THE SNAKEBOARD AS A MECHANICAL CONTROL AND CONSTRAINED SYSTEM

A THESIS SUBMITTED TO
THE GRADUATE SCHOOL OF NATURAL AND APPLIED SCIENCES
OF
ÇANKAYA UNIVERSITY

BY

ADNAN BiLGEN

IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR
THE DEGREE OF MASTER OF SCIENCE
IN
MATHEMATICS AND COMPUTER SCIENCE

AUGUST 2006

Title of the Thesis : **The Snakeboard as a Mechanical Control and Constrained System**

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This is to certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality, as a thesis for the degree of Master of Science.

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ABSTRACT

THE SNAKEBOARD AS A MECHANICAL CONTROL AND CONSTRAINED SYSTEM

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August 2006, 31 pages

The snakeboard problem is investigated as a constrained, control system using the Hamilton-Jacobi formulation. Equations of motion are solved numerically in Lagrangian and Hamiltonian approaches.

Keywords: Constrained systems,Snakeboard,Hamilton-Jacobi formulation

ÖZ

MEKANiK KONTROL VE BAĞIL BiR SiSTEM OLARAK KAYKAY

Bilgen, Adnan

Yüksek Lisans, Matematik ve Bilgisayar Bölümü

Tez Yöneticisi: Prof. Dr. Yurdahan Güler

Ağustos 2006, 31 sayfa

Kaykay problemi bağıl ve kontrol sistem olarak araştırıldı; hareket denklemleri Hamilton-Jacobi formülasyonu kullanılarak bulundu ve nümerik olarak çözüldü.

Anahtar Kelimeler: Bağıl sistemler, Kaykay,Hamilton-Jacobi formülasyonu

ACKNOWLEDGMENTS

I would like to express my sincere gratitude to my thesis advisor Prof. Dr. Yurdahan Güler, who has encouraged me and guided throughout this thesis patiently.

I had a lot of useful discussion with Prof. Dr. Ahmet Eriş. I would like to thank him cordially for his valuable comments.

I would like to dedicate my graceful thanks to Assist. Prof. Emre Sermutlu who has made me love the magical world of Mathematics.

I owe many thanks to TUBiTAK for supporting me during the Ms. program.

I wish to thank the examining committee for their kindness during the presentation of this thesis.

I would like to express my deep gratitude to my family for their endless and continuous encourage and support throughout the years.

Finally, I would like to thank all my close friends with whom we have shared good and bad times for many years.

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CHAPTER 1

INTRODUCTION

1.1 Nonlinear Control Systems

A finite dimensional nonlinear control system on a smooth n-manifold M is defined by a first order linear differential equation in the form

$$\dot{x} = f(x,u) = \frac{dx}{dt} \tag{1.1}$$

where f is C^∞ (smooth) or C^W (Analytic) [1]. $u(t)$ is defined as a mapping from positive real numbers to a set of constraints Ω, i.e.

$$u(t) : R^+ \to \Omega \subset R^M \tag{1.2}$$

is labelled as an input function. If $u(t)$ is piecewise analytic, then it is admissible.

Generally speaking one has two aims in control theory :

1. To "drive" the given system in M,
2. To "stabilize" a control system around an equilibrium manifold.

There are numerous books and articles on this subject approaching the problem from different point of views.[2],[3],[4]. However, the main concern is whether one can really "drive" a system. In other words, mathematically speaking, one should prove that the system is "controllable". To simplify the problem of controllability we define an affine nonlinear control system as

$$\frac{dx}{dt} = \dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u(t) \tag{1.3}$$

where f and g_i, $i = 1, \ldots, m$, are smooth vector fields on M. For such a system the controllability is valid if there is a control function $u(t)$ defined on the time scale $[0, T]$ such that the system reaches from an initial point $x_i \in M$ to the final point $x_f \in M$ during this time interval.

1.2 Lagrangian Formulation of Mechanical Systems

There are basically two approaches in analytic mechanics, Lagrangian and Hamiltonian methods. Generally, the Lagrangian function is defined on the configuration space consists of generalized coordinates q_i and generalized velocities $\dot{q}_i$ as $L(q, \dot{q}, t)$. For conservative systems the Lagrangian is given as

$$L = T - V \tag{1.4}$$

where T is the kinetic energy and V is the potential energy of the system.

The variational principle states that variation of action

$$S = \int_i^f L(q, \dot{q})dt \tag{1.5}$$

should be zero. Mathematically speaking the problem can be stated as the determination of smooth curves such that

$$\delta S = 0 \tag{1.6}$$

Variational principle leads us to the Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0, \qquad \forall i = 1, \ldots, n. \tag{1.7}$$

These equations are second order ordinary differential equations and are valid for systems in which all forces are conservative, i.e., derivable from a potential.

In some physical systems, there may be some constraints which restrict the motion of the system. One can classify the constraints as,holonomic and nonholonomic . If the equations of constraints are in the form

$$f(q_i, t) = 0 \tag{1.8}$$

then the constraints are called holonomic. Constraints which are not expressible as (1.8) are called non-holonomic. The motion of a mass (m) on an inclined plane is an example of a holonomic constraint and the motion of a vertical disc on a horizontal plane is an example of a nonholonomic constraint [4].

Nonholonomic constraint systems can be investigated by a variational method, if constraints can be put in the form

$$\sum_{k=1} a_{lk}(q, t)dq_k + a_{lt}(q, t)dt = 0, \qquad \forall l = 1, \ldots, m. \tag{1.9}$$

Since we are interested in virtual displacements (not in t),then,the varied paths should satisfy

$$\sum_{k=1}^{n} a_{lk}(q, t)dq_k = 0, \qquad \forall l = 1, \ldots, m. \tag{1.10}$$

The method of undetermined multipliers will be used to restrict the variations to independent displacements. To do this, the term

$$\sum_{l,k} \lambda_l a_{lk} \delta q_k = 0, \qquad l = 1, \ldots, m; k = 1, \ldots, n \tag{1.11}$$

will be added to the action and the variation should be considered. This way, one gets

$$\int_i^f \sum_{k=1}^n \left\{ \left(\frac{\partial L}{\partial q_i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}_k} \right) + \sum_l \lambda_l a_{lk} \right\} \delta q_k dt = 0. \tag{1.12}$$

Choosing $(n - m)$,δq_k as independent variations one obtains

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_k} \right) - \frac{\partial L}{\partial q_k} = \sum_{l=1}^m \lambda_l a_{lk}, \qquad k = 1, \ldots, n. \tag{1.13}$$

To determine m undetermined multipliers λ_l, and q_i, it is sufficient to solve differential equations (1.13) and constraint equations

$$\sum_{k=1}^n a_{lk} \dot{q}_k + a_{lt} = 0 \tag{1.14}$$

simultaneously. The physical meaning of the quantity

$$\sum_{l=1}^m \lambda_l a_{lk} = Q_k \tag{1.15}$$

is the generalized force component of constraints.

Variational method for nonholonomic constraints can easily be extended to holonomic systems. In fact, if there is a holonomic constraint

$$f(q_1, \ldots, q_n, t) = 0 \tag{1.16}$$

then it can be expressed as

$$df = \sum_{k=1}^n \frac{\partial l}{\partial q_k} dq_k + \frac{\partial f}{\partial t} dt = 0 \tag{1.17}$$

which gives

$$a_{lk} = \frac{\partial f}{\partial q_k} \qquad , \qquad a_{lt} = \frac{\partial f}{\partial t} \tag{1.18}$$

The vertical disc rolling on a plane will be studied as an example of a nonholonomic system.

