MATHEMATICAL ASPECTS OF SUPERINTEGRABLE SYSTEMS

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I certify that this thesis satisfies all the requirements as a thesis for the degree of Master of Science.

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This is to certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality, as a thesis for the degree of Master of Science.

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ABSTRACT

MATHEMATICAL ASPECTS OF SUPERINTEGRABLE SYSTEMS

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Polynomial invariants and the geometric aspects of superintegrable systems in two-dimensional space were analyzed. Killing tensors and Killing-Yano tensors corresponding to a set of four two dimensional superintegrable systems were found. The geometries obtained by adding a suitable term involving the components of the angular momentum to the corresponding free Lagrangians. Killing vectors, Killing-Yano and Killing tensors of the obtained manifolds were investigated.

Keywords: Superintegrable systems, polynomial invariants, Killing tensors, Killing-Yano tensors.

ÖZ

MATEMATİKSEL YÖNLERİYLE SÜPERİNTEGRALLENEBİLİR SİSTEMLER

Defterli, Özlem

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Polinom değişmezler ve geometrik yönleriyle iki boyutlu uzayda süperintegrallenebilir sistemler analiz edilmiştir. İki boyutlu süperintegrallenebilir sistemlerden dördüne ilişkin Killing tansörleri ve Killing-Yano tansörleri hesaplanmıştır. Verilen bağımsız bir Lagrangian'a açısal momentin bileşenlerini içeren uygun bir terimin eklenmesiyle geometriler elde edilmiştir. Bu şekilde oluşturulan manifoldların Killing vektörleri, Killing-Yano tansörleri ve Killing tansörleri incelenmiştir.

Anahtar sözcükler: Süperintegrallenebilir sistemler, polinom değişmezler, Killing tansörleri, Killing-Yano tansörleri.

Dedicated To My PARENTS...

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CHAPTER 1

INTRODUCTION

The superintegrability in two dimensions represents today one of the hot topics in applied mathematics. On the other hand, it represents the laboratory where we tested some techniques and find some important results and we hope that we will apply something in higher dimensions.

Extensive studies exist about systems with second-order integrals of motion, either in Euclidean space or in spaces with nonzero-constant curvature.

Highly symmetric systems are often integrable, and in special cases, superintegrable and exactly solvable, Wojciechowski [47], Kalnins [33].

Superintegrable systems on the two-dimensional Euclidean spaces have been classified in Evans [18], and also extended to the two dimensional spheres, Grosche [25].

Recent classifications of superintegrable systems for these two dimensional Riemannian spaces can be found in Ranada [39], Kalnins [34], Kalnins [35].

A Hamiltonian system of N degrees of freedom is said to be completely integrable, in the Liouville-Arnold sense, if it possesses functionally independent globally defined and single valued N integrals of motion in involution Arnold [1], Goldstein [23]. It is called superintegrable if it admits more than N integrals of motion. Not all the integrals of superintegrable system can be in involution, but they must be functionally independent otherwise the extra invariants are trivial. In analogy to the classical mechanics, a quantum mechanical system described in N-dimensional Euclidean space by a stationary Hamiltonian operator H is called to be completely integrable if there exists a set of N - 1 (together with H, N) algebraically independent linear operators X_i, $i = 1, 2, ..., N-1$ commuting with H and among each other, Fris [21]-Tempesta [44]. If there exist k additional operators Y_j , $j = 1, 2, ..., k$ where $0 < k \leq N-1$, commuting with H it is said to be superintegrable. The superintegrability is said to be minimal if $k = 1$ and maximal if $k = N-1$.

The plan of my thesis is as follows:

Basic definitions of the concept of superintegrability with the classifications of the polynomial invariants are given in Chapter II.

Chapter III covers the problem of superintegrability on a curved manifold in two dimensions.

In Chapter IV, non-generic(hidden) symmetries of two dimensional superintegrable manifolds were calculated.

Chapter V is dedicated to my concluding remarks.

CHAPTER 2

INVARIANTS OF TWO DIMENSIONAL SUPERINTEGRABLE SYSTEMS

In the first part of this chapter, the fundamental notions of superintegrability are given and the types of transformations that preserve integrability are discussed. The polynomial invariants of the two dimensional superintagrable systems are classified in the second part.

2.1 Basic definitions

The concept of integrability can be formulated for the Lagrange method but we will use the Hamiltonian approach with Poisson bracket. In that case a Lagrangian $L = L(x_i, x_i')$, where $x' = dx/dt$ is given and the equations of motion follow from

$$\frac{d}{dt}\frac{\partial L}{\partial x_i'} - \frac{\partial L}{\partial x_i} = 0, \qquad i = 1, \ldots, D. \tag{2.1}$$

Definition 1 *Let the Lagrangian $L = L(x_i, x_i')$ be given. Consider a function $I = I(x_i, x_i')$. Its total time derivative is given by $dI/dt = x_i' I_{x_i} + x_i'' I_{x_i'}$. If dI/dt vanishes when x_i'' are eliminated using (2.1) then I is a constant of motion.*

For higher-dimensional systems there can be several invariants and the definitions generalize as follows, Hietarinta [28].

Definition 2 *A set of functions I_n is said to be involution if $\{I_k, I_m\} = 0$, for all k, m.*

Definition 3 *A D-dimensional Hamiltonian system H is said to be Liouville integrable if there exists a system of D functionally independent functions $I_n \in F(P)$ $[I_1 = H]$ which are in involution.*

The system is **superintegrable** if a further m integrals $\{Y_1, \ldots, Y_m, 1 \leq m \leq n-1\}$ exist such that the set of constants $\{I_1 = H, I_2, \ldots, I_n, Y_1, \ldots, Y_m\}$ are functionally independent. The additional integrals have vanishing Poisson bracket with H, but not necessarily with each other or with the I_i's, Kalnins [33].

This definition is motivated by a famous theorem of Liouville, which states that systems satisfying the above definition of Liouville integrability can in fact be integrated by quadratures. The following theorem proves that the property of being in involution is preserved under generalized canonical transformations.

Theorem 1 *A set of functions in involution will stay in involution under the transformation $X_i = f_i(x_1, \ldots, x_D, p_1, \ldots, p_D)$, $P_i = g_i(x_1, \ldots, x_D, p_1, \ldots, p_D)$ for which $\{f_j, f_k\} = \{g_j, g_k\} = 0$ and $\{f_j, g_k\} = \delta_{jk} Z$, where Z is some function of the p's and x's.*

For ordinary canonical transformations Z=1. In most applications below Z will be a constant, but it could in principle be a more general function.

Proof. Let us consider two commuting functions $K(X, P)$ and $L(X, P)$, and their transformed counterparts $k(x,p) = K(f(x,p), g(x,p))$ and $l(x,p) = L(f(x,p), g(x,p))$. By the chain rule and the commutation properties of the f's and the g's we have

$$\{k, l\}_{(x,p)} = \left(\frac{\partial K}{\partial X_i}\frac{\partial L}{\partial P_j} - \frac{\partial K}{\partial P_j}\frac{\partial L}{\partial X_i}\right)\{f_i, g_j\} = \{K, L\}_{(X,P)} Z \tag{2.2}$$

and thus k and l commute if and only if K and L commute, Hietarinta [28].

The rest of this section is devoted to a discussion of certain simple transformations that preserve integrability and to demonstrating how they can be used to eliminate at least some of the nonessential degrees of freedom.

The first simplification in the search of any kind of constant of motion is obtained by considering its invariance when $p \to -p$.

Definition 4 *A function K of the p's and x's is said to have a good time reflection parity if $K(-p,x) = cK(p,x)$. K is said to be even if $c = 1$ and odd if $c = -1$.*

Theorem 2 *If for a nontrivial pair of functions K, L we have $\{K, L\} = 0$, and K has good time reflection parity, then there exists a nontrivial function L_* such that it has good time reflection and $\{K, L_*\} = 0$.*

Proof. Applying the Theorem 1 with the transformation $f_i(x,p) = x_i$ and $g_i(x,p) = -p_i$ gives $Z = -1$. Thus if $K(p,x)$ and $L(x,p)$ commute, so do $K(-p,x) = cK(p,x)$ and $L(x,-p)$. Therefore K commutes with $L_+(p,x) = \frac{1}{2}\{L(p,x) + L(-p,x)\}$ and $L_-(p,x) = \frac{1}{2}\{L(p,x) - L(-p,x)\}$ as well. Both of these have good time reflection parity, and since they cannot vanish simultaneously, if $L(p,x)$ is nonzero we have found that L_* is the invariant, Hietarinta [28].

Note that if we apply this to the Hamiltonian $H = \frac{1}{2}\sum(p^2)_i + V(x)$ then all the invariants must be either even or odd in momenta.

The version of Theorem 2 for weighted homogeneous functions is given as follows:

Theorem 3 *If we have two commuting functions $K(p,x)$, $L(p,x)$, of which K is weighted homogeneous and L is a polynomial, then L can be written as a sum $L = \sum_{i=1}^{M} L_i(p,x)$, where each L_i is weighted homogeneous with different degree, and each L_i commutes with K.*

Proof. Assume that K is weighted homogeneous for the scalings $x \to c^n x$, $p \to c^m p$. Each monomial of L will get a definite factor c^k in this scaling. Collecting terms with the same k together we can write $L(p,x) = \sum_{i=1}^{M} L_i(p,x)$, where $L_i(c^m p, c^n x) = c^{k_i} L_i(p,x)$ and $k_i \neq k_j$ for $i \neq j$. Each L_i is weighted homogeneous and can be shown to commute with H as follows. Since H is invariant in the above scaling it commutes with $L(c^m p, c^n x)$ for any c. Let us now take M different c's and consider the sum

$$\sum_{j=1}^{M} d_j L(c_j^m p, c_j^n x) = \sum_{j=1}^{M} d_j \sum_{i=1}^{M} c_j^{k_i} L_i(p,x) = \sum_{i=1}^{M} \{\sum_{j=1}^{M} d_j (c_j)^{k_i}\} L_i, \tag{2.3}$$

which commutes with K as well. Now any given L_m can be picked up from this sum choosing the d_j's so that they solve the set of equations

$$\sum_{j=1}^{M} d_j c_j^{k_i} = \delta_{im}, \qquad i = 1, \ldots, M. \tag{2.4}$$

This set has a nontrivial solution because the c_j's can be chosen so that $det_{ij}\{c_j^{k_i}\}$ does not vanish, Hietarinta [28].

As a corollary we find that in the search for integrable weighted homogeneous Hamiltonians we may assume that a polynomial invariant is weighted homogeneous.

Let us consider, Hietarinta [28], a homogeneous polynomial of x and y (or of p_x and p_y)

$$P(x,y) = \sum_{n=0}^{N} a_n x^{N-n} y^n. \tag{2.5}$$

Since integrability is preserved under the rotations

$$\begin{aligned} x &= coswX + sinwY, & p_x &= coswP_X + sinwP_Y, \\ y &= -sinwX + coswY, & p_y &= -sinwP_X + coswP_Y, \end{aligned} \tag{2.6}$$

we can use (2.6) to transform the polynomial (2.5) to a more suitable form.

If we substitute (2.6) into (2.5), then we obtain

$$P(x,y) = P'(X,Y) = X^N\{\sum_{n=0}^{N} a_n cosw^{N-n}(-sinw)^n\}$$

$$+X^{N-1}Y\{\sum_{n=0}^{N} cosw^{N-n}(-sinw)^n[(n+1)a_{n+1} - (N-n+1)a_{n-1}]\} + \ldots \quad (2.7)$$

If the coefficient $a_{N-1} = 0$ we can make the $X^{N-1}Y$ term vanish by choosing $cosw = 0$, otherwise we divide out by $cosw^N$ and then the second term vanishes if

$$\sum_{n=0}^{N}(-tanw)^n[(n+1)a_{n+1} - (N-n+1)a_{n-1}] = 0. \quad (2.8)$$

This method was used in search of integrable cubic potentials.

