taken from the bivariate distribution of the random variables as given above. The log likelihood function for the sample can then be expressed as

$$\ln L = -n\ \ln(\sigma_j) - \nu_j^* \sum_{i=1}^{n} \ln\left(\frac{1 + z_{ji}^2}{\nu_j}\right) - n\ \ln(\sigma_{k\cdot j}) - \nu_k^* \sum_{i=1}^{n} \ln\left(\frac{1 + z_{ki}^2}{\nu_k}\right), \tag{4}$$

where $\nu_k^* = (\nu_k + 1)/\nu_k$.

The likelihood equations are equations expressed in terms of nonlinear functions $g_j(z_{ji}) = z_{ji}/(1 + z_{ji}^2/\nu_j)$ and $g_k(z_{ki}) = z_{ki}/(1 + z_{ki}^2/\nu_k)(1 \le i \le n)$ and solving them by iteration is fraught with difficulties as mentioned earlier. Instead, we first modify the likelihood equations by replacing $z_{ji}$ and $z_{ki}$ with corresponding ordered variates $z_{j(i)}$ and $z_{k(i)}$. The functions $g_j$ and $g_k$ are then replaced by their linear approximations $g_j(z_{j(i)}) \cong \alpha_{ji} + \beta_{ji}z_{j(i)}$ and $g_k(z_{k(i)}) \cong \alpha_{ki} + \beta_{ki}z_{k(i)}$ obtained from the first two terms of a Taylor series expansion about the $i$th quantiles $t_{ji}$ and $t_{ki}$ of

Student's $t$ distributions with $\nu_j$ and $\nu_k$ DFs, respectively [10,11,13,15]. So,

$$\alpha_{ji} = \frac{(2t_{ji}^3/\nu_j)}{(1+t_{ji}^2/\nu_j)^2}, \quad \beta_{ji} = \frac{(1-t_{ji}^2/\nu_j)}{(1+t_{ji}^2/\nu_j)^2} \tag{5}$$

and

$$\alpha_{ki} = \frac{(2t_{ki}^3/\nu_k)}{(1+t_{ki}^2/\nu_k)^2}, \quad \beta_{ki} = \frac{(1-t_{ki}^2/\nu_k)}{(1+t_{ki}^2/\nu_k)^2}. \tag{6}$$

For a given $\delta_j$, let $\varepsilon_{ji} = y_{ji} - \delta_j x_i$ and let $\varepsilon_{j(i)}$ be the ordered (ascending) $\varepsilon_{ji}$. Denote the observation $(x_i, y_{ji})$ associated with $\varepsilon_{j(i)}$ as $(x_{[i]}, y_{j[i]})$ and call it concomitant of $\varepsilon_{j(i)}$. The linearized likelihood equations are then solvable analytically providing closed-form estimates (MMLE) of the parameters $\gamma_j$, $\delta_j$, and $\sigma_j (1 \le j \le q)$ as follows [13]:

$$\hat{\gamma}_j = \bar{y}_{j[\cdot]} - \hat{\delta}_j \bar{x}_{j[\cdot]}, \tag{7}$$

$$\hat{\delta}_j = K_j + D_j \hat{\sigma}_j, \tag{8}$$

$$\hat{\sigma}_j = \frac{(B_j + \sqrt{B_j^2 + 4nC_j})}{2n}, \tag{9}$$

where

$$\bar{x}_{j[\cdot]} = \frac{\sum_{i=1}^n \beta_{ji} x_{[i]}}{m_j}, \quad \bar{y}_{j[\cdot]} = \frac{\sum_{i=1}^n \beta_{ji} y_{j[i]}}{m_j}, \quad m_j = \sum_{i=1}^n \beta_{ji},$$

$$K_j = \frac{\sum_{i=1}^n \beta_{ji}(x_{[i]} - \bar{x}_{j[\cdot]}) y_{j[i]}}{\sum_{i=1}^n \beta_{ji}(x_{[i]} - \bar{x}_{j[\cdot]})^2}, \quad D_j = \frac{\sum_{i=1}^n \alpha_{ji}(x_{[i]} - \bar{x}_{j[\cdot]})}{\sum_{i=1}^n \beta_{ji}(x_{[i]} - \bar{x}_{j[\cdot]})^2},$$

$$B_j = \nu_j^* \sum_{i=1}^n \alpha_{ji}\{(y_{j[i]} - \bar{y}_{j[\cdot]}) - K_j(x_{[i]} - \bar{x}_{j[\cdot]})\},$$

$$C_j = \nu_j^* \sum_{i=1}^n \beta_{ji}\{(y_{j[i]} - \bar{y}_{j[\cdot]}) - K_j(x_{[i]} - \bar{x}_{j[\cdot]})\}^2.$$

Now consider the concomitant $(w_{j[i]}, x_{[i]}, y_{k[i]}, z_{j[i]})$ of $\varepsilon_{k(i)}$ as the observation $(w_{ji}, x_i, y_{ji})$, $w_{ji} = 1/(1 + z_{ji}^2/\nu_j))$, $z_{ji} = (y_{ji} - \gamma_j - \delta_j x_i)/\sigma_j$, associated with ordered (ascending) $\varepsilon_{ki} = \sqrt{w_{ji}}(y_{ki} - \gamma_k - \delta_k x_i) - \theta_{kj}\sigma_j z_{ji}$. The solution of the linearized likelihood equations will then provide the MML estimate of the correlation coefficients $\rho_{kj} (2 \le k \le q,\ 1 \le j \le k-1)$ and is given as

$$\hat{\rho}_{kj} = \frac{\hat{\theta}_{kj}\hat{\sigma}_j}{\hat{\sigma}_k}, \tag{10}$$

$$\text{where} \quad \hat{\theta}_{kj} = \frac{(K + D\hat{\sigma}_{k\cdot j}/\sqrt{\nu_j^*})}{\hat{\sigma}_j}, \tag{11}$$

$$\hat{\sigma}_{k\cdot j} = \frac{B_k + \sqrt{B_k^2 + 4nC_k}}{2n}, \tag{12}$$

$$\bar{x}_{k[\cdot]} = \frac{\sum_{i=1}^n \beta_{ki}^* x_{[i]}}{m_k}, \quad \bar{y}_{k[\cdot]} = \frac{\sum_{i=1}^n \beta_{ki}^* y_{k[i]}}{m_k}, \quad \bar{z}_{jk[\cdot]} = \frac{\sum_{i=1}^n \beta_{ki}^* z_{j[i]}}{m_k},$$

$$m_k = \sum_{i=1}^n \beta_{ki}^*, \quad \beta_{ki}^* = w_{j[i]}\beta_{ki},$$

$$\alpha_{ki}^* = \sqrt{w_{j[i]}}\alpha_{ki}, \quad K = \frac{\sum_{i=1}^n \beta_{ki}^* z_{j[i]}(y_{k[i]} - \bar{y}_{k[\cdot]}) - K_k \sum_{i=1}^n \beta_{ki}^* z_{j[i]}(x_{[i]} - \bar{x}_{k[\cdot]})}{\sum_{i=1}^n \beta_{ki}^* z_{j[i]}(z_{j[i]} - \bar{z}_{jk[\cdot]}) - R_k \sum_{i=1}^n \beta_{ki}^* z_{j[i]}(x_{[i]} - \bar{x}_{k[\cdot]})},$$

$$K_k = \frac{\sum_{i=1}^n \beta_{ki}^*(x_{[i]} - \bar{x}_{k[\cdot]})y_{k[i]}}{\sum_{i=1}^n \beta_{ki}^*(x_{[i]} - \bar{x}_{k[\cdot]})^2}, \quad R_k = \frac{\sum_{i=1}^n \beta_{ki}^*(x_{[i]} - \bar{x}_{k[\cdot]})(z_{j[i]} - \bar{z}_{jk[\cdot]})}{\sum_{i=1}^n \beta_{ki}^*(x_{[i]} - \bar{x}_{k[\cdot]})^2},$$

$$D = \frac{\sum_{i=1}^n \alpha_{ki}^*(z_{j[i]} - \bar{z}_{jk[\cdot]}) - D_k \sum_{i=1}^n \beta_{ki}^* z_{j[i]}(x_{[i]} - \bar{x}_{k[\cdot]})}{\sum_{i=1}^n \beta_{ki}^* z_{j[i]}(z_{j[i]} - \bar{z}_{jk[\cdot]}) - R_k \sum_{i=1}^n \beta_{ki}^* z_{j[i]}(x_{[i]} - \bar{x}_{k[\cdot]})}, \quad D_k = \frac{\sum_{i=1}^n \alpha_{ki}^*(x_{[i]} - \bar{x}_{k[\cdot]})}{\sum_{i=1}^n \beta_{ki}^*(x_{[i]} - \bar{x}_{k[\cdot]})^2},$$

$$B_k = \nu_k^* \sqrt{\nu_j^*} \sum_{i=1}^n \alpha_{ki}^*[(y_{k[i]} - \bar{y}_{k[\cdot]}) - K_k(x_{[i]} - \bar{x}_{k[\cdot]}) - K\{(z_{j[i]} - \bar{z}_{jk[\cdot]}) - R_k(x_{[i]} - \bar{x}_{k[\cdot]})\}],$$

$$C_k = \nu_k^* \nu_j^* \sum_{i=1}^n \beta_{ki}^*[(y_{k[i]} - \bar{y}_{k[\cdot]}) - K_k(x_{[i]} - \bar{x}_{k[\cdot]}) - K\{(z_{j[i]} - \bar{z}_{jk[\cdot]}) - R_k(x_{[i]} - \bar{x}_{k[\cdot]})\}]^2.$$