1.3 The Vertical Rolling Disk

The vertical rolling disk is a basic and simple example of a system subject to nonholonomic constraints (Fig.1.1)[1]. In the first instance we consider the "vertical" disk, a disk that, unphysically of course, may not tilt away from the vertical. It is not difficult to generalize the situation to the "falling" disk. It is helpful to think of a coin such as penny, since we are concerned with orientation and the roll angle of the disk.

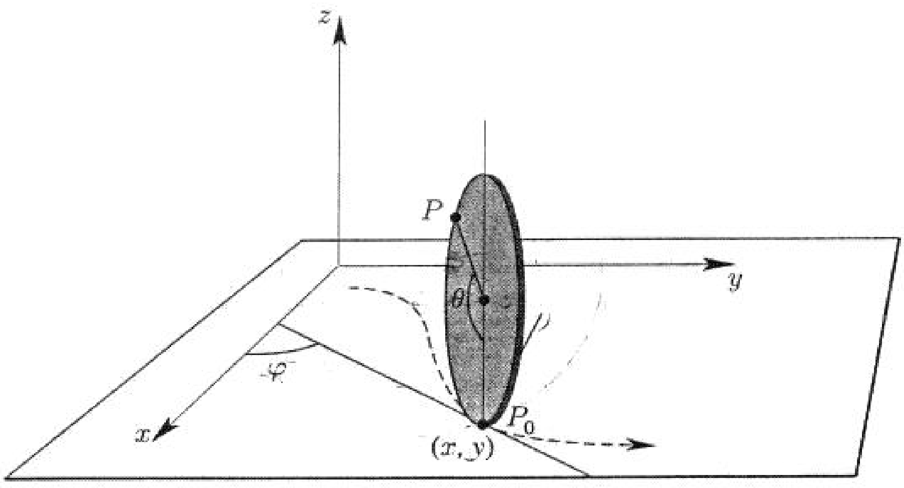

Figure 1.1: The geometry of the rolling disc.

Let S^1 denote the circle of radius 1 in the plane. It may be parameterized by an angular variable, a variable that is 2π periodic. The configuration space for the vertical rolling disk is $Q = \mathbb{R}^2 \times S^1 \times S^1$ and is parameterized by the generalized coordinates $q = (x, y, \theta, \varphi)$, denoting the position of the contact point in the xy-plane,the rotation angle of the disk, and the orientation of the disc respectively, as in figure (1.1).

The variables (x, y, φ) may also be regarded as giving the translational position of the disk together with a rotational position. That is, we may regard (x, y, φ) as an element of the *Euclidean group* in the plane. This group denoted by $SE(2)$, is the group of translations and rotations in the plane, that is, the group of rigid motions in the plane. Thus, $SE(2) = \mathbb{R}^2 \times S^1$. This group and its three-dimensional counterpart in space, $SE(3)$, play an important role for nonholonomic mechanics.

In summary the configuration space of the vertical rolling disk is given by

$$\theta = SE(2) \times S^1 \tag{1.19}$$

and this space has (generalized coordinates) given by $((x, y, \varphi), \theta)$. The Lagrangian for the vertical rolling disk is the total kinetic energy of the system, namely

$$L = (x, y, \varphi, \theta, \dot{x}, \dot{y}, \dot{\varphi}, \dot{\theta}) = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}J\dot{\varphi}^2 \tag{1.20}$$

where m is the mass of the disk, I is the moment of inertia of the disk about the axis perpendicular to the plane of the disk, and J is the moment of inertia about an axis in the plane of the disk, both axes passing through the center of the disc.

If R is the radius of disk, the nonholonomic constraints of rolling without slipping are

$$\begin{aligned} \dot{x} &= R(\cos\varphi)\dot{\theta} \\ \dot{y} &= R(\sin\varphi)\dot{\theta} \end{aligned} \tag{1.21}$$

We can write the constraint equations as

$$\dot{x} - R(\cos\varphi)\dot{\theta} = 0$$
$$\dot{y} - R(\sin\varphi)\dot{\theta} = 0 \tag{1.22}$$

or as

$$a^1(\dot{x}, \dot{y}, \dot{\varphi}, \dot{\theta})^T = 0$$
$$a^2(\dot{x}, \dot{y}, \dot{\varphi}, \dot{\theta})^T = 0 \tag{1.23}$$

where T denotes the transpose and

$$a^1 = (1, 0, 0, -R\cos\varphi), \qquad a^2 = (0, 1, 0, -R\sin\varphi).$$

In a compact form, the equations expressed as

$$\sum_{k=1} a_k^j(q^i)\dot{q}^k = 0, \qquad j = 1, 2. \tag{1.24}$$

Here,

$$a_1^1 = 1, a_2^1 = 0, a_3^1 = 0, a_4^1 = -R\cos\varphi$$

and

$$a_1^2 = 0, a_2^2 = 1, a_3^2 = 0, a_4^2 = -R\sin\varphi.$$

The Euler-Lagrange equations for this system are

$$\begin{aligned}
\frac{d}{dt}\frac{\partial L}{\partial \dot{x}} - \frac{\partial L}{\partial x} &= 0 \Rightarrow m\ddot{x} = 0\\
\frac{d}{dt}\frac{\partial L}{\partial \dot{y}} - \frac{\partial L}{\partial y} &= 0 \Rightarrow m\ddot{y} = 0\\
\frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} - \frac{\partial L}{\partial \varphi} &= 0 \Rightarrow J\ddot{\varphi} = 0\\
\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}} - \frac{\partial L}{\partial \theta} &= 0 \Rightarrow I\ddot{\theta} = 0.
\end{aligned} \tag{1.25}$$

1.4 The Variational Controlled System

The variational system is obtained by using Lagrange multipliers with the Lagrangian rather than Lagrange multipliers with the equations. Namely, we consider the Lagrangian

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}I\dot{\theta}^2 + \frac{1}{2}J\dot{\varphi}^2 + \mu_1(\dot{x} - R\dot{\theta}\cos\varphi) + \mu_2(\dot{y} - R\dot{\theta}\sin\varphi). \tag{1.26}$$

We write down the Euler-Lagrange equations for this Lagrangian and determine the multipliers from the constraints and initial conditions to the extent possible.

Now we write the Euler-Lagrange equations and find the following equations :

$$m\ddot{x} + \dot{\mu}_1 = 0 \tag{1.27}$$

$$m\ddot{y} + \dot{\mu}_2 = 0 \tag{1.28}$$

$$J\ddot{\varphi} - \mu_1 R\dot{\theta}\sin\varphi + \mu_2 R\dot{\theta}\cos\varphi = u_\varphi \tag{1.29}$$

$$I\ddot{\theta} - R\dot{\mu}_1\cos\varphi + R\mu_1\dot{\varphi}\sin\varphi - R\dot{\mu}_2\sin\varphi - R\mu_2\varphi\dot{\cos}\varphi = u_\theta. \tag{1.30}$$

Constraints are given as :

$$\dot{x} = R(\cos\varphi)\dot{\theta}, \qquad \dot{y} = R(\sin\varphi)\dot{\theta}. \tag{1.31}$$

From constraints (1.31) we get the following equations ,

$$\ddot{x} = R\ddot{\theta}\cos\varphi - R\dot{\theta}\dot{\varphi}\sin\varphi \tag{1.32}$$

and

$$\ddot{y} = R\ddot{\theta}\sin\varphi - R\dot{\theta}\dot{\varphi}\cos\varphi. \tag{1.33}$$

Substituting (1.32) in (1.27) one obtains

$$\dot{\mu}_1 = -mR\dot{\theta}\cos\varphi + A. \tag{1.34}$$

After some calculations we arrive at the following result

$$\ddot{\theta}(I + mR^2) = R\dot{\varphi}(B\cos\varphi - A\sin\varphi) + u_\theta. \tag{1.35}$$

Here A, B are constants.