Now, we will discuss the translations in the coordinates: $x \to x+u$, $y \to y+v$, where u and v are constants. Both in a general search for integrable systems and in the identification of integrable potentials we are again faced with the problem of fixing the translational degree of freedom.

In the case where the invariant has leading part consisting some angular momentum terms, the translation freedom can be used to fix some of the free parameters.

Example: If the invariant is linear in space and momentum coordinates it must be of the form $I = (ax+b)p_y + (-ay+c)p_x$. If $a \neq 0$ we make the translation $x \to x - b/a$, $y \to y + c/a$ to bring I into the form $I = a(xp_y - yp_x)$. And if $a = 0$ we would make a rotation and scale to bring the invariant into $I = p_x$ or $I = p_x \pm ip_y$. In this simple example the equations for the integrability of the Hamiltonian can be solved in principle, however in more complicated cases it is necessary to break down the investigation into simple subcases like the ones above.

The last type of transformation which is called gauge transformation is applicable only for Hamiltonians that have terms linear in p, i.e.

$$H = \frac{1}{2}(p_x^2 + p_y^2) + A(x,y)p_x + B(x,y)p_y + C(x,y). \quad (2.9)$$

A Hamiltonian in this form will stay form-invariant under the canonical transformation

$$p_x \rightarrow p_x + F(x,y)_x, \qquad p_y \rightarrow p_y + F(x,y)_y, \tag{2.10}$$

where $F(x,y)$ is an arbitrary function and the subscripts indicate partial derivatives. The functions A, B and C will the change as

$$A \rightarrow A + F_x, \qquad B \rightarrow B + F_y,$$

$$C \rightarrow C + AF_x + BF_y + \frac{1}{2}(F_x^2 + F_y^2). \tag{2.11}$$

From these equations we see that the quantities U and W, defined by

$$U = A_y - B_x, \qquad W = C - \frac{1}{2}(A^2 + B^2). \tag{2.12}$$

are gauge-invariant and therefore suitable to characterize the model.

2.2 Polynomial invariants

In this section the systems having the polynomial invariants are discussed. We assumed that the Hamiltonian is of the form

$$H = \frac{1}{2}(p_x^2 + p_y^2) + V(x,y). \tag{2.13}$$

The aim is to construct a function which is in involution with H. We know that the Hamiltonian is even in momenta and so the second invariant is either even or odd in momenta. Hence the general form of the invariant I is :

$$I = \sum_{n=0}^{[N/2]} \sum_{m=0}^{N-2n} p_x^m p_y^{N-2n-m} d^{m,N-2n}(x,y), \tag{2.14}$$

where the functions d are the unknowns. The problem is how to find these unknown functions d. Taking the Poisson bracket of H with I is the right way of it.

2.2.1 Linear invariants

The invariant for the Hamiltonian in (2.13) must be either odd or even when $p_i \to -p_i$(as it is shown in Section II.1). As potential invariants, we first turn our attention to functions which are linear in momenta,

$$I = A(x,y)p_x + B(x,y)p_y. \tag{2.15}$$

We require I to be in involution with H. So, computing the Poisson bracket of (2.13) with (2.15) and equating to zero gives the following set of equations Hietarinta [28]:

$$A_x = 0, \tag{2.16}$$

$$A_y - B_x = 0, \tag{2.17}$$

$$B_y = 0, \tag{2.18}$$

$$AV_x + BV_y = 0. \tag{2.19}$$

The solutions of these equations (2.16), (2.17) and (2.18) are :

$$A = ay + b, \qquad B = -ax + c. \tag{2.20}$$

In connection with (2.19) two cases arise:

Case (1): $a = 0$. Then (2.19) can be integrated with the result

$$V = f(cx - by), \qquad I = bp_x + cp_y. \tag{2.21}$$

Case (2): $a \neq 0$. Let us take $a = 1$. To simplify A and B make the translation $X = x + c$, $Y = y - b$, $P_Y = p_y$, $P_X = p_x$ which is a canonical transformation. Then (2.19) becomes $yV_x - xV_y = 0$, and the integration gives

$$V = f(x^2 + y^2), \qquad I = yp_x - xp_y. \tag{2.22}$$

By combining the two cases we obtained the solution

$$V = f(\tfrac{1}{2}a(x^2 + y^2) + cx - by)$$

$$I = a(yp_x - xp_y) + bp_x + cp_y. \tag{2.23}$$

2.2.2 Quadratic invariants

In the case of the quadratic invariants the second invariant has the following form with the coefficients A, B, C and D which are functions of x and y.

$$I = Ap_xp_x + Bp_xp_y + Cp_yp_y + D. \tag{2.24}$$

The following set of equations are obtained in Hietarinta [28] by taking the Poisson bracket (2.24) with (2.13) :

$$A_x = 0, \qquad A_y + B_x = 0, \qquad B_y + C_x = 0, \qquad C_y = 0, \tag{2.25}$$

$$D_x = 2AV_x + BV_y, \qquad D_y = BV_x + 2CV_y. \tag{2.26}$$

The equations (2.25) have the solution

$$\begin{aligned} A &= ay^2 + by + c, \\ B &= -2axy - bx - dy - e, \\ C &= ax^2 + dx + f. \end{aligned} \tag{2.27}$$

When (2.27) is substituted into (2.26) we get the condition in below for the integrability of D :

$$(2axy + bx + dy + e)(V_{xx} - V_{yy}) - 2[a(x^2 - y^2) + dx - by + f - c]V_{xy}$$

$$+ 3(2ay + b)V_x - 3(2ax + d)V_y = 0. \tag{2.28}$$

The equation (2.28) is solved in the following cases :

(1) $a \neq 0$. Take $a = 1$ and integrate it by Darboux,

(2) $a \neq 0$, but $c = 0$,

(3) $a \neq 0$, $c \neq 0$, $e^2+c^2 = 0$, then $e = \mp ic$.

(4) $a = 0$, but b or $d \neq 0$. Let us investigate this case in details. If we do the rotation as $b = 1$, $d = 0$ and the translations so that $f = 0$ and $c = 0$, $e = 0$. So, the equation(2.28) becomes

$$2yV_{xy} + 3V_x + x(V_{xx} - V_{yy}) = 0, \tag{2.29}$$

with the solution

$$V = [f(r+y) + g(r-y)]/r,$$

$$I = (yp_x - xp_y)p_x + [(r+y)g(r-y) - (r-y)f(r+y)]/r. \tag{2.30}$$

(5) $a = 0$, $b \neq 0$ but $b^2 + d^2 = 0$,

(6) when the constants e, c and f are eliminated,

(7) a=0, $b = 0$ and $d = 0$,

(8) a=0, $b = 0$, $d = 0$,$f = 0$, $c \neq 0$ but $e^2 + c^2 = 0$.

Superintegrability: Observe that we have two independent invariants other than H, i.e. *superintegrability*, when a potential is two of the following potential types simultaneously; homogeneous, polynomial or in the form $V = (x^2+y^2)^2 + Ax^2 + By^2$ (order doubling, see Hietarinta [28], page 107-108). For example in Fris [21]:

$$V = a(x^2+y^2) + bx^{-2} + cy^{-2} \tag{2.31}$$

belongs simultaneously to the cases (2) (where $V = g(r) + f(x/y)r^{-2}$) and (7) (where $V = f(x) + g(y)$),

$$V = ax^2 + by^2 + cx^{-2} \tag{2.32}$$

to the cases (4)(where $V = [f(r+y) + g(r-y)]/r$) and (7), and

$$V = a/r + [b/(r+y) + c/(r-y)]/r \tag{2.33}$$

to the cases (2) and (4).

2.2.3 Cubic invariants

The invariant cubic in momenta has the form

$$I_2 = Ap_x^3 + Bp_x^2 p_y + Cp_x p_y^2 + Dp_y^3 + Fp_x + Gp_y. \tag{2.34}$$

where $A, \ldots, G$ are functions of x and y. Taking the Poisson bracket (2.13) with (2.34) and then collecting the coefficients of the terms $p_x^n p_y^m$ gives the following set of equations, Hietarinta [28]:

$$A_x = 0, \tag{2.35}$$

$$B_x + A_y = 0, \tag{2.36}$$

$$C_x + B_y = 0, \tag{2.37}$$

$$D_x + C_y = 0, \tag{2.38}$$

$$D_y = 0, \tag{2.39}$$

$$F_x - 3AV_x - BV_y = 0, \tag{2.40}$$

$$G_x + F_y - 2BV_x - 2CV_y = 0, \tag{2.41}$$

$$G_y - CV_x - 3DV_y = 0, \tag{2.42}$$

$$FV_x + GV_y = 0. \tag{2.43}$$

Equations (2.35) - (2.39) have the polynomial solutions

$$A = a_0 + a_1 y + a_2 y^2 + a_3 y^3, \tag{2.44}$$

$$B = b_0 + b_1 y + d_2 y^2 - a_1 x - 2a_2 xy - 3a_3 xy^2, \tag{2.45}$$

$$C = c_0 - b_1 x + a_2 x^2 - d_1 y - 2d_2 xy + 3a_3 x^2 y, \tag{2.46}$$

$$D = d_0 + d_1 x + d_2 x^2 - a_3 x^3. \tag{2.47}$$

There are many ways to complete the computation but in Holt [32] the following method was used. Firstly, the solution of (2.43) is found by

$$F = V_x Z, \qquad G = -V_y Z, \tag{2.48}$$

where Z is a new unknown function of x and y. Then, by substituting this to (2.40) - (2.42) results that

$$ZV_{xy} + Z_x V_y - 3AV_x - BV_y = 0, \tag{2.49}$$

$$Z(V_{yy} - V_{xx}) + Z_y V_y - Z_x V_x - 2BV_x - 2CV_y, \tag{2.50}$$

$$-ZV_{xy} - Z_y V_x - 3DV_y - CV_x = 0. \tag{2.51}$$

Now we have three equations for two unknown functions and ten free constants. Then adding (2.49) and (2.51) gives that

$$V_x(3A + CZ_y) + V_y(3D + B - Z_x) = 0, \tag{2.52}$$

from which Z can be solved as

$$Z = \Phi(V) + Y, \tag{2.53}$$

$$\begin{aligned} Y &= z_0 + (3d_0 + b_0)x + (3d_1 - a_1)x^2/2 + d_2x^3 - (3a_0 + c_0)y - (3a_1 - d_1)y^2/2 \\ &\quad -a_2y^3 - a_2x^2y + d_2xy^2 + b_1xy - 3a_3(x^2 + y^2)^2/4. \end{aligned} \tag{2.54}$$

When these values of Z and Y are put into (2.49) and (2.50) then

$$YV_{xy} + 3DV_y - 3AV_x = -\Phi(V)_{xy}, \tag{2.55}$$

$$Y(V_{yy} - V_{xx}) - 3(A + C)V_y - 3(D + B)V_x = \Phi(V)_{xx} - \Phi(V)_{yy}. \tag{2.56}$$

The equations (2.55) and (2.56) can be solved in three cases which are the following with their results :

(1) when $\Phi(V) \neq 0$

$$H = \frac{1}{2}(p_x^2 + p_y^2) + (x^2 - y^2)^{-2/3}, \tag{2.57}$$

$$I_1 = (p_x^2 - p_y^2)(xp_y - yp_x) - 4(yp_x + xp_y)(x^2 - y^2)^{-2/3}. \tag{2.58}$$

(2) assume $\Phi(V) = 0$, then (2.55) and (2.56) are linear in V. Therefore a linear superposition rule is applied to the integrable potentials.