*Remark* If the coefficients $\beta_{ji}$ and $\beta_{ki}$ are positive for $i = 1$, then both of them are positive for all other values of $i$ and the estimates $\hat{\sigma}_j$, $\hat{\sigma}_{k\cdot j}$, and $\hat{\sigma}_k$ are all real and positive. However, for small $\nu_j$ and/or $\nu_k$ and large $n$, perhaps the first and few other such coefficients in order may have negative values and as a consequence $\hat{\sigma}_j$, $\hat{\sigma}_{k\cdot j}$ can cease to be real. In such situations we propose to modify the coefficients as follows in order to have $\hat{\sigma}_j$, $\hat{\sigma}_{k\cdot j}$, and $\hat{\sigma}_k$ always real and positive [17]

$$\alpha_{ji} = \frac{t_{ji}^3/\nu_j}{(1 + t_{ji}^2/\nu_j)^2}, \quad \beta_{ji} = \frac{1}{(1 + t_{ji}^2/\nu_j)^2}, \quad \alpha_{ki} = \frac{t_{ki}^3/\nu_k}{(1 + t_{ki}^2/\nu_k)^2}, \quad \beta_{ki} = \frac{1}{(1 + t_{ki}^2/\nu_k)^2}.$$

This does not alter the asymptotic properties of the MMLE since all of the coefficients and the functions $g_j$ and $g_k$ are bounded.

*Computation.* In order to compute the MMLE, we require the concomitants corresponding to $\varepsilon_{j(i)}$ and $\varepsilon_{k(i)}(1 \le i \le n)$, respectively. For $\varepsilon_{j(i)}$, first arrange the residuals $\tilde{\varepsilon}_{ji} = y_{ji} - \tilde{\delta}_j x_i$ ($\tilde{\delta}_j$ is the least-square estimate (LSE) given in Equation (14)) in ascending order and denote as $\tilde{\varepsilon}_{j(i)}$. Then pick up the observations corresponding to $\tilde{\varepsilon}_{j(i)}$ as concomitants $(x_{[i]}, y_{j[i]})$. The MMLE $\hat{\gamma}_j$, $\hat{\delta}_j$, and $\hat{\sigma}_j$ in Equations (7)–(9) are then computed to generate new residuals $\hat{\varepsilon}_{ji} = y_{ji} - \hat{\delta}_j x_i$ and a new set of concomitants are thus obtained to be used to compute the final MMLE. The estimates are stabilized well in just these two iterations. Similarly, for $\varepsilon_{k(i)}$ first order (ascending) the residuals $\tilde{\varepsilon}_{ki} = \sqrt{\tilde{w}_{ji}}(y_{ki} - \tilde{\gamma}_k - \tilde{\delta}_k x_i) - \tilde{\theta}_{kj}\tilde{\sigma}_j\tilde{z}_{ji}(\tilde{w}_{ji} = 1/(1 + \tilde{z}_{ji}^2/\nu_j))$, $\tilde{z}_{ji} = (y_{ji} - \tilde{\gamma}_j - \tilde{\delta}_j x_i)/\tilde{\sigma}_j$; $\tilde{\gamma}_j, \tilde{\delta}_j, \tilde{\sigma}_j, \tilde{\delta}_k, \tilde{\theta}_{kj}$ are LSE given in Equations (13)–(17) and denote by $\tilde{\varepsilon}_{k(i)}$. Then take the observations corresponding to $\tilde{\varepsilon}_{k(i)}$ as concomitants $(w_{j[i]}, x_{[i]}, y_{k[i]}, z_{j[i]})$. The MMLE (7)–(12) are then computed using these concomitants and new residuals $\hat{\varepsilon}_{k(i)} = \sqrt{\hat{w}_{ji}}(y_{ki} - \hat{\gamma}_k - \hat{\delta}_k x_i) - \hat{\theta}_{kj}\hat{\sigma}_j\hat{z}_{ji}$ are obtained. This process is repeated one more time in order to get the final MMLE using Equations (10)–(12).

### 2.2 *Least-square estimates*

The LSEs are obtained by minimizing
$\sum_{i=1}^n (y_{ji} - \gamma_j - \delta_j x_i)^2$ and $\sum_{i=1}^n w_{ji}\{(y_{ki} - \gamma_k - \delta_k x_i) - \theta_{kj}\sigma_j z_{ji}\}^2$, $2 \le k \le q, 1 \le j \le k-1$, and making scale adjustments. They are ($\nu_j > 2$)

$$\tilde{\gamma}_j = \bar{y}_j - \tilde{\delta}_j \bar{x}, \tag{13}$$

$$\tilde{\delta}_j = \frac{\sum_{i=1}^n (x_i - \bar{x})y_{ji}}{\sum_{i=1}^n (x_i - \bar{x})^2}, \tag{14}$$

$$\tilde{\sigma}_j = \sqrt{\frac{\nu_j - 2}{\nu_j}} \sqrt{\frac{\sum_{i=1}^{n} \{(y_{ji} - \bar{y}_j) - \tilde{\delta}_j (x_i - \bar{x})\}^2}{n-2}}, \tag{15}$$

$$\tilde{\rho}_{kj} = \frac{\tilde{\theta}_{kj} \tilde{\sigma}_j}{\tilde{\sigma}_k}, \tag{16}$$

$$\tilde{\theta}_{kj} = \frac{K}{\tilde{\sigma}_j}, \tag{17}$$

$$\tilde{\sigma}_{k \cdot j} = \sqrt{\frac{\nu_j - 2}{\nu_j}} \sqrt{\frac{\nu_k - 2}{\nu_k}} \sqrt{\frac{\sum_{i=1}^{n} \tilde{w}_{ji} [(y_{ki} - \bar{y}_k) - K_k (x_i - \bar{x}_j) - K\{(\tilde{z}_{ji} - \bar{z}_j) - R_k (x_i - \bar{x}_j)\}]^2}{n-2}}. \tag{18}$$

Here

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i}{n}, \quad \bar{y}_j = \frac{\sum_{i=1}^{n} y_{ji}}{n}, \quad K = \frac{\sum_{i=1}^{n} \tilde{w}_{ji} (y_{ki} - \bar{y}_k) \tilde{z}_{ji} - K_k \sum_{i=1}^{n} \tilde{w}_{ji} (x_i - \bar{x}_j) \tilde{z}_{ji}}{\sum_{i=1}^{n} \tilde{w}_{ji} (\tilde{z}_{ji} - \bar{z}_j)^2 - R_k \sum_{i=1}^{n} \tilde{w}_{ji} (x_i - \bar{x}_j) \tilde{z}_{ji}},$$

$$K_k = \frac{\sum_{i=1}^{n} \tilde{w}_{ji} (x_i - \bar{x}_j) y_{ki}}{\sum_{i=1}^{n} \tilde{w}_{ji} (x_i - \bar{x}_j)^2}, \quad R_k = \frac{\sum_{i=1}^{n} \tilde{w}_{ji} (x_i - \bar{x}_j) \tilde{z}_{ji}}{\sum_{i=1}^{n} \tilde{w}_{ji} (x_i - \bar{x}_j)^2}, \quad \bar{x}_j = \frac{\sum_{i=1}^{n} \tilde{w}_{ji} x_i}{\sum_{i=1}^{n} \tilde{w}_{ji}}, \quad \bar{y}_k = \frac{\sum_{i=1}^{n} \tilde{w}_{ji} y_{ki}}{\sum_{i=1}^{n} \tilde{w}_{ji}},$$

$$\bar{z}_j = \frac{\sum_{i=1}^{n} \tilde{w}_{ji} \tilde{z}_{ji}}{\sum_{j=1}^{n} \tilde{w}_{ji}}, \quad \tilde{z}_{ji} = \frac{(y_{ji} - \tilde{\gamma}_j - \tilde{\delta}_j x_i)}{\tilde{\sigma}_j}, \quad \tilde{w}_{ji} = \frac{1}{1 + \tilde{z}_{ji}^2 / \nu_j}.$$

## 3. Multivariate *t*-distribution

Now consider error variable $e$ to be a $q$-variate random vector having a multivariate $t$-distribution with $\nu$ DF and scatter matrix $\Omega$. The density function of $e$ is, therefore, given as

$$f(e; \Omega, \nu) \propto |\Omega|^{-1/2} \left\{ 1 + \frac{(e)' \Omega^{-1} (e)}{\nu} \right\}^{-(\nu+q)/2}. \tag{19}$$

It is known that any distinct pair $(e_j, e_k)(2 \le k \le q,\ 1 \le j \le k)$ of random variables $e_j$ and $e_k$ in $e$ is distributed as bivariate $t$ with $\nu$ DF and having corresponding scale parameters $\sigma_j$ and $\sigma_k$ and covariances $\rho_{kj} \sigma_k \sigma_j (\sigma_{jk} = \sigma_{kj})$ with $\rho_{kj} (= \rho_{jk})$ being the Pearson correlation coefficient between random variables $e_j$ and $e_k$. The MMLE and the LSE of these parameters are given by Equations (7)–(12) and (13)–(18), respectively, with $\nu_j = \nu$ and $\nu_k = \nu + 1$.