1.5 The Hamiltonian Formulation of Mechanical Systems

The Hamiltonian formulation is described on the phase space, which is composed of $2n$ variables (q_i, p_i). If the matrix $\partial^2 L/\partial \dot{q}_i \partial \dot{q}_j$ is nonsingular, we call L a *nondegenerate* or *regular* Lagrangian, and in this case we can make the change of variables from $(q_i, \dot{q}_i)$ to the variables (q_i, p_i), where the momentum is defined as

$$p_i = \frac{\partial L}{\partial \dot{q}_i}, \qquad \forall i = 1, ..., n. \tag{1.36}$$

This change of variables is commonly referred to as the *Legendre transformations.* Introducing the Hamiltonian as

$$H(q_i, p_i) = \sum_{i=1}^{n} p_i \dot{q}_i - L(q_i, \dot{q}_i), \qquad \forall i = 1, ..., n \tag{1.37}$$

we obtain the *Hamilton's equations* as

$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \dot{p}_i = -\frac{\partial H}{\partial q_i}, \qquad \forall i = 1, ..., n. \tag{1.38}$$

If one introduces the *Poisson bracket* of two functions K,L on the phase space by the definition

$$\{K, L\} = \sum_{i=1}^{n} \frac{\partial K}{\partial q_i}\frac{\partial L}{\partial p_i} - \frac{\partial L}{\partial q_i}\frac{\partial K}{\partial p_i} \tag{1.39}$$

the Hamilton's equations may be written as

$$\dot{F} = \{F, H\} \tag{1.40}$$

for all functions F. In particular, since the Poisson bracket is clearly skew symmetric in K,L, we see that $\{H, H\} = 0$, and so H has zero time derivative (conservation of energy). If the canonical variables (q_i, p_i) are not all independent, i.e. if there are constraints in the form

$$\psi_k(q_i, p_i, t) = 0, \qquad k = 1, \ldots, m \tag{1.41}$$

the canonical equations can be expressed as

$$\begin{aligned} \frac{\partial H}{\partial p_i} + \sum_{k=1}^{m} \frac{\partial \psi_k}{\partial p_i} &= \dot{q}_i \\ \frac{\partial H}{\partial q_i} + \sum_{k=1}^{m} \lambda_k \frac{\partial \psi_k}{\partial q_i} &= -\dot{p}_i. \end{aligned} \tag{1.42}$$

1.6 Lagrangian and Hamiltonian Control Systems

The main idea is to consider a mechanical system, expressed by a Lagrangian L or a Hamiltonian H, exerted by external forces u_i, $i = 1, \ldots, m$. Thus, the Euler-Lagrange equations are

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} &= u_i, \qquad i = 1, \ldots, m \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} &= 0, \qquad i = m+1, \ldots, n. \end{aligned} \tag{1.43}$$

In general the control input u_i can be a force or a velocity. More generally, we have the system expressed by the following Euler-Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L(q, \dot{q}, u)}{\partial \dot{q}_i}\right) - \frac{\partial L(q, \dot{q}, u)}{\partial q_i} = 0 \tag{1.44}$$

for $q \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$.

In the same way the Hamilton's equations of a control system are given as

$$\begin{aligned} \dot{q}_i &= \frac{\partial H(q,p,u)}{\partial p_i} \\ \dot{p}_i &= -\frac{\partial H(q,p,u)}{\partial q_i}. \end{aligned} \tag{1.45}$$

To generalize this concept on a symplectic manifold M one generally works with affine Hamiltonian control system which is expressed as

$$\dot{x} = X_{H_0}(x) + \sum_{j=1}^{m} X_{H_j}(x)u_i \tag{1.46}$$

where $x \in M$,X_{H_j} is the Hamiltonian vector field corresponding to H_j.

To illustrate Lagrangian control systems, the controlled disc and the Heisenberg systems will be examined.

1.7 Dynamics of the Controlled Disk

We consider the case where we have two controls,one that can steer the disk and another that determines the roll torque. Now we shall use the general equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = \sum_{j=1}^{m} \lambda_j a_i^j + F_i^e, \qquad \forall i = 1, ..., n \tag{1.47}$$

where $i = 1, \ldots, n$, to write down the equations for the controlled vertical rolling disk. Here F^e denotes the external forces. According to these equations, we add the forces to the right-hand side of the Euler-Lagrange equations for the given Lagrangian along with Lagrange multipliers to enforce the constraints and to represent the reaction forces. In our case, L is given as eq.(1.20) is cyclic in the configuration variables $q = (x, y, \varphi, \theta)$. We have the vectors a^1 and a^2 so the summation at right hand side of the equation is for j=1 and j=2. The controls are in the directions of the two angles φ and θ respectively. u_φ and u_θ are control functions, so the external control forces are $F = u_\varphi f^\varphi + u_\theta f^\theta$ and λ_1, λ_2 are Lagrange multipliers. In our case $q = (x, y, \varphi, \theta)$ and the dynamical equation becomes

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) = u_\varphi f^\varphi + u_\theta f^\theta + \lambda^1 a_1 + \lambda^2 a_2 \tag{1.48}$$

$$\frac{\partial L}{\partial \dot{q}} = (m\dot{x}, m\dot{y}, j\dot{\varphi}, I\dot{\theta}). \tag{1.49}$$

Here

$$f^\varphi = (0, 0, 1, 0), f^\theta = (0, 0, 0, 1)$$

Thus eq.(1.48) gives explicitly,

$$m\ddot{x} = \lambda_1$$

$$m\ddot{y} = \lambda_2$$

$$u_\varphi = j\ddot{\varphi}$$

$$u_\theta - \lambda_1 R\cos\varphi - \lambda_2 R\sin\varphi = I\ddot{\theta} \tag{1.50}$$

$$\lambda_1 = m\ddot{x} = mR[-(\sin\varphi)\dot{\varphi}\dot{\theta} + \cos\varphi\ddot{\theta}] \tag{1.51}$$

$$\lambda_2 = m\ddot{y} = mR[(\cos\varphi)\dot{\varphi}\dot{\theta} + \sin\varphi\ddot{\theta}] \tag{1.52}$$