(3) taking $\Phi = 0$ in each case for the certain choices of parameters and then solving the equations (2.55) and (2.56) the following results are obtained in Holt [32]:

$$V = \frac{3}{4}y^{4/3} + x^2y^{-2/3} + \delta y^{-2/3}, \tag{2.59}$$

$$I_2 = 2p_x^3 + 3p_xp_y^2 + 3p_x(-3y^{4/3} + 2x^2y^{-2/3} + 2\delta y^{-2/3}) + 18p_yxy^{1/3}, \tag{2.60}$$

$$V = x^2 + 4y^2 + \delta x^{-2}, \tag{2.61}$$

$$I_3 = p_x^2 p_y + 8xyp_x + 2(-x^2 + \delta x^{-2})p_y. \tag{2.62}$$

In the last case the system has a third invariant which means that it is *superintegrable.* Another *superintegrable* case was found in Fokas [20] with

$$V = x^2/2 + y^2/18, \tag{2.63}$$

$$I_3 = (xp_y - yp_x)p_y^2 + y^3 p_x/27 - xy^2 p_y/3. \tag{2.64}$$

2.2.4 Higher order invariants

There are few results about the invariants of order higher than four. Some examples are given :

I. *Holt - type potentials* :

From Painlevé analysis in Grammaticos [24] and Hietarinta [29] the potential $V = 12x^{4/3} + y^2 x^{-2/3}$ is integrable and a search at p^6 produced a positive result. This potential can have additional terms, in Hietarinta [30] it was found that

$$V = 12x^{4/3} + (y^2 + d)x^{-2/3} + Ly^{-2}. \tag{2.65}$$

is also integrable with

$$\begin{aligned} I = {} & p_y^6 + 3p_x^2 p_y^4 + 72x^{1/3} y p_x p_y^3 + 6p_y^4(3x^{4/3} + (y^2 + d)x^{-2/3}) + 648x^{2/3}y^2 p_y^2 \\ & +648y^4 + L[12y^{-2}p_x^2 p_y^2 + 6y^{-2}p_y^4 + 12Ly^{-4}p_x^2 + 144x^{1/3}y^{-1}p_x p_y \\ & +12(6x^{4/3}y^{-2} + 2dx^{-2/3}y^{-2} + 2x^{-2/3} + Ly^{-4})p_y^2 + 8(54x^{2/3} \\ & +9Lx^{4/3}y^{-4} + 3dLx^{-2/3}y^{-4} + 3Lx^{-2/3}y^{-2} + L^2 y^{-6})]. \end{aligned} \tag{2.66}$$

II. *Toda - type potentials* :

The Toda - type potential has the form

$$V = e^{[\sqrt{3}x - y]/2} + e^y + e^{[-\sqrt{3}x + y]/2}. \tag{2.67}$$

There are some generalizations of (2.67) that are integrable with a sixth order invariant. If we denote the cubic invariant of first two terms of (2.67) with I_{3T}, then the sixth order results are as follows :

$$V_1 = e^{([\sqrt{3}x-y]/2)} + e^{y} + e^{(-\sqrt{3}x)}. \tag{2.68}$$

$$\begin{aligned} I \;=\; & I_{3T}^2 + e^{(-\sqrt{3}x)}\{6[p_x^4 - 4p_x^2p_y^2 + 3p_y^4] + 12(2e^{([\sqrt{3}x-y]/2)} \\ & + e^{(-\sqrt{3}x)} - 4e^{y})p_x^2 + 12\sqrt{3}([\sqrt{3}x - y]/2)p_xp_y \\ & + 12(-3e^{([\sqrt{3}x-y]/2)} - 2e^{(-\sqrt{3}x)} + 6e^{y})p_y^2 \\ & + 8[-9e^{([\sqrt{3}x+y]/2)} + 3e^{(-[\sqrt{3}x+y]/2)} \\ & - 6e^{(-\sqrt{3}x+y)} + e^{(-2\sqrt{3}x)} + 9e^{(2y)}]\}, \end{aligned} \tag{2.69}$$

$$V_2 = e^{([\sqrt{3}x-y]/2)} + e^{y} + e^{(-\sqrt{3}x/3)}, \tag{2.70}$$

$$\begin{aligned} I \;=\; & I_{3T}^2 + e^{(-\sqrt{3}x/3)}\{6[p_x^4 - 4p_x^2p_y^2 + 3p_y^4] + 12(2e^{([\sqrt{3}x-y]/2)} \\ & + e^{(-\sqrt{3}x/3-4e^{y})}p_x^2 + 36\sqrt{3}e^{([\sqrt{3}x-y]/2)}p_xp_y \\ & + 12(3e^{([\sqrt{3}x-y]/2)} - 2e^{(-\sqrt{3}x/3)} + 6e^{y})p_y^2 \\ & + 8[9e^{([\sqrt{3}x+y]/2)} + 3e^{([\sqrt{3}x-3y]/6)} \\ & - 6e^{(-\sqrt{3}x/3+y)} + e^{(-2\sqrt{3}x/3)} + 9e^{(2y)}]\}. \end{aligned} \tag{2.71}$$

V_1 can be identified as V_{G2} of Bogoyavlensky [12] while for V_2 similar Lie-algebraic identifications were given in Yoshida [46] and Dorizzi [17]. The above potentials are also integrable when the e^y term is omitted. Both potentials can then be rotated to

$$V_3 = e^{([\sqrt{3}x-3y]/2)} + e^{y}, \tag{2.72}$$

$$I \;=\; p_x^6 - 6p_x^4p_y^2 + 9p_x^2p_y^4 + 6e^{([-\sqrt{3}x-3y]/2)}(p_x^4 + 3\sqrt{3}p_x^3p_y$$

$$+5p_x^2p_y^2 - \sqrt{3}p_xp_y^3) + 12e^y(-p_x^4 + p_x^2p_y^2) - 3(5e^{(\sqrt{3}x-3y)}$$

$$+4e^{([\sqrt{3}x-y]/2)} - 12e^{(2y)})p_x^2 - 18\sqrt{3}(e^{(\sqrt{3}x-3y)}$$

$$+2e^{([\sqrt{3}x-y]/2)})p_xp_y + 3e^{(\sqrt{3}x-3y)}p_y^2 + 8(3e^{(\sqrt{3}x-2y)}$$

$$+e^{(3[\sqrt{3}x-3y]/2)}). \tag{2.73}$$

III. *The Calogero system* :

The Calogero-Moser (C-M) system, Ranada [41], is a completely integrable system of n particles with interaction force, between every two particles, given by the inverse of the square of their relative distance. The Lagrangian is given by

$$L_{CM} = \frac{1}{2}\sum_{j=1}^{n} v_j^2 - \sum_{i<j} V_{ij}, \qquad V_{ij} = \frac{c_0^2}{q_{ij}^2}, \tag{2.74}$$

where $q_{ij} = q_i - q_j, i, j = 1, 2, \ldots, n$, and c_0 is an arbitrary constant (the masses of the particles are set equal to unity).

Moser proved in Moser [38] that this system can be presented as a Lax equation

$$\frac{dA}{dt} = \{A, B\}, \qquad A = A_d + ic_0A_n, \qquad B = ic_0(B_d - B_n), \tag{2.75}$$

where A_d and B_d denote the diagonal matrices

$$A_d = diagonal[v_1, v_2, \ldots, v_n], B_d = diagonal[\sum_{j\neq 1} x_{1j}^2, \sum_{j\neq 2} x_{2j}^2, \ldots, \sum_{j\neq n} x_{nj}^2], \tag{2.76}$$

and A_n and B_n take the form

$$A_n = [(1-\delta_{ij})x_{ij}], \qquad B_n = [(1-\delta_{ij})x_{ij}^2], \tag{2.77}$$

where $x_{ij} = 1/q_{ij}$. The important point is that, because of the Lax equation, the traces of the powers of the matrix A are constants of motion

$$I_m = (\frac{1}{m})trA^m, \qquad \frac{d}{dt}I_m = 0, \qquad m = 1, 2, \ldots, n. \tag{2.78}$$

These n functions, which are globally defined, independent, and in involution, take the form

$$I_m = (\frac{1}{m})(v_1^m + v_2^m + \ldots + v_n^m)$$
$$+\text{terms of lower order in the velocities.} \quad (2.79)$$

The main objective of this example is the study of a Lax representation with the following three main characteristics: (i) it depends on two parameters, (ii) it is time dependent, and (iii) it includes, as a particular case, the standard(time-independent) Lax representation.

We begin by introducing the following notation: K will represent a linear polnomial in the time t ,

$$K = k_0 + k_1 t, \quad (2.80)$$

and Q the diagonal matrix defined as

$$Q = \text{diagonal}[q_1, q_2, \ldots, q_n]. \quad (2.81)$$

Next we introduce the following time-dependent matrix:

$$A^t = KA - k_1 Q. \quad (2.82)$$

In the following the two coefficients, k_0, k_1, taken as parameters.

Proposition 1 *The two matrices (A^t, B) are a Lax pair for the Calogero-Moser system.*

Proof. The time derivative of A^t is given by

$$\frac{d}{dt}A^t = k_1 A + K\frac{d}{dt}A - k_1\frac{d}{dt}Q. \quad (2.83)$$

The time-evolution of Q can be written as follows:

$$\frac{d}{dt}Q = \{Q, B\} + A. \tag{2.84}$$

Thus we obtain

$$\frac{d}{dt}A^t = k_1 A + K\{A, B\} - k_1(\{Q, B\} + A) = \{KA - k_1 Q, B\} = \{A^t, B\}, \tag{2.85}$$

so the proposition is proved Ranada [41].

The eigenvalues of $A^t(t)$ or, alternatively, the traces of the powers of the matrix $A^t(t)$ are constants of motion

$$K_m = (\frac{1}{m})tr(A^t)^m, \qquad \frac{d}{dt}K_m = 0, \qquad m = 1, 2, \dots, n. \tag{2.86}$$

They take the form

$$K_m = (\frac{1}{m})[k_0^m \sum_{i=1}^{n} v_i^m + \dots + k_1^m \sum_{i=1}^{n} (tv_i - q_i)^m] + \dots. \tag{2.87}$$

Consequently, this two-parameter dependent Lax equation extend, and include as a particular case, the standard Lax representation discussed in the beginning of this model. The case ($k_0 \neq 0$, $k_1 = 0$) reduces to the classical time-independent case studied by Moser. The other particular case ($k_0 = 0$, $k_1 \neq 0$) leads to the following set of integrals, Ranada [41]:

$$J_m = (\frac{1}{m})tr(tA - Q)^m, \qquad \frac{d}{dt}J_m = 0, \qquad m = 1, 2, \dots, n. \tag{2.88}$$

Notice that they have the form

$$\begin{aligned} J_m &= (\frac{1}{m})[(tv_1 - q_1)^m + \dots + (tv_n - q_n)^m] \\ &\quad + \text{terms of lower degree in the velocities.} \end{aligned} \tag{2.89}$$

The general expression for K_m will be

$$K_m = k_0^m I_m + \sum_{s=1}^{m-1} k_0^r k_1^s K_{rs} + k_1^m J_m, \qquad r = m - s. \tag{2.90}$$

Since k_0 and k_1 are arbitrary, the m functions $\{K_{rs}, r+s=m\}$ are, all of them, integrals of motion.

