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Fractional curve flows and solitonic hierarchies in gravity and geometric mechanics

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Methods from the geometry of nonholonomic manifolds and Lagrange–Finsler spaces are applied in fractional calculus with Caputo derivatives and for elaborating models of fractional gravity and fractional Lagrange mechanics. The geometric data for such models are encoded into (fractional) bi–Hamiltonian structures and associated solitonic hierarchies. The constructions yield horizontal/vertical pairs of fractional vector sine–Gordon equations and fractional vector mKdV equations when the hierarchies for corresponding curve fractional flows are described in explicit forms by fractional wave maps and analogs of Schrödinger maps. © *2011 American Institute of Physics*. [doi:10.1063/1.3589964]

I. INTRODUCTION

The goal of this paper is to show how fractional solitonic hierarchies can be canonically generated in various models of fractional gravity and geometric Lagrange mechanics. Such constructions are possible in explicit form for a class of fractional derivatives resulting in zero for actions on constants, for instance, for the Caputo fractional derivative.[1–6] This property is crucial for constructing geometric models of theories with fractional calculus even, after corresponding nonholonomic deformations, we may prefer to work with another type of fractional derivatives.

This is the second partner work of paper[7] (we also recommend readers to consult in advance the papers[8–11] on details, notation conventions and bibliography) where we proved an important result that via nonholonomic deformations on fractional manifolds and bundle spaces, determined by a generating fundamental Lagrange/Finsler, or an Einstein metric, we can construct linear connections with constant coefficient curvature. For such fractional "covariant" connections, it is possible to provide a formal encoding of integer and non–integer gravitational dynamics, Ricci flow evolution and constrained Lagrange/Hamilton mechanics into hierarchies of solitonic equations.

The most important consequence of such geometric studies is that using bi–Hamilton models and related solitonic systems we can study analytically and numerically as well to try to construct some analogous mechanical systems, with the aim to mimic a nonlinear/fractional nonholonomic dynamics/evolution and even to provide certain schemes of quantization, such as in the "fractional" Fedosov approach.[9,12]

This work is organized in the form: In Sec. II, we remember the most important formulas on Caputo fractional derivatives and nonlinear connections. Section III is devoted to definition of basic equations for fractional curve flows. The main theorem on fractional bi–Hamiltonians and solitonic hierarchies is formulated and proved in Sec. IV. Finally, we derive in general form the corresponding nonholonomic fractional solitonic hierarchies in Sec. V.

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II. CAPUTO FRACTIONAL DERIVATIVES AND N–CONNECTIONS

We summarize some important formulas on fractional calculus for nonholonomic manifold elaborated both in global and coordinate free forms, as well as with important local integro–differential parametrizations, in Refs. 7,8,10, and 11. Readers are recommended to study in advance those works (and references therein) on fractional differential geometry and applications. Such a calculus is nonlocal both on space and/or time coordinates when the algebra of fractional derivatives does not have the same properties as in the integer case. Nevertheless, having well–defined concepts of integral calculus for curved nonholonomic manifolds and bundles of integer dimension, we can introduce such algebras in local forms and then globalize the constructions using corresponding charts on atlases covering corresponding spaces.

Our geometric arena consists from an abstract fractional manifold $\overset{\alpha}{\mathbf{V}}$ (we shall also use the term "fractional space" as an equivalent one enabled with certain fundamental geometric structures) with prescribed nonholonomic distribution modeling both the fractional calculus and the non–integrable dynamics of interactions. We note that in our works a corresponding system of notations is elaborated in a form to unify the approaches on fractional calculus, nonholonomic bundle spaces, nonlinear connection (N–connection) formalism, etc. They are considered boldface symbols, over/under and left up/low labels, etc., as we shall explain below.

Let us consider that $f(x)$ is a derivable function $f : [\,_1x,\ _2x] \to \mathbb{R}$, for $\mathbb{R} \ni \alpha > 0$, and denote the derivative on x as $\partial_x = \partial/\partial x$. We note by $_1x$ and $_2x$ (with left low labels, respectively, 1 and 2) two ends of a real line interval. The fractional left, respectively, right Caputo derivatives are denoted in the form,

$$ {}_{_1x}\overset{\alpha}{\underline{\partial}}_x f(x) := \frac{1}{\Gamma(s-\alpha)} \int\limits_{_1x}^{x} (x-\ x')^{s-\alpha-1} \left(\frac{\partial}{\partial x'}\right)^s f(x')\,dx'; \tag{1} $$

$$ {}_x\overset{\alpha}{\underline{\partial}}_{\,_2x} f(x) := \frac{1}{\Gamma(s-\alpha)} \int\limits_{x}^{_2x} (x'-x)^{s-\alpha-1} \left(-\frac{\partial}{\partial x'}\right)^s f(x')\,dx'\,. $$

For instance, we emphasize that the integral is considered from $_1x$ to x in the symbol of partial derivative ${}_{_1x}\overset{\alpha}{\underline{\partial}}_x$. We shall always put α over a symbol (or up-left/-right to such a symbol) in order to emphasize that the constructions are considered for a fractional calculus with $\alpha \in (0, 1)$. There will be underlined some corresponding symbols if their definition is strictly related to the concept of Caputo derivative. Using operators (1) generalized on $\mathbb{R}^n$, where $n = 1, 2\ldots,$ and the same fractional α is associated to any such integer dimension, we can construct the fractional absolute differential $\overset{\alpha}{d} := (dx^j)^\alpha\ \ {}_0\overset{\alpha}{\underline{\partial}}_j$ when $\overset{\alpha}{d}x^j = (dx^j)^\alpha \frac{(x^j)^{1-\alpha}}{\Gamma(2-\alpha)}$, where we consider $_1x^i = 0$.

We denote a fractional tangent bundle in the form $\overset{\alpha}{\underline{T}}M$ for $\alpha \in (0, 1)$, associated to a manifold M of necessary smooth class and integer $\dim M = n$. The symbol T is underlined in order to emphasize that we shall associate the approach to a fractional Caputo derivative. Locally, both the integer and fractional local coordinates are written in the form $u^\beta = (x^j, y^a)$, where indices to coordinates run values $i, j, \ldots = 1, 2, \ldots n$ (for coordinates on base manifold) and $a, b, \ldots = n + 1, n + 2, \ldots, n + n$ (for typical fiber coordinates) for integer dimensions but keep in mind that the local derivatives are of fractional type (1), associated, respectively, to any such integer coordinate. A fractional frame basis $\overset{\alpha}{\underline{e}}_\beta = e^{\beta'}_{\ \beta}(u^\beta)\overset{\alpha}{\underline{\partial}}_{\beta'}$ on $\overset{\alpha}{\underline{T}}M$ is connected via a vierlbein transform $e^{\beta'}_{\ \beta}(u^\beta)$ with a fractional local coordinate basis

$$ \overset{\alpha}{\underline{\partial}}_{\beta'} = \left(\overset{\alpha}{\underline{\partial}}_{j'} = {}_{_1x^{j'}}\ \overset{\alpha}{\underline{\partial}}_{j'}, \overset{\alpha}{\underline{\partial}}_{b'} = {}_{_1y^{b'}}\ \overset{\alpha}{\underline{\partial}}_{b'} \right), \tag{2} $$

for $j' = 1, 2, \ldots, n$ and $b' = n+1, n+2, \ldots, n+n$. The fractional co–bases are written $\overset{\alpha}{\underline{e}}{}^{\beta} = e^{\beta}_{\ \beta'}(u^{\beta})\overset{\alpha}{d}u^{\beta'}$, where the fractional local coordinate co–basis is

$$\overset{\alpha}{d}u^{\beta'} = \left((dx^{i'})^{\alpha}, (dy^{a'})^{\alpha}\right). \tag{3}$$

It is possible to define a nonlinear connection (N–connection) $\overset{\alpha}{\mathbf{N}}$ for a fractional space $\overset{\alpha}{\mathbf{V}}$ by a nonholonomic distribution (Whitney sum) with conventional h– and v–subspaces, $\underline{h}\overset{\alpha}{\mathbf{V}}$ and $\underline{v}\overset{\alpha}{\mathbf{V}}$,

$$\overset{\alpha}{\underline{T}}\overset{\alpha}{\mathbf{V}} = \underline{h}\overset{\alpha}{\mathbf{V}} \oplus \underline{v}\overset{\alpha}{\mathbf{V}}. \tag{4}$$

Locally, such a fractional N–connection is characterized by its local coefficients $\overset{\alpha}{\mathbf{N}} = \{ {}^{\alpha}N^{a}_{i}\}$, when $\overset{\alpha}{\mathbf{N}} = {}^{\alpha}N^{a}_{i}(u)(dx^{i})^{\alpha} \otimes \overset{\alpha}{\underline{\partial}}_{a}$.

On $\overset{\alpha}{\mathbf{V}}$, it is convenient to work with N–adapted fractional (co) frames,

$${}^{\alpha}\mathbf{e}_{\beta} = \left[{}^{\alpha}\mathbf{e}_{j} = \overset{\alpha}{\underline{\partial}}_{j} - {}^{\alpha}N^{a}_{j}\overset{\alpha}{\underline{\partial}}_{a},\ {}^{\alpha}e_{b} = \overset{\alpha}{\underline{\partial}}_{b} \right], \tag{5}$$

$${}^{\alpha}\mathbf{e}^{\beta} = [\, {}^{\alpha}e^{j} = (dx^{j})^{\alpha},\ {}^{\alpha}\mathbf{e}^{b} = (dy^{b})^{\alpha} + {}^{\alpha}N^{b}_{k}(dx^{k})^{\alpha}]. \tag{6}$$

A fractional metric structure (d–metric) $\overset{\alpha}{\mathbf{g}} = \{ {}^{\alpha}g_{\underline{\alpha\beta}}\} = \left[{}^{\alpha}g_{kj},\ {}^{\alpha}g_{cb}\right]$ on $\overset{\alpha}{\mathbf{V}}$ can be represented in different equivalent forms,

$$\begin{aligned} \overset{\alpha}{\mathbf{g}} &= {}^{\alpha}g_{\underline{\gamma\beta}}(u)(du^{\underline{\gamma}})^{\alpha} \otimes (du^{\underline{\beta}})^{\alpha} \\ &= {}^{\alpha}g_{kj}(x,y)\,{}^{\alpha}e^{k} \otimes {}^{\alpha}e^{j} + {}^{\alpha}g_{cb}(x,y)\,{}^{\alpha}\mathbf{e}^{c} \otimes {}^{\alpha}\mathbf{e}^{b} \\ &= \eta_{k'j'}\,{}^{\alpha}e^{k'} \otimes {}^{\alpha}e^{j'} + \eta_{c'b'}\,{}^{\alpha}\mathbf{e}^{c'} \otimes {}^{\alpha}\mathbf{e}^{b'}, \end{aligned} \tag{7}$$

where matrices $\eta_{k'j'} = diag[\pm 1, \pm 1, \ldots, \pm 1]$ and $\eta_{a'b'} = diag[\pm 1, \pm 1, \ldots, \pm 1]$, for the signature of a "prime" spacetime $\mathbf{V}$, are obtained by frame transforms $\eta_{k'j'} = e^{k}_{\ k'}\ e^{j}_{\ j'}\ {}^{\alpha}g_{kj}$ and $\eta_{a'b'} = e^{a}_{\ a'}\ e^{b}_{\ b'}\ {}^{\alpha}g_{ab}$.

We can adapt geometric objects on $\overset{\alpha}{\mathbf{V}}$ with respect to a given N–connection structure $\overset{\alpha}{\mathbf{N}}$, calling them as distinguished objects (d–objects). For instance, a distinguished connection (d–connection) $\overset{\alpha}{\mathbf{D}}$ on $\overset{\alpha}{\mathbf{V}}$ is defined as a linear connection preserving under parallel transports the Whitney sum (4). There is an associated N–adapted differential 1–form,

$${}^{\alpha}\boldsymbol{\Gamma}^{\tau}_{\ \beta} = {}^{\alpha}\boldsymbol{\Gamma}^{\tau}_{\ \beta\gamma}\,{}^{\alpha}\mathbf{e}^{\gamma}, \tag{8}$$

parametrizing the coefficients (with respect to (6) and (5)) in the form ${}^{\alpha}\boldsymbol{\Gamma}^{\gamma}_{\ \tau\beta} = \left({}^{\alpha}L^{i}_{\ jk}, {}^{\alpha}L^{a}_{bk}, {}^{\alpha}C^{i}_{jc}, {}^{\alpha}C^{a}_{bc}\right)$.