$$u_\varphi = j\ddot{\varphi}. \tag{1.53}$$

Substituting (1.50) and (1.51) in (1.52), we get

$$u_\theta = \ddot{\theta}(I + mR^2).$$

Now we have

$$J\ddot{\varphi} = u_\varphi \tag{1.54}$$

$$(I + mR^2)\ddot{\theta} = u_\theta \tag{1.55}$$

$$\dot{x} = R(\cos\varphi)\dot{\theta} \tag{1.56}$$

$$\dot{y} = R(\sin\varphi)\dot{\theta}. \tag{1.57}$$

The free equations,in which we set $u_\varphi = u_\theta = 0$, are

$$J\ddot{\varphi} = 0 \tag{1.58}$$

and

$$(I + mR^2)\ddot{\theta} = 0 \tag{1.59}$$

since $J \neq 0$ and $I + mR^2 \neq 0$, we have

$$\varphi = Wt + \phi_0 \tag{1.60}$$

$$\theta = \Omega t + \theta_0 \tag{1.61}$$

where W and Ω are constants. Using (1.60) and (1.61) in eq. (1.21) we get

$$\dot{x} = R[\cos(Wt + \phi_0)]\Omega \tag{1.62}$$

$$x = \int R\Omega\cos(Wt + \phi_0)dt \tag{1.63}$$

and

$$x = \frac{R\Omega}{W}\ \sin(Wt + \phi_0) + x_0. \tag{1.64}$$

Again using the second constraint in eq. (1.21) we obtain

$$\dot{y} = R(\sin(Wt + \phi_0)) \tag{1.65}$$

$$y = \int R\Omega\sin(Wt + \phi_0)dt \tag{1.66}$$

and

$$y = -\frac{R\Omega}{W}\ \cos(Wt + \phi_0) + y_o. \tag{1.67}$$

Here $x_0, y_0, \phi_0, \theta_0$ are constants.

1.8 The Heisenberg System

The Heisenberg system has played a significant role as an example in both nonlinear control and nonholonomic mechanics [1]. The dynamic Heisenberg system comes in two forms, one associated with the Lagrange-d'Alembert principle and one with an optimal control problem. The equations in each case are different. We consider the following Lagrangian on Euclidean three-space $\mathbb{R}^3$

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \tag{1.68}$$

subject to the constraint

$$\dot{z} = y\dot{x} - x\dot{y}. \tag{1.69}$$

Controls u_1 and u_2 are given in the x and y directions. Letting $q = (x, y, z)^T$,the dynamic nonholonomic control system is

$$\ddot{q} = u_1 X_1 + u_2 X_2 + \lambda W \tag{1.70}$$

where $X_1 = (1, 0, 0)^T$ and $X_2 = (0, 1, 0)^T$ and $W = (-y, x, 1)^T$.

From $\dot{z} - y\dot{x} + x\dot{y} = 0$ we obtain $a^{'} = (-y, x, 1)$

The general Euler-Lagrange equation read as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial \dot{q}_i} = \lambda_1 a^1 + f_1 u_1 + f_2 u_2 \tag{1.71}$$

where $f_1 = (1, 0, 0)$, $f_2 = (0, 1, 0)$. From the Lagrangian we get

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial \dot{q}_i} = (\ddot{x}, \ddot{y}, \ddot{z}) \tag{1.72}$$

and give

$$\ddot{z} = \lambda_1 \tag{1.73}$$

$$\ddot{y} = \lambda_1 x + u_2 \tag{1.74}$$

$$\ddot{x} = -\lambda_1 y + u_1. \tag{1.75}$$

However, the constraint can also be expressed as

$$\ddot{z} = y\ddot{x} - x\ddot{y}. \tag{1.76}$$

After some calculations we obtain

$$\begin{aligned}
\ddot{x}(1 + x^2 + y^2) &= xyu_2 + u_1(1 + x^2) \\
\ddot{y}(1 + x^2 + y^2) &= xyu_1 + (1 + y^2)u_2 \\
\ddot{z}(1 + x^2 + y^2) &= u_1 y - u_2 x.
\end{aligned} \tag{1.77}$$

1.9 The Hamilton-Jacobi Formulation of Constrained Systems

The Hamiltonian formulation of constrained systems is first discussed by Dirac [5],[6]. If the rank of the Hessian matrix

$$\frac{\partial^2 L}{\partial \dot{q}_i \partial \dot{q}_j}, \qquad i, j = 1, \ldots, n \tag{1.78}$$

is $n - r$, $r < n$, then one can produce r functionally independent constraints,

$$H'_\mu(q, p) \approx 0, \qquad \mu = 1, \ldots, r \tag{1.79}$$

which are called primary constraints. The total Hamiltonian is defined as

$$H_T = H_0 + v_\mu H'_\mu \tag{1.80}$$

where H_0 is the standard Hamiltonian

$$H_0 = -L + \sum_{i=1}^{n} p_i \dot{q}_i \tag{1.81}$$

and v_μ are coefficients.

Consistency conditions are given as

$$\dot{H}'_\mu = \frac{dH'_\mu}{dt} = \{H'_\mu, H_0\} + v_\ell\{H'_\mu, H'_\ell\} \approx 0. \tag{1.82}$$

These conditions may be identically satisfied as a result of the primary constraints, or they lead to new conditions which are called the secondary constraints. Primary and secondary constraints can be classified as first class and second class. First-class constraints are those which have vanishing Poisson brackets with all other constraints and second-class constraints are those which have non-vanishing Poisson brackets. Second-class constraints could be used to eliminate some of p's and q's from the theory.

Recently, a second method, which is called the Hamilton-Jacobi method is introduced [7],[8],[9],[10],[11]. The equivalent Lagrangian method is used to obtain the set of Hamilton-Jacobi partial differential equations (HJPDE) as

$$H'_\alpha\left(t_\beta, q_a, \frac{\partial S}{\partial q_a}, \frac{\partial S}{\partial t_\beta}\right) = 0, \qquad \alpha, \beta = 0, 1, \ldots, r, \qquad a = 1, \ldots, n-r, \tag{1.83}$$

where

$$H'_\alpha = H_\alpha + P_\alpha. \tag{1.84}$$

The equations of motion are given as total differential equations in variables t_β,

$$dq_a = \sum \frac{\partial H'_\alpha}{\partial p_a} dt_\alpha, \qquad dp_a = \sum -\frac{\partial H'_\alpha}{\partial q_a} dt_\alpha \tag{1.85}$$

$$dp_\mu = \sum -\frac{\partial H'_\alpha}{\partial q_\mu} dt_\alpha, \qquad dz = \sum \left(-H_\alpha + p_a \frac{\partial H'_\alpha}{\partial p_a}\right) dt_\alpha, \tag{1.86}$$

where

$$z = S(t_\alpha, q_a). \tag{1.87}$$

Since equations are total differential equations, integrability conditions should be checked. Equations of motion are integrable if variations of H'_α vanish identically. If they do not vanish identically we consider them as new constraints. This procedure continues until a complete system is obtained.

CHAPTER 2

THE SNAKEBOARD

2.1 The Snakeboard as a Constrained System

The snakeboard is a mechanical system such that the rider can move it without touching the ground (Fig.2.1). It consists of a platform (board) on which the rider

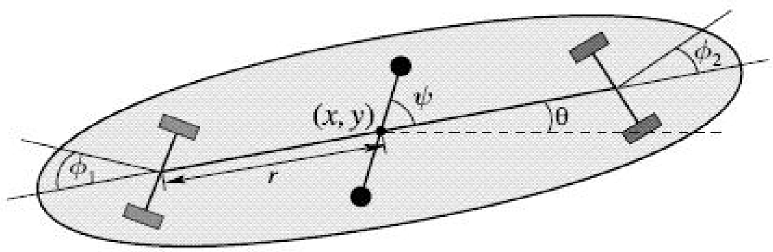

Figure 2.1: The geometry of the snakeboard.

stands, two wheels, rear and front and a rotor. The motion of the snakeboard is accomplished by the torques applied by the rider turning his (her) feet in and out. To describe the motion, the configuration space is parameterized by $x, y, \theta, \phi_1, \phi_2$ and ψ. Here, the parameters (x, y, θ) describe the position and the orientation of the center of the solid body, board; ψ is the angle of the momentum wheel or rotor with respect to the board; ϕ_1 and ϕ_2 are angles of the back and front wheels again with respect to the board. The distance between the center of the board and the wheels is r. Besides, J_0 is the inertia of the rotor; J_1 and J_2 are the inertias of the rear and front wheels, and J is the inertia of the board. The following simplifications will be used in calculations:

1. $J_1 = J_2$
2. $\phi_1 = -\phi_2 = \phi$
3. $J + J_0 + J_1 + J_2 = mr^2$ where m is the mass of the board.