### 3.1 *Efficiencies of estimates*

We have conducted a simulation study taking $q = 4$, $n = 30,\ 50,\ 100$, and $\nu = 3,\ 4,\ 5$. We have simulated from $[100{,}000/n]$ (integer value) Monte Carlo runs the means and variances of the estimators while generating random samples $(e_{ji})$ $(1 \le j \le q, 1 \le i \le n)$ from a multivariate $t$-distribution, taking scales $\sigma_j = 1$ $(1 \le j \le q)$ and correlation coefficients $\rho_{kj} = 0.5$ $(2 \le k \le q,\ 1 \le j \le k)$, without loss of generality. A set of design points $x_i (1 \le i \le n)$ is randomly generated from a Uniform(0, 1) distribution and is common to all samples. The observations $y_{ji} (1 \le j \le q, 1 \le i \le n)$ are thus obtained from the model (1) by taking $\gamma_j = 0$ and $\delta_j = 1$ $(1 \le j \le q)$, without loss of generality. The quantities $z_{ji}$ in Equation (2) and $z_{ki}$ in Equation (3) are thus random observations having Student's $t$-distributions with $\nu$ and $\nu + 1$ DF, respectively. The MMLE and LSE are obtained from Equations (7)–(12) and (13)–(18), respectively.

The performance of MMLE is compared with that of LSE by evaluating the relative efficiencies (RE) as follows:

$$\mathrm{RE(LSE)} = 100 \times \left( \frac{\text{Variance of MMLE}}{\text{Variance of LSE}} \right). \tag{20}$$

The results are presented in Tables 1 and 2 for $v$ equal to 3 and 5, respectively. The biases in all the estimates are found to be negligible. It is also found that the MMLE are substantially more efficient than LSE. Contrary to common belief, the efficiencies of LSE decrease with increase in sample size.

*Note*: In this and the subsequent sections, the computer programs used for simulation are written in MATLAB, version 7.8.0.347 and several standard functions available in its Statistical Tool Box are invoked; particularly the function for generating the random deviates from a multivariate $t$-distribution. These programs are available from the first author.

### 3.2 *Robustness of estimates*

In practice, the exact values of DF $v$ might be somewhat different from the assumed one. Furthermore, the sample might contain anomalous observations such as outliers. The least-square estimators are known to be very sensitive to such anomalies and consequently may produce biased and inefficient estimates. Robustness of the estimates with respect to such anomalies is, therefore, very much desired. We consider an estimator to be robust if it is fully efficient for an assumed model and maintains high efficiency for plausible deviations from the assumed model [16]. We have investigated the robustness of the MMLE by taking the assumed distribution as multivariate $t$ with four DF ($v = 4$) and considering the following alternative situations.

(a) Misspecified models:
  (a) Model 1: Samples are generated from multivariate $t$ with $v = 3$ DF,
  (b) Model 2: Samples are generated from multivariate $t$ with $v = 5$ DF,
(b) Contaminated model:
  (a) Model 3: 90% sample values are drawn from the assumed model ($v = 4$) and 10% are taken from a multivariate normal distribution with the same mean vector and the covariance matrix as for the assumed model.
(c) Mixture models:
  (a) Model 4: 90% sample values are drawn from the assumed model ($v = 4$) and 10% are taken from a multivariate $t$ distribution with $v = 7$.
(d) Outliers model:
  (a) Model 5: 10% outliers in the samples are generated by selecting observations randomly and multiplying these by a factor of 4.

For the two models in (a) the results are presented in Tables 3 and 4, respectively. Clearly, the MMLE are not only less biased but also substantially more efficient and hence robust as compared with the LSE.

Similar results are obtained for cases (b), (c) and (d) and are shown in Tables 5–7, respectively.

## 4. Test of hypothesis

It is of interest to test for the significance of individual coefficients $\delta_j$ using the null hypotheses $H_{0j} : \delta_j = 0$ $(1 \le j \le q)$. In this regard it should be noted that the MMLE $\hat{\delta}_j$ is minimum variance bound estimator and is asymptotically normally distributed with mean $\delta_j$ and variance $\sigma_j^2 / \sum_{i=1}^{n} \beta_{ji}(x_{[i]} - \bar{x}_{j[\cdot]})^2$ (see [14]). Furthermore, the distribution of $(n-2)\hat{\sigma}_j^2/\sigma_j^2$ is chi-square

Table 1. Simulated mean and variance of estimates ($v = 3$) and REs.

| | $n = 30$ | | | | $n = 50$ | | | | $n = 100$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Mean | | | | Mean | | | | Mean | | | |
| Par | LSE | MMLE | $n \times$var MMLE | RE (%) of LSE | LSE | MMLE | $n \times$var MMLE | RE (%) of LSE | LSE | MMLE | $n \times$var MMLE | RE (%) of LSE |
| $\gamma_1$ | 0.004 | −0.001 | 8.246 | 55 | 0.013 | 0.005 | 7.008 | 56 | −0.001 | 0.001 | 6.710 | 52 |
| $\gamma_2$ | −0.003 | 0.001 | 8.514 | 55 | 0.019 | 0.028 | 6.960 | 56 | −0.022 | −0.015 | 6.918 | 48 |
| $\gamma_3$ | 0.017 | 0.009 | 8.824 | 58 | 0.014 | 0.010 | 7.201 | 53 | 0.000 | −0.004 | 7.060 | 55 |
| $\gamma_4$ | −0.009 | −0.005 | 8.639 | 54 | 0.026 | 0.021 | 6.754 | 52 | 0.001 | −0.007 | 6.865 | 52 |
| $\delta_1$ | 0.999 | 1.001 | 16.860 | 53 | 0.969 | 0.984 | 6.776 | 56 | 1.020 | 1.001 | 18.191 | 52 |
| $\delta_2$ | 1.006 | 1.001 | 17.507 | 52 | 0.962 | 0.950 | 6.878 | 56 | 1.033 | 1.022 | 18.386 | 50 |
| $\delta_3$ | 0.969 | 0.979 | 17.855 | 54 | 0.972 | 0.982 | 7.270 | 53 | 1.001 | 1.010 | 20.167 | 53 |
| $\delta_4$ | 1.022 | 1.011 | 17.739 | 53 | 0.951 | 0.973 | 6.194 | 52 | 0.996 | 1.004 | 18.215 | 53 |
| $\sigma_1$ | 0.923 | 1.076 | 2.791 | 69 | 0.938 | 1.063 | 2.113 | 48 | 0.953 | 1.038 | 1.839 | 26 |
| $\sigma_2$ | 0.931 | 1.082 | 3.240 | 65 | 0.942 | 1.061 | 2.060 | 47 | 0.954 | 1.038 | 1.849 | 25 |
| $\sigma_3$ | 0.936 | 1.085 | 3.376 | 64 | 0.941 | 1.065 | 2.225 | 48 | 0.956 | 1.039 | 2.161 | 18 |
| $\sigma_4$ | 0.929 | 1.081 | 3.343 | 66 | 0.938 | 1.060 | 2.267 | 47 | 0.947 | 1.035 | 1.744 | 30 |
| $\rho_{21}$ | 0.500 | 0.496 | 1.029 | 69 | 0.499 | 0.494 | 0.911 | 57 | 0.501 | 0.495 | 0.875 | 43 |
| $\rho_{31}$ | 0.495 | 0.490 | 1.049 | 71 | 0.499 | 0.496 | 0.938 | 61 | 0.501 | 0.495 | 0.893 | 45 |
| $\rho_{32}$ | 0.503 | 0.497 | 0.980 | 68 | 0.499 | 0.494 | 0.973 | 55 | 0.500 | 0.493 | 0.918 | 46 |
| $\rho_{41}$ | 0.501 | 0.498 | 0.943 | 69 | 0.501 | 0.496 | 0.903 | 58 | 0.499 | 0.495 | 0.845 | 49 |
| $\rho_{42}$ | 0.497 | 0.495 | 1.009 | 70 | 0.500 | 0.492 | 0.982 | 61 | 0.500 | 0.493 | 0.851 | 51 |
| $\rho_{43}$ | 0.503 | 0.498 | 0.987 | 70 | 0.502 | 0.493 | 0.919 | 58 | 0.499 | 0.497 | 0.868 | 45 |

Notes: Par, parameter; var, variance; RE, relative efficiency.