The absolute fractional differential ${}^{\alpha}\mathbf{d} = {}_{1x}\overset{\alpha}{d}_{x} + {}_{1y}\overset{\alpha}{d}_{y}$ acts on fractional differential forms in N–adapted form. This is a fractional distinguished operator, d–operator, when the value ${}^{\alpha}\mathbf{d} := {}^{\alpha}\mathbf{e}^{\beta}\,{}^{\alpha}\mathbf{e}_{\beta}$ splits into exterior h- and v–derivatives when

$${}_{1x}\overset{\alpha}{d}_{x} := (dx^{i})^{\alpha}\ \ {}_{1x}\overset{\alpha}{\underline{\partial}}_{i} = {}^{\alpha}e^{j}\,{}^{\alpha}\mathbf{e}_{j} \text{ and } {}_{1y}\overset{\alpha}{d}_{y} := (dy^{a})^{\alpha}\ \ {}_{1x}\overset{\alpha}{\underline{\partial}}_{a} = {}^{\alpha}\mathbf{e}^{b}\,{}^{\alpha}e_{b}.$$

Using such differentials, we can compute in explicit form the torsion and curvature (as fractional two d–forms derived for (8)) of a fractional d–connection $\overset{\alpha}{\mathbf{D}} = \{ {}^{\alpha}\boldsymbol{\Gamma}^{\tau}_{\ \beta\gamma}\}$,

$$\begin{aligned} {}^{\alpha}\mathcal{T}^{\tau} &\doteqdot \overset{\alpha}{\mathbf{D}}\,{}^{\alpha}\mathbf{e}^{\tau} = {}^{\alpha}\mathbf{d}\,{}^{\alpha}\mathbf{e}^{\tau} + {}^{\alpha}\boldsymbol{\Gamma}^{\tau}_{\ \beta} \wedge {}^{\alpha}\mathbf{e}^{\beta} \text{ and} \\ {}^{\alpha}\mathcal{R}^{\tau}_{\beta} &\doteqdot \overset{\alpha}{\mathbf{D}}\,{}^{\alpha}\Gamma^{\tau}_{\ \beta} = {}^{\alpha}\mathbf{d}\,{}^{\alpha}\Gamma^{\tau}_{\ \beta} - {}^{\alpha}\boldsymbol{\Gamma}^{\gamma}_{\ \beta} \wedge {}^{\alpha}\boldsymbol{\Gamma}^{\tau}_{\ \gamma} = {}^{\alpha}\mathbf{R}^{\tau}_{\ \beta\gamma\delta}\,{}^{\alpha}\mathbf{e}^{\gamma} \wedge {}^{\alpha}\mathbf{e}^{\delta}. \end{aligned} \tag{9}$$

Contracting, respectively, the indices, we can compute the fractional Ricci tensor ${}^{\alpha}\mathcal{R}ic = \{ {}^{\alpha}\mathbf{R}_{\alpha\beta} \doteqdot {}^{\alpha}\mathbf{R}^{\tau}{}_{\alpha\beta\tau} \}$ with components,

$$ {}^{\alpha}R_{ij} \doteqdot {}^{\alpha}R^{k}{}_{ijk}, \quad {}^{\alpha}R_{ia} \doteqdot -{}^{\alpha}R^{k}{}_{ika}, \quad {}^{\alpha}R_{ai} \doteqdot {}^{\alpha}R^{b}{}_{aib}, \quad {}^{\alpha}R_{ab} \doteqdot {}^{\alpha}R^{c}{}_{abc} \tag{10} $$

and the scalar curvature of fractional d–connection $\overset{\alpha}{\mathbf{D}}$,

$$ {}_{s}^{\alpha}\mathbf{R} \doteqdot {}^{\alpha}\mathbf{g}^{\tau\beta}\, {}^{\alpha}\mathbf{R}_{\tau\beta} = {}^{\alpha}R + {}^{\alpha}S, \ {}^{\alpha}R = {}^{\alpha}g^{ij}\, {}^{\alpha}R_{ij}, \ \ {}^{\alpha}S = {}^{\alpha}g^{ab}\, {}^{\alpha}R_{ab}, \tag{11} $$

with ${}^{\alpha}\mathbf{g}^{\tau\beta}$ being the inverse coefficients to a d–metric (7). For applications in modern gravity and geometric mechanics, we can consider more special classes of d–connections:

- There is a unique canonical metric compatible fractional d–connection ${}^{\alpha}\widehat{\mathbf{D}} = \{ {}^{\alpha}\widehat{\mathbf{\Gamma}}^{\gamma}{}_{\alpha\beta} = \left({}^{\alpha}\widehat{L}^{i}{}_{jk}, {}^{\alpha}\widehat{L}^{a}_{bk}, {}^{\alpha}\widehat{C}^{i}{}_{jc}, {}^{\alpha}\widehat{C}^{a}_{bc} \right) \}$, when ${}^{\alpha}\widehat{\mathbf{D}}\ ({}^{\alpha}\mathbf{g}) = 0$, satisfying the conditions ${}^{\alpha}\widehat{T}^{i}{}_{jk} = 0$ and ${}^{\alpha}\widehat{T}^{a}_{bc} = 0$, but ${}^{\alpha}\widehat{T}^{i}{}_{ja}$, ${}^{\alpha}\widehat{T}^{a}{}_{ji}$, and ${}^{\alpha}\widehat{T}^{a}_{bi}$ are not zero. The N–adapted coefficients are explicitly determined by the coefficients of (7),

$$ {}^{\alpha}\widehat{L}^{i}{}_{jk} = \frac{1}{2}\, {}^{\alpha}g^{ir} \left({}^{\alpha}\mathbf{e}_{k}\, {}^{\alpha}g_{jr} + {}^{\alpha}\mathbf{e}_{j}\, {}^{\alpha}g_{kr} - {}^{\alpha}\mathbf{e}_{r}\, {}^{\alpha}g_{jk} \right), $$

$$ {}^{\alpha}\widehat{L}^{a}_{bk} = {}^{\alpha}e_{b}({}^{\alpha}N^{a}_{k}) + \frac{1}{2}\, {}^{\alpha}g^{ac}({}^{\alpha}\mathbf{e}_{k}\, {}^{\alpha}g_{bc} - {}^{\alpha}g_{dc}\ {}^{\alpha}e_{b}\, {}^{\alpha}N^{d}_{k} - {}^{\alpha}g_{db}\ {}^{\alpha}e_{c}\, {}^{\alpha}N^{d}_{k}), $$

$$ {}^{\alpha}\widehat{C}^{i}{}_{jc} = \frac{1}{2}\, {}^{\alpha}g^{ik}\, {}^{\alpha}e_{c}\, {}^{\alpha}g_{jk}, \quad {}^{\alpha}\widehat{C}^{a}_{bc} = \frac{1}{2}\, {}^{\alpha}g^{ad} ({}^{\alpha}e_{c}\, {}^{\alpha}g_{bd} + {}^{\alpha}e_{c}\, {}^{\alpha}g_{cd} - {}^{\alpha}e_{d}\, {}^{\alpha}g_{bc}). $$

- The fractional Levi–Civita connection ${}^{\alpha}\nabla = \{ {}^{\alpha}\Gamma^{\gamma}{}_{\alpha\beta} \}$ can be defined in standard from but for the fractional Caputo left derivatives acting on the coefficients of a fractional metric (7).

The Einstein tensor of any metric compatible $\overset{\alpha}{\mathbf{D}}$, when $\overset{\alpha}{\mathbf{D}}_{\tau}\, {}^{\alpha}\mathbf{g}^{\tau\beta} = 0$, is defined ${}^{\alpha}\mathcal{E}ns = \{ {}^{\alpha}\mathbf{G}_{\alpha\beta} \}$, where

$$ {}^{\alpha}\mathbf{G}_{\alpha\beta} := {}^{\alpha}\mathbf{R}_{\alpha\beta} - \frac{1}{2}\, {}^{\alpha}\mathbf{g}_{\alpha\beta}\ {}_{s}^{\alpha}\mathbf{R}. \tag{12} $$

The regular fractional mechanics defined by a fractional Lagrangian $\overset{\alpha}{L}$ can be equivalently encoded into canonical geometric data $({}_{L}\overset{\alpha}{\mathbf{N}},\ {}_{L}\overset{\alpha}{\mathbf{g}},\ {}_{c}^{\alpha}\mathbf{D})$, where we put the label L in order to emphasize that such geometric objects are induced by a fractional Lagrangian as we provided in Refs. 7, 8, 10, and 11. We also note that it is possible to "arrange" on $\overset{\alpha}{\mathbf{V}}$ such nonholonomic distributions when a d–connection ${}_{0}\overset{\alpha}{\mathbf{D}} = \{ {}_{0}^{\alpha}\widetilde{\mathbf{\Gamma}}^{\gamma'}{}_{\alpha'\beta'} \}$ is described by constant matrix coefficients, see details in Refs. 13 and 14 for integer dimensions, and Ref. 7 for fractional dimensions.

III. BASIC EQUATIONS FOR FRACTIONAL CURVE FLOWS

In symbolic, abstract index form, the constructions for nonholonomic fractional spaces with correspondingly defined distributions are similar to those for the Riemannian symmetric–spaces soldered to Klein geometry of "integer" dimension. The fractional structure is encoded into the local Caputo derivatives.

Following the introduced Cartan–Killing parametrizations, we analyze the flow $\gamma(\tau, \mathbf{l})$ of a non–stretching curve in $\mathbf{V_N} = \mathbf{G}/SO(n) \oplus\ SO(m)$ extended to $\overset{\alpha}{\mathbf{V}}_{\mathbf{N}} = \mathbf{G}/SO(n) \oplus\ SO(m)$. We use an isomorphism between the real space $\mathfrak{so}(n)$ and the Lie algebra of $n \times n$ skew–symmetric matrices. This allows us to establish an isomorphism between $h\mathfrak{p} \simeq \mathbb{R}^n$ and the tangent spaces $T_x M = \mathfrak{so}(n+1)/\,\mathfrak{so}(n)$ of the Riemannian manifold $M = SO(n+1)/\,SO(n)$ as described by the following canonical decomposition:

$$ h\mathfrak{g} = \mathfrak{so}(n+1) \supset h\mathfrak{p} \in \begin{bmatrix} 0 & h\mathbf{p} \\ -h\mathbf{p}^{T} & h\mathbf{0} \end{bmatrix} \ \text{for } h\mathbf{0} \in h\mathfrak{h} = \mathfrak{so}(n) $$

with $h\mathbf{p} = \{p^{i'}\} \in \mathbb{R}^n$ being the h–component of the d–vector $\mathbf{p} = (p^{i'}, p^{a'})$ and $h\mathbf{p}^T$ mean the transposition of the row $h\mathbf{p}$. In our approach, $T_x M \to \underline{T}_x M$, with Caputo fractional derivatives. The Cartan–Killing inner product on $h\mathfrak{g}$ is

$$h\mathbf{p}\cdot h\mathbf{p} = \left\langle \begin{bmatrix} 0 & h\mathbf{p} \\ -h\mathbf{p}^T & h\mathbf{0} \end{bmatrix}, \begin{bmatrix} 0 & h\mathbf{p} \\ -h\mathbf{p}^T & h\mathbf{0} \end{bmatrix} \right\rangle$$
$$\doteqdot \frac{1}{2} tr \left\{ \begin{bmatrix} 0 & h\mathbf{p} \\ -h\mathbf{p}^T & h\mathbf{0} \end{bmatrix}^T \begin{bmatrix} 0 & h\mathbf{p} \\ -h\mathbf{p}^T & h\mathbf{0} \end{bmatrix} \right\},$$