Since the parameters ψ, ϕ_1 and ϕ_2 are related to rider input, they are generally labelled a "controlled" variables.

As a manifold Q is given as

$$Q = SE(2) \times S^1 \times S^1 \tag{2.1}$$

where the Euclidean group $SE(2)$ is the group of rigid motion in plane which describes the motion of the board. Since the rear and front wheels roll without sliding there are nonholonomic constraints expressed as

$$G_1 = -\sin(\theta + \phi)\dot{x} + \cos(\theta + \phi)\dot{y} - r\cos\phi\dot{\theta} = 0 \tag{2.2}$$

$$G_2 = -\sin(\theta - \phi)\dot{x} + \cos(\theta - \phi)\dot{y} + r\cos\phi\dot{\theta} = 0. \tag{2.3}$$

They can be expressed as constraint 1-forms

$$w_1(q) = -\sin(\theta + \phi)dx + \cos(\theta + \phi)dy - r\cos\phi d\theta \tag{2.4}$$

$$w_2(q) = -\sin(\theta - \phi)dx + \cos(\theta - \phi)dy + r\cos\phi d\theta. \tag{2.5}$$

Notice that constraints are invariant under the Euclidean group $SE(2)$.

2.2 The Lagrangian Formulation

The Lagrangian of a snakeboard in the configuration space specified above is

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}mr^2\dot{\theta}^2 + \frac{1}{2}J_0\dot{\psi}^2 + J_0\dot{\psi}\dot{\theta} + J_1\dot{\phi}^2. \tag{2.6}$$

It is obvious that L is invariant under $SE(2)$. To obtain the equations of motion we start with the extended Lagrangian L_1 which is defined as

$$L_1 = L + \dot{\lambda}_1 G_1 + \dot{\lambda}_2 G_2. \tag{2.7}$$

Explicitly L_1 is

$$\begin{aligned} L_1 \quad = \quad & \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}mr^2\dot{\theta}^2 + \frac{1}{2}J_0\dot{\psi}^2 + J_0\dot{\psi}\dot{\theta} + J_1\dot{\phi}^2 \\ & +[-\sin(\theta+\phi)\dot{x} + \cos(\theta+\phi)\dot{y} - r\cos\phi\dot{\theta}]\dot{\lambda}_1 \\ & +[-\sin(\theta-\phi)\dot{x} + \cos(\theta-\phi)\dot{y} + r\cos\phi\dot{\theta}]\dot{\lambda}_2. \end{aligned} \tag{2.8}$$

There are seven generalized coordinates

$$q_1 = x, q_2 = y, q_3 = \theta, q_4 = \psi, q_5 = \phi, q_6 = \lambda_1, q_7 = \lambda_2. \tag{2.9}$$

The equations of motion are given by the Euler-Lagrange equations which are expressed as

$$\frac{d}{dt}\left(\frac{\partial L_1}{\partial \dot{q}_i}\right) - \frac{\partial L_1}{\partial q_i} = 0, \qquad (i = 1, ..., 7). \tag{2.10}$$

Since $x, y, \psi, \lambda_1, \lambda_2$ are cyclic coordinates, the corresponding generalized momenta are conserved in time. In other words,

$$\frac{\partial L_1}{\partial \dot{x}} = c_1, \frac{\partial L_1}{\partial \dot{y}} = c_2, \frac{\partial L_1}{\partial \dot{\psi}} = c_4, \frac{\partial L_1}{\partial \dot{\lambda}_1} = c_6, \frac{\partial L_1}{\partial \dot{\lambda}_2} = c_7 \tag{2.11}$$

where c_1, c_2, c_4, c_6, c_7 are constants. Simultaneous solutions of equations (2.11), in terms of c_1, c_2, c_4, c_6, c_7 give

$$\dot{x} = -\csc 2\phi[\cos(\theta - \phi)c_6 - \cos(\theta + \phi)c_7 + 2r\cos\theta\cos^2\phi\dot{\theta}] \tag{2.12}$$

$$\dot{y} = -\csc 2\phi[\sin(\theta - \phi)c_6 - \sin(\theta + \phi)c_7 + 2r\sin\theta cos^2\phi\dot{\theta}] \tag{2.13}$$

$$\dot{\psi} = \frac{c_4}{J_0} - \dot{\theta} \tag{2.14}$$

$$\begin{aligned} \dot{\lambda}_1 &= -\csc^2 2\phi\{sin2\phi[\cos(\theta - \phi)c_1 + \sin(\theta - \phi)c_2] \\ &+ m[c_6 - \cos 2\phi c_7 + 2r\cos^3\phi\dot{\theta}]\} \end{aligned} \tag{2.15}$$

$$\begin{aligned} \dot{\lambda}_2 &= -\csc^2 2\phi\{\sin 2\phi[\cos(\theta + \phi)c_1 + \sin(\theta + \phi)c_2] \\ &+ m[\cos 2\phi c_6 - c_7 + 2r\cos^3\phi\dot{\theta}]\}. \end{aligned} \tag{2.16}$$

One should notice that equations from (2.12) to (2.16), are expressed in terms of generalized coordinates θ and ϕ. Thus, if we are able to determine them using the Euler-Lagrange equations,

$$\frac{d}{dt}(\frac{\partial L_1}{\partial \dot{\theta}}) - \frac{\partial L_1}{\partial \theta} = 0 \tag{2.17}$$

$$\frac{d}{dt}(\frac{\partial L_1}{\partial \dot{\phi}}) - \frac{\partial L_1}{\partial \phi} = 0 \tag{2.18}$$

then, we will solve the problem. In fact, equations (2.17) and (2.18) are

$$\begin{aligned} &\cos\phi[(\cos\theta\dot{x} + \sin\theta\dot{y})(\dot{\lambda}_1 + \dot{\lambda}_2) + r(-\ddot{\lambda}_1 + \ddot{\lambda}_2)] \\ - &\sin\phi(\sin\theta\dot{x} - \cos\theta\dot{y} - r\dot{\phi})(\dot{\lambda}_1 - \dot{\lambda}_2) + mr^2\ddot{\theta} + J_0\ddot{\psi} = 0 \end{aligned} \tag{2.19}$$

$$\begin{aligned} &[\cos(\theta + \phi)\dot{x} + \sin(\theta + \phi)\dot{y} - r\sin\phi\dot{\theta}]\dot{\lambda}_1 \\ - &[\cos(\theta - \phi)\dot{x} + \sin(\theta - \phi)\dot{y} - r\sin\phi\dot{\theta}]\dot{\lambda}_2 + 2J_1\ddot{\phi} = 0. \end{aligned} \tag{2.20}$$