Table 2. Simulated mean and variance of estimates ($v = 5$) and REs.

| | $n = 30$ | | | | $n = 50$ | | | | $n = 100$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Mean | | | | Mean | | | | Mean | | | |
| Par | LSE | MMLE | $n\times$var MMLE | RE (%) of LSE | LSE | MMLE | $n\times$ var MMLE | RE (%) of LSE | LSE | MMLE | $n\times$ var MMLE | RE (%) of LSE |
| $\gamma_1$ | 0.013 | 0.013 | 7.063 | 84 | −0.007 | −0.003 | 6.246 | 83 | 0.007 | 0.003 | 5.576 | 79 |
| $\gamma_2$ | −0.003 | 0.002 | 7.170 | 84 | −0.000 | −0.001 | 5.934 | 81 | 0.010 | 0.007 | 5.348 | 82 |
| $\gamma_3$ | 0.009 | 0.007 | 7.196 | 83 | −0.014 | −0.014 | 6.223 | 83 | 0.013 | 0.008 | 5.648 | 76 |
| $\gamma_4$ | 0.004 | 0.008 | 6.803 | 84 | −0.008 | −0.008 | 6.173 | 83 | 0.005 | −0.001 | 5.636 | 80 |
| $\delta_1$ | 0.983 | 0.982 | 14.488 | 84 | 1.012 | 1.006 | 15.156 | 83 | 0.994 | 1.000 | 15.182 | 80 |
| $\delta_2$ | 1.001 | 0.992 | 15.035 | 83 | 0.997 | 1.001 | 14.163 | 81 | 0.987 | 0.989 | 14.582 | 81 |
| $\delta_3$ | 0.990 | 0.994 | 14.902 | 82 | 1.028 | 1.028 | 15.140 | 83 | 0.986 | 0.992 | 15.955 | 77 |
| $\delta_4$ | 0.987 | 0.981 | 14.926 | 84 | 1.007 | 1.008 | 14.306 | 84 | 0.998 | 1.006 | 15.001 | 82 |
| $\sigma_1$ | 0.973 | 0.961 | 0.845 | 69 | 0.987 | 0.964 | 1.027 | 59 | 0.992 | 0.965 | 0.973 | 57 |
| $\sigma_2$ | 0.971 | 0.960 | 0.842 | 73 | 0.988 | 0.964 | 0.991 | 59 | 0.989 | 0.961 | 0.998 | 59 |
| $\sigma_3$ | 0.980 | 0.966 | 0.863 | 66 | 0.983 | 0.959 | 0.920 | 60 | 0.994 | 0.965 | 1.015 | 59 |
| $\sigma_4$ | 0.974 | 0.962 | 0.870 | 70 | 0.987 | 0.964 | 0.972 | 62 | 0.991 | 0.964 | 0.920 | 61 |
| $\rho_{21}$ | 0.495 | 0.494 | 0.835 | 88 | 0.495 | 0.496 | 0.787 | 80 | 0.500 | 0.500 | 0.794 | 75 |
| $\rho_{31}$ | 0.496 | 0.495 | 0.842 | 91 | 0.497 | 0.497 | 0.814 | 83 | 0.498 | 0.497 | 0.784 | 76 |
| $\rho_{32}$ | 0.493 | 0.494 | 0.829 | 88 | 0.493 | 0.491 | 0.811 | 81 | 0.503 | 0.502 | 0.757 | 76 |
| $\rho_{41}$ | 0.497 | 0.497 | 0.810 | 87 | 0.495 | 0.493 | 0.812 | 83 | 0.500 | 0.502 | 0.729 | 81 |
| $\rho_{42}$ | 0.489 | 0.490 | 0.825 | 85 | 0.497 | 0.497 | 0.784 | 81 | 0.500 | 0.500 | 0.775 | 81 |
| $\rho_{43}$ | 0.490 | 0.490 | 0.823 | 87 | 0.497 | 0.496 | 0.764 | 80 | 0.505 | 0.504 | 0.752 | 80 |

Notes: Par, parameter; var, variance; RE, relative efficiency.

Table 3. Simulated mean and variance of estimates and REs (Model 1).

| | $n=30$ | | | | $n=50$ | | | | $n=100$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Mean | | | | Mean | | | | Mean | | | |
| Par | LSE | MMLE | $n\times$var MMLE | RE (%) of LSE | LSE | MMLE | $n\times$ var MMLE | RE (%) of LSE | LSE | MMLE | $n\times$ var MMLE | RE (%) of LSE |
| $\gamma_1$ | 0.004 | −0.003 | 8.249 | 55 | 0.013 | 0.006 | 6.910 | 55 | −0.001 | 0.001 | 6.651 | 51 |
| $\gamma_2$ | −0.003 | 0.002 | 8.469 | 55 | 0.019 | 0.029 | 6.921 | 56 | −0.022 | −0.014 | 6.855 | 47 |
| $\gamma_3$ | 0.017 | 0.008 | 8.772 | 58 | 0.014 | 0.009 | 7.174 | 53 | 0.000 | −0.005 | 7.082 | 55 |
| $\gamma_4$ | −0.009 | −0.004 | 8.565 | 54 | 0.026 | 0.022 | 6.679 | 52 | 0.001 | −0.007 | 6.778 | 51 |
| $\delta_1$ | 0.999 | 1.004 | 16.843 | 53 | 0.969 | 0.983 | 16.486 | 55 | 1.020 | 0.999 | 18.064 | 52 |
| $\delta_2$ | 1.006 | 1.000 | 17.287 | 52 | 0.962 | 0.948 | 16.745 | 56 | 1.033 | 1.021 | 18.301 | 50 |
| $\delta_3$ | 0.969 | 0.981 | 17.683 | 54 | 0.972 | 0.983 | 17.270 | 53 | 1.001 | 1.012 | 20.303 | 54 |
| $\delta_4$ | 1.022 | 1.010 | 17.565 | 53 | 0.951 | 0.970 | 16.100 | 52 | 0.996 | 1.005 | 18.073 | 53 |
| $\sigma_1$ | 1.130 | 1.103 | 3.764 | 60 | 1.149 | 1.102 | 3.374 | 45 | 1.167 | 1.087 | 3.532 | 27 |
| $\sigma_2$ | 1.140 | 1.113 | 4.641 | 59 | 1.153 | 1.100 | 3.290 | 44 | 1.168 | 1.087 | 3.595 | 26 |
| $\sigma_3$ | 1.146 | 1.116 | 4.862 | 58 | 1.152 | 1.104 | 3.518 | 45 | 1.171 | 1.088 | 4.584 | 22 |
| $\sigma_4$ | 1.137 | 1.111 | 4.718 | 59 | 1.149 | 1.097 | 3.565 | 44 | 1.160 | 1.085 | 3.311 | 31 |
| $\rho_{21}$ | 0.497 | 0.495 | 1.090 | 78 | 0.497 | 0.495 | 0.989 | 66 | 0.499 | 0.496 | 0.979 | 52 |
| $\rho_{31}$ | 0.492 | 0.491 | 1.073 | 76 | 0.496 | 0.497 | 1.036 | 71 | 0.499 | 0.497 | 1.014 | 55 |
| $\rho_{32}$ | 0.501 | 0.498 | 1.047 | 78 | 0.496 | 0.494 | 1.070 | 67 | 0.497 | 0.495 | 1.004 | 53 |
| $\rho_{41}$ | 0.497 | 0.497 | 0.991 | 76 | 0.498 | 0.498 | 1.000 | 69 | 0.497 | 0.496 | 0.965 | 60 |
| $\rho_{42}$ | 0.494 | 0.495 | 1.056 | 78 | 0.497 | 0.494 | 1.050 | 70 | 0.497 | 0.493 | 0.989 | 63 |
| $\rho_{43}$ | 0.500 | 0.498 | 1.042 | 78 | 0.499 | 0.496 | 0.992 | 68 | 0.497 | 0.497 | 1.009 | 58 |

Notes: Par, parameter; var, variance; RE, relative efficiency.

Table 4. Simulated mean and variance of estimates and REs (Model 2).

| | $n = 30$ | | | | $n = 50$ | | | | $n = 100$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Mean | | | | Mean | | | | Mean | | | |
| Par | LSE | MMLE | $n\times$var MMLE | RE (%) of LSE | LSE | MMLE | $n\times$ var MMLE | RE (%) of LSE | LSE | MMLE | $n\times$ var MMLE | RE (%) of LSE |
| $\gamma_1$ | 0.013 | 0.013 | 7.093 | 84 | −0.007 | −0.003 | 6.273 | 84 | 0.007 | 0.002 | 5.563 | 79 |
| $\gamma_2$ | −0.003 | 0.003 | 7.190 | 84 | −0.000 | −0.001 | 5.959 | 82 | 0.010 | 0.007 | 5.378 | 82 |
| $\gamma_3$ | 0.009 | 0.006 | 7.189 | 83 | −0.014 | −0.014 | 6.243 | 83 | 0.013 | 0.009 | 5.646 | 76 |
| $\gamma_4$ | 0.004 | 0.009 | 6.819 | 84 | −0.008 | −0.007 | 6.189 | 84 | 0.005 | −0.001 | 5.661 | 80 |
| $\delta_1$ | 0.983 | 0.983 | 14.570 | 85 | 1.012 | 1.006 | 15.242 | 83 | 0.994 | 1.001 | 15.177 | 80 |
| $\delta_2$ | 1.001 | 0.990 | 15.094 | 84 | 0.997 | 1.001 | 14.183 | 81 | 0.987 | 0.988 | 14.692 | 82 |
| $\delta_3$ | 0.990 | 0.995 | 14.878 | 82 | 1.028 | 1.029 | 15.206 | 84 | 0.986 | 0.991 | 15.954 | 77 |
| $\delta_4$ | 0.987 | 0.979 | 14.983 | 84 | 1.007 | 1.008 | 14.308 | 84 | 0.998 | 1.005 | 15.088 | 82 |
| $\sigma_1$ | 0.888 | 0.926 | 0.947 | 69 | 0.901 | 0.931 | 1.129 | 59 | 0.905 | 0.933 | 1.247 | 54 |
| $\sigma_2$ | 0.887 | 0.924 | 0.934 | 70 | 0.902 | 0.931 | 1.120 | 60 | 0.903 | 0.929 | 1.276 | 54 |
| $\sigma_3$ | 0.895 | 0.931 | 0.965 | 69 | 0.897 | 0.926 | 1.080 | 60 | 0.908 | 0.933 | 1.287 | 57 |
| $\sigma_4$ | 0.889 | 0.927 | 0.964 | 69 | 0.901 | 0.931 | 1.102 | 62 | 0.905 | 0.932 | 1.200 | 56 |
| $\rho_{21}$ | 0.496 | 0.495 | 0.859 | 87 | 0.495 | 0.496 | 0.795 | 78 | 0.500 | 0.500 | 0.804 | 73 |
| $\rho_{31}$ | 0.497 | 0.495 | 0.863 | 90 | 0.497 | 0.497 | 0.820 | 82 | 0.498 | 0.496 | 0.787 | 74 |
| $\rho_{32}$ | 0.493 | 0.494 | 0.848 | 87 | 0.493 | 0.491 | 0.823 | 79 | 0.504 | 0.501 | 0.756 | 74 |
| $\rho_{41}$ | 0.498 | 0.497 | 0.833 | 87 | 0.495 | 0.493 | 0.825 | 82 | 0.500 | 0.501 | 0.724 | 78 |
| $\rho_{42}$ | 0.490 | 0.491 | 0.848 | 85 | 0.497 | 0.497 | 0.808 | 80 | 0.500 | 0.499 | 0.763 | 77 |
| $\rho_{43}$ | 0.490 | 0.490 | 0.849 | 87 | 0.497 | 0.496 | 0.775 | 78 | 0.505 | 0.504 | 0.763 | 78 |