where tr denotes the trace of product of matrices. This product identifies canonically $h\mathfrak{p} \simeq \mathbb{R}^n$ with its dual $h\mathfrak{p}^* \simeq \mathbb{R}^n$. In a similar form, we can consider

$$v\mathfrak{g} = \mathfrak{so}(m+1) \supset v\mathfrak{p} \in \begin{bmatrix} 0 & v\mathbf{p} \\ -v\mathbf{p}^T & v\mathbf{0} \end{bmatrix} \text{ for } v\mathbf{0} \in v\mathfrak{h} = \mathfrak{so}(m)$$

with $v\mathbf{p} = \{p^{a'}\} \in \mathbb{R}^m$ being the v–component of the d–vector $\mathbf{p} = (p^{i'}, p^{a'})$ and define the Cartan–Killing inner product $v\mathbf{p}\cdot v\mathbf{p} \doteqdot \frac{1}{2} tr\{...\}$. In general, we can consider the Cartan–Killing N–adapted inner product $\mathbf{p}\cdot\mathbf{p} = h\mathbf{p}\cdot h\mathbf{p} + v\mathbf{p}\cdot v\mathbf{p}$. This extension is defined via a coframe ${}^{\alpha}\mathbf{e} \in \underline{T}^*_{\gamma}\mathbf{V_N} \otimes (h\mathfrak{p} \oplus v\mathfrak{p})$, which is a N–adapted $(SO(n) \oplus SO(m))$–parallel basis along γ, and its associated canonical d–connection 1–form ${}^{\alpha}\mathbf{\Gamma} \in \underline{T}^*_{\gamma}\mathbf{V_N} \otimes (\mathfrak{so}(n) \oplus \mathfrak{so}(m))$. Such d–objects are parametrized,

$${}^{\alpha}\mathbf{e_X} = {}^{\alpha}\mathbf{e}_{h\mathbf{X}} + {}^{\alpha}\mathbf{e}_{v\mathbf{X}},$$

where (for $(1, \overrightarrow{0}) \in \mathbb{R}^n$, $\overrightarrow{0} \in \mathbb{R}^{n-1}$ and $(1, \overleftarrow{0}) \in \mathbb{R}^m$, $\overleftarrow{0} \in \mathbb{R}^{m-1}$),

$${}^{\alpha}\mathbf{e}_{h\mathbf{X}} = \gamma_{h\mathbf{X}} \rfloor h\, {}^{\alpha}\mathbf{e} = \begin{bmatrix} 0 & (1, \overrightarrow{0}) \\ -(1, \overrightarrow{0})^T & h\mathbf{0} \end{bmatrix}$$

and

$${}^{\alpha}\mathbf{e}_{v\mathbf{X}} = \gamma_{v\mathbf{X}} \rfloor v\, {}^{\alpha}\mathbf{e} = \begin{bmatrix} 0 & (1, \overleftarrow{0}) \\ -(1, \overleftarrow{0})^T & v\mathbf{0} \end{bmatrix}.$$

We also introduce

$${}^{\alpha}\mathbf{\Gamma} = [\mathbf{\Gamma}_{h\mathbf{X}}, \mathbf{\Gamma}_{v\mathbf{X}}]$$

for

$$\mathbf{\Gamma}_{h\mathbf{X}} = \gamma_{h\mathbf{X}} \rfloor \mathbf{L} = \begin{bmatrix} 0 & (0, \overrightarrow{0}) \\ -(0, \overrightarrow{0})^T & \mathbf{L} \end{bmatrix} \in \mathfrak{so}(n+1),$$

where $\mathbf{L} = \begin{bmatrix} 0 & \overrightarrow{v} \\ -\overrightarrow{v}^T & h\mathbf{0} \end{bmatrix} \in \mathfrak{so}(n)$, $\overrightarrow{v} \in \mathbb{R}^{n-1}$, $h\mathbf{0} \in \mathfrak{so}(n-1)$, and

$$\mathbf{\Gamma}_{v\mathbf{X}} = \gamma_{v\mathbf{X}} \rfloor \mathbf{C} = \begin{bmatrix} 0 & (0, \overleftarrow{0}) \\ -(0, \overleftarrow{0})^T & \mathbf{C} \end{bmatrix} \in \mathfrak{so}(m+1),$$

where $\mathbf{C} = \begin{bmatrix} 0 & \overleftarrow{v} \\ -\overleftarrow{v}^T & v\mathbf{0} \end{bmatrix} \in \mathfrak{so}(m)$, $\overleftarrow{v} \in \mathbb{R}^{m-1}$, $v\mathbf{0} \in \mathfrak{so}(m-1)$.

There are decompositions of horizontal $SO(n+1)/\,SO(n)$ matrices,

$$h\mathfrak{p} \ni \begin{bmatrix} 0 & h\mathbf{p} \\ -h\mathbf{p}^T & h\mathbf{0} \end{bmatrix} = \begin{bmatrix} 0 & \left(h\mathbf{p}_{\parallel},\ \overrightarrow{0}\right) \\ -\left(h\mathbf{p}_{\parallel},\ \overrightarrow{0}\right)^T & h\mathbf{0} \end{bmatrix} + \begin{bmatrix} 0 & \left(0,\, h\overrightarrow{\mathbf{p}}_{\perp}\right) \\ -\left(0,\, h\overrightarrow{\mathbf{p}}_{\perp}\right)^T & h\mathbf{0} \end{bmatrix},$$

into tangential and normal parts relative to ${}^{\alpha}\mathbf{e}_{h\mathbf{X}}$ via corresponding decompositions of h–vectors $h\mathbf{p} = (h\mathbf{p}_{\parallel}, h\overrightarrow{\mathbf{p}}_{\perp}) \in \mathbb{R}^n$ relative to $\left(1,\ \overrightarrow{0}\right)$, when $h\mathbf{p}_{\parallel}$ is identified with $h\mathfrak{p}_C$ and $h\overrightarrow{\mathbf{p}}_{\perp}$ is identified with $h\mathfrak{p}_{\perp} = h\mathfrak{p}_{C^{\perp}}$. In a similar form, it is possible to decompose vertical $SO(m+1)/\,SO(m)$ matrices

$$v\mathfrak{p} \ni \begin{bmatrix} 0 & v\mathbf{p} \\ -v\mathbf{p}^T & v\mathbf{0} \end{bmatrix} = \begin{bmatrix} 0 & \left(v\mathbf{p}_{\parallel},\ \overleftarrow{0}\right) \\ -\left(v\mathbf{p}_{\parallel},\ \overleftarrow{0}\right)^T & v\mathbf{0} \end{bmatrix} + \begin{bmatrix} 0 & \left(0,\, v\overleftarrow{\mathbf{p}}_{\perp}\right) \\ -\left(0,\, v\overleftarrow{\mathbf{p}}_{\perp}\right)^T & v\mathbf{0} \end{bmatrix},$$

into tangential and normal parts relative to ${}^{\alpha}\mathbf{e}_{v\mathbf{X}}$ via corresponding decompositions of h–vectors $v\mathbf{p} = (v\mathbf{p}_{\parallel}, v\overleftarrow{\mathbf{p}}_{\perp}) \in \mathbb{R}^m$ relative to $\left(1,\ \overleftarrow{0}\right)$, when $v\mathbf{p}_{\parallel}$ is identified with $v\mathfrak{p}_C$ and $v\overleftarrow{\mathbf{p}}_{\perp}$ is identified with $v\mathfrak{p}_{\perp} = v\mathfrak{p}_{C^{\perp}}$. Locally, we consider, for instance, instead of $\mathbb{R}^n$ the fractional space ${}^{\alpha}\mathbb{R}^n$ local (co) vectors defined by Caputo fractional derivatives and their duals.

The canonical d–connection ${}^{\alpha}\widehat{\mathbf{D}} = \{{}^{\alpha}\widehat{\mathbf{\Gamma}}^{\tau}{}_{\beta\gamma}\}$ (Ref. 7) induces matrices decomposed with respect to the fractional flow direction. In the h–direction, we parametrize

$${}^{\alpha}\mathbf{e}_{h\mathbf{Y}} = \gamma_{\tau}\rfloor h\,{}^{\alpha}\mathbf{e} = \begin{bmatrix} 0 & \left(h\mathbf{e}_{\parallel},\, h\overrightarrow{\mathbf{e}}_{\perp}\right) \\ -\left(h\mathbf{e}_{\parallel},\, h\overrightarrow{\mathbf{e}}_{\perp}\right)^T & h\mathbf{0} \end{bmatrix},$$

when ${}^{\alpha}\mathbf{e}_{h\mathbf{Y}} \in h\mathfrak{p}$, $\left(h\mathbf{e}_{\parallel},\, h\overrightarrow{\mathbf{e}}_{\perp}\right) \in {}^{\alpha}\mathbb{R}^n$ and $h\overrightarrow{\mathbf{e}}_{\perp} \in {}^{\alpha}\mathbb{R}^{n-1}$, and

$$\mathbf{\Gamma}_{h\mathbf{Y}} = \gamma_{h\mathbf{Y}}\rfloor\mathbf{L} = \begin{bmatrix} 0 & (0,\ \overrightarrow{0}) \\ -(0,\ \overrightarrow{0})^T & h\varpi_{\tau} \end{bmatrix} \in \mathfrak{so}(n+1), \tag{13}$$

where $h\varpi_{\tau} = \begin{bmatrix} 0 & \overrightarrow{\varpi} \\ -\overrightarrow{\varpi}^T & h\Theta \end{bmatrix} \in \mathfrak{so}(n)$, $\overrightarrow{\varpi} \in {}^{\alpha}\mathbb{R}^{n-1}$, $h\Theta \in \mathfrak{so}(n-1)$.

In the v–direction, we have

$$\mathbf{e}_{v\mathbf{Y}} = \gamma_{\tau}\rfloor v\mathbf{e} = \begin{bmatrix} 0 & \left(v\mathbf{e}_{\parallel},\, v\overleftarrow{\mathbf{e}}_{\perp}\right) \\ -\left(v\mathbf{e}_{\parallel},\, v\overleftarrow{\mathbf{e}}_{\perp}\right)^T & v\mathbf{0} \end{bmatrix},$$

when $\mathbf{e}_{v\mathbf{Y}} \in v\mathfrak{p}$, $\left(v\mathbf{e}_{\parallel},\, v\overleftarrow{\mathbf{e}}_{\perp}\right) \in {}^{\alpha}\mathbb{R}^m$ and $v\overleftarrow{\mathbf{e}}_{\perp} \in {}^{\alpha}\mathbb{R}^{m-1}$, and

$$\mathbf{\Gamma}_{v\mathbf{Y}} = \gamma_{v\mathbf{Y}}\rfloor\mathbf{C} = \begin{bmatrix} 0 & (0,\ \overleftarrow{0}) \\ -(0,\ \overleftarrow{0})^T & v\varpi_{\tau} \end{bmatrix} \in \mathfrak{so}(m+1),$$

where $v\varpi_{\tau} = \begin{bmatrix} 0 & \overleftarrow{\varpi} \\ -\overleftarrow{\varpi}^T & v\Theta \end{bmatrix} \in \mathfrak{so}(m)$, $\overleftarrow{\varpi} \in {}^{\alpha}\mathbb{R}^{m-1}$, $v\Theta \in \mathfrak{so}(m-1)$.