Next we illustrate this situation with the $n=3$ particular case as an example:

$$L_{CM} = \frac{1}{2}(v_1^2 + v_2^2 + v_3^2) - (V_{12} + V_{23} + V_{13}). \tag{2.91}$$

In this simple case the three time-dependent functions K_m, $m = 1, 2, 3$, have the following expressions :

$$\begin{aligned} K_1 &= k_0 I_1 + k_1 J_1, \\ K_2 &= k_0^2 I_2 + k_0 k_1 K_{11} + k_1^2 J_2, \\ K_3 &= k_0^3 I_3 + k_0^2 k_1 K_{21} + k_0 k_1^2 K_{12} + k_1^3 J_3, \end{aligned} \tag{2.92}$$

where I_m, m = 1, 2, 3, are the three time-independent standard constants of motion,

$$\begin{aligned} I_1 &= v_1 + v_2 + v_3, \\ I_2 &= (1/2)(v_1^2 + v_2^2 + v_3^2) + (V_{12} + V_{23} + V_{13}), \\ I_3 &= (1/3)(v_1^3 + v_2^3 + v_3^3) + (V_{12} + V_{13})v_1 \\ &\quad + (V_{21} + V_{32})v_2 + (V_{13} + V_{32})v_3, \end{aligned} \tag{2.93}$$

in which J_m, m = 1, 2, 3, are the three time-dependent constants corresponding to the parameter k_1,

$$\begin{aligned} J_1 &= t(v_1 + v_2 + v_3) - (q_1 + q_2 + q_3), \\ J_2 &= (1/2)[(tv_1 - q_1)^2 + (tv_2 - q_2)^2 + (tv_3 - q_3)^2] + t^2(V_{12} + V_{23} + V_{13}), \\ J_3 &= (1/3) + [(tv_1 - q_1)^3 + (tv_2 - q_2)^3 + (tv_3 - q_3)^3] + t^2[(tv_1 - q_1)(V_{12} + V_{13}) \\ &\quad + (tv_2 - q_2)(V_{12} + V_{32}) + (tv_3 - q_3)(V_{32} + V_{13})], \end{aligned} \tag{2.94}$$

and K_{11}, K_{21}, K_{12} are given by

$$
\begin{aligned}
K_{11} &= 2tI_2 - (q_1v_1 + q_2v_2 + q_3v_3), \\
K_{21} &= 3tI_3 - [q_1(v_1^2 + V_{12} + V_{13}) + q_2(v_2^2 + V_{12} + V_{23}) + q_3(v_3^2 + V_{23} + V_{13})], \\
K_{12} &= 3t^2I_3 - 2t[q_1(v_1^2 + V_{12} + V_{13}) + q_2(v_2^2 + V_{12} + V_{23}) + q_3(v_3^2 + V_{23} + V_{13})] \\
&\quad + (q_1^2v_1 + q_2^2v_2 + q_3^2v_3). \qquad (2.95)
\end{aligned}
$$

As a conclusion, the *superintegrability* of a (time-independent) system means two facts : first, integrability in the Liouville-Arnold sense, second, existence of an additional independent family of integrals.

The C-M system is a ***superintegrable*** system that, in addition, also possesses time-dependent constants of motion.

CHAPTER 3

GEOMETRICAL ASPECTS OF SUPERINTEGRABILITY IN TWO DIMENSIONAL SPACE OF NON - CONSTANT CURVATURE

The problem of when a two dimensional Riemannian space admits more than one quadratic constant is discussed and the results given by Darboux and Koenigs are listed. By considering the Darboux space of type one, the method of separation of variables in three different coordinate systems is applied for the Schrödinger equation that corresponds to the free Hamilton-Jacobi equation. The last section deals with the potentials that gives superintegrability.

3.1 Preliminaries

If we consider a Riemannian space in two dimensions with the following infinitesimal distance and classical Hamiltonian

$$ds^2 = g_{ij}(u)du^i du^j, \qquad i, j = 1, 2, \tag{3.1}$$

$$H = g_{ij}p_i p_j + V(u), \tag{3.2}$$

respectively, Kalnins [33]. Then the corresponding Schrödinger equation is of the form

$$\hat{H}\Psi = -\frac{1}{2\sqrt{g}}\partial_{u^i}(\sqrt{g}g^{ik}\partial_{u^k}\Psi) + V(u)\Psi = E\Psi, \tag{3.3}$$

where $g = det(g_{ij})$. Now, the problem is to investigate the potentials $V(u)$ of (3.2) and Riemannian spaces generated by the metric g_{ij} such that they admit at least two extra (other than H) functionally independent constants of motion of the form

$$\lambda_1 = a^{ij}(u)p_i p_j + b(u) \tag{3.4}$$

or

$$\lambda_2 = a^i(u)p_i + c(u). \tag{3.5}$$

The corresponding Hamilton-Jacobi equation to be solved is obtained from the equation $H = E$ by substituting $p_i = \frac{\partial S}{\partial u^i}$ as

$$H = \frac{1}{2}g^{ij}\frac{\partial S}{\partial u^i}\frac{\partial S}{\partial u^j} + V(u) = E. \tag{3.6}$$

The method of separation of variables can sometimes solve the equation (3.6) by additive separation

$$S = S_1(u^1, \alpha, E) + S_2(u^2, \alpha, E) \tag{3.7}$$

and can solve the corresponding Schrödinger equation by the product separation

$$\Psi = \psi_1(u^1, \lambda, E)\psi_2(u^2, \lambda, E). \tag{3.8}$$

If the condition $\{\lambda_i, H\} = 0$ is satisfied then the quantities λ_i are constants of the motion. This condition implies for λ_2 that $a^i(u)$ is a Killing vector and $a^i(u)p_i$ is a symmetry of the free Hamiltonian and also $c(u) = 0$, and for λ_1, $a^{ij}(u)$ is a Killing tensor.

3.2 On geodesics with quadratic integrals

G. Koenigs answered the problem of when does the free Hamiltonian of a two-dimensional Riemannian space admit more than one quadratic constant of the motion

Koenigs [36]. For a general two-dimensional Riemannian space he took the infinitesimal distance as

$$ds^2 = 4f(x,y)dxdy, \tag{3.9}$$

which can always be done in two dimensions over C. Then the corresponding free Hamiltonian has the form

$$H = \frac{1}{2f(x,y)} p_x p_y. \tag{3.10}$$

Darboux and Koenigs construct the following propositions, Kalnins [33], by assuming the existence of a second order Killing tensor $\lambda = a^{ij}(u)p_i p_j$:

1. Any two-dimensional Riemannian space that admits more than one Killing vector must be a space of constant curvature and admit three linearly independent Killing vectors.

2. Any two-dimensional Riemannian space that admits more than three Killing tensors is a space of constant curvature. It then actually admits five linearly independent Killing tensors which are all bilinear expressions in the Killing vectors. The sixth bilinear combination is the Hamiltonian itself.

3. Any two-dimensional Riemannian space that admits precisely three linearly independent Killing tensors will be a Riemannian space of revolution. In fact there will be one Killing vector and two Killing tensors.

These kind of two-dimensional Riemannian spaces were classified by the following four types of infinitesimal distances

$$\begin{aligned}
(I) \quad ds^2 &= (x+y)dxdy. \\
(II) \quad ds^2 &= \left(\frac{a}{(x-y)^2} + b\right)dxdy. \\
(III) \quad ds^2 &= \left(ae^{-\frac{x+y}{2}} + be^{-x-y}\right)dxdy.
\end{aligned}$$

$$(IV) \quad ds^2 = \frac{a(e^{\frac{x-y}{2}} + e^{\frac{y-x}{2}}) + b}{(e^{\frac{x-y}{2}} - e^{\frac{y-x}{2}})^2} dxdy. \tag{3.11}$$

These are called Darboux spaces denoted by D_1, D_2, D_3 and D_4 respectively.

3.3 The free particle and separation of variables in a Darboux space of type one

In this section the first infinitesimal distance in (3.11) is investigated. By making the change of variables as $x = u + iv$, $y = u - iv$, we obtain the infinitesimal distance

$$ds^2 = 2u(du^2 + dv^2), \tag{3.12}$$

and the Hamiltonian

$$H = \frac{1}{4u}(p_u^2 + p_v^2), \tag{3.13}$$

correspondingly. There are three integrals of the free motion given as,

$$\begin{aligned} K &= p_v, \\ X_1 &= p_u p_v - \frac{v}{2u}(p_u^2 + p_v^2), \\ X_2 &= p_v(vp_u - up_v) - \frac{v^2}{4u}(p_u^2 + p_v^2). \end{aligned} \tag{3.14}$$

which fulfill the polynomial Poisson algebra relations,

$$\{K, X_1\} = 2H, \qquad \{K, X_2\} = -X_1, \qquad \{X_1, X_2\} = 2K^3. \tag{3.15}$$

The existence of the following relation implies these integrals can not be functionally independent.

$$4HX_2 + X_1^2 + K^4 = 0 \tag{3.16}$$

Lets consider the following operators for the quantum problem

$$\begin{aligned}\hat{H} &= -\frac{1}{4u}(\partial_u^2 + \partial_v^2), \\ \hat{K} &= -i\partial_v, \\ \hat{X}_1 &= -\partial_u\partial_v + \frac{v}{2u}(\partial_u^2 + \partial_v^2), &(3.17)\\ \hat{X}_2 &= -\frac{1}{2}[\partial_v, v\partial_u - u\partial_v] + \frac{v^2}{4u}(\partial_u^2 + \partial_v^2), &(3.18)\end{aligned}$$

where $[A, B]_+ = AB + BA$. The quantum versions of the quadratic constants are obtained by the formula

$$\hat{\lambda} = -\frac{1}{\sqrt{g}}\partial_i(a^{ij}\sqrt{g}\partial_j). \tag{3.19}$$

The operators given above have the same relations in (3.15) with the commutator bracket instead of the Poisson bracket as

$$[\hat{K}, \hat{X}_1] = 2i\hat{H}, \qquad [\hat{K}, \hat{X}_2] = -i\hat{X}_1, \qquad [\hat{X}_1, \hat{X}_2] = -2i\hat{K}^3 \tag{3.20}$$

and also

$$4\hat{H}\hat{X}_2 + \hat{X}_1{}^2 + \hat{K}^4 = 0. \tag{3.21}$$

When we consider classically, if we have a general quadratic first integral λ with the free Hamiltonian

$$H = g_{ij}(u)p_ip_j, \tag{3.22}$$

and the characteristic equation,

$$|a^{ij} - \rho g^{ij}| = 0, \tag{3.23}$$

which has two distinct roots ρ_1 and ρ_2, then the Hamiltonian will have Liouville form as follows

$$H = \frac{\sigma(\rho_1)p_{\rho_1}^2 + \tau(\rho_2)p_{\rho_2}^2}{\rho_1 + \rho_2}. \tag{3.24}$$

Now, the separation of variables can solve both classical and quantum systems. If all different separable coordinate systems for a given Hamiltonian is classified then it is needed to know that how many different quadratic first integrals are possible. Here, by the notion of equivalence we mean that two quadratic integrals are equivalent if they are related by a motion of this group. As a result the most general quadratic constant has the form

$$\lambda = aX_1 + bX_2 + cK^2. \tag{3.25}$$

The second order elements X_i transform under the adjoint action as

$$\begin{aligned} X_i &\rightarrow e^{\alpha K} X_i e^{-\alpha K} \\ &= e^{\alpha Ad(K)} X_i \\ &= X_i + \alpha\{K, X_i\} + \frac{1}{2}\alpha^2\{K,\{K,X_i\}\} + \ldots \end{aligned} \tag{3.26}$$

or

$$\begin{aligned} X_1 &\rightarrow X_1 + 2\alpha H, \\ X_2 &\rightarrow X_2 - \alpha X_1 - \alpha^2 H. \end{aligned} \tag{3.27}$$

In the following there are typical representatives of the three classes of possible quadratic first integrals and for each of them the construction of the separable coordinates is done.

$$X_1 + aK^2, \qquad X_2 + aK^2, \qquad K^2. \tag{3.28}$$

I. *Separating coordinates associated with* $X_1 + aK^2$

The first representative is

$$L = X_1 + \sinh cK^2, \tag{3.29}$$

with new variables

$$r = \rho_1 = -2(Cu+v),$$
$$s = \rho_2 = \frac{2}{C}(u - Cv), \qquad C = e^{-c}, \tag{3.30}$$

where ρ_1, ρ_2 are the roots of the characteristic equation. Then we can rewrite the Hamiltonian and the corresponding quadratic constant in term of these new coordinates as

$$H = \frac{2(C^2+1)^2}{C(s-r)}(\frac{1}{C^2}p_s^2 + p_r^2), \tag{3.31}$$

$$L = 2\frac{(C^2+1)^2}{C(s-r)}(\frac{r}{C^2}p_s^2 + sp_r^2), \tag{3.32}$$

respectively.