Notes: Par, parameter; var, variance; RE, relative efficiency.

Table 5. Simulated mean and variance of estimates and REs (Model 3).

| | $n = 30$ | | | | $n = 50$ | | | | $n = 100$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Mean | | | | Mean | | | | Mean | | | |
| Par | LSE | MMLE | $n\times$var MMLE | RE (%) of LSE | LSE | MMLE | $n\times$ var MMLE | RE (%) of LSE | LSE | MMLE | $n\times$ var MMLE | RE (%) of LSE |
| $\gamma_1$ | −0.016 | −0.016 | 6.826 | 77 | 0.004 | −0.004 | 5.863 | 74 | 0.004 | −0.000 | 6.154 | 79 |
| $\gamma_2$ | −0.012 | −0.003 | 6.982 | 77 | −0.009 | −0.015 | 6.495 | 76 | 0.002 | −0.001 | 6.055 | 75 |
| $\gamma_3$ | 0.002 | 0.001 | 7.185 | 76 | −0.009 | −0.008 | 6.204 | 74 | 0.004 | 0.000 | 6.358 | 76 |
| $\gamma_4$ | −0.011 | −0.012 | 6.979 | 75 | −0.003 | −0.004 | 5.969 | 75 | 0.010 | 0.012 | 6.236 | 78 |
| $\delta_1$ | 1.024 | 1.022 | 14.113 | 78 | 0.994 | 1.009 | 14.091 | 76 | 0.986 | 0.990 | 16.549 | 78 |
| $\delta_2$ | 1.019 | 1.007 | 14.426 | 78 | 1.017 | 1.027 | 14.536 | 76 | 0.985 | 0.991 | 16.231 | 72 |
| $\delta_3$ | 0.998 | 0.996 | 14.637 | 76 | 1.028 | 1.019 | 14.584 | 73 | 0.989 | 0.998 | 17.300 | 75 |
| $\delta_4$ | 1.019 | 1.023 | 14.313 | 75 | 1.003 | 1.007 | 14.749 | 75 | 0.985 | 0.975 | 17.348 | 75 |
| $\sigma_1$ | 0.944 | 0.962 | 1.252 | 64 | 0.947 | 0.960 | 1.075 | 57 | 0.952 | 0.953 | 1.030 | 55 |
| $\sigma_2$ | 0.939 | 0.958 | 1.304 | 64 | 0.949 | 0.959 | 1.104 | 55 | 0.954 | 0.954 | 1.190 | 45 |
| $\sigma_3$ | 0.942 | 0.961 | 1.305 | 65 | 0.952 | 0.963 | 1.208 | 55 | 0.957 | 0.957 | 1.097 | 52 |
| $\sigma_4$ | 0.937 | 0.956 | 1.294 | 64 | 0.958 | 0.965 | 1.187 | 55 | 0.951 | 0.954 | 1.150 | 50 |
| $\rho_{21}$ | 0.501 | 0.498 | 0.863 | 81 | 0.495 | 0.495 | 0.850 | 76 | 0.497 | 0.495 | 0.838 | 70 |
| $\rho_{31}$ | 0.499 | 0.495 | 0.821 | 80 | 0.493 | 0.492 | 0.817 | 76 | 0.495 | 0.494 | 0.813 | 73 |
| $\rho_{32}$ | 0.491 | 0.490 | 0.897 | 82 | 0.497 | 0.493 | 0.794 | 73 | 0.497 | 0.498 | 0.820 | 69 |
| $\rho_{41}$ | 0.501 | 0.499 | 0.863 | 82 | 0.495 | 0.494 | 0.840 | 76 | 0.497 | 0.496 | 0.831 | 71 |
| $\rho_{42}$ | 0.493 | 0.491 | 0.912 | 84 | 0.497 | 0.495 | 0.869 | 80 | 0.496 | 0.494 | 0.765 | 66 |
| $\rho_{43}$ | 0.495 | 0.492 | 0.885 | 84 | 0.499 | 0.496 | 5.863 | 77 | 0.493 | 0.492 | 0.792 | 72 |

Notes: Par, parameter; var, variance; RE, relative efficiency.

Table 6. Simulated mean and variance of estimates and REs (Model 4).

| | $n = 30$ | | | | $n = 50$ | | | | $n = 100$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Mean | | | | Mean | | | | Mean | | | |
| Par | LSE | MMLE | $n\times$var MMLE | RE (%) of LSE | LSE | MMLE | $n\times$ var MMLE | RE (%) of LSE | LSE | MMLE | $n\times$ var MMLE | RE (%) of LSE |
| $\gamma_1$ | 0.015 | 0.012 | 7.420 | 74 | 0.002 | 0.007 | 6.198 | 74 | 0.002 | 0.007 | 6.198 | 74 |
| $\gamma_2$ | 0.003 | 0.009 | 7.431 | 75 | 0.003 | 0.005 | 6.217 | 71 | 0.003 | 0.005 | 6.217 | 71 |
| $\gamma_3$ | 0.010 | 0.012 | 7.703 | 77 | −0.004 | −0.002 | 6.095 | 72 | −0.004 | −0.002 | 6.095 | 72 |
| $\gamma_4$ | 0.008 | −0.000 | 7.242 | 74 | 0.002 | 0.007 | 6.129 | 74 | 0.002 | 0.007 | 6.129 | 74 |
| $\delta_1$ | 0.985 | 0.988 | 15.416 | 74 | 0.997 | 0.992 | 15.419 | 74 | 0.997 | 0.992 | 15.419 | 74 |
| $\delta_2$ | 1.005 | 0.997 | 15.282 | 74 | 1.000 | 0.997 | 15.198 | 71 | 1.000 | 0.997 | 15.198 | 71 |
| $\delta_3$ | 0.988 | 0.985 | 15.823 | 76 | 1.008 | 1.005 | 14.692 | 73 | 1.008 | 1.005 | 14.692 | 73 |
| $\delta_4$ | 0.999 | 1.008 | 15.104 | 74 | 1.006 | 0.998 | 14.819 | 75 | 1.006 | 0.998 | 14.819 | 75 |
| $\sigma_1$ | 0.950 | 0.969 | 1.176 | 68 | 0.963 | 0.974 | 1.185 | 57 | 0.963 | 0.974 | 1.185 | 57 |
| $\sigma_2$ | 0.955 | 0.971 | 1.131 | 66 | 0.964 | 0.975 | 1.256 | 53 | 0.964 | 0.975 | 1.256 | 53 |
| $\sigma_3$ | 0.953 | 0.972 | 1.122 | 67 | 0.963 | 0.972 | 1.222 | 56 | 0.963 | 0.972 | 1.222 | 56 |
| $\sigma_4$ | 0.956 | 0.973 | 1.142 | 66 | 0.962 | 0.973 | 1.230 | 54 | 0.962 | 0.973 | 1.230 | 54 |
| $\rho_{21}$ | 0.489 | 0.488 | 0.893 | 84 | 0.495 | 0.496 | 0.824 | 77 | 0.495 | 0.496 | 0.824 | 77 |
| $\rho_{31}$ | 0.489 | 0.488 | 0.935 | 83 | 0.495 | 0.495 | 0.837 | 80 | 0.495 | 0.495 | 0.837 | 80 |
| $\rho_{32}$ | 0.490 | 0.487 | 0.911 | 84 | 0.493 | 0.495 | 0.864 | 73 | 0.493 | 0.495 | 0.864 | 73 |
| $\rho_{41}$ | 0.488 | 0.489 | 0.902 | 85 | 0.494 | 0.493 | 0.844 | 78 | 0.494 | 0.493 | 0.844 | 78 |
| $\rho_{42}$ | 0.491 | 0.488 | 0.946 | 83 | 0.501 | 0.500 | 0.792 | 73 | 0.501 | 0.500 | 0.792 | 73 |
| $\rho_{43}$ | 0.489 | 0.488 | 0.927 | 84 | 0.498 | 0.495 | 0.876 | 74 | 0.499 | 0.495 | 0.876 | 74 |

Notes: Par, parameter; var, variance; RE, relative efficiency.