The components $h\mathbf{e}_{\parallel}$ and $h\overrightarrow{\mathbf{e}}_{\perp}$ correspond to the decomposition

$${}^{\alpha}\mathbf{e}_{h\mathbf{Y}} = h\mathbf{g}(\gamma_{\tau}, \gamma_{l})\,{}^{\alpha}\mathbf{e}_{h\mathbf{X}} + (\gamma_{\tau})_{\perp}\rfloor h\mathbf{e}_{\perp}$$

into tangential and normal parts relative to ${}^{\alpha}\mathbf{e}_{h\mathbf{X}}$. In a similar form, one considers $v\mathbf{e}_{\parallel}$ and $v\overleftarrow{\mathbf{e}}_{\perp}$ corresponding to the decomposition,

$$ {}^{\alpha}\mathbf{e}_{v\mathbf{Y}} = v\mathbf{g}(\gamma_{\tau}, \gamma_{l})\,{}^{\alpha}\mathbf{e}_{v\mathbf{X}} + (\gamma_{\tau})_{\perp}\rfloor v\mathbf{e}_{\perp}. $$

Working with such matrix parametrizations, we define

$$ \left[{}^{\alpha}\mathbf{e}_{h\mathbf{X}},\ {}^{\alpha}\mathbf{e}_{h\mathbf{Y}}\right] = -\begin{bmatrix} 0 & 0 \\ 0 & h\mathbf{e}_{\perp} \end{bmatrix} \in \mathfrak{so}(n+1), \tag{14} $$

$$ \text{for } h\mathbf{e}_{\perp} = \begin{bmatrix} 0 & h\overrightarrow{\mathbf{e}}_{\perp} \\ -(h\overrightarrow{\mathbf{e}}_{\perp})^{T} & h\mathbf{0} \end{bmatrix} \in \mathfrak{so}(n); $$

$$ \left[\mathbf{\Gamma}_{h\mathbf{Y}},\ {}^{\alpha}\mathbf{e}_{h\mathbf{Y}}\right] = -\begin{bmatrix} 0 & \left(0, \overrightarrow{\varpi}\right) \\ -\left(0, \overrightarrow{\varpi}\right)^{T} & 0 \end{bmatrix} \in h\mathfrak{p}_{\perp}; $$

$$ \left[\mathbf{\Gamma}_{h\mathbf{X}},\ {}^{\alpha}\mathbf{e}_{h\mathbf{Y}}\right] = -\begin{bmatrix} 0 & \left(-\overrightarrow{v}\cdot h\overrightarrow{\mathbf{e}}_{\perp}, h\mathbf{e}_{\parallel}\overrightarrow{v}\right) \\ -\left(-\overrightarrow{v}\cdot h\overrightarrow{\mathbf{e}}_{\perp}, h\mathbf{e}_{\parallel}\overrightarrow{v}\right)^{T} & h\mathbf{0} \end{bmatrix} \in h\mathfrak{p}; $$

and

$$ \left[{}^{\alpha}\mathbf{e}_{v\mathbf{X}},\ {}^{\alpha}\mathbf{e}_{v\mathbf{Y}}\right] = -\begin{bmatrix} 0 & 0 \\ 0 & v\mathbf{e}_{\perp} \end{bmatrix} \in \mathfrak{so}(m+1), \tag{15} $$

$$ \text{for } v\mathbf{e}_{\perp} = \begin{bmatrix} 0 & v\overleftarrow{\mathbf{e}}_{\perp} \\ -(v\overleftarrow{\mathbf{e}}_{\perp})^{T} & v\mathbf{0} \end{bmatrix} \in \mathfrak{so}(m); $$

$$ \left[\mathbf{\Gamma}_{v\mathbf{Y}},\ {}^{\alpha}\mathbf{e}_{v\mathbf{Y}}\right] = -\begin{bmatrix} 0 & \left(0, \overleftarrow{\varpi}\right) \\ -\left(0, \overleftarrow{\varpi}\right)^{T} & 0 \end{bmatrix} \in v\mathfrak{p}_{\perp}; $$

$$ \left[\mathbf{\Gamma}_{v\mathbf{X}},\ {}^{\alpha}\mathbf{e}_{v\mathbf{Y}}\right] = -\begin{bmatrix} 0 & \left(-\overleftarrow{v}\cdot v\overleftarrow{\mathbf{e}}_{\perp}, v\mathbf{e}_{\parallel}\overleftarrow{v}\right) \\ -\left(-\overleftarrow{v}\cdot v\overleftarrow{\mathbf{e}}_{\perp}, v\mathbf{e}_{\parallel}\overleftarrow{v}\right)^{T} & v\mathbf{0} \end{bmatrix} \in v\mathfrak{p}. $$

We use formulas (14) and (15) in order to write the structure equations in terms of N–adapted curve fractional flow operators soldered to the geometry Klein N–anholonomic spaces, the formulas are "fractional" extensions of the respective ones in Refs. 13 and 14 . This way, it is possible to construct, respectively, the $\mathbf{G}$ –invariant N–adapted torsion and curvature generated by the fractional canonical d–connection,

$$ \begin{aligned} {}^{\alpha}\widehat{\mathbf{T}}(\gamma_{\tau}, \gamma_{l}) &= \left({}^{\alpha}\widehat{\mathbf{D}}_{\mathbf{X}}\gamma_{\tau} - {}^{\alpha}\widehat{\mathbf{D}}_{\mathbf{Y}}\gamma_{l}\right)\rfloor\,{}^{\alpha}\mathbf{e} \\ &= {}^{\alpha}\widehat{\mathbf{D}}_{\mathbf{X}}\,{}^{\alpha}\mathbf{e}_{\mathbf{Y}} - {}^{\alpha}\widehat{\mathbf{D}}_{\mathbf{Y}}\,{}^{\alpha}\mathbf{e}_{\mathbf{X}} + \left[{}^{\alpha}\widehat{\mathbf{\Gamma}}_{\mathbf{X}},\ {}^{\alpha}\mathbf{e}_{\mathbf{Y}}\right] - \left[{}^{\alpha}\widehat{\mathbf{\Gamma}}_{\mathbf{Y}},\ {}^{\alpha}\mathbf{e}_{\mathbf{X}}\right] \end{aligned} \tag{16} $$

and

$$ \begin{aligned} {}^{\alpha}\widehat{\mathbf{R}}(\gamma_{\tau}, \gamma_{l})\,{}^{\alpha}\mathbf{e} &= \left[{}^{\alpha}\widehat{\mathbf{D}}_{\mathbf{X}},\ {}^{\alpha}\widehat{\mathbf{D}}_{\mathbf{Y}}\right]{}^{\alpha}\mathbf{e} \\ &= {}^{\alpha}\widehat{\mathbf{D}}_{\mathbf{X}}\,{}^{\alpha}\widehat{\mathbf{\Gamma}}_{\mathbf{Y}} - {}^{\alpha}\widehat{\mathbf{D}}_{\mathbf{Y}}\,{}^{\alpha}\widehat{\mathbf{\Gamma}}_{\mathbf{X}} + \left[{}^{\alpha}\widehat{\mathbf{\Gamma}}_{\mathbf{X}},\ {}^{\alpha}\widehat{\mathbf{\Gamma}}_{\mathbf{Y}}\right], \end{aligned} \tag{17} $$

where ${}^{\alpha}\mathbf{e}_{\mathbf{X}} \doteqdot \gamma_{l}\rfloor\,{}^{\alpha}\mathbf{e}$, ${}^{\alpha}\mathbf{e}_{\mathbf{Y}} \doteqdot \gamma_{\tau}\rfloor\,{}^{\alpha}\mathbf{e}$, ${}^{\alpha}\widehat{\mathbf{\Gamma}}_{\mathbf{X}} \doteqdot \gamma_{l}\rfloor\,{}^{\alpha}\widehat{\mathbf{\Gamma}}$, and ${}^{\alpha}\widehat{\mathbf{\Gamma}}_{\mathbf{Y}} \doteqdot \gamma_{\tau}\rfloor\,{}^{\alpha}\widehat{\mathbf{\Gamma}}$.

Applying a d–connection ${}^{\alpha}\mathbf{D}$ (in particular, we can take ${}^{\alpha}\widehat{\mathbf{D}}$) instead of the Levi–Civita one ${}^{\alpha}\nabla$, we get

$$
\begin{aligned}
0 &= ({}^{\alpha}\mathbf{D}_{h\mathbf{X}}\gamma_{\tau} - {}^{\alpha}\mathbf{D}_{h\mathbf{Y}}\gamma_{l})\rfloor h\,{}^{\alpha}\mathbf{e} \\
&= {}^{\alpha}\mathbf{D}_{h\mathbf{X}}\,{}^{\alpha}\mathbf{e}_{h\mathbf{Y}} - {}^{\alpha}\mathbf{D}_{h\mathbf{Y}}\,{}^{\alpha}\mathbf{e}_{h\mathbf{X}} + \left[{}^{\alpha}\mathbf{L}_{h\mathbf{X}},\,{}^{\alpha}\mathbf{e}_{h\mathbf{Y}}\right] - \left[{}^{\alpha}\mathbf{L}_{h\mathbf{Y}},\,{}^{\alpha}\mathbf{e}_{h\mathbf{X}}\right]; \\
0 &= ({}^{\alpha}\mathbf{D}_{v\mathbf{X}}\gamma_{\tau} - {}^{\alpha}\mathbf{D}_{v\mathbf{Y}}\gamma_{l})\rfloor v\,{}^{\alpha}\mathbf{e} \\
&= {}^{\alpha}\mathbf{D}_{v\mathbf{X}}\,{}^{\alpha}\mathbf{e}_{v\mathbf{Y}} - {}^{\alpha}\mathbf{D}_{v\mathbf{Y}}\,{}^{\alpha}\mathbf{e}_{v\mathbf{X}} + \left[{}^{\alpha}\mathbf{C}_{v\mathbf{X}},\,{}^{\alpha}\mathbf{e}_{v\mathbf{Y}}\right] - \left[{}^{\alpha}\mathbf{C}_{v\mathbf{Y}},\,{}^{\alpha}\mathbf{e}_{v\mathbf{X}}\right], \\
& h\,{}^{\alpha}\mathbf{R}(\gamma_{\tau},\gamma_{l})h\,{}^{\alpha}\mathbf{e} = \left[{}^{\alpha}\mathbf{D}_{h\mathbf{X}},\,{}^{\alpha}\mathbf{D}_{h\mathbf{Y}}\right]h\,{}^{\alpha}\mathbf{e} \\
&= {}^{\alpha}\mathbf{D}_{h\mathbf{X}}\,{}^{\alpha}\mathbf{L}_{h\mathbf{Y}} - {}^{\alpha}\mathbf{D}_{h\mathbf{Y}}\,{}^{\alpha}\mathbf{L}_{h\mathbf{X}} + \left[{}^{\alpha}\mathbf{L}_{h\mathbf{X}},\,{}^{\alpha}\mathbf{L}_{h\mathbf{Y}}\right] \\
& v\,{}^{\alpha}\mathbf{R}(\gamma_{\tau},\gamma_{l})v\,{}^{\alpha}\mathbf{e} = \left[{}^{\alpha}\mathbf{D}_{v\mathbf{X}},\,{}^{\alpha}\mathbf{D}_{v\mathbf{Y}}\right]v\,{}^{\alpha}\mathbf{e} \\
&= {}^{\alpha}\mathbf{D}_{v\mathbf{X}}\,{}^{\alpha}\mathbf{C}_{v\mathbf{Y}} - {}^{\alpha}\mathbf{D}_{v\mathbf{Y}}\,{}^{\alpha}\mathbf{C}_{v\mathbf{X}} + \left[{}^{\alpha}\mathbf{C}_{v\mathbf{X}},\,{}^{\alpha}\mathbf{C}_{v\mathbf{Y}}\right].
\end{aligned}
\tag{18}
$$

Following N–adapted curve flow parametrizations (14) and (15), Eqs. (18) are written

$$
\begin{aligned}
& 0 = {}^{\alpha}\mathbf{D}_{h\mathbf{X}}h\mathbf{e}_{\parallel} + \overrightarrow{v}\cdot h\overrightarrow{\mathbf{e}}_{\perp},\, 0 = {}^{\alpha}\mathbf{D}_{v\mathbf{X}}v\mathbf{e}_{\parallel} + \overleftarrow{v}\cdot v\overleftarrow{\mathbf{e}}_{\perp}; \\
& 0 = \overrightarrow{\varpi} - h\mathbf{e}_{\parallel}\overrightarrow{v} + {}^{\alpha}\mathbf{D}_{h\mathbf{X}}h\overrightarrow{\mathbf{e}}_{\perp},\, 0 = \overleftarrow{\varpi} - v\mathbf{e}_{\parallel}\overleftarrow{v} + {}^{\alpha}\mathbf{D}_{v\mathbf{X}}v\overleftarrow{\mathbf{e}}_{\perp}; \\
& {}^{\alpha}\mathbf{D}_{h\mathbf{X}}\overrightarrow{\varpi} - {}^{\alpha}\mathbf{D}_{h\mathbf{Y}}\overrightarrow{v} + \overrightarrow{v}\rfloor h\Theta = h\overrightarrow{\mathbf{e}}_{\perp}, \\
& {}^{\alpha}\mathbf{D}_{v\mathbf{X}}\overleftarrow{\varpi} - {}^{\alpha}\mathbf{D}_{v\mathbf{Y}}\overleftarrow{v} + \overleftarrow{v}\rfloor v\Theta = v\overleftarrow{\mathbf{e}}_{\perp}; \\
& {}^{\alpha}\mathbf{D}_{h\mathbf{X}}h\Theta - \overrightarrow{v}\otimes\overrightarrow{\varpi} + \overrightarrow{\varpi}\otimes\overrightarrow{v} = 0, \\
& {}^{\alpha}\mathbf{D}_{v\mathbf{X}}v\Theta - \overleftarrow{v}\otimes\overleftarrow{\varpi} + \overleftarrow{\varpi}\otimes\overleftarrow{v} = 0.
\end{aligned}
\tag{19}
$$