II. *Separating coordinates associated with* $X_2 + aK^2$

In that case the representative is $L = X_2 + aK^2$, and the relation between the new variables ξ, η and the roots ρ_i is given by

$$\rho_1 = \eta^2(2a - \eta^2), \qquad \rho_2 = -\xi^2(2a + \xi^2). \tag{3.33}$$

Then the corresponding classical Hamiltonian and constant of motion has the form

$$H = \frac{p_\xi^2 + p_\eta^2}{2(\xi^2+\eta^2)(\xi^2-\eta^2+2a)}, \tag{3.34}$$

$$L = \frac{\eta^2(2a-\eta^2)p_\xi^2 - \xi^2(2a+\xi^2)p_\eta^2}{2(\xi^2+\eta^2)(\xi^2-\eta^2+2a)}. \tag{3.35}$$

The coordinates u and v can be express in terms of the new coordinates ξ, η as

$$u = \frac{1}{2}(\xi^2 - \eta^2) + a, \qquad v = \xi\eta. \tag{3.36}$$

III. *Separating coordinates associated with* K^2

For the last representative K^2, we need only the coordinates u, v to recognise the fact that $K = p_v$.

Now, let us discuss the solutions to the free particle and free Schrödinger equation of these three cases.

Case I: Lets make the choice of variables as

$$u = r\cos\theta + s\sin\theta, \qquad v = -r\sin\theta + s\cos\theta. \tag{3.37}$$

Then, the classical Hamilton-Jacobi equation becomes

$$H = \frac{(\frac{\partial S}{\partial r})^2 + (\frac{\partial S}{\partial s})^2}{4(r\cos\theta + s\sin\theta)} = E \tag{3.38}$$

with the general separable solution

$$S = S_1(r) + S_2(s) = \frac{(4Er\cos\theta - \lambda)^{3/2}}{6E\cos\theta} + \frac{(4Es\sin\theta - \lambda)^{3/2}}{6E\sin\theta}. \tag{3.39}$$

The corresponding free Schrödinger equation

$$\hat{H}\Psi = -\frac{1}{(4r\cos\theta + s\sin\theta)}(\partial_r^2 + \partial_s^2)\Psi = E\Psi \tag{3.40}$$

has the typical product solutions

$$\begin{aligned}\Psi &= \sqrt{(r - \frac{\mu}{4E\cos\theta})(s + \frac{\mu}{4E\sin\theta})} I_{\frac{1}{3}}(\frac{2}{3}\sqrt{4E\cos\theta}(r - \frac{\mu}{4E\cos\theta})^{3/2}) \\ &\quad \times I_{\frac{1}{3}}(\frac{2}{3}\sqrt{4E\sin\theta}(s + \frac{\mu}{4E\sin\theta})^{3/2}),\end{aligned} \tag{3.41}$$

where $I_v(z)$ is a solution of the Bessel's equation.

Case II: The classical Hamilton-Jacobi equation with the general solution S is

$$H = \frac{(\frac{\partial S}{\partial \xi})^2 + (\frac{\partial S}{\partial \eta})^2}{2(\xi^2 + \eta^2)(\xi^2 - \eta^2 + 2c)} = E, \tag{3.42}$$

$$S = \int \sqrt{2E\xi^4 + 2Ec\xi^2 - \lambda}\, d\xi + \int \sqrt{-2E\eta^4 + 2Ec\eta^2 + \lambda}\, d\eta. \tag{3.43}$$

The corresponding Schrödinger equation has a solution of the form $\Psi = \psi_1(\xi)\psi_2(\eta)$ where the ψ_i satisfy the following equations

$$(\partial_\xi^2 + 2E\xi^4 + 4Ec\xi^2 + \lambda)\psi_1(\xi)=0,$$

$$(\partial_\eta^2 - 2E\eta^4 + 4Ec\eta^2 - \lambda)\psi_2(\eta) = 0. \tag{3.44}$$

Case III: The classical Hamilton-Jacobi equation with the separable solutions S is

$$H = \frac{1}{4u}((\frac{\partial S}{\partial u})^2 + (\frac{\partial S}{\partial v})^2) = E, \tag{3.45}$$

$$S = \frac{1}{6E}(4Eu - k^2)^{3/2} + kv. \tag{3.46}$$

The free Schrödinger equation with the separable solutions Ψ is given by

$$-\frac{1}{4u}(\partial_u^2 + \partial_v^2)\Psi = E\Psi \tag{3.47}$$

having the solution

$$\Psi = \sqrt{u - \frac{m^2}{4E}} I_{\frac{1}{3}}(\frac{2}{3}\sqrt{4E}(u - \frac{m^2}{4E})^{3/2})e^{imv}. \tag{3.48}$$

In classical motion or the corresponding Schrödinger equation, their solutions depends on which real manifold we are considering.

3.4 Integrable and superintegrable systems for the Darboux space of type one

This section deals with the problem of superintegrability for the following type of the Hamiltonian

$$H = \frac{1}{4u}(p_u^2 + p_v^2), \tag{3.49}$$

so we search for potentials $V(u, v)$ such that (3.50) admits at least two extra quadratic integrals.

$$\bar{H} = H + V(u, v) \tag{3.50}$$

To solve this problem, we assume first that one quadratic first integral already exists and has the form

$$\bar{L} = a(u,v)p_u^2 + b(u,v)p_u p_v + c(u,v)p_v^2 + d(u,v) \tag{3.51}$$

The quadratic part of $\bar{L}$ in (3.51) must relate with one of the cases discussed in the Section III. 3. The separation of variables can be applied in coordinates α, β where $u = u(\alpha,\beta)$,$v = v(\alpha,\beta)$ for each of these cases. The addition of a potential implies that separation is preserved and hence we can write $\bar{H}$ and $\bar{L}$ as

$$\bar{H} = \frac{p_\alpha^2 + p_\beta^2 + f(\alpha) + g(\beta)}{\sigma(\alpha) + \tau(\beta)}, \tag{3.52}$$

$$\bar{L} = \frac{\sigma(\alpha)(p_\beta^2 + g(\beta)) - \tau(\beta)(p_\alpha^2 + f(\alpha))}{\sigma(\alpha) + \tau(\beta)}. \tag{3.53}$$

We force the existence of a further quadratic first integral by putting some conditions on the functions $f(\alpha)$ and $g(\beta)$. So we reach the following three cases:

I.

$$H = \frac{p_u^2 + p_v^2}{4u} + \frac{b_1(4u^2 + v^2)}{4u} + \frac{b_2}{u} + \frac{b_3}{uv^2}. \tag{3.54}$$

The additional constants of motion have the form

$$R_1 = X_2 - \frac{b_1 v^4}{4u} - \frac{b_2 v^2}{u} - \frac{b_3(4u^2 + v^2)}{v^2 u}\frac{b_3}{uv^2}. \tag{3.55}$$

$$R_2 = K^2 + b_1 v^2 + \frac{4b_3}{v^2} \tag{3.56}$$

and the corresponding quadratic algebra Daskaloyannis [16], Létourneau [37] relations are determined by

$$\begin{aligned}
\{R, R_1\} &= 8HR_1 + 6R_2^2 + 16b_2R_2 - 32b_1b_3, \\
\{R, R_2\} &= -8HR_2 - 16b_1R_1, \\
R^2 &= -16HR_1R_2 - 4R_2^3 - 16b_2R_2^2 - 64b_3H^2 - 16b_1R_1^2 + 64b_1b_3R^2 \\
&\quad +256b_1b_2b_3,
\end{aligned} \tag{3.57}$$

where $R = \{R_1, R_2\}$. By changing the coordinates as $u = \frac{1}{2}(\xi^2 - \eta^2) + a$, $v = \xi\eta$, the Hamiltonian, that separates also in these coordinates, is

$$H = \frac{p_\xi^2 + p_\eta^2}{2(\xi^2+\eta^2)(\xi^2-\eta^2+2a)} + \frac{b_1((\xi^2-\eta^2+2a)^2+\xi^2\eta^2)+4b_2+\frac{4b_3}{\xi^2\eta^2}}{2(\xi^2-\eta^2+2a)} \quad (3.58)$$

with the corresponding quadratic quantum algebra relations

$$\begin{aligned}
[\hat{R}, \hat{R}_1] &= -6\hat{R}_2^2 - 8\hat{H}\hat{R}_1 + 16b_2\hat{R}_2 + 2b_1(3+16b_3), \\
[\hat{R}, \hat{R}_2] &= 8\hat{H}\hat{R}_2 - 16b_1\hat{R}_1, \\
\hat{R}^2 &= +4\hat{R}_2^3 - 8\hat{H}[\hat{R}_1, \hat{R}_2]_+ - 16b_2\hat{R}_2^2 - 16b_1\hat{R}_1^2 \\
&\quad -4b_1(11+16b_3)\hat{R}_2 - 4(3+16b_3)\hat{H}^2 \\
&\quad +16b_1b_2(3+16b_3), \quad (3.59)
\end{aligned}$$

where $\hat{R} = [\hat{R}_1, \hat{R}_2]$.

II.

$$H = \frac{p_u^2 + p_v^2}{4u} + \frac{a_1}{u} + \frac{a_2 v}{u} + \frac{a_3(u^2v^2)}{u}. \quad (3.60)$$

The additional constants of the motion have the form

$$\begin{aligned}
R_1 &= X_1 - \frac{2a_1 v}{u} + \frac{2a_2(u^2-v^2)}{u} + \frac{2a_3 v(u^2-v^2)}{u}, \\
R_2 &= K^2 + 4a_2 v + 4a_3 v^2 \quad (3.61)
\end{aligned}$$

and the corresponding quadratic algebra relations are determined by

$$\begin{aligned}
\{R, R_1\} &= 8H^2 + 16a_3R_2 + 8(a_2^2 + 4a_1a_3), \\
\{R, R_2\} &= 16a_2H - 16a_3R_1, \\
R^2 &= 16H^2R_2 - 16a_3R_2^2 + 32a_2HR_1 - 16a_3R_1^2 \\
&\quad - 16(a_2^2 + 4a_1a_3)R_2 - 64a_1a_2^2. \quad (3.62)
\end{aligned}$$

By making the change of coordinates as $u = r\cos\theta + s\sin\theta$, $v = -r\sin\theta + s\cos\theta$ the Hamiltonian has the form

$$H = \frac{p_r^2 + p_s^2 + 4a_1 + 4a_2(-r\sin\theta + s\cos\theta) + 4a_3(r^2+s^2)}{4(r\cos\theta + s\sin\theta)}, \tag{3.63}$$

which clearly also separates in these coordinates.