Evaluating the second derivatives $\ddot{\psi}, \ddot{\lambda}_1, \ddot{\lambda}_2$ from eqs. (2.12)-(2.16), and inserting them in eqs. (2.19) and (2.20), we get

$$
\begin{aligned}
& -r\cot\phi c_1(\sin\theta\cdot\dot{\theta} + \cos\theta\cot\phi\cdot\dot{\phi}) + r\cot\phi c_2(\cos\theta\cdot\dot{\theta} - \cot\phi\sin\theta\cdot\dot{\phi}) \\
& +[\cos(\theta+\phi)\dot{x} + \sin(\theta+\phi)\dot{y}]\dot{\lambda}_1 + [\cos(\theta-\phi)\dot{x} + \sin(\theta-\phi)\dot{y}]\dot{\lambda}_2 \\
& -\frac{1}{2}r\dot{\phi}\{m\cot\phi\csc\phi[2\cot\phi(c_6-c_7) + r(3+\cos 2\phi)\csc\phi\dot{\theta}] \\
& +2\sin\phi(-\dot{\lambda}_1 + \dot{\lambda}_2)\} + (mr^2\csc^2\phi - J_0)\ddot{\theta} = 0
\end{aligned} \tag{2.21}
$$

$$
\begin{aligned}
& [\cos(\theta+\phi)\dot{x} + \sin(\theta+\phi)\dot{y} - r\sin\phi\cdot\dot{\theta}]\dot{\lambda}_1 \\
& -[\cos(\theta-\phi)\dot{x} + \sin(\theta-\phi)\dot{y} - r\sin\phi\cdot\dot{\theta}]\dot{\lambda}_2 + 2J_1\ddot{\phi} = 0.
\end{aligned} \tag{2.22}
$$

Inserting $\dot{x}, \dot{y}, \dot{\psi}, \dot{\lambda}_1, \dot{\lambda}_2$ in (2.21) and (2.22) from eqs. (2.12)-(2.16) we get the following two second order non-linear, ordinary differential equations in the generalized coordinates θ and ϕ.

$$
\begin{aligned}
- & \frac{1}{4}\csc^3\phi(mr((3+\cos 2\phi)(c_6-c_7) + 8r\cos\phi\dot{\theta})\dot{\phi} \\
+ & 2\sin\phi c_1((-\sin(\theta-\phi)c_6 + \sin(\theta+\phi)c_7)\tan\phi \\
+ & 2r\cos\theta\dot{\phi}) + 2\sin\phi c_2((\cos(\theta-\phi)c_6 \\
- & \cos(\theta+\phi)c_7)\tan\phi + 2r\sin\theta\dot{\phi}) \\
+ & 4\sin\phi(-mr^2 + \sin^2\phi J_0)\ddot{\theta}) = 0
\end{aligned} \tag{2.23}
$$

$$
\begin{aligned}
& \csc^3 2\phi(c_6 - \cos 2\phi c_7 + 2r\cos\phi\dot{\theta})(\sin 2\phi(\cos(\theta+\phi)c_1 \\
+ \; & \sin(\theta+\phi)c_2) + m(\cos 2\phi c_6 - c_7 + 2r\cos^3\phi\dot{\theta})) \\
+ \; & \csc^3 2\phi(\cos 2\phi c_6 - c_7 + 2r\cos\phi\dot{\theta})(\sin 2\phi(\cos\theta - \phi c_1 \\
+ \; & \sin(\theta-\phi)c_2) + m(c_6 - \cos 2\phi c_7 + 2r\cos^3\phi\dot{\theta})) + 2J_1\ddot{\phi} = 0.
\end{aligned} \tag{2.24}
$$

Since, the analytic solutions of the coupled equations (2.23),(2.24) are extremely difficult, we prefer numerical solutions. To achieve this goal we set

$$c_1 = 1kg.m/s, c_2 = 1kg.m/s, c_4 = 1kg.m/s, c_6 = 1kg.m/s, \tag{2.25}$$
$$c_7 = 1kg.m/s, r = \frac{1}{2}m, J_0 = 1kg.m^2, J_1 = 1kg.m^2, m = 3kg$$

in the time interval $[0, 10]$. The graphs of seven generalized coordinates are given below :

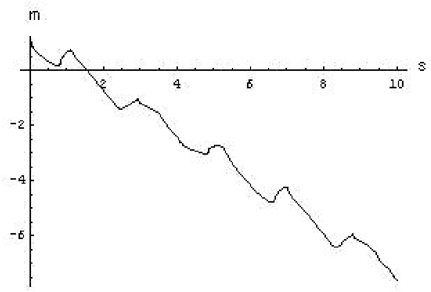

Figure 2.2: graph of x

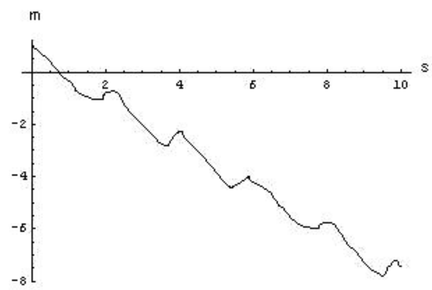

Figure 2.3: graph of y

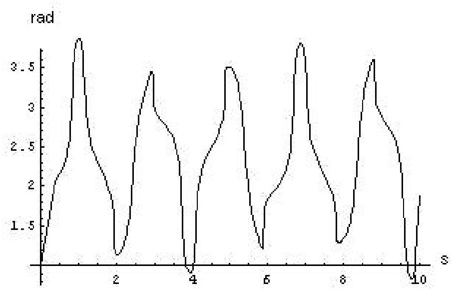

Figure 2.4: graph of θ

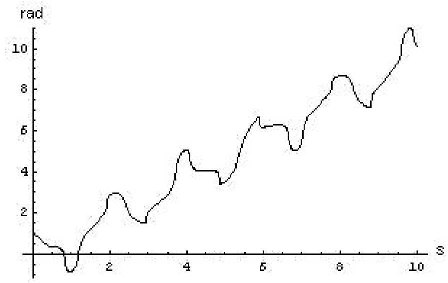

Figure 2.5: graph of ψ

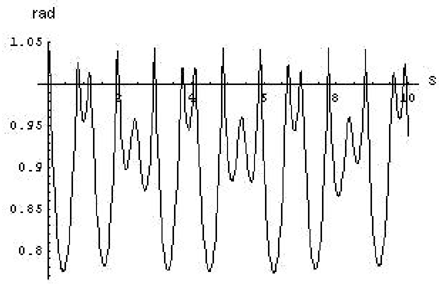

Figure 2.6: graph of ϕ

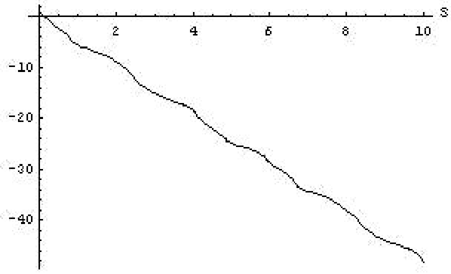

Figure 2.7: graph of λ_1

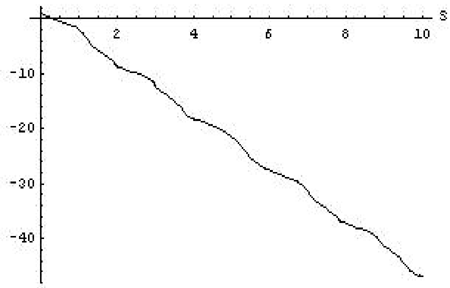

Figure 2.8: graph of λ_2

2.3 The Hamiltonian Formulation

To determine the Hamiltonian function of the snakeboard problem in terms of generalized coordinates q_i and generalized momenta p_i, $i = 1, \ldots, 7$, we express the Lagrangian L_1 as