For such fractional spaces, the tensor and interior products, for instance, for the h–components, are defined in the form: $\otimes$ denotes the outer product of pairs of vectors ($1\times n$ row matrices), producing $n\times n$ matrices $\overrightarrow{A}\otimes\overrightarrow{B} = \overrightarrow{A}^{T}\overrightarrow{B}$, and $\rfloor$ denotes multiplication of $n\times n$ matrices on vectors ($1\times n$ row matrices); one holds the properties $\overrightarrow{A}\rfloor\left(\overrightarrow{B}\otimes\overrightarrow{C}\right) = \left(\overrightarrow{A}\cdot\overrightarrow{B}\right)\overrightarrow{C}$ which is the transpose of the standard matrix product on column vectors, and $\left(\overrightarrow{B}\otimes\overrightarrow{C}\right)\overrightarrow{A} = \left(\overrightarrow{C}\cdot\overrightarrow{A}\right)\overrightarrow{B}$. As basic vectors, we use the Caputo fractional derivatives. Similar formulas hold for the v–components but, for instance, we have to change, correspondingly, $n\to m$ and $\overrightarrow{A}\to\overleftarrow{A}$; for such constructions the fractional differentials have to be used.

Lemma 3.1: On nonholonomic fractional manifolds with constant curvature matrix coefficients for a d–connection, there are N–adapted fractional Hamiltonian symplectic operators,

$$
h\,{}^{\alpha}\mathcal{J} = {}^{\alpha}\mathbf{D}_{h\mathbf{X}} + {}^{\alpha}\mathbf{D}_{h\mathbf{X}}^{-1}\left(\overrightarrow{v}\cdot\right)\overrightarrow{v} \ \ \textit{and} \ \ v\,{}^{\alpha}\mathcal{J} = {}^{\alpha}\mathbf{D}_{v\mathbf{X}} + {}^{\alpha}\mathbf{D}_{v\mathbf{X}}^{-1}\left(\overleftarrow{v}\cdot\right)\overleftarrow{v}, \tag{20}
$$

and cosymplectic operators

$$
h\,{}^{\alpha}\mathcal{H} \doteqdot {}^{\alpha}\mathbf{D}_{h\mathbf{X}} + \overrightarrow{v}\rfloor\,{}^{\alpha}\mathbf{D}_{h\mathbf{X}}^{-1}\left(\overrightarrow{v}\wedge\right) \ \ \textit{and} \ \ v\,{}^{\alpha}\mathcal{H} \doteqdot {}^{\alpha}\mathbf{D}_{v\mathbf{X}} + \overleftarrow{v}\rfloor\,{}^{\alpha}\mathbf{D}_{v\mathbf{X}}^{-1}\left(\overleftarrow{v}\wedge\right), \tag{21}
$$

where, for instance, $\overrightarrow{A}\wedge\overrightarrow{B} = \overrightarrow{A}\otimes\overrightarrow{B} - \overrightarrow{B}\otimes\overrightarrow{A}$.

Proof: We sketch some key steps of the proof. The variables $\mathbf{e}_{\parallel}$ and Θ, written in h– and v–components, can be expressed correspondingly in terms of variables $\overrightarrow{v}$, $\overrightarrow{\varpi}$, $h\overrightarrow{\mathbf{e}}_{\perp}$ and $\overleftarrow{v}$, $\overleftarrow{\varpi}$, $v\overleftarrow{\mathbf{e}}_{\perp}$ (see Eqs. (19)),

$$
h\mathbf{e}_{\parallel} = -\,{}^{\alpha}\mathbf{D}_{h\mathbf{X}}^{-1}(\overrightarrow{v}\cdot h\overrightarrow{\mathbf{e}}_{\perp}),\, v\mathbf{e}_{\parallel} = -\,{}^{\alpha}\mathbf{D}_{v\mathbf{X}}^{-1}(\overleftarrow{v}\cdot v\overleftarrow{\mathbf{e}}_{\perp}),
$$

and $h\Theta = {}^{\alpha}\mathbf{D}_{h\mathbf{X}}^{-1}\left(\overrightarrow{v} \otimes \overrightarrow{\varpi} - \overrightarrow{\varpi} \otimes \overrightarrow{v}\right)$, $v\Theta = {}^{\alpha}\mathbf{D}_{v\mathbf{X}}^{-1}\left(\overleftarrow{v} \otimes \overleftarrow{\varpi} - \overleftarrow{\varpi} \otimes \overleftarrow{v}\right)$. Substituting these values, respectively, in Eqs. (19), we express

$$\overrightarrow{\varpi} = -\,{}^{\alpha}\mathbf{D}_{h\mathbf{X}} h\overrightarrow{\mathbf{e}}_{\perp} - {}^{\alpha}\mathbf{D}_{h\mathbf{X}}^{-1}(\overrightarrow{v} \cdot h\overrightarrow{\mathbf{e}}_{\perp})\overrightarrow{v},$$
$$\overleftarrow{\varpi} = -\,{}^{\alpha}\mathbf{D}_{v\mathbf{X}} v\overleftarrow{\mathbf{e}}_{\perp} - {}^{\alpha}\mathbf{D}_{v\mathbf{X}}^{-1}(\overleftarrow{v} \cdot v\overleftarrow{\mathbf{e}}_{\perp})\overleftarrow{v}$$

contained in the h– and v–flow equations, respectively, on $\overrightarrow{v}$ and $\overleftarrow{v}$, considered as scalar components when ${}^{\alpha}\mathbf{D}_{h\mathbf{Y}}\overrightarrow{v} = \overrightarrow{v}_{\tau}$ and ${}^{\alpha}\mathbf{D}_{h\mathbf{Y}}\overleftarrow{v} = \overleftarrow{v}_{\tau}$,

$$\overrightarrow{v}_{\tau} = {}^{\alpha}\mathbf{D}_{h\mathbf{X}}\overrightarrow{\varpi} - \overrightarrow{v}\rfloor {}^{\alpha}\mathbf{D}_{h\mathbf{X}}^{-1}\left(\overrightarrow{v} \otimes \overrightarrow{\varpi} - \overrightarrow{\varpi} \otimes \overrightarrow{v}\right) - {}^{\alpha}\overrightarrow{R}\, h\overrightarrow{\mathbf{e}}_{\perp},$$
$$\overleftarrow{v}_{\tau} = {}^{\alpha}\mathbf{D}_{v\mathbf{X}}\overleftarrow{\varpi} - \overleftarrow{v}\rfloor {}^{\alpha}\mathbf{D}_{v\mathbf{X}}^{-1}\left(\overleftarrow{v} \otimes \overleftarrow{\varpi} - \overleftarrow{\varpi} \otimes \overleftarrow{v}\right) - {}^{\alpha}\overleftarrow{S}\, v\overleftarrow{\mathbf{e}}_{\perp}, \tag{22}$$

where ${}^{\alpha}\overrightarrow{R}$ and ${}^{\alpha}\overleftarrow{S}$ are the scalar curvatures of chosen d–connection. For symmetric Riemannian spaces such as $SO(n+1)/SO(n) \simeq S^n$, the value $\overrightarrow{R}$ is just the scalar curvature $\chi = 1$. On N–anholonomic fractional manifolds, it is possible to define such d–connections that ${}^{\alpha}\overrightarrow{R}$ and ${}^{\alpha}\overleftarrow{S}$ are certain zero or nonzero constants. □

The properties of operators (20) and (21) are defined by

Theorem 3.1: *The fractional d–operators* ${}^{\alpha}\mathcal{J} = (h\,{}^{\alpha}\mathcal{J}, v\,{}^{\alpha}\mathcal{J})$ *and* ${}^{\alpha}\mathcal{H} = (h\,{}^{\alpha}\mathcal{H}, v\,{}^{\alpha}\mathcal{H})$ *are, respectively,* $(O(n-1), O(m-1))$ *–invariant Hamiltonian symplectic and cosymplectic d–operators with respect to the fractional Hamiltonian d–variables* $\left(\overrightarrow{v}, \overleftarrow{v}\right)$. *This class of d–operators defines the Hamiltonian form for the curve fractional flow equations on N–anholonomic fractional manifolds with constant d–connection curvature: the fractional h–flows are given by*

$$\overrightarrow{v}_{\tau} = h\,{}^{\alpha}\mathcal{H}\left(\overrightarrow{\varpi}\right) - {}^{\alpha}\overrightarrow{R}\, h\overrightarrow{\mathbf{e}}_{\perp} = h\,{}^{\alpha}\mathfrak{R}\left(h\overrightarrow{\mathbf{e}}_{\perp}\right) - {}^{\alpha}\overrightarrow{R}\, h\overrightarrow{\mathbf{e}}_{\perp},$$
$$\overrightarrow{\varpi} = h\,{}^{\alpha}\mathcal{J}\left(h\overrightarrow{\mathbf{e}}_{\perp}\right); \tag{23}$$

the fractional v–flows are given by

$$\overleftarrow{v}_{\tau} = v\,{}^{\alpha}\mathcal{H}\left(\overleftarrow{\varpi}\right) - {}^{\alpha}\overleftarrow{S}\, v\overleftarrow{\mathbf{e}}_{\perp} = v\,{}^{\alpha}\mathfrak{R}\left(v\overleftarrow{\mathbf{e}}_{\perp}\right) - {}^{\alpha}\overleftarrow{S}\, v\overleftarrow{\mathbf{e}}_{\perp},$$
$$\overleftarrow{\varpi} = v\,{}^{\alpha}\mathcal{J}\left(v\overleftarrow{\mathbf{e}}_{\perp}\right), \tag{24}$$

where the so–called (fractional) hereditary recursion d–operator has the respective h– and v–components,

$$h\,{}^{\alpha}\mathfrak{R} = h\,{}^{\alpha}\mathcal{H} \circ h\,{}^{\alpha}\mathcal{J} \ \text{ and } \ v\,{}^{\alpha}\mathfrak{R} = v\,{}^{\alpha}\mathcal{H} \circ v\,{}^{\alpha}\mathcal{J}. \tag{25}$$