Table 7. Simulated mean and variance of estimates and REs (Model 5).

| | $n = 30$ | | | | $n = 50$ | | | | $n = 100$ | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | Mean | | | | Mean | | | | Mean | | | |
| Par | LSE | MMLE | $n\times$var MMLE | RE (%) of LSE | LSE | MMLE | $n\times$ var MMLE | RE (%) of LSE | LSE | MMLE | $n\times$ var MMLE | RE (%) of LSE |
| $\gamma_1$ | 0.035 | 0.009 | 10.226 | 40 | −0.009 | −0.001 | 8.548 | 38 | 0.019 | 0.019 | 8.253 | 41 |
| $\gamma_2$ | 0.035 | 0.015 | 10.888 | 42 | 0.022 | 0.018 | 8.242 | 36 | −0.003 | −0.006 | 8.318 | 42 |
| $\gamma_3$ | 0.027 | 0.003 | 10.254 | 41 | −0.000 | 0.010 | 8.272 | 36 | −0.001 | 0.000 | 7.973 | 41 |
| $\gamma_4$ | 0.014 | 0.002 | 10.641 | 41 | 0.014 | 0.009 | 8.195 | 39 | 0.019 | 0.010 | 8.133 | 42 |
| $\delta_1$ | 0.965 | 0.994 | 21.063 | 40 | 1.009 | 0.999 | 20.834 | 39 | 0.972 | 0.965 | 22.704 | 42 |
| $\delta_2$ | 0.969 | 0.989 | 22.229 | 41 | 0.979 | 0.977 | 19.333 | 37 | 0.987 | 0.999 | 21.534 | 43 |
| $\delta_3$ | 0.970 | 1.003 | 21.359 | 40 | 1.003 | 0.984 | 20.014 | 37 | 0.992 | 0.992 | 21.249 | 39 |
| $\delta_4$ | 0.983 | 0.997 | 21.741 | 39 | 0.984 | 0.992 | 19.539 | 38 | 0.976 | 0.988 | 21.844 | 41 |
| $\sigma_1$ | 1.468 | 1.357 | 9.673 | 61 | 1.480 | 1.340 | 11.984 | 47 | 1.499 | 1.316 | 14.094 | 39 |
| $\sigma_2$ | 1.459 | 1.351 | 10.244 | 60 | 1.508 | 1.356 | 13.220 | 46 | 1.517 | 1.325 | 15.278 | 37 |
| $\sigma_3$ | 1.461 | 1.353 | 9.867 | 61 | 1.483 | 1.341 | 11.700 | 47 | 1.511 | 1.322 | 14.602 | 38 |
| $\sigma_4$ | 1.470 | 1.358 | 11.148 | 58 | 1.477 | 1.340 | 11.099 | 50 | 1.504 | 1.320 | 14.295 | 39 |
| $\rho_{21}$ | 0.495 | 0.502 | 1.321 | 69 | 0.488 | 0.495 | 2.289 | 67 | 0.497 | 0.495 | 1.188 | 59 |
| $\rho_{31}$ | 0.493 | 0.496 | 1.271 | 69 | 0.490 | 0.491 | 1.366 | 63 | 0.500 | 0.502 | 1.264 | 50 |
| $\rho_{32}$ | 0.490 | 0.494 | 1.276 | 70 | 0.493 | 0.495 | 1.290 | 63 | 0.499 | 0.501 | 1.232 | 51 |
| $\rho_{41}$ | 0.495 | 0.497 | 1.294 | 71 | 0.485 | 0.490 | 1.349 | 65 | 0.497 | 0.498 | 1.163 | 55 |
| $\rho_{42}$ | 0.486 | 0.495 | 1.314 | 70 | 0.499 | 0.500 | 1.266 | 62 | 0.506 | 0.503 | 1.233 | 55 |
| $\rho_{43}$ | 0.494 | 0.497 | 1.388 | 75 | 0.493 | 0.496 | 1.275 | 61 | 0.508 | 0.503 | 1.160 | 48 |

Notes: Par, parameter; var, variance; RE, relative efficiency.

with $(n-2)$ DF. Hence, we can propose the following test-statistic for testing the above hypothesis:

$$T_M = \sqrt{\sum_{i=1}^{n} \beta_{ji}(x_{[i]} - \bar{x}_{j[\cdot]})^2} \left( \frac{\hat{\delta}_j}{\hat{\sigma}_j} \right), \tag{21}$$

and assuming it to be distributed as Student's $t$ with $(n-2)$ DF. Also for LSE, a similar test-statistic can be taken as

$$T_L = \sqrt{\sum_{i=1}^{n} (x_i - \bar{x})^2} \left( \frac{\tilde{\delta}_j}{\tilde{\sigma}_j} \right). \tag{22}$$

The null hypothesis is rejected in favor of alternate hypothesis $H_{1j} : \delta_j > 0$ $(1 \leq j \leq q)$ if large values of the test statistics are observed.

The simulated power for the two test statistics, $T_{\mathrm{L}}$ and $T_{\mathrm{M}}$, for $\delta_1$ $(q = 4; \delta_2 = \delta_3 = \delta_4 = 1)$ is reported in Table 8, for an assumed test-size 0.05. It is observed that the test based upon $T_{\mathrm{M}}$ is much more powerful than the test based upon $T_{\mathrm{L}}$. Similar results are also obtained for other $\delta$-parameters, although not reported here for conciseness.

In order to study the robustness, simulated values of the power of the tests for Model 1–Model 5 (Section 3.2) are obtained and it is observed that the Type I error of the $T_L$-test is generally higher than the $T_M$-test and its power is lower. The results are not provided here for conciseness.

## 5. Empirical application

We take the weekly returns data using the historical indices for the period from 2 January 2005 to 30 December 2011 for five stock markets (NYSE 100, CAC 40, DAX, Hang Seng, and Nikkei 225) selected from the USA, Europe, and Asia/Pacific regions. The returns are calculated according to the following formula:

$$R_{ji} = \frac{(I_{ji} - I_{ji-1})}{I_{ji-1}}, \tag{23}$$

where $R_{ji}$ is the return for market $j$ ($j$ =0, 1, 2, 3, 4) on week $i$ ($i = 1, 2, \ldots, 364$), $I_{ji}$ is the index of market $j$ on week $i$ and $I_{ji-1}$ is the index of the same market on week $i - 1$.

We take as independent variable $X = R_0$ (market returns for NYSE 100), considering it to be a benchmark. Then there are four dependent variables $Y_j = R_j$ $(j = 1, 2, 3, 4)$ representing the market returns for the four markets CAC 40, DAX, Hang Seng, and Nikkei 225, respectively. Furthermore, we are using the multivariate linear regression model (1) with $n = 364$ and $q = 4$.

The marginal residuals $\tilde{e}_{ji}$ $(1 \leq j \leq q, 1 \leq i \leq n)$ expressed in terms of the LSE (13)–(15) $\tilde{z}_{ji} = \tilde{e}_{ji}/\tilde{\sigma}_j = (y_{ji} - \tilde{\gamma}_j - \tilde{\delta}_j x_i)/\tilde{\sigma}_j, \tilde{\gamma}_j, \tilde{\delta}_j, \tilde{\sigma}_j$, show very high kurtosis and the normality assumptions are rejected (we use Lillilifors test for the null hypothesis that the sample comes from a distribution in the normal family, against the alternative that it does not come from a normal distribution) for all marginal residuals except $\tilde{e}_4$. Furthermore, the $Q$–$Q$ Plots (Figures A1–A4, Appendix) for the marginal residuals $\tilde{z}_{ji}$ $(1 \leq j \leq q)$ suggest the best fitted univariate $t$-distributions with the DFs given in Table 9.

Similarly, the conditional residuals $\tilde{e}_{ki}|\tilde{e}_{ji}$ are expressed in terms of the LSE (13)–(18) $\tilde{z}_{ki} = \sqrt{\nu_j^*}\sqrt{\tilde{w}_{ji}}((\tilde{e}_{ki} - \tilde{\theta}_{kj}\tilde{e}_{ji})/\tilde{\sigma}_{k\cdot j})$ $(2 \leq k \leq q, 1 \leq j \leq k-1, 1 \leq i \leq n)$. Again the normality assumptions are violated for $\tilde{z}_k$ (see Table 10) and the $Q$–$Q$ Plots (Figures A5–A10, Appendix) suggest using univariate $t$-distributions with corresponding DFs. Hence, the use of multivariate $t$-distribution with varying marginal/conditional DFs is recommended.