The commutation relations of the corresponding quantum algebra are

$$\begin{aligned}
[\hat{R}, \hat{R}_1] &= 16a_3\hat{R}_2 + 8\hat{H}^2 - 8(a_2^2 + 4a_1a_3), \\
[\hat{R}, \hat{R}_2] &= -16a_3\hat{R}_1 + 16a_2\hat{H}, \\
\hat{R}^2 &= -16a_3\hat{R}_2^2 - 16a_3\hat{R}_1^2 + 16\hat{H}^2\hat{R}_2 + 32a_2\hat{H}\hat{R}_1 \\
&\quad -16(a_2^2 + 4a_1a_3)\hat{R}_2 + 64(a_3^2 - a_1a_2^2).
\end{aligned} \tag{3.64}$$

III.

$$H = \frac{p_u^2 + p_v^2}{4u} + \frac{a}{u}. \tag{3.65}$$

There are three extra constants associated with this Hamiltonian,

$$R_1 = X_1 - \frac{2av}{u}, \qquad R_2 = X_2 - \frac{av^2}{u} \quad \text{and} \quad K. \tag{3.66}$$

and the associated Poisson bracket relations are

$$\begin{aligned}
\{K, R_1\} &= 2H, \\
\{K, R_2\} &= -R_1, \\
\{R_1, R_2\} &= 2K(K^2 + 2a)
\end{aligned} \tag{3.67}$$

with

$$4HR_2 + R_1^2 + K^4 + 4aK^2 = 0. \tag{3.68}$$

In the following there are commutation relations of the corresponding quantum problem and the relation between the operators, respectively.

$$[\hat{K}, \hat{R}_1] = 2i\hat{H},$$

$$[\hat{K}, \hat{R}_2] = -i\hat{R}_1,$$

$$[\hat{R}_1, \hat{R}_2] = -2i\hat{K}(\hat{K}^2 - 2a), \quad (3.69)$$

$$4\hat{H}\hat{R}_2 + \hat{R}_1^2 + \hat{K}^4 - 4a\hat{K}^2 = 0. \quad (3.70)$$

So while constructing of the various *superintegrable* potentials by multiplying the equation $H = E$ by a suitable factor, we reobtain one of the *superintegrable* systems that is already classified for spaces of constant(or zero) curvature. For the first potential above, the equation $H = E$ may be written

$$p_u^2 + p_v^2 + b_1(4u^2 + v^2) + 4b_2 + \frac{4b_3}{v^2} - 4Eu = 0. \quad (3.71)$$

This equation is known to have separable solutions in coordinates u, v and associated parabolic coordinates ξ, η given by $u = \frac{1}{2}(\xi^2 - \eta^2)$, $v = \xi\eta$. With the second potential, $H = E$ becomes

$$p_u^2 + p_v^2 + 4a_3(u^2 + v^2) + 4a_1 + 4a_2 v - 4Eu = 0 \quad (3.72)$$

and the third,

$$p_u^2 + p_v^2 - 4Eu + 4a = 0. \quad (3.73)$$

All three of the above systems are special cases of the *superintegrable* systems found in E_2, Fris [21], Sheftel [43].

CHAPTER 4

APPLICATIONS

In this chapter, I calculate the Killing tensors and Killing-Yano tensors for the four types of metrics listed by Darboux, observing that the two dimensional systems represented by these four metrics are superintegrable and admit Killing tensors and Killing-Yano tensors. In the second application, the total time derivative term which includes the components of the angular momentum is added to a given free Lagrangian. Finally, I investigate the Killing vectors, Killing-Yano tensors and Killing tensors for these newly constructed spaces both in non-singular and singular case. The case of motion on a sphere is considered, as well.

4.1 Killing tensors and Killing-Yano tensors for superintegrable systems in two dimensions

The Killing-Yano (KY) tensors, that were first introduced by Yano [45] in a purely mathematical setting, have profound implications for the supersymmetric classical and quantum mechanics on curved manifolds where such tensors exist, Gibbons [22]. KY tensors, Baleanu [2]-Baleanu [8], play an important role in theories with spin and especially in the Dirac theory on curved spacetimes where they produce first order differential operators, called Dirac-type operators, which anticommute with the standard Dirac one, Carter [13]. Another virtue of the KY tensors is that they enter as square roots in the structure of several second rank Stäckel-Killing tensors that generate conserved quantities in classical mechanics or conserved operators which commute with Dirac operator. The symmetric Stäckel-Killing tensor $k_{\mu\nu}$ involved in the constant of motion quadratic in the four-momentum p_μ

$$L = \frac{1}{2}k^{\mu\nu}p_\mu p_\nu \tag{4.1}$$

has a certain square root in terms of KY tensors $f_{\mu\eta}$, Baleanu [9, 10]:

$$k_{\mu\nu} = f_{\mu\lambda}f^{\lambda}_{\nu}. \tag{4.2}$$

The KY tensor here is a 2-form $f_{\mu\nu} = -f_{\nu\mu}$ which satisfies the equation

$$f_{\mu\nu;\lambda} + f_{\mu\lambda;\nu} = 0. \tag{4.3}$$

Another method of obtaining a Killing tensor is to solve the corresponding equations

$$k_{\nu\lambda;\mu} + k_{\lambda\mu;\nu} + k_{\mu\nu;\lambda} = 0, \tag{4.4}$$

where $k_{\mu\nu}$ is a symmetric tensor.

There exist some two dimensional *superintegrable* systems which admit Killing tensors and KY tensors. In the following, the metrics in Kalnins [33] that describe these *superintegrable* systems are given with the corresponding calculated Killing tensors and KY tensors.

The first metric is given by

$$g^1_{ij} = \begin{pmatrix} 0 & x+y \\ x+y & 0 \end{pmatrix}. \tag{4.5}$$

If we do the change of coordinates as $(x, y) \to (u, v)$, where $x = u + iv$ and $y = u - iv$, then we reobtain the metric g^1_{ij} as follows

$$g^1_{ij} = \begin{pmatrix} 2u & 0 \\ 0 & 2u \end{pmatrix}. \tag{4.6}$$

The Killing tensor for the metric (4.6) is calculated as

$$\begin{aligned} k_{11} &= 1/2((C_1v^2+2C_2v+2C_3)u), \\ k_{12} &= -u^2(C_1v+C_2), \\ k_{22} &= 1/2(u(C_1v^2+2C_2v+4C_1u^2+2C_3+2C_4u)). \end{aligned} \tag{4.7}$$

and the KY tensor is

$$f_{12} = C_1u. \tag{4.8}$$

For the second metric having the form

$$g_{ij}^2 = \begin{pmatrix} (a/(-4v^2))+b & 0 \\ 0 & (a/(-4v^2))+b \end{pmatrix}, \tag{4.9}$$

the components of the Killing tensor were found as follows

$$k_{11} = \frac{(\frac{-a}{4}+bv^2)((\frac{C_1u^2}{2}+C_2u+C_3-4bC_4-\frac{C_1v^2}{2})a+16b^2C_4v^2+2C_1v^4b)}{bv^4}, \tag{4.10}$$

$$\begin{aligned} k_{12} &= \frac{C_1ua^2-8C_1uabv^2+16C_1ub^2v^4+C_2a^2-8C_2abv^2+16C_2b^2v^4}{8bv^3}, \\ k_{22} &= \frac{(C_1u^2+2C_2u+2C_3)(-a+4bv^2)}{2v^2} \end{aligned} \tag{4.11}$$

with the KY tensor

$$f_{12} = \frac{C_1(-a+4bv^2)}{v^2}. \tag{4.12}$$

The next metric has the following form

$$g_{ij}^3 = \begin{pmatrix} ae^{-u}+be^{-2u} & 0 \\ 0 & ae^{-u}+be^{-2u} \end{pmatrix}, \tag{4.13}$$

and the components of its corresponding Killing tensors are given by

$$k_{11} = \frac{(C_1+C_2sin(v)+C_3cos(v))(a+be^{-u})^2}{(ae^u+b)},$$

$$
\begin{aligned}
k_{12} &= -[(a^3e^{-u}C_2 + 2a^2(e^{-u})^2C_2b + a(e^{-u})^3C_2b^2 + be^{-2u}C_2a^2 \\
&\quad +2b^2e^{-2u}C_2ae^{-u} + b^3e^{-2u}C_2(e^{-u})^2)cos(v) + (-a^3e^{-u}C_3 \\
&\quad -2a^2(e^{-u})^2C_3b - a(e^{-u})^3C_3b^2 - be^{-2u}C_3a^2 - 2b^2e^{-2u}C_3ae^{-u} \\
&\quad -b^3e^{-2u}C_3(e^{-u})^2)sin(v)]/((ae^u+b)ae^{-u}),
\end{aligned}
$$

$$
\begin{aligned}
k_{22} &= [(-2ae^{-3u}b^5C_2 - 9a^2e^{-2u}b^4C_2 - 6ba^5e^uC_2 - 16a^3e^{-u}b^3C_2 \\
&\quad -14a^4b^2C2 - a^6e^{2u}C_2)sin(v) + (-6ba^5e^uC_3 - 9b^4a^2e^{-2u}C_3 \\
&\quad -2ae^{-3u}b^5C_3 - a^6e^{2u}C_3 - 14b^2a^4C_3 - 16a^3e^{-u}b^3C_3)cos(v) \\
&\quad +e^u(a^7C_4 + 5a^5bC_1) + e^{-u}(14a^3C_1b^3 + 10a^5b^2C_4) \\
&\quad +e^{-2u}(10b^3a^4C_4 + 11b^4C_1a^2) + e^{-3u}(5b^5C_1a + 5b^4C_4a^3) \\
&\quad +e^{-4u}(b^6C_1 + b^5C_4a^2) + a^6C_1e^{2u} + 5a^6bC_4 \\
&\quad +11a^4b^2C_1]/(a^2(ae^u+b)^3) \qquad (4.14)
\end{aligned}
$$

and the corresponding KY tensor is

$$
f_{12} = C_1(ae^u + b)e^{-2u}. \qquad (4.15)
$$

Finally, for the fourth metric

$$
g^4_{ij} = \begin{pmatrix} (2acos(v)+b)/(4(sin(v))^2) & 0 \\ 0 & (2acos(v)+b)/(4(sin(v))^2) \end{pmatrix}, \qquad (4.16)
$$

we calculated the Killing tensors as

$$
\begin{aligned}
k_{11} &= 4[(a-\frac{b}{2})C_2tan(\frac{v}{2})^4 + (-C_2b + \frac{C_1}{2})tan(\frac{v}{2})^2 \\
&\quad -(\frac{b}{2}+a)C_2)(tan(\frac{v}{2})^2+1)((a-\frac{b}{2})tan(\frac{v}{2})^2 - a - \frac{b}{2})]/tan(\frac{v}{2})^4, \\
k_{12} &= 0, \\
k_{22} &= \frac{C_1(2atan(\frac{v}{2})^4 - 2a - btan(\frac{v}{2})^4 - 2btan(\frac{v}{2})^2 - b)}{tan(\frac{v}{2})^2} \qquad (4.17)
\end{aligned}
$$

with the KY tensor

$$f_{12} = \frac{C_1(2atan(\frac{v}{2})^4 - 2a - btan(\frac{v}{2})^4 - 2btan(\frac{v}{2})^2 - b)}{tan(\frac{v}{2})^2}. \quad (4.18)$$

It was observed that the first three metrics admits Killing tensors with three non-zero components but the last one has only two.

4.2 Killing-Yano tensors and angular momentum

In the following part, the generic(standard) and non-generic(hidden) symmetries of the extended Lagrangians are investigated with symmetries of the geometries induced by the motion on a sphere.

4.2.1 Extended Lagrangians and their corresponding geometries

Let us assume that a given free Lagrangian $L(\dot{q}^i, q^i)$ admits a set of constants of motion denoted by L_i, $i = 1,2,3$. If we add the components of the angular momentum corresponding to L, the extended Lagrangian

$$L' = L + \dot{\lambda}^i L_i, \qquad i = 1,2,3 \quad (4.19)$$

can be rewritten as $L' = \frac{1}{2}a_{ij}\dot{q}^i\dot{q}^j$. In this context the second term in (4.19) is a total time derivative and the Lagrangians L and L' are equivalent. Since a_{ij} is symmetric by construction, the issue is to find a way to construct induced manifolds. In other words we are looking to find whether a_{ij} is singular or not. If the matrix a_{ij} is singular L' corresponds to a singular system in Güler [27]. Assuming that a_{ij} is a singular nxn matrix of rank n-1 we obtain non-singular symmetric matrices of order (n-1)x(n-1), where n will be 3,5 and 6. The final step is to consider the obtained matrices as metrics on the extended space and to investigate their generic(standard) and non-generic(hidden) symmetries.