Proof: Such a proof follows from the Lemma and (22). □

IV. FRACTIONAL BI–HAMILTONIANS AND SOLITONIC HIERARCHIES

The fractional recursion h–operator from (25),

$$h\,{}^{\alpha}\mathfrak{R} = {}^{\alpha}\mathbf{D}_{h\mathbf{X}}\left({}^{\alpha}\mathbf{D}_{h\mathbf{X}} + {}^{\alpha}\mathbf{D}_{h\mathbf{X}}^{-1}\left(\overrightarrow{v}\cdot\right)\overrightarrow{v}\right) + \overrightarrow{v}\rfloor {}^{\alpha}\mathbf{D}_{h\mathbf{X}}^{-1}\left(\overrightarrow{v} \wedge {}^{\alpha}\mathbf{D}_{h\mathbf{X}}\right)$$
$$= {}^{\alpha}\mathbf{D}_{h\mathbf{X}}^{2} + |\,{}^{\alpha}\mathbf{D}_{h\mathbf{X}}|^2 + {}^{\alpha}\mathbf{D}_{h\mathbf{X}}^{-1}\left(\overrightarrow{v}\cdot\right)\overrightarrow{v}_{\mathbf{l}} - \overrightarrow{v}\rfloor {}^{\alpha}\mathbf{D}_{h\mathbf{X}}^{-1}(\overrightarrow{v}_{\mathbf{l}}\wedge), \tag{26}$$

induces a horizontal hierarchy of commuting Hamiltonian vector fields $h\overrightarrow{\mathbf{e}}_{\perp}^{(k)}$ starting from $h\overrightarrow{\mathbf{e}}_{\perp}^{(0)} = \overrightarrow{v}_{\mathbf{l}}$. Such vector fields are given by the infinitesimal generator of $\mathbf{l}$–translations in terms of arclength $\mathbf{l}$ along the curve. A vertical hierarchy of commuting vector fields $v\overleftarrow{\mathbf{e}}_{\perp}^{(k)}$ starting from $v\overleftarrow{\mathbf{e}}_{\perp}^{(0)} = \overleftarrow{v}_{\mathbf{l}}$ is generated by the recursion v–operator,

$$v\,{}^{\alpha}\mathfrak{R} = {}^{\alpha}\mathbf{D}_{v\mathbf{X}}\left({}^{\alpha}\mathbf{D}_{v\mathbf{X}} + {}^{\alpha}\mathbf{D}_{v\mathbf{X}}^{-1}\left(\overleftarrow{v}\cdot\right)\overleftarrow{v}\right) + \overleftarrow{v}\rfloor {}^{\alpha}\mathbf{D}_{v\mathbf{X}}^{-1}\left(\overleftarrow{v} \wedge {}^{\alpha}\mathbf{D}_{v\mathbf{X}}\right)$$
$$= {}^{\alpha}\mathbf{D}_{v\mathbf{X}}^{2} + |\,{}^{\alpha}\mathbf{D}_{v\mathbf{X}}|^2 + {}^{\alpha}\mathbf{D}_{v\mathbf{X}}^{-1}\left(\overleftarrow{v}\cdot\right)\overleftarrow{v}_{\mathbf{l}} - \overleftarrow{v}\rfloor {}^{\alpha}\mathbf{D}_{v\mathbf{X}}^{-1}(\overleftarrow{v}_{\mathbf{l}}\wedge). \tag{27}$$

We can associate fractional hierarchies generated by adjoint operators ${}^{\alpha}\mathfrak{R}^{*} = (h\,{}^{\alpha}\mathfrak{R}^{*},\ v\,{}^{\alpha}\mathfrak{R}^{*})$ of involuntive fractional Hamiltonian h–covector fields $\overrightarrow{\varpi}^{(k)} = \delta\left(h\,{}^{\alpha}H^{(k)}\right)/\delta\overrightarrow{v}$ in terms of fractional Hamiltonians $h\,{}^{\alpha}H = h\,{}^{\alpha}H^{(k)}(\overrightarrow{v}, \overrightarrow{v}_{\mathbf{l}}, \overrightarrow{v}_{2\mathbf{l}}, \ldots)$ starting from $\overrightarrow{\varpi}^{(0)} = \overrightarrow{v}$, $h\,{}^{\alpha}H^{(0)} = \frac{1}{2}|\overrightarrow{v}|^{2}$ and of involutive fractional Hamiltonian v–covector fields $\overleftarrow{\varpi}^{(k)} = \delta\left(v\,{}^{\alpha}H^{(k)}\right)/\delta\overleftarrow{v}$ in terms of Hamiltonians $v\,{}^{\alpha}H = v\,{}^{\alpha}H^{(k)}(\overleftarrow{v}, \overleftarrow{v}_{\mathbf{l}}, \overleftarrow{v}_{2\mathbf{l}}, \ldots)$ starting from $\overleftarrow{\varpi}^{(0)} = \overleftarrow{v}$, $v\,{}^{\alpha}H^{(0)} = \frac{1}{2}|\overleftarrow{v}|^{2}$. The relations between fractional hierarchies is given by formulas,

$$h\overrightarrow{\mathbf{e}}_{\perp}^{(k)} = h\,{}^{\alpha}\mathcal{H}\left(\overrightarrow{\varpi}^{(k)}, \overrightarrow{\varpi}^{(k+1)}\right) = h\,{}^{\alpha}\mathcal{J}\left(h\overrightarrow{\mathbf{e}}_{\perp}^{(k)}\right),$$

$$v\overleftarrow{\mathbf{e}}_{\perp}^{(k)} = v\,{}^{\alpha}\mathcal{H}\left(\overleftarrow{\varpi}^{(k)}, \overleftarrow{\varpi}^{(k+1)}\right) = v\,{}^{\alpha}\mathcal{J}\left(v\overleftarrow{\mathbf{e}}_{\perp}^{(k)}\right),$$

where $k = 0, 1, 2, \ldots$. All hierarchies (horizontal, vertical, and their adjoint ones) have a typical mKdV scaling symmetry, for instance, $\mathbf{l} \to \lambda\mathbf{l}$ and $\overrightarrow{v} \to \lambda^{-1}\overrightarrow{v}$ under which the values $h\overrightarrow{\mathbf{e}}_{\perp}^{(k)}$ and $h\,{}^{\alpha}H^{(k)}$ have scaling weight $2 + 2k$, while $\overrightarrow{\varpi}^{(k)}$ has scaling weight $1 + 2k$.

Corollary 4.1: There are N–adapted fractional hierarchies of distinguished horizontal and vertical commuting bi–Hamiltonian fractional flows, correspondingly, on $\overrightarrow{v}$ and $\overleftarrow{v}$, associated to the recursion d–operator (25) given by $O(n-1) \oplus O(m-1)$–invariant d–vector evolution equations,

$$\overrightarrow{v}_{\tau} = h\overrightarrow{\mathbf{e}}_{\perp}^{(k+1)} - {}^{\alpha}\overrightarrow{R}\, h\overrightarrow{\mathbf{e}}_{\perp}^{(k)} = h\,{}^{\alpha}\mathcal{H}\left(\delta\left(h\,{}^{\alpha}H^{(k,\overrightarrow{R})}\right)/\delta\overrightarrow{v}\right)$$
$$= (h\,{}^{\alpha}\mathcal{J})^{-1}\left(\delta\left(h\,{}^{\alpha}H^{(k+1,\overrightarrow{R})}\right)/\delta\overrightarrow{v}\right)$$

with horizontal fractional Hamiltonians

$$h\,{}^{\alpha}H^{(k+1,\overrightarrow{R})} = h\,{}^{\alpha}H^{(k+1,\overrightarrow{R})} - {}^{\alpha}\overrightarrow{R}\, h\,{}^{\alpha}H^{(k,\overrightarrow{R})}$$

and

$$\overleftarrow{v}_{\tau} = v\overleftarrow{\mathbf{e}}_{\perp}^{(k+1)} - {}^{\alpha}\overleftarrow{S}\, v\overleftarrow{\mathbf{e}}_{\perp}^{(k)} = v\,{}^{\alpha}\mathcal{H}\left(\delta\left(v\,{}^{\alpha}H^{(k,\overleftarrow{S})}\right)/\delta\overleftarrow{v}\right)$$
$$= (v\,{}^{\alpha}\mathcal{J})^{-1}\left(\delta\left(v\,{}^{\alpha}H^{(k+1,\overleftarrow{S})}\right)/\delta\overleftarrow{v}\right)$$

with vertical fractional Hamiltonians

$$v\,{}^{\alpha}H^{(k+1,\overleftarrow{S})} = v\,{}^{\alpha}H^{(k+1,\overleftarrow{S})} - {}^{\alpha}\overleftarrow{S}\, v\,{}^{\alpha}H^{(k,\overleftarrow{S})},$$

for $k = 0, 1, 2, \ldots$ The fractional d–operators ${}^{\alpha}\mathcal{H}$ and ${}^{\alpha}\mathcal{J}$ are N–adapted and mutually compatible from which one can construct an alternative (explicit) fractional Hamilton d–operator ${}^{\alpha}\mathcal{H} = {}^{\alpha}\mathcal{H} \circ {}^{\alpha}\mathcal{J} \circ {}^{\alpha}\mathcal{H} = {}^{\alpha}\mathfrak{R} \circ {}^{\alpha}\mathcal{H}$.

Proof: It follows from above presented considerations. □

A. Formulation of the main theorem

Our goal is to prove that the geometric data for any fractional metric (in a model of fractional gravity or geometric mechanics) naturally define a N–adapted fractional bi–Hamiltonian flow hierarchy inducing anholonomic fractional solitonic configurations.

Theorem 4.1: *For any N–anholonomic fractional manifold with prescribed fractional d–metric structure, there is a hierarchy of bi-Hamiltonian N–adapted fractional flows of curves $\gamma(\tau, \mathbf{l}) = h\gamma(\tau, \mathbf{l}) + v\gamma(\tau, \mathbf{l})$ described by geometric nonholonomic fractional map equations. The 0 fractional flows are defined as convective (traveling wave) maps*

$$\gamma_{\tau} = \gamma_{\mathbf{l}},\ \textit{distinguished}\ \ (h\gamma)_{\tau} = (h\gamma)_{h\mathbf{X}}\ \textit{and}\ \ (v\gamma)_{\tau} = (v\gamma)_{v\mathbf{X}}. \qquad (28)$$

There are fractional $+1$ *flows defined as non–stretching mKdV maps*

$$-(h\gamma)_\tau = {}^{\alpha}\mathbf{D}^2_{h\mathbf{X}}(h\gamma)_{h\mathbf{X}} + \frac{3}{2}\left|{}^{\alpha}\mathbf{D}_{h\mathbf{X}}(h\gamma)_{h\mathbf{X}}\right|^2_{h\mathbf{g}}(h\gamma)_{h\mathbf{X}}, \tag{29}$$

$$-(v\gamma)_\tau = {}^{\alpha}\mathbf{D}^2_{v\mathbf{X}}(v\gamma)_{v\mathbf{X}} + \frac{3}{2}\left|{}^{\alpha}\mathbf{D}_{v\mathbf{X}}(v\gamma)_{v\mathbf{X}}\right|^2_{v\mathbf{g}}(v\gamma)_{v\mathbf{X}},$$

and fractional $+2, \ldots$ *flows as higher order analogs. Finally, the fractional –1 flows are defined by the kernels of recursion fractional operators (26) and (27) inducing non–stretching fractional maps*

$${}^{\alpha}\mathbf{D}_{h\mathbf{Y}}(h\gamma)_{h\mathbf{X}} = 0 \textit{ and } {}^{\alpha}\mathbf{D}_{v\mathbf{Y}}(v\gamma)_{v\mathbf{X}} = 0. \tag{30}$$

Proof: It is given below in Sec. IV B. □

B. Proof of the main theorem

We generalize for fractional spaces a similar proof from Refs. 13 and 14 sketching the key steps for horizontal flows. The vertical constructions are similar but with respective changing of h–variables/objects into v-variables/objects. By corresponding nonholonomic constraints, we can emphasize certain h– and v–evolutions which are inter–related.