Using the marginal and conditional $t$-distributions suggested above, the MMLE (7)–(10) and LSE (13)–(16) for the parameters are obtained and given in Table 11. The efficiencies of these

Table 8. Simulated size and power of tests.

| | $n = 30$ | | | $n = 50$ | | | $n = 100$ | |
|---|---|---|---|---|---|---|---|---|
| $\delta_1$ | $T_L$ | $T_M$ | $\delta_1$ | $T_L$ | $T_M$ | $\delta_1$ | $T_L$ | $T_M$ |
| | | | | $v = 3$ | | | | |
| 0.00 | 0.050 | 0.050 | 0.00 | 0.050 | 0.050 | 0.00 | 0.051 | 0.051 |
| 0.40 | 0.125 | 0.138 | 0.40 | 0.148 | 0.178 | 0.30 | 0.122 | 0.147 |
| 0.80 | 0.222 | 0.268 | 0.60 | 0.222 | 0.297 | 0.50 | 0.227 | 0.293 |
| 1.00 | 0.309 | 0.368 | 0.80 | 0.304 | 0.414 | 0.70 | 0.335 | 0.478 |
| 1.30 | 0.422 | 0.520 | 1.00 | 0.409 | 0.542 | 0.80 | 0.424 | 0.547 |
| 1.50 | 0.502 | 0.599 | 1.20 | 0.526 | 0.692 | 1.00 | 0.543 | 0.737 |
| 1.80 | 0.623 | 0.738 | 1.40 | 0.618 | 0.782 | 1.20 | 0.673 | 0.852 |
| 2.10 | 0.715 | 0.824 | 1.80 | 0.787 | 0.921 | 1.30 | 0.738 | 0.905 |
| 2.40 | 0.803 | 0.904 | 2.00 | 0.861 | 0.955 | 1.60 | 0.842 | 0.970 |
| 3.50 | 0.955 | 0.992 | 2.40 | 0.921 | 0.993 | 1.80 | 0.898 | 0.993 |
| | | | | $v = 4$ | | | | |
| 0.00 | 0.050 | 0.050 | 0.00 | 0.052 | 0.051 | 0.00 | 0.051 | 0.051 |
| 0.40 | 0.122 | 0.135 | 0.20 | 0.104 | 0.111 | 0.20 | 0.111 | 0.129 |
| 0.70 | 0.213 | 0.254 | 0.50 | 0.216 | 0.235 | 0.40 | 0.185 | 0.243 |
| 1.00 | 0.335 | 0.408 | 0.70 | 0.332 | 0.357 | 0.60 | 0.359 | 0.446 |
| 1.20 | 0.431 | 0.517 | 0.90 | 0.445 | 0.517 | 0.80 | 0.520 | 0.612 |
| 1.40 | 0.523 | 0.613 | 1.10 | 0.561 | 0.641 | 1.00 | 0.657 | 0.773 |
| 1.70 | 0.674 | 0.770 | 1.30 | 0.697 | 0.777 | 1.20 | 0.788 | 0.891 |
| 2.00 | 0.778 | 0.864 | 1.50 | 0.765 | 0.841 | 1.40 | 0.876 | 0.971 |
| 2.40 | 0.874 | 0.931 | 1.80 | 0.876 | 0.940 | 1.60 | 0.943 | 0.984 |
| 3.10 | 0.970 | 0.993 | 2.30 | 0.964 | 0.992 | 1.80 | 0.971 | 0.997 |
| | | | | $v = 5$ | | | | |
| 0.00 | 0.051 | 0.050 | 0.00 | 0.049 | 0.049 | 0.00 | 0.050 | 0.049 |
| 0.40 | 0.131 | 0.143 | 0.40 | 0.146 | 0.148 | 0.20 | 0.119 | 0.131 |
| 0.70 | 0.240 | 0.251 | 0.60 | 0.246 | 0.269 | 0.40 | 0.244 | 0.278 |
| 1.00 | 0.380 | 0.404 | 0.80 | 0.372 | 0.406 | 0.60 | 0.371 | 0.438 |
| 1.20 | 0.485 | 0.512 | 0.90 | 0.418 | 0.476 | 0.70 | 0.461 | 0.541 |
| 1.40 | 0.569 | 0.617 | 1.10 | 0.569 | 0.631 | 0.80 | 0.583 | 0.630 |
| 1.60 | 0.673 | 0.727 | 1.30 | 0.680 | 0.745 | 0.90 | 0.677 | 0.723 |
| 1.80 | 0.760 | 0.804 | 1.50 | 0.789 | 0.841 | 1.10 | 0.829 | 0.882 |
| 2.10 | 0.852 | 0.894 | 1.70 | 0.878 | 0.926 | 1.40 | 0.939 | 0.963 |
| 3.00 | 0.978 | 0.993 | 2.30 | 0.982 | 0.995 | 1.70 | 0.982 | 0.992 |

Table 9. Non-normality tests for marginal distributions and DF for associated $t$-distributions.

| Marginal residual | Kurtosis | $p$-Value | DF |
|---|---|---|---|
| $\tilde{z}_1$ | 5.450 | 0.001 | 3.5 |
| $\tilde{z}_2$ | 5.183 | 0.002 | 4.8 |
| $\tilde{z}_3$ | 4.528 | 0.005 | 5.1 |
| $\tilde{z}_4$ | 6.302 | 0.058 | 7.0 |

Note: DF, degrees of freedom.

estimates are also compared with the LSE (Table 11) by generating bootstrap samples from the sample under study. The MMLE are found to be unbiased and highly efficient.

*Remark*: In the context of using a multivariate $t$-distribution (symmetric distribution) we have explored here the kurtosis in the data while leaving aside the possibility of having high skewness. However, the MMLE can be derived for any location-scale type distribution and selecting a

Table 10. Non-normality tests for conditional distributions and DF for associated $t$-distributions.

| Conditional residual | Kurtosis | $p$-Value | DF |
|---|---|---|---|
| $\tilde{z}_2\|\tilde{z}_1$ | 6.845 | 0.002 | 4.6 |
| $\tilde{z}_3\|\tilde{z}_1$ | 4.054 | 0.017 | 7.6 |
| $\tilde{z}_4\|\tilde{z}_1$ | 4.770 | 0.068 | 7.4 |
| $\tilde{z}_3\|\tilde{z}_2$ | 3.920 | 0.129 | 7.8 |
| $\tilde{z}_4\|\tilde{z}_2$ | 5.065 | 0.500 | 9.2 |
| $\tilde{z}_4\|\tilde{z}_3$ | 7.523 | 0.005 | 5.8 |

Note: DF, degrees of freedom.

Table 11. Estimates and REs.

| | Estimate | | |
|---|---|---|---|
| Parameter | LS | MML | RE (%) of LSE |
| $\gamma_1$ | 0.001 | −0.000 | 61 |
| $\gamma_2$ | −0.001 | −0.002 | 68 |
| $\gamma_3$ | −0.001 | −0.001 | 65 |
| $\gamma_4$ | 0.001 | −0.000 | 63 |
| $\delta_1$ | 1.088 | 1.074 | 72 |
| $\delta_2$ | 1.093 | 1.046 | 70 |
| $\delta_3$ | 0.800 | 0.820 | 71 |
| $\delta_4$ | 0.720 | 0.558 | 76 |
| $\sigma_1$ | 0.012 | 0.012 | 53 |
| $\sigma_2$ | 0.014 | 0.013 | 55 |
| $\sigma_3$ | 0.021 | 0.021 | 56 |
| $\sigma_4$ | 0.023 | 0.022 | 52 |
| $\rho_{21}$ | 0.670 | 0.738 | 78 |
| $\rho_{31}$ | 0.212 | 0.232 | 76 |
| $\rho_{32}$ | 0.243 | 0.230 | 75 |
| $\rho_{41}$ | 0.179 | 0.198 | 79 |
| $\rho_{42}$ | 0.303 | 0.264 | 73 |
| $\rho_{43}$ | 0.347 | 0.386 | 72 |

suitable skew-leptokurtic distribution such as Generalized Logistic [4] can accommodate the excess skewness and kurtosis both.

## 6. Conclusion

We have tackled the problem of estimation in multivariate linear regression models with elliptically distributed errors, particularly the case when errors follow a multivariate $t$-distribution. We have provided the estimates (MMLE) that are unbiased and are expressions in a closed form, i.e. easily computable. We have also established their superiority over the traditional LSE even if the LSE are explicitly corrected for bias that arises from the violation of the normality assumption. Furthermore, it is shown that MMLE are inherently robust to plausible deviations from assumed distributions and also many data anomalies that are usually present in almost all real life samples. Test of hypothesis for the regression coefficients is provided and it is shown that the tests based upon MMLE are enormously more powerful than the tests based upon LSE. Moreover, our method of estimation allows us to work with the multivariate $t$-distribution even if the resulting marginal (and conditional) $t$-distributions have varying DFs. This gives us flexibility in dealing with real-life data that may exhibit such an implicit behavior, e.g. financial data. We have shown the applicability of our method in international stock market returns data. It is observed that these data do not follow a multivariate $t$-distribution strictly; rather it suggests using a flexible form of the distribution as mentioned above. The MMLE so obtained are found to be more reliable (efficient) than their counterpart; i.e. the LSE.

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## Appendix

This section contains the following $Q$–$Q$ Plots drawn for marginal (Figures A1–A4) and conditional (Figures A5–A10) residuals given in Equations (2) and (3), respectively.

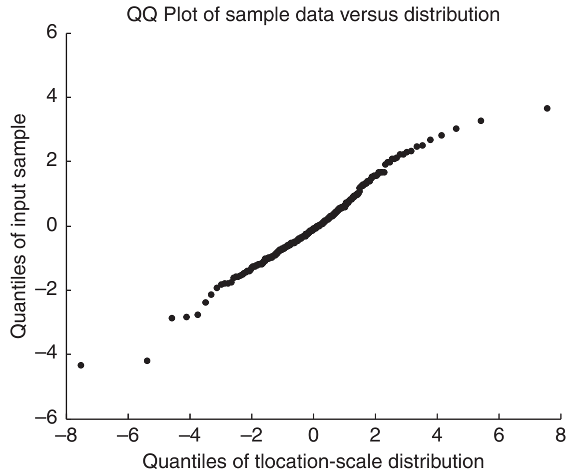


Figure A1. $Q$–$Q$ Plot for marginal residual $\tilde{z}_1$.