4.2.1.1 The nonsingular case

As a starting point let us consider the following Lagrangian

$$L' = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \dot{\lambda}_3(x\dot{y} - y\dot{x}) \tag{4.20}$$

From (4.20) we obtain $L' = \frac{1}{2}a_{ij}\dot{q}^i\dot{q}^j$, where a_{ij} is given by

$$a_{ij} = \begin{pmatrix} 1 & 0 & -y \\ 0 & 1 & x \\ -y & x & 0 \end{pmatrix}. \tag{4.21}$$

The corresponding Killing vector is V=(y, -x, 0).

A KY is an antisymmetric tensor defined as

$$f_{\mu\nu;\lambda} + f_{\lambda\nu;\mu} = 0. \tag{4.22}$$

Solving (4.22) corresponding to (4.21) we obtained the following KY tensor

$$f_{12} = 0, \qquad f_{23} = -Cx\sqrt{x^2 + y^2}, \qquad f_{13} = Cy\sqrt{x^2 + y^2}, \tag{4.23}$$

where C is a constant in Baleanu [11].

If a KY tensor exists, then a Killing tensor of order two is generated as

$$K_{\mu\nu} = f_{\mu\lambda}f^{\lambda}_{\nu}. \tag{4.24}$$

Using (4.23) and (4.24) a Killing tensor is constructed as

$$K_{ij} = \begin{pmatrix} y^2 & -xy & -y(y^2 + x^2) \\ -xy & x^2 & x(x^2 + y^2) \\ -y(y^2 + x^2) & x(x^2 + y^2) & 0 \end{pmatrix} \tag{4.25}$$

Solving (4.4) corresponding to (4.21) we obtain a class of solutions given by Baleanu [11]

$$k_{11} = \frac{1}{2}y^2(C_2\lambda + C_3) + C_1,$$

$$
\begin{aligned}
k_{12} &= -\frac{1}{2}xy(C_2\lambda + C_3), \\
k_{13} &= -\frac{y}{4}[(x^2+y^2)(C_2 \arctan(\frac{x}{y}) - 4C_4) + 4C_1], \\
k_{22} &= \frac{1}{2}x^2(\lambda C_2 + C_3) + C_1, \\
k_{23} &= \frac{x}{4}[(x^2+y^2)(C_2 \arctan(\frac{x}{y}) - 4C_4) + 4C_1], \\
k_{33} &= 0.
\end{aligned}
\tag{4.26}
$$

We observed that if $C_1 = C_2 = C_4 = 0, C_3 = \frac{1}{2}$ we reobtain the solution from (4.25). Choosing the appropriate values of the constants $C_i, i = 1, \cdots 4$, we obtain a set of non-singular Killing tensors. These Killing tensors can be considered as manifolds and we have so called geometric duality (for more details see Refs. Rietdijk [42], Hinterleitner [31]). If $C_2 = 0$ the dual metrics have the following forms, Baleanu [11]:

$$
\begin{aligned}
k_{11}^{-1} &= \frac{x^2}{(x^2+y^2)C_1}, \\
k_{12}^{-1} &= \frac{xy}{(x^2+y^2)C_1}, \\
k_{13}^{-1} &= \frac{y}{(x^2+y^2)[C_4(x^2+y^2) - C_1]}, \\
k_{22}^{-1} &= \frac{x^2}{(y^2+y^2)C_1}, \\
k_{23}^{-1} &= \frac{x}{(x^2+y^2)[C_4(x^2+y^2) - C_1]}, \\
k_{33}^{-1} &= -\frac{1}{2}\frac{(x^2+y^2)C_3 + 2C_1}{(x^2+y^2)(C_1 - C_4(x^2+y^2))^2},
\end{aligned}
\tag{4.27}
$$

and the scalar curvatures corresponding to (4.27) are

$$
R = \frac{2C_1C_4[5C_4(x^2+y^2) + 2C_1]}{[-C_1 + C_4(x^2+y^2)]^2}. \tag{4.28}
$$

Let us add two components of the angular momentum at a free, three-dimensional Lagrangian. The extended Lagrangian becomes

$$
L' = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \dot{\lambda}_1(y\dot{z} - z\dot{y}) + \dot{\lambda}_2(z\dot{x} - x\dot{z}) \tag{4.29}
$$

and from (4.29) we identify a_{ij} as the following non-singular matrix

$$a_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 & z \\ 0 & 1 & 0 & -z & 0 \\ 0 & 0 & 1 & y & -x \\ 0 & -z & y & 0 & 0 \\ z & 0 & -x & 0 & 0 \end{pmatrix}. \tag{4.30}$$

The metric (4.30) admits three Killing vectors as

$$V_1 = (y, -x, 0, 0, 0), \qquad V_2 = (0, -z, y, 0, 0), \qquad V_3 = (z, 0, -x, 0, 0). \tag{4.31}$$

In this case KY tensors components are given in Baleanu [11] as

$$\begin{aligned} f_{15} &= -Gxy, & f_{14} = G(z^2 + y^2), \\ f_{24} &= -Gxy, & f_{34} = -Gxz, \\ f_{25} &= G(x^2 + z^2), & f_{35} = \frac{-Gxzy}{x}, \\ f_{12} &= Cz, & f_{13} = -Cy, \end{aligned} \tag{4.32}$$

others zero. Here C and G are constants. The corresponding Killing tensor has the following form

$$K = \begin{pmatrix} G(-2C+G)(z^2+y^2) & GDxy & GDzx & 0 & G^2r^2z \\ GDxy & -GD(x^2+z^2) & GDzy & -r^2zG^2 & 0 \\ GDzx & GDzy & -GD(y^2+x^2) & G^2r^2y & -G^2r^2x \\ 0 & -G^2zr^2 & G^2yr^2 & 0 & 0 \\ G^2zr^2 & 0 & -G^2xr^2 & 0 & 0 \end{pmatrix} \tag{4.33}$$

where D =2C+G and $r^2 = x^2 + y^2 + z^2$.

4.2.1.2 The singular case

The final step is to add all angular momentum components at the Lagrangian of the free particle in three-dimensions. In this case $L^{'}$ is given by

$$L^{'} = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + \dot{\lambda}_1(y\dot{z} - z\dot{y}) + \dot{\lambda}_2(z\dot{x} - x\dot{z}) + \dot{\lambda}_3(x\dot{y} - y\dot{x}) \tag{4.34}$$

In compact form (4.34) is written as $L^{'} = \frac{1}{2}a_{ij}\dot{q}^i\dot{q}^j$, where a_{ij} is singular having the form

$$a_{ij} = \begin{pmatrix} 1 & 0 & 0 & 0 & z & -y \\ 0 & 1 & 0 & -z & 0 & x \\ 0 & 0 & 1 & y & -x & 0 \\ 0 & -z & y & 0 & 0 & 0 \\ z & 0 & -x & 0 & 0 & 0 \\ -y & x & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{4.35}$$

Since the rank of (4.35) is 5 we obtained three non-singular symmetric matrices corresponding to three non-zero minors. The first one is given by (4.30) and the other two are as follows:

$$b^{(2)}_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 & -y \\ 0 & 1 & 0 & -z & x \\ 0 & 0 & 1 & y & 0 \\ 0 & -z & y & 0 & 0 \\ -y & x & 0 & 0 & 0 \end{pmatrix} \tag{4.36}$$

and

$$b^{(3)}_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & z & -y \\ 0 & 1 & 0 & 0 & x \\ 0 & 0 & 1 & -x & 0 \\ z & 0 & -x & 0 & 0 \\ -y & x & 0 & 0 & 0 \end{pmatrix}. \tag{4.37}$$

By direct calculations, Baleanu [11], we observed that (4.36) and (4.37) admit three Killing vectors given by (4.31) and a KY tensor having the following non-zero components

$$f_{12} = z, \qquad f_{13} = -y, \qquad f_{23} = x. \tag{4.38}$$

4.2.2 The motion on a sphere and its induced geometries

It was proved in Curtright [15] that the motion on a sphere admits four constants of motion, the Hamiltonian and three components of the angular momentum. In the following using the surface term we will generate four -dimensional manifolds. In this case the Lagrangian is given by

$$\begin{aligned} L' &= \frac{1}{2}(1+\frac{x^2}{u})\dot{x}^2 + \frac{1}{2}(1+\frac{y^2}{u})\dot{y}^2 + \frac{xy}{u}\dot{x}\dot{y} - \frac{xy}{\sqrt{u}}\dot{\lambda_1}\dot{x} + (\frac{x^2}{\sqrt{u}} + \sqrt{u})\dot{\lambda_2}\dot{x} \\ &- (\frac{y^2}{\sqrt{u}} + \sqrt{u})\dot{\lambda_1}\dot{y} + \frac{xy}{\sqrt{u}}\dot{\lambda_2}\dot{y} + x\dot{\lambda_3}\dot{y} - y\dot{\lambda_3}\dot{x}, \end{aligned} \tag{4.39}$$

where $u = 1 - x^2 - y^2$. From (4.39) we identify the singular matrix a_{ij} as

$$a_{ij} = \begin{pmatrix} 1+\frac{x^2}{u} & \frac{xy}{u} & -\frac{xy}{\sqrt{u}} & \frac{x^2}{\sqrt{u}}+\sqrt{u} & -y \\ \frac{xy}{u} & 1+\frac{y^2}{u} & -\frac{y^2}{\sqrt{u}}-\sqrt{u} & \frac{xy}{\sqrt{u}} & x \\ -\frac{xy}{\sqrt{u}} & -\frac{y^2}{\sqrt{u}}-\sqrt{u} & 0 & 0 & 0 \\ \frac{x^2}{\sqrt{u}}+\sqrt{u} & \frac{xy}{\sqrt{u}} & 0 & 0 & 0 \\ -y & x & 0 & 0 & 0 \end{pmatrix}. \tag{4.40}$$

Because (4.40) is a singular matrix of rank 4 we identify three symmetric minors of order four. If we treat these minors as a metric we observed that they are not conformaly flat but their scalar curvatures are zero.

The first metric is given by

$$g^{(1)}_{\mu\nu} = \begin{pmatrix} 1+\frac{x^2}{u} & \frac{xy}{u} & \sqrt{u}+\frac{x^2}{\sqrt{u}} & -y \\ \frac{xy}{u} & 1+\frac{y^2}{u} & \frac{xy}{\sqrt{u}} & x \\ \sqrt{u}+\frac{x^2}{\sqrt{u}} & \frac{xy}{\sqrt{u}} & 0 & 0 \\ -y & x & 0 & 0 \end{pmatrix}. \tag{4.41}$$

The Killing vectors of (4.41) has the following components, Baleanu [11]

$$\begin{aligned} V_1 &= (y, -x, 0, 0), \\ V_2 &= (\sqrt{1-x^2-y^2}+\frac{x^2}{1-x^2-y^2}, \frac{xy}{1-x^2-y^2}, 0, 0), \\ V_3 &= (-\frac{xy}{1-x^2-y^2}, -\sqrt{1-x^2-y^2}-\frac{y^2}{1-x^2-y^2}, 0, 0). \end{aligned} \tag{4.42}$$

The next step is to investigate its KY tensors. Solving (4.22) we obtain the following set of solutions:

a. One-solution is $f_{21} = \frac{C_1}{\sqrt{1-x^2-y^2}}$, others zero.

b. Two-by-two solution has the form: $f_{31} = f_{42} = C$,

c. Three-by-three solution is $f_{21} = \frac{C_1}{\sqrt{-1+x^2+y^2}}$ and $f_{31} = f_{42} = C$, where C and C_1 are constants.