$$L_1 = \sum_{i,j=1}^{7} a_{ij} \dot{q}_i \dot{q}_j \tag{2.26}$$

where the symmetric matrix a_{ij} is defined as

$$a_{ij} = \begin{pmatrix} m & 0 & 0 & 0 & 0 & -\sin(\theta+\phi) & -\sin(\theta-\phi) \\ 0 & m & 0 & 0 & 0 & \cos(\theta+\phi) & \cos(\theta-\phi) \\ 0 & 0 & mr^2 & J_0 & 0 & -r\cos\phi & r\cos\phi \\ 0 & 0 & J_0 & J_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2J_1 & 0 & 0 \\ -\sin(\theta+\phi) & \cos(\theta+\phi) & -r\cos\phi & 0 & 0 & 0 & 0 \\ -\sin(\theta-\phi) & \cos(\theta-\phi) & r\cos\phi & 0 & 0 & 0 & 0 \end{pmatrix} \tag{2.27}$$

Since the matrix is invertible, the Hamiltonian function is defined as

$$H = \sum_{i,j=1}^{7} \frac{1}{2} a_{ij}^{-1} p_i p_j. \tag{2.28}$$

The corresponding equations of motion in the phase space are given as

$$\begin{aligned} \dot{q}_i &= \frac{\partial H}{\partial p_i}, \\ & \qquad\qquad\qquad (i = 1, \ldots, 7) \\ \dot{p}_i &= -\frac{\partial H}{\partial q_i}. \end{aligned} \tag{2.29}$$

One should solve these fourteen first order equations simultaneously.Since $x, y, \psi, \lambda_1, \lambda_2$ are cyclic, corresponding momenta p_1, p_2, p_4, p_6, p_7 are constants, i.e.

$$p_1 = c_1, p_2 = c_2, p_4 = c_4, p_6 = c_6, p_7 = c_7. \tag{2.30}$$

Explicitly,

$$p_1 = \frac{\partial L_1}{\partial \dot{x}} = m\dot{x} - \sin(\theta+\phi)\dot{\lambda}_1 - \sin(\theta-\phi)\dot{\lambda}_2 = c_1 \tag{2.31}$$

$$p_2 = \frac{\partial L_1}{\partial \dot{y}} = m\dot{y} + \cos(\theta+\phi)\dot{\lambda}_1 + \cos(\theta-\phi)\dot{\lambda}_2 = c_2 \tag{2.32}$$

$$p_4 = \frac{\partial L_1}{\partial \dot{\psi}} = J_0(\dot{\theta} + \dot{\psi}) = c_4 \tag{2.33}$$

$$p_6 = \frac{\partial L_1}{\partial \dot{\lambda}_1} = -\sin(\theta+\phi)\dot{x} + \cos(\theta+\phi)\dot{y} - r\cos\phi \cdot \dot{\theta} = c_6 \tag{2.34}$$

$$p_7 = \frac{\partial L_1}{\partial \dot{\lambda}_2} = -\sin(\theta-\phi)\dot{x} + \cos(\theta-\phi)\dot{y} + r\cos\phi \cdot \dot{\theta} = c_7 \tag{2.35}$$

$$p_3 = \frac{\partial L_1}{\partial \dot{\theta}} = mr^2\dot{\theta} + J_0\dot{\psi} + r\cos\phi(\dot{\lambda}_2 - \dot{\lambda}_1) \tag{2.36}$$

$$p_5 = \frac{\partial L_1}{\partial \dot{\phi}} = 2J_1\dot{\phi}. \tag{2.37}$$

One should notice that equation (2.31-2.35), are linear in $\dot{x}, \dot{y}, \dot{\psi}, \dot{\lambda}_1$ and $\dot{\lambda}_2$. Thus, we can solve them as,

$$\dot{x} = -\csc 2\phi[\cos(\theta-\phi)c_6 - \cos(\theta+\phi)c_7 + 2r\cos\theta\cos^2\phi\dot{\theta}] \tag{2.38}$$

$$\dot{y} = -\csc 2\phi[\sin(\theta-\phi)c_6 - \sin(\theta+\phi)c_7 + 2r\sin\theta cos^2\phi\dot{\theta}] \tag{2.39}$$

$$\dot{\psi} = \frac{c_4}{J_0} - \dot{\theta} \tag{2.40}$$

$$\begin{aligned} \dot{\lambda}_1 &= -\csc^2 2\phi\{sin2\phi[\cos(\theta-\phi)c_1 \\ &+ \sin(\theta-\phi)c_2] + m[c_6 - \cos 2\phi c_7 + 2r\cos^3\phi\dot{\theta}]\} \end{aligned} \tag{2.41}$$

$$\begin{aligned} \dot{\lambda}_2 &= -\csc^2 2\phi\{\sin 2\phi[\cos(\theta+\phi)c_1 \\ &+ \sin(\theta+\phi)c_2] + m[\cos 2\phi c_6 - c_7 + 2r\cos^3\phi\dot{\theta}]\}. \end{aligned} \tag{2.42}$$

Besides, the equations of motion for θ and ϕ are

$$\dot{\theta} = \frac{\partial H}{\partial p_3} \quad , \quad \dot{\phi} = \frac{\partial H}{\partial p_5} \tag{2.43}$$

Taking the time derivatives of both sides and using equations (2.38-2.42) we obtain

$$
\begin{aligned}
\ddot{\theta} \quad &= \frac{1}{4mr^2 - 4\sin^2\phi \cdot J_0}\{2\sin\phi[-(\sin\theta \cdot c_1 - \cos\theta \cdot c_2)(c_6 - c_7) \\
&+(\cos\theta \cdot c_1 + \sin\theta \cdot c_2)(c_6 + c_7)\tan\phi] \qquad (2.44) \\
&+r[4\cos\theta \cdot c_1 + 4\sin\theta \cdot c_2 + m(3+\cos 2\phi)\csc\phi \cdot (c_6 - c_7) + 8mr\cot\phi \cdot \dot{\theta}]\dot{\phi}\} \\
\ddot{\phi} \quad &= \frac{1}{2J_1}\{\csc^3 2\phi[m(-2\cos 2\phi \cdot c_6^2 + 3c_6c_7 + \cos 4\phi \cdot c_6c_7 - 2\cos 2\phi \cdot c_7^2 \\
&-2r\cos^3\phi(3+\cos 2\phi)(c_6 - c_7)\dot{\theta} - 8r^2\cos^4\phi\dot{\theta}^2) \\
&+4\cos\phi\sin\phi \cdot c_2(-\cos\theta\sin^3\phi(c_6 + c_7) \qquad (2.45) \\
&-\cos^2\phi\sin\theta(\cos\phi(c_6 - c_7) + 2r\dot{\theta})) + c_1(4\cos\phi\sin\theta\sin^4\phi(c_6 + c_7) \\
&-4\cos\theta\cos^3\phi\sin\phi(\cos\phi(c_6 - c_7) + 2r\dot{\theta}))]\}.
\end{aligned}
$$