We get a fractional vector mKdV equation up to a convective term (which can be absorbed by redefinition of coordinates) defining the fractional +1 flow for $h\overrightarrow{\mathbf{e}}_\perp = \overrightarrow{v}_\mathbf{l}$,

$$\overrightarrow{v}_\tau = \overrightarrow{v}_{3\mathbf{l}} + \frac{3}{2}|\overrightarrow{v}|^2 - {}^{\alpha}\overrightarrow{R}\,\overrightarrow{v}_\mathbf{l},$$

when the fractional $+(k+1)$ flow gives a vector mKdV equation of higher order $3+2k$ on $\overrightarrow{v}$ and there is a 0 h–flow $\overrightarrow{v}_\tau = \overrightarrow{v}_\mathbf{l}$ arising from $h\overrightarrow{\mathbf{e}}_\perp = 0$ and $h\overrightarrow{\mathbf{e}}_\parallel = 1$ belonging outside the hierarchy generated by $h\,{}^{\alpha}\mathfrak{R}$. Such fractional flows correspond to N–adapted horizontal motions of the curve $\gamma(\tau, \mathbf{l}) = h\gamma(\tau, \mathbf{l}) + v\gamma(\tau, \mathbf{l})$, given by

$$(h\gamma)_\tau = f\left((h\gamma)_{h\mathbf{X}}, {}^{\alpha}\mathbf{D}_{h\mathbf{X}}(h\gamma)_{h\mathbf{X}}, {}^{\alpha}\mathbf{D}^2_{h\mathbf{X}}(h\gamma)_{h\mathbf{X}}, \ldots\right)$$

subject to the non–stretching condition $|(h\gamma)_{h\mathbf{X}}|_{h\mathbf{g}} = 1$, when the equation of fractional motion is to be derived from the identifications

$$(h\gamma)_\tau \longleftrightarrow {}^{\alpha}\mathbf{e}_{h\mathbf{Y}},\ {}^{\alpha}\mathbf{D}_{h\mathbf{X}}(h\gamma)_{h\mathbf{X}} \longleftrightarrow {}^{\alpha}\mathcal{D}_{h\mathbf{X}}\,{}^{\alpha}\mathbf{e}_{h\mathbf{X}} = \left[{}^{\alpha}\mathbf{L}_{h\mathbf{X}},\ {}^{\alpha}\mathbf{e}_{h\mathbf{X}}\right]$$

and so on, which maps the constructions from the tangent fractional space of the curve to the space $h\mathfrak{p}$. For such identifications, we have

$$\left[{}^{\alpha}\mathbf{L}_{h\mathbf{X}},\ {}^{\alpha}\mathbf{e}_{h\mathbf{X}}\right] = -\begin{bmatrix} 0 & \left(0, \overrightarrow{v}\right) \\ -\left(0, \overrightarrow{v}\right)^T & h\mathbf{0} \end{bmatrix} \in h\mathfrak{p},$$

$$\left[{}^{\alpha}\mathbf{L}_{h\mathbf{X}}, \left[{}^{\alpha}\mathbf{L}_{h\mathbf{X}},\ {}^{\alpha}\mathbf{e}_{h\mathbf{X}}\right]\right] = -\begin{bmatrix} 0 & \left(|\overrightarrow{v}|^2, \overrightarrow{0}\right) \\ -\left(|\overrightarrow{v}|^2, \overrightarrow{0}\right)^T & h\mathbf{0} \end{bmatrix}$$

and so on, see similar calculus in (14). Stating for the fractional $+1$ h–flow

$$h\overrightarrow{\mathbf{e}}_\perp = \overrightarrow{v}_\mathbf{l} \text{ and } h\overrightarrow{\mathbf{e}}_\parallel = -\,{}^{\alpha}\mathbf{D}^{-1}_{h\mathbf{X}}\left(\overrightarrow{v}\cdot\overrightarrow{v}_\mathbf{l}\right) = -\frac{1}{2}|\overrightarrow{v}|^2,$$

we compute

$$
\begin{aligned}
{}^{\alpha}\mathbf{e}_{h\mathbf{Y}} &= \begin{bmatrix} 0 & \left(h\mathbf{e}_{\parallel}, h\overrightarrow{\mathbf{e}}_{\perp}\right) \\ -\left(h\mathbf{e}_{\parallel}, h\overrightarrow{\mathbf{e}}_{\perp}\right)^{T} & h\mathbf{0} \end{bmatrix} \\
&= -\frac{1}{2}|\overrightarrow{v}|^{2} \begin{bmatrix} 0 & \left(1, \overrightarrow{\mathbf{0}}\right) \\ -\left(0, \overrightarrow{\mathbf{0}}\right)^{T} & h\mathbf{0} \end{bmatrix} + \begin{bmatrix} 0 & \left(0, \overrightarrow{v}_{h\mathbf{X}}\right) \\ -\left(0, \overrightarrow{v}_{h\mathbf{X}}\right)^{T} & h\mathbf{0} \end{bmatrix} \\
&= {}^{\alpha}\mathbf{D}_{h\mathbf{X}}\left[{}^{\alpha}\mathbf{L}_{h\mathbf{X}}, {}^{\alpha}\mathbf{e}_{h\mathbf{X}}\right] + \frac{1}{2}\left[{}^{\alpha}\mathbf{L}_{h\mathbf{X}}, \left[{}^{\alpha}\mathbf{L}_{h\mathbf{X}}, {}^{\alpha}\mathbf{e}_{h\mathbf{X}}\right]\right] \\
&= -{}^{\alpha}\mathcal{D}_{h\mathbf{X}}\left[{}^{\alpha}\mathbf{L}_{h\mathbf{X}}, {}^{\alpha}\mathbf{e}_{h\mathbf{X}}\right] - \frac{3}{2}|\overrightarrow{v}|^{2}\, {}^{\alpha}\mathbf{e}_{h\mathbf{X}}.
\end{aligned}
$$

The above identifications are related to the first and second terms, when

$$
\begin{aligned}
|\overrightarrow{v}|^{2} = &< \left[{}^{\alpha}\mathbf{L}_{h\mathbf{X}}, {}^{\alpha}\mathbf{e}_{h\mathbf{X}}\right], \\
&\left[{}^{\alpha}\mathbf{L}_{h\mathbf{X}}, {}^{\alpha}\mathbf{e}_{h\mathbf{X}}\right] >_{h\mathfrak{p}} \longleftrightarrow h\, {}^{\alpha}\mathbf{g}\left({}^{\alpha}\mathbf{D}_{h\mathbf{X}}(h\gamma)_{h\mathbf{X}}, {}^{\alpha}\mathbf{D}_{h\mathbf{X}}(h\gamma)_{h\mathbf{X}}\right) \\
&= \left|{}^{\alpha}\mathbf{D}_{h\mathbf{X}}(h\gamma)_{h\mathbf{X}}\right|^{2}_{h\,{}^{\alpha}\mathbf{g}},
\end{aligned}
$$

and allow us to identify ${}^{\alpha}\mathcal{D}_{h\mathbf{X}}[{}^{\alpha}\mathbf{L}_{h\mathbf{X}}, {}^{\alpha}\mathbf{e}_{h\mathbf{X}}]$ to ${}^{\alpha}\mathbf{D}^{2}_{h\mathbf{X}}(h\gamma)_{h\mathbf{X}}$. As a result, we have

$$
-{}^{\alpha}\mathbf{e}_{h\mathbf{Y}} \longleftrightarrow {}^{\alpha}\mathbf{D}^{2}_{h\mathbf{X}}(h\gamma)_{h\mathbf{X}} + \frac{3}{2}\left|{}^{\alpha}\mathbf{D}_{h\mathbf{X}}(h\gamma)_{h\mathbf{X}}\right|^{2}_{h\,{}^{\alpha}\mathbf{g}}(h\gamma)_{h\mathbf{X}}
$$

which is just the fractional equation (29) in Theorem 4.1 defining a non–stretching mKdV map h–equation induced by the h–part of the fractional canonical d–connection.

To derive the higher order terms of hierarchies, we use the adjoint representation $ad(\cdot)$ acting in the Lie algebra $h\mathfrak{g} = h\mathfrak{p} \oplus \mathfrak{so}(n)$, with

$$
ad\left(\left[{}^{\alpha}\mathbf{L}_{h\mathbf{X}}, {}^{\alpha}\mathbf{e}_{h\mathbf{X}}\right]\right){}^{\alpha}\mathbf{e}_{h\mathbf{X}} = \begin{bmatrix} 0 & \left(0, \overrightarrow{\mathbf{0}}\right) \\ -\left(0, \overrightarrow{\mathbf{0}}\right)^{T} & \overrightarrow{\mathbf{v}} \end{bmatrix} \in \mathfrak{so}(n+1),
$$

where $\overrightarrow{\mathbf{v}} = -\begin{bmatrix} 0 & \overrightarrow{v} \\ -\overrightarrow{v}^{T} & h\mathbf{0} \end{bmatrix} \in \mathfrak{so}(n)$. Applying again $ad([{}^{\alpha}\mathbf{L}_{h\mathbf{X}}, {}^{\alpha}\mathbf{e}_{h\mathbf{X}}])$, we get

$$
\begin{aligned}
ad\left(\left[{}^{\alpha}\mathbf{L}_{h\mathbf{X}}, {}^{\alpha}\mathbf{e}_{h\mathbf{X}}\right]\right)^{2}{}^{\alpha}\mathbf{e}_{h\mathbf{X}} &= -|\overrightarrow{v}|^{2} \begin{bmatrix} 0 & \left(1, \overrightarrow{\mathbf{0}}\right) \\ -\left(1, \overrightarrow{\mathbf{0}}\right)^{T} & \mathbf{0} \end{bmatrix} \\
&= -|\overrightarrow{v}|^{2}\, {}^{\alpha}\mathbf{e}_{h\mathbf{X}},
\end{aligned}
$$

when the fractional equation (29) can be represented in alternative form,

$$
-(h\gamma)_{\tau} = {}^{\alpha}\mathbf{D}^{2}_{h\mathbf{X}}(h\gamma)_{h\mathbf{X}} - \frac{3}{2}\, {}^{\alpha}\overrightarrow{R}^{-1} ad\left({}^{\alpha}\mathbf{D}_{h\mathbf{X}}(h\gamma)_{h\mathbf{X}}\right)^{2}(h\gamma)_{h\mathbf{X}}.
$$

Finally, we consider a fractional –1 flow contained in the h–hierarchy derived from the property that $h\overrightarrow{\mathbf{e}}_{\perp}$ is annihilated by the fractional h–operator $h\ {}^{\alpha}\mathcal{J}$ and mapped into $h\ {}^{\alpha}\mathfrak{R}(h\overrightarrow{\mathbf{e}}_{\perp}) = 0$. This means that $h\ {}^{\alpha}\mathcal{J}(h\overrightarrow{\mathbf{e}}_{\perp}) = \overrightarrow{\varpi} = 0$. Such properties together with (13) and fractional equations (22) imply ${}^{\alpha}\mathbf{L}_{\tau} = 0$ and hence

$$
h\ {}^{\alpha}\mathcal{D}_{\tau}\ {}^{\alpha}\mathbf{e}_{h\mathbf{X}} = [\ {}^{\alpha}\mathbf{L}_{\tau},\ {}^{\alpha}\mathbf{e}_{h\mathbf{X}}] = 0 \text{ for } h\ {}^{\alpha}\mathcal{D}_{\tau} = h\ {}^{\alpha}\mathbf{D}_{\tau} + [\ {}^{\alpha}\mathbf{L}_{\tau}, \cdot].
$$

We obtain the equation of fractional motion for the h–component of curve, $h\gamma(\tau, \mathbf{l})$, following the correspondences ${}^{\alpha}\mathbf{D}_{h\mathbf{Y}} \longleftrightarrow h\ {}^{\alpha}\mathcal{D}_{\tau}$ and $h\gamma_{\mathbf{l}} \longleftrightarrow {}^{\alpha}\mathbf{e}_{h\mathbf{X}}$, ${}^{\alpha}\mathbf{D}_{h\mathbf{Y}}(h\gamma(\tau, \mathbf{l})) = 0$, which is just the first fractional equation in (30).

V. FRACTIONAL NONHOLONOMIC mKdV AND SG HIERARCHIES

In this section, we consider explicit constructions when fractional solitonic hierarchies are derived following the conditions of Theorem 4.1.