From (4.39) another two metrics can be identified as

$$g^{(2)}_{\mu\nu} = \begin{pmatrix} 1+\frac{x^2}{u} & \frac{xy}{u} & -\frac{xy}{\sqrt{u}} & -y \\ \frac{xy}{u} & 1+\frac{y^2}{u} & -\sqrt{u}-\frac{y^2}{\sqrt{u}} & x \\ -\frac{xy}{\sqrt{u}} & -\sqrt{u}-\frac{y^2}{\sqrt{u}} & 0 & 0 \\ -y & x & 0 & 0 \end{pmatrix} \tag{4.43}$$

and

$$g^{(3)}_{\mu\rho} = \begin{pmatrix} 1+\frac{x^2}{u} & \frac{xy}{u} & -\frac{xy}{\sqrt{u}} & \frac{x^2}{\sqrt{u}}+\sqrt{u} \\ \frac{xy}{u} & 1+\frac{y^2}{u} & -\frac{y^2}{\sqrt{u}}-\sqrt{u} & \frac{xy}{\sqrt{u}} \\ -\frac{xy}{\sqrt{u}} & -\frac{y^2}{\sqrt{u}}-\sqrt{u} & 0 & 0 \\ \frac{x^2}{\sqrt{u}}+\sqrt{u} & \frac{xy}{\sqrt{u}} & 0 & 0 \end{pmatrix}. \tag{4.44}$$

By direct calculations we obtained that (4.43) and (4.44) admit the same Killing vector as in (4.42). Solving (4.22) corresponding to (4.43) and (4.44) we find one non-zero component of KY tensor as

$$f_{21} = \frac{C_1}{\sqrt{1-x^2-y^2}}. \tag{4.45}$$

CHAPTER 5

CONCLUSION

In this thesis, I presented the classical and geometrical aspects of the superintegrable systems in two dimensions.

In Chapter II, the definition of integrability and superintegrability is given. The types of transformations that preserve the integrability and their usage to find the canonical forms of the partial differential equations are presented. The systems that have polynomial invariants, with the existence of superintegrability, are classified with respect to the degree of the invariants. Some types of potentials that give superintegrable Hamiltonian systems are presented as examples.

The superintegrability of two dimensional space of non-constant curvature was investigated in Chapter III. The classification was done by Darboux and Koenigs. The forms of the four superintegrable systems were discussed, the separation of variables in a Darboux space of type one as well as the integrability and superintegrability for this case were presented.

Chapter IV contains my original contribution. In the first part of this chapter,

I calculated the Killing tensors and Killing-Yano tensors corresponding to the four types of metrics which were discussed in Chapter III. Then as a second application, integrable geometries were reported by adding a total time derivative involving the components of the angular momentum to a given free Lagrangian. The existence of

Killing vectors, Killing-Yano and Killing tensors is investigated and in all cases a solution is presented. The first step was to add, to a free two-dimensional Lagrangian, a total time derivative involving the third component of the angular momentum. In this case a three-dimensional metric was induced. This metric is conformaly flat but its duals are not.

Increasing the number of dimensions to three and adding a total time derivative involving two components of the angular momentum we obtained geometries, in four and five dimensions. The obtained induced manifolds are not conformaly flat but all of them have Ricci scalar zero.

If we add a total time derivative involving all components of the angular momentum to a three dimensional free Lagrangian we observed that a singular matrix a_{ij} arises. We identify three symmetric minors of this metric and we investigated the existence of Killing vectors, KY and Killing tensors corresponding to those induced manifolds. We observed that the obtained manifolds admit the same Killing vectors but different KY solutions.

The geometries induced by the motion on a sphere are investigated and a four dimensional induced manifolds were obtained. As in the previous case the manifolds admit the same Killing vectors but different KY tensors.

REFERENCES

[1] Arnold, V. I., *Mathematical Methods of Classical Mechanics*, Second Edition, Springer-Verlag, New York, (1989).

[2] Baleanu, D., "New non-generic symmetries on extended Taub-NUT space-time", *Helv. Acta. Phys.*, **71 (3)**, 343, (1998).

[3] Baleanu, D., "Geodesic motion on extended Taub-NUT spinning space", *Gen. Rel. Grav.*, **30 (2)**, 195, (1998).

[4] Baleanu, D., Başkal, S., "Geometrization of the Lax pair tensors", *Mod. Phys. Lett.A*, **15 (24)**, 1503, (2000).

[5] Baleanu, D., "Symmetries of NUT-Kerr-Newman dual metrics", *Nuovo Cimento B*, **116 (2)**, 205, (2001).

[6] Baleanu, D., Başkal, S., "Dual metrics for a class of radiative space-times", *Mod. Phys. Lett.A*, **16 (3)**, 135, (2001).

[7] Baleanu, D., "Symmetries of the dual metrics", *Int. Journ. Mod. Phys.D*, **11 (3)**, 405, (2002).

[8] Baleanu, D., Başkal, S., "Dual metrics and nongeneric supersymmetries for a class of Siklos space-times",*Int. Journ. Mod. Phys.A*, **17**, 3737, (2002).

[9] Baleanu, D., "Killing-Yano tensors and nambu tensors", *Nuovo Cimento B*, **114 (9)**, 1065, (1999).

[10] Baleanu, D., Başkal, S., "Killing-Yano symmetry for a class of spacetimes admitting parallel null 1-planes", *Nuovo Cimento B*, **117 (5)**, 501, (2002).

[11] Baleanu, D., Defterli, Ö., "Killing-Yano tensors and angular momentum", *Czech. Journ. Phys.*, **54**, 157, (2004).

[12] Bogoyavlensky, O. I., "On perturbations of the periodic Toda lattice", *Commun. Math. Phys.*, **51**, 210, (1976).

[13] Carter, B., and McLenaghan, R. G., "Generalized total angular momentum operator for the Dirac equation in curved space-time", *Phys. Rev.D*, **19**, 1093, (1979).

[14] Cotaescu, I. I., Visinescu, M., "Symmetries of the Dirac operators associated with covariantly constant Killing-Yano tensors", *Class. Quant. Grav.*, **21**, 11, (2004).

[15] Curtright, T. L., Zachos, C. K., "Deformation quantization of superintegrable systems in Nambu mechanics", *New J. Phys.*, **4**, 83, (2002).

[16] Daskaloyannis, C., "Quadratic Poisson algebras of two-dimensional classical superintegrable systems and quadratic associative algebras of quantum superintegrable systems", *J. Math. Phys.*, **42**, 1100, (2001).

[17] Dorizzi, B., Grammaticos, B., Padjen, R., Papageorgiou, V., "Integrals of motion for Toda systems with unequal masses", *J. Math. Phys.*, **25**, 2200, (1984).

[18] Evans, N. W., "Superintegrability in classical mechanics", *Phys. Rev.A*, **41**, 5666, (1990).

[19] Evans, N. W., "Super-integrability of the Winternitz system", *Phys. Lett.A*, **147**, 483, (1990).

[20] Fokas, A. S., Lagerstrom, P. A., "Quadratic and cubic invariants in classical mechanics", *J. Math. Anal. Appl.*, **74**, 325, (1980).

[21] Fris, J., Mandrosov, V., Smorodinsky, Ya. A., Uhlir, M., Winternitz, P., "On higher symmetries in quantum mechanics", *Phys. Lett.*, **16**, 354, (1965).

[22] Gibbons, G. W., Rietdijk, R. H., and van Holten, J. W., "SUSY in the sky", *Nucl.Phys.B*, **404**, 42, (1993).

[23] Goldstein, H., *Classical Mechanics*, Second Edition, Addison-Wesley, Massachusetts, (1980).

[24] Grammaticos, B., Dorizzi, B., Ramani, A., "Hamiltonians with higher order integrals and the "weak-Painleve" concept", *J. Math. Phys.*, **25**, 3470, (1984).

[25] Grosche, C., Pogosyan, G. S., Sissakian, A. N., "Path-integral discussion for Smorodinsky-Winternitz potentials.2. The 2-dimensional and 3-dimensional sphere", *Fortschr. Phys.*, **43**, 523, (1995).

[26] Grosche, C., Pogosyan, G. S., Sissakian, A. N., "Path-integral discussion for Smorodinsky-Winternitz potentials.1. 2-dimensional and 3-dimensional Euclidean-space", *Fortschr. Phys.*, **43**, 453, (1995).

[27] Güler, Y., Baleanu, D., Cenk, M., "Surface terms, angular momentum and Hamilton - Jacobi formalism", *Nuovo Cimento B*, **118**, 293, (2003).

[28] Hietarinta, J., "Direct methods for the search of the second invariant", *Phys. Rep.*, **147**, 87, (1987).

[29] Hietarinta, J., "Integrable families of Hénon-Heiles type Hamiltonians and a new duality", *Phys. Rev.A*, **28**, 3670, (1983).

[30] Hietarinta, J., "Classical versus quantum integrability", *J. Math. Phys.*, **25**, 1833, (1984).

[31] Hinterleitner, F., "Killing tensors as space-time metrics", *Ann. Phys.*, **271**, 23, (1999).

[32] Holt, C. R., "Construction of new integrable Hamiltonians in two degrees of freedom", *J. Math. Phys.*, **23**, 1037, (1982).

[33] Kalnins, E. G., Kress, J. M., Winternitz, P., "Superintegrability in a two-dimensional space of non-constant curvature", *J. Math. Phys.*, **43**, 970, (2002).

[34] Kalnins, E. G., Miller, W. Jr., Pogosyan, G. S., "Completeness of multiseparability on the complex 2-sphere", *J. Phys.A: Math. Gen.*, **33**, 6791, (2000).

[35] Kalnins, E. G., Kress, J. M., Pogosyan, G. S., Miller, W. Jr., "Completeness of multiseparability in two-dimensional constant curvature spaces", *J. Phys.A: Math. Gen.*, **34**, 4705, (2001).

[36] Koenigs, G., *Sur les géodésiques a intégrales quadratiques.* A note appearing in *Leçons sur la théorie générale des surfaces*, G. Darboux, (Chelsea Publishing, vol.4 1972) p. 368-404.

[37] Létourneau, P., Vinet, L., "Superintegrable systems: polynomial algebras and quasi-exactly solvable Hamiltonians", *Ann. Phys.*, **243**, 144, (1995).

[38] Moser, J., "Three integrable Hamiltonian systems connected with isospectral deformations", *Adv. Math.*, **16**, 197, (1975).

[39] Ranada, M. F., Santander, M., "Superintegrable systems on two-dimensional sphere S-2 and the hyperbolic plane H-2", *J. Math. Phys.*, **40**, 5026, (1999).

[40] Ranada, M. F., "Superintegrable n=2 systems, quadratic constants of motion, and potentials of Dirach", *J. Math. Phys.*, **38**, 4165, (1997).

[41] Ranada, M. F.,"On a second Lax structure for the Calogero-Moser system: time-dependent constants and superintegrability", *Phys. Lett.A*, **277**, 219, (2000).

[42] Rietdijk, R. H., van Holten, J. W., "Killing tensors and a new geometric duality", *Nuc. Phys.B*, **472**, 427, (1996).

[43] Sheftel, M. B., Tempesta, P., Winternitz, P., "Superintegrable systems in quantum mechanics and classical Lie theory", *J. Math. Phys.*, **42**, 659, (2001).

[44] Tempesta, P., Turbiner, A. V., Winternitz, P., "Exact solvability of superintegrable systems", *J. Math. Phys.*, **42**, 4248, (2001).

[45] Yano, K., "Some Remarks on Tensor Fields and Curvature", *Ann. Math.*, **55**, 328, (1952).

[46] Yoshida, H., "Integrability of generalized Toda lattice systems and singularities in the complex t-plane", in: *Nonlinear Integrable Systems - Classical Theory and Quantum Theory*, (Eds.), Jimbo, M., Miwa, T., (World Science Publishing Co. , Singapore, 1983a) p. 273-289.

[47] Wojciechowski, S., "Superintegrability of the Calogero - Moser system", *Phys. Lett.A*, **95**, 279, (1983).