These second order, ordinary, non-linear differential equations again will be solved numerically setting the constants as before, in the time interval $[0, 10]$: Inserting the solutions for θ and ϕ in equations (2.38-2.42), one gets the following graphs for $x, y, \theta, \psi, \phi, \lambda_1, \lambda_2$

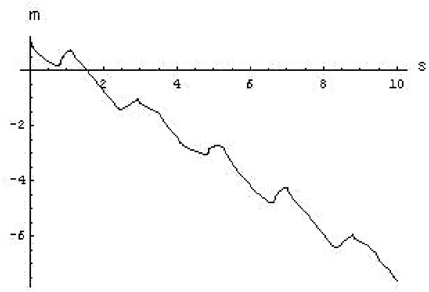

Figure 2.9: graph of x

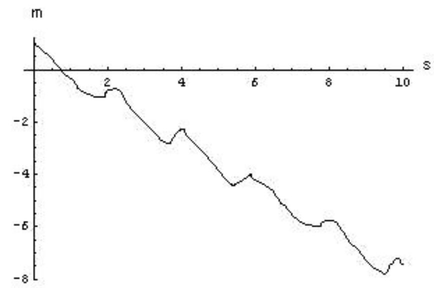

Figure 2.10: graph of y

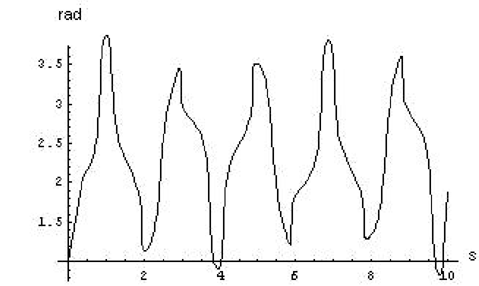

Figure 2.11: graph of θ

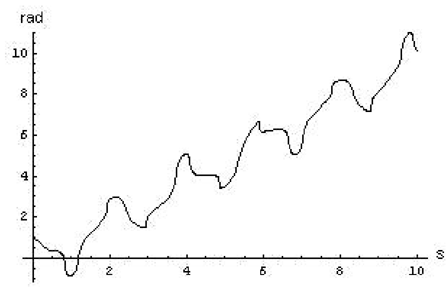

Figure 2.12: graph of ψ

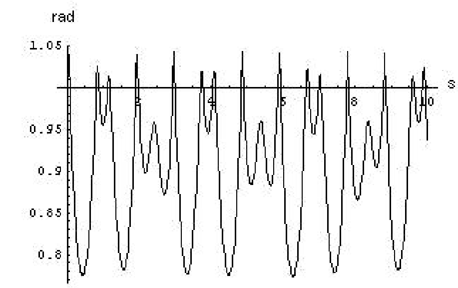

Figure 2.13: graph of ϕ

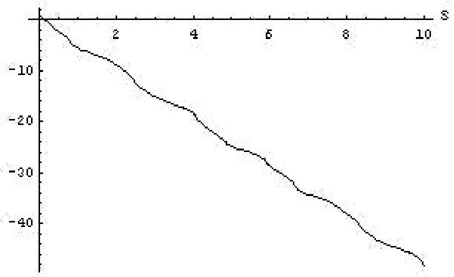

Figure 2.14: graph of λ_1

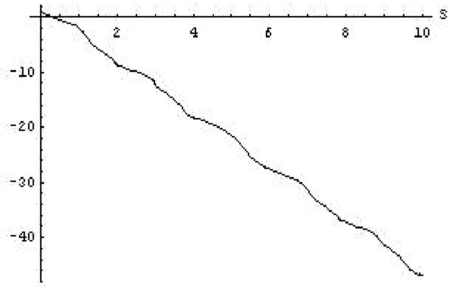

Figure 2.15: graph of λ_2

In the same way inserting the solutions in p_3 and p_5 we get them as

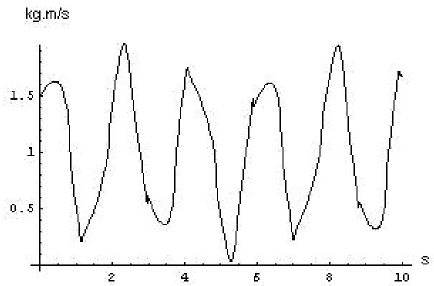

Figure 2.16: graph of p_3

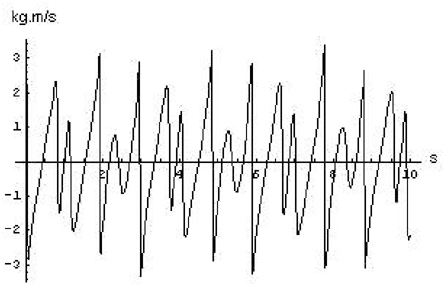

Figure 2.17: graph of p_5

CHAPTER 3

CONCLUSION

The snakeboard problem was first studied by Lewis, Ostrowski, Murray and Burdick [12]. It is an instructive example of a non-holonomic control system. They proved that the system is locally controllable. The same problem was further studied by Ostrowski using symmetry and constraints in mechanical systems [14]. He used the method which was used by Bloch, Krisnaprasad, Marsden, and Murray [12].

The snakeboard problem has been studied as a locomotion device by several authors [12]. By definition, the locomotion is to generate motions by periodic variations of certain control variables. Walking, running are examples of locomotion. The snakeboard gives the motivation to investigate the relation between locomotion and nonholonomic systems.

The Kinematic controllability was studied as an affine connection problem. The method is basically based on determination of a set of decoupling vector fields. In other words, if the system has such a set, then the controllability is given by integral curves.

In this thesis the snakeboard problem was investigated as a nonholonomic system using the Hamilton-Jacobi approach. The constraints were added to the original Lagrangian via Lagrange multipliers $\dot{\lambda}_1$ and $\dot{\lambda}_2$. Euler-Lagrange equations were solved numerically. The Hamiltonian of the system was obtained and the equations of motion were solved numerically and the same graphs were obtained as the Lagrangian approach.

Detailed investigation of graphs of numerical solutions of the generalized coordinates x, y, θ, ψ and ϕ gives some information about the motion of the snakeboard. Although the time interval $[0, 10]$ is not sufficient to describe the motion. One may have a general view. First point which should be emphasized is that the generalized coordinates (x, y), which describe the position of the center of the solid body, are approximately linear. Besides, the angular coordinates θ, ψ and ϕ are almost periodical.This implies that the motion of a snakeboard obeying the Euler-Lagrange equations without external torques (forces) has almost the pattern described above. In fact, in reference [12], the same problem studied as a control system. The authors introduce the input torques in ϕ and ψ. Using sinusoidal ϕ and ψ and proved that the system is controllable. Of course they solved the Euler- Lagrange equations

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = \sum_{j=1}^{2} \lambda_j w_{ij} + F_i, \qquad i = 1, \ldots, 6 \tag{3.1}$$

numerically. Here w_1 and w_2 are constraint one forms and F_i are external forces.

Thus our results are in agreement with the solutions of ref. [12] in the general pattern. Investigation of the snakeboard problem with the control variables using the Hamilton-Jacobi formulation will be the subject of a future work.

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