The fractional h–flow and v–flow equations resulting from (30) are

$$\overrightarrow{v}_\tau = -\ {}^{\alpha}\overrightarrow{R}\, h\overrightarrow{\mathbf{e}}_\perp \ \text{ and } \ \overleftarrow{v}_\tau = -\ {}^{\alpha}\overleftarrow{S}\, v\overleftarrow{\mathbf{e}}_\perp, \tag{31}$$

when, respectively,

$$0 = \overrightarrow{\varpi} = -\ {}^{\alpha}\mathbf{D}_{h\mathbf{X}} h\overrightarrow{\mathbf{e}}_\perp + h\mathbf{e}_\parallel \overrightarrow{v},\ \ {}^{\alpha}\mathbf{D}_{h\mathbf{X}} h\mathbf{e}_\parallel = h\overrightarrow{\mathbf{e}}_\perp \cdot \overrightarrow{v}$$

and

$$0 = \overleftarrow{\varpi} = -\ {}^{\alpha}\mathbf{D}_{v\mathbf{X}} v\overleftarrow{\mathbf{e}}_\perp + v\mathbf{e}_\parallel \overleftarrow{v},\ \ {}^{\alpha}\mathbf{D}_{v\mathbf{X}} v\mathbf{e}_\parallel = v\overleftarrow{\mathbf{e}}_\perp \cdot \overleftarrow{v}.$$

The fractional d–flow equations possess horizontal and vertical conservation laws

$${}^{\alpha}\mathbf{D}_{h\mathbf{X}}\left((h\mathbf{e}_\parallel)^2 + |h\overrightarrow{\mathbf{e}}_\perp|^2\right) = 0,$$

for $(h\mathbf{e}_\parallel)^2 + |h\overrightarrow{\mathbf{e}}_\perp|^2 = < h\mathbf{e}_\tau, h\mathbf{e}_\tau >_{h\mathfrak{p}} = |\,(h\gamma)_\tau\,|^2_{h\ {}^{\alpha}\mathbf{g}}$, and

$${}^{\alpha}\mathbf{D}_{v\mathbf{Y}}\left((v\mathbf{e}_\parallel)^2 + |v\overleftarrow{\mathbf{e}}_\perp|^2\right) = 0,$$

for $(v\mathbf{e}_\parallel)^2 + |v\overleftarrow{\mathbf{e}}_\perp|^2 = < v\mathbf{e}_\tau, v\mathbf{e}_\tau >_{v\mathfrak{p}} = |\,(v\gamma)_\tau\,|^2_{v\ {}^{\alpha}\mathbf{g}}$. This corresponds to

$${}^{\alpha}\mathbf{D}_{h\mathbf{X}}|\,(h\gamma)_\tau\,|^2_{h\ {}^{\alpha}\mathbf{g}} = 0 \ \text{ and } \ {}^{\alpha}\mathbf{D}_{v\mathbf{X}}|\,(v\gamma)_\tau\,|^2_{v\ {}^{\alpha}\mathbf{g}} = 0.$$

We can also rescale conformally the variable τ in order to get $|\,(h\gamma)_\tau\,|^2_{h\ {}^{\alpha}\mathbf{g}} = 1$ and (it could be for other rescaling) $|\,(v\gamma)_\tau\,|^2_{v\ {}^{\alpha}\mathbf{g}} = 1$, i.e., to have

$$(h\mathbf{e}_\parallel)^2 + |h\overrightarrow{\mathbf{e}}_\perp|^2 = 1 \ \text{ and } \ (v\mathbf{e}_\parallel)^2 + |v\overleftarrow{\mathbf{e}}_\perp|^2 = 1.$$

Then, we express $h\mathbf{e}_\parallel$ and $h\overrightarrow{\mathbf{e}}_\perp$ in terms of $\overrightarrow{v}$ and its derivatives and, similarly, we express $v\mathbf{e}_\parallel$ and $v\overleftarrow{\mathbf{e}}_\perp$ in terms of $\overleftarrow{v}$ and its derivatives, which follows from (31). The N–adapted fractional wave map equations describing the –1 flows reduce to a system of two independent nonlocal fractional evolution equations for the h– and v–components,

$$\overrightarrow{v}_\tau = -\ {}^{\alpha}\mathbf{D}^{-1}_{h\mathbf{X}}\left(\sqrt{\ {}^{\alpha}\overrightarrow{R}^2 - |\overrightarrow{v}_\tau|^2}\,\overrightarrow{v}\right),$$

$$\overleftarrow{v}_\tau = -\ {}^{\alpha}\mathbf{D}^{-1}_{v\mathbf{X}}\left(\sqrt{\ {}^{\alpha}\overleftarrow{S}^2 - |\overleftarrow{v}_\tau|^2}\,\overleftarrow{v}\right).$$

We can rescale the equations on τ to the case when the terms ${}^{\alpha}\overrightarrow{R}^2,\ {}^{\alpha}\overleftarrow{S}^2 = 1$, and the fractional evolution equations transform into a system of hyperbolic d–vector equations,

$${}^{\alpha}\mathbf{D}_{h\mathbf{X}}(\overrightarrow{v}_\tau) = -\sqrt{1 - |\overrightarrow{v}_\tau|^2}\,\overrightarrow{v} \ \text{ and } \ {}^{\alpha}\mathbf{D}_{v\mathbf{X}}(\overleftarrow{v}_\tau) = -\sqrt{1 - |\overleftarrow{v}_\tau|^2}\,\overleftarrow{v}, \tag{32}$$

where ${}^{\alpha}\mathbf{D}_{h\mathbf{X}} = {}^{\alpha}\partial_{h\mathbf{l}}$ and ${}^{\alpha}\mathbf{D}_{v\mathbf{X}} = {}^{\alpha}\partial_{v\mathbf{l}}$ are usual partial derivatives on direction $\mathbf{l} = h\mathbf{l} + v\mathbf{l}$ with $\overrightarrow{v}_\tau$ and $\overleftarrow{v}_\tau$ considered as scalar functions for the covariant derivatives ${}^{\alpha}\mathbf{D}_{h\mathbf{X}}$ and ${}^{\alpha}\mathbf{D}_{v\mathbf{X}}$ defined by the fractional canonical d–connection. It also follows that $h\overrightarrow{\mathbf{e}}_\perp$ and $v\overleftarrow{\mathbf{e}}_\perp$ obey corresponding fractional vector sine–Gordon (SG) equations,

$$\left(\sqrt{(1 - |h\overrightarrow{\mathbf{e}}_\perp|^2)^{-1}}\ {}^{\alpha}\partial_{h\mathbf{l}}(h\overrightarrow{\mathbf{e}}_\perp)\right)_\tau = -h\overrightarrow{\mathbf{e}}_\perp \tag{33}$$

$$\left(\sqrt{(1 - |v\overleftarrow{\mathbf{e}}_\perp|^2)^{-1}}\ {}^{\alpha}\partial_{v\mathbf{l}}(v\overleftarrow{\mathbf{e}}_\perp)\right)_\tau = -v\overleftarrow{\mathbf{e}}_\perp. \tag{34}$$

Conclusion 5.1: The recursion fractional d–operator $\ ^{\alpha}\mathfrak{R} = (h\ ^{\alpha}\mathfrak{R}, h\ ^{\alpha}\mathfrak{R})$ *(25), see (26) and (27), generates two hierarchies of fractional vector mKdV symmetries: the first one is horizontal,*

$$\overrightarrow{v}^{(0)}_{\tau} = \overrightarrow{v}_{h\mathbf{l}},\ \overrightarrow{v}^{(1)}_{\tau} = h\ ^{\alpha}\mathfrak{R}(\overrightarrow{v}_{h\mathbf{l}}) = \overrightarrow{v}_{3h\mathbf{l}} + \frac{3}{2}|\overrightarrow{v}|^2\overrightarrow{v}_{h\mathbf{l}}, \tag{35}$$

$$\overrightarrow{v}^{(2)}_{\tau} = h\ ^{\alpha}\mathfrak{R}^2(\overrightarrow{v}_{h\mathbf{l}}) = \overrightarrow{v}_{5h\mathbf{l}} + \frac{5}{2}\left(|\overrightarrow{v}|^2\overrightarrow{v}_{2h\mathbf{l}}\right)_{h\mathbf{l}}$$
$$+\frac{5}{2}\left((|\overrightarrow{v}|^2)_{h\mathbf{l}h\mathbf{l}} + |\overrightarrow{v}_{h\mathbf{l}}|^2 + \frac{3}{4}|\overrightarrow{v}|^4\right)\overrightarrow{v}_{h\mathbf{l}} - \frac{1}{2}|\overrightarrow{v}_{h\mathbf{l}}|^2\overrightarrow{v},$$
$$\ldots,$$

with all such terms commuting with the fractional –1 flow

$$(\overrightarrow{v}_{\tau})^{-1} = h\overrightarrow{\mathbf{e}}_{\perp} \tag{36}$$

associated to the fractional vector SG equation (33); the second one is vertical,

$$\overleftarrow{v}^{(0)}_{\tau} = \overleftarrow{v}_{v\mathbf{l}},\ \overleftarrow{v}^{(1)}_{\tau} = v\ ^{\alpha}\mathfrak{R}(\overleftarrow{v}_{v\mathbf{l}}) = \overleftarrow{v}_{3v\mathbf{l}} + \frac{3}{2}|\overleftarrow{v}|^2\overleftarrow{v}_{v\mathbf{l}}, \tag{37}$$

$$\overleftarrow{v}^{(2)}_{\tau} = v\ ^{\alpha}\mathfrak{R}^2(\overleftarrow{v}_{v\mathbf{l}}) = \overleftarrow{v}_{5v\mathbf{l}} + \frac{5}{2}\left(|\overleftarrow{v}|^2\overleftarrow{v}_{2v\mathbf{l}}\right)_{v\mathbf{l}}$$
$$+\frac{5}{2}\left((|\overleftarrow{v}|^2)_{v\mathbf{l}v\mathbf{l}} + |\overleftarrow{v}_{v\mathbf{l}}|^2 + \frac{3}{4}|\overleftarrow{v}|^4\right)\overleftarrow{v}_{v\mathbf{l}} - \frac{1}{2}|\overleftarrow{v}_{v\mathbf{l}}|^2\overleftarrow{v},$$
$$\ldots,$$

with all such terms commuting with the fractional –1 flow

$$(\overleftarrow{v}_{\tau})^{-1} = v\overleftarrow{\mathbf{e}}_{\perp} \tag{38}$$

associated to the fractional vector SG equation (34).

Proof. It follows from the above, in this section, and Corollary 4.1. □

Finally, using the above Conclusion, we derive that the adjoint fractional d–operator $\ ^{\alpha}\mathfrak{R}^* = \ ^{\alpha}\mathcal{J} \circ \ ^{\alpha}\mathcal{H}$ generates a horizontal hierarchy of fractional Hamiltonians,

$$h\ ^{\alpha}H^{(0)} = \frac{1}{2}|\overrightarrow{v}|^2,\ h\ ^{\alpha}H^{(1)} = -\frac{1}{2}|\overrightarrow{v}_{h\mathbf{l}}|^2 + \frac{1}{8}|\overrightarrow{v}|^4, \tag{39}$$

$$h\ ^{\alpha}H^{(2)} = \frac{1}{2}|\overrightarrow{v}_{2h\mathbf{l}}|^2 - \frac{3}{4}|\overrightarrow{v}|^2|\overrightarrow{v}_{h\mathbf{l}}|^2 - \frac{1}{2}\left(\overrightarrow{v}\cdot\overrightarrow{v}_{h\mathbf{l}}\right) + \frac{1}{16}|\overrightarrow{v}|^6, \ldots,$$

and vertical hierarchy of fractional Hamiltonians

$$v\ ^{\alpha}H^{(0)} = \frac{1}{2}|\overleftarrow{v}|^2,\ v\ ^{\alpha}H^{(1)} = -\frac{1}{2}|\overleftarrow{v}_{v\mathbf{l}}|^2 + \frac{1}{8}|\overleftarrow{v}|^4, \tag{40}$$

$$v\ ^{\alpha}H^{(2)} = \frac{1}{2}|\overleftarrow{v}_{2v\mathbf{l}}|^2 - \frac{3}{4}|\overleftarrow{v}|^2|\overleftarrow{v}_{v\mathbf{l}}|^2 - \frac{1}{2}\left(\overleftarrow{v}\cdot\overleftarrow{v}_{v\mathbf{l}}\right) + \frac{1}{16}|\overleftarrow{v}|^6, \ldots,$$

all of which are conserved densities for respective horizontal and vertical fractional –1 flows and determining higher conservation laws for the corresponding hyperbolic fractional equations (33) and (34).

Finally we note that two concrete examples of fractional solitonic solutions in gravity are studied in Sec. 5.3 of Ref. 11. The "integer" solitonic hierarchies from Refs. 13 and 14 can be similarly generated in "fractional" forms by using corresponding fractional Caputo derivative operators.

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