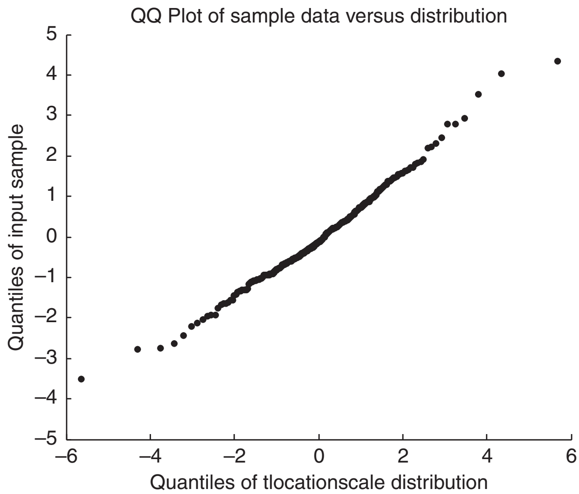


Figure A2. $Q$–$Q$ Plot for marginal residual $\tilde{z}_2$.

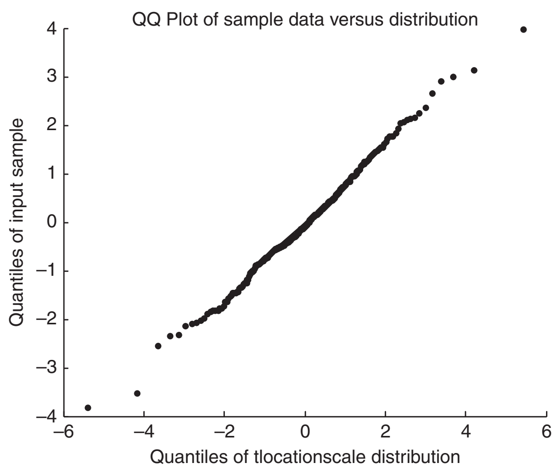


Figure A3. $Q$–$Q$ Plot for marginal residual $\tilde{z}_3$.

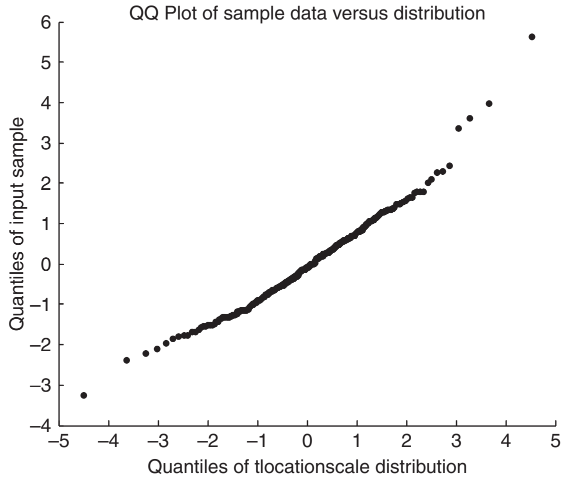


Figure A4. *Q*–*Q* Plot for marginal residual $\tilde{z}_4$.

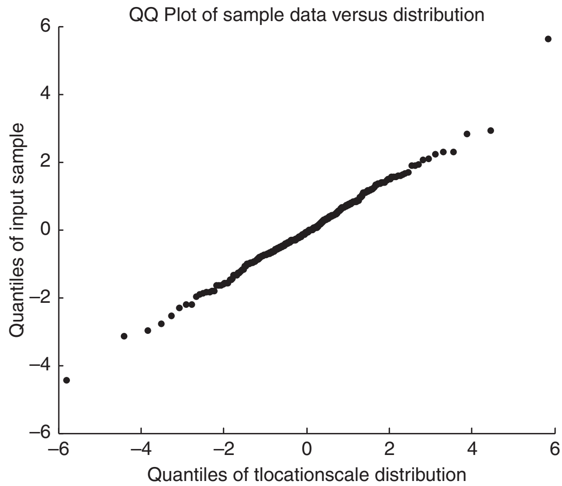


Figure A5. *Q*–*Q* Plot for conditional residual $\tilde{z}_2|\tilde{z}_1$.

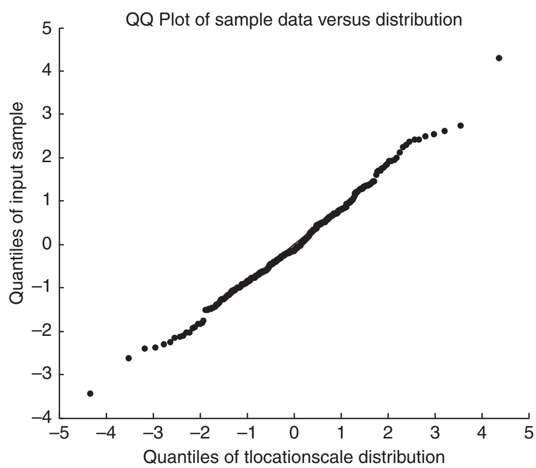


Figure A6. *Q*–*Q* Plot for conditional residual $\tilde{z}_3|\tilde{z}_1$.

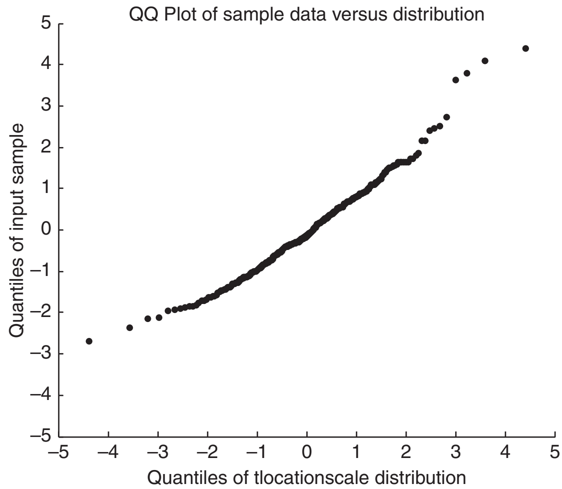


Figure A7. $Q$–$Q$ Plot for conditional residual $\tilde{z}_4|\tilde{z}_1$.

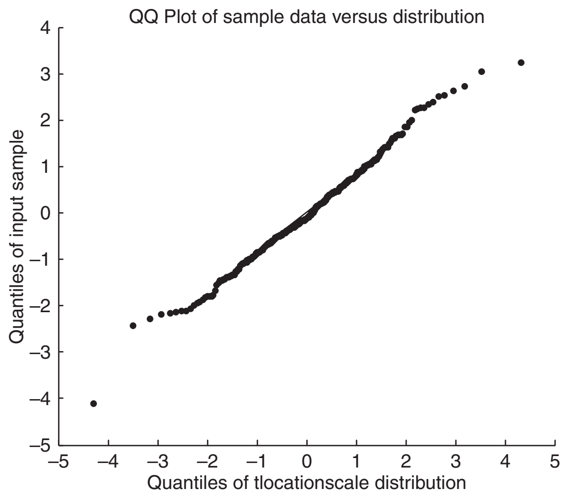


Figure A8. $Q$–$Q$ Plot for conditional residual $\tilde{z}_3|\tilde{z}_2$.

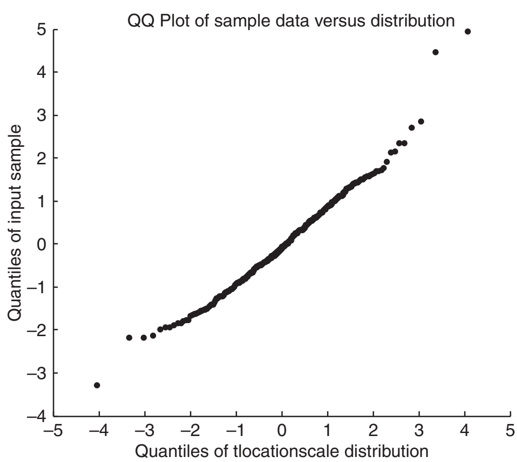


Figure A9. $Q$–$Q$ Plot for conditional residual $\tilde{z}_4|\tilde{z}_2$.

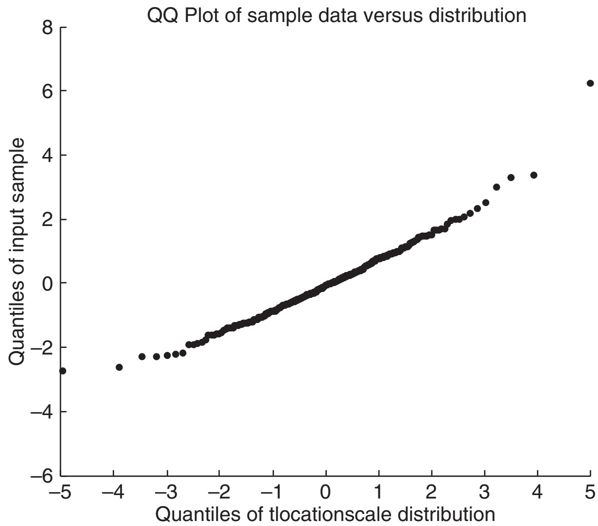


Figure A10. *Q–Q* Plot for conditional residual $\tilde{z}_4|\tilde{z}_3$.