VARIATIONAL ITERATION METHOD FOR FRACTIONAL DAVEY-STEWARTSON EQUATIONS

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TUĞBA YILMAZ

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Submitted by **Tuğba YILMAZ**

Approval of the Graduate School of Natural and Applied Sciences, Çankaya University

Prof. Dr. Taner ALTUNOK
Director

I certify that this thesis satisfies all the requirements as a thesis for the degree of Master of Science

Billûr K

Prof. Dr. Billur KAYMAKÇALAN
Head of Department

This is to certify that we have read this thesis and that in our opinion it is fully adequate, in scope and quality, as a thesis for the degree of Master of Science.

Asst. Prof. Dr. Fahd JARAD
Co-Supervisor

Baleanu

Asst. Prof. Dr. Dumitru BALEANU
Supervisor

Examination Date : 26.01.2012

Examining Committee Members:

Prof. Dr. Billur KAYMAKÇALAN Billûr K

Asst. Prof. Dr. Dumitru BALEANU Baleanu

Asst. Prof. Dr. Rajeh EID

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Name, Last Name : Tuğba YILMAZ

Signature :

Date : 09.04.2012

ABSTRACT

VARIATIONAL ITERATION METHOD FOR FRACTIONAL DAVEY-STEWARTSON EQUATIONS

YILMAZ, Tuğba

M. S., Department of Mathematics and Computer Science

Supervisor :Asst. Prof. Dr. Dumitru BALEANU

Co-Supervisor: Asst. Prof. Dr. Fahd JARAD

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The Variational iteration method is a new and powerful method to solve both linear and nonlinear differential equations. The Variational iteration method was applied to the fractional Davey-Stewartson equations within the Caputo sense and approximate analytical solutions were obtained.

Keywords: Varaitional Iteration Method, Fractional Davey - Stewartson Equations, Caputo Derivative.

ÖZ

KESİRLİ DAVEY-STEWARTSON DENKLEMLERİ İÇİN VARYASYONEL İTERASYON METODU

YILMAZ, Tuğba

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Varyasyonel iterasyon metodu lineer ve lineer olmayan denklemlerin çözümü için yeni ve etkili bir yöntemdir. Varyasyonel iterasyon metodu kesirli Davey-Stewartson denklemlerine uygulanmış ve yaklaşık analitik çözümler elde edilmiştir ve kesirli türevler Caputo anlamı ile verilmiştir.

Anahtar Kelimeler: Varyasyonel Iterasyon Metodu, Kesirli Davey-Stewartson, Caputo Türevi.

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CHAPTER 1

INTRODUCTION

Fractional calculus deals with the integral and derivatives of any order as well as with their applications [1, 2, 3, 4, 5, 6, 7]. The applications of fractional calculus successfully concentrated in several areas of science and engineering [1, 2, 3, 4, 5, 6, 7].

For example, the fractional variational principles are applied successfully to several systems of physical interest as well as to the control area [8, 9, 10, 11, 12, 13].

The fractional differential geometry and its applications have recently been investigated intensely [14, 15, 16]. A new application of the fractional calculus in Nuclear magnetic resonance (NMR) is reported in [17]. Some interesting applications of the fractional calculus in Physics are presented in [18, 19, 20]. In addition, the research of fractional calculus involves areas of mathematics as it is given in the references [21, 22, 23]. Moreover it was shown that fractional-order calculus arises as an alternative calculus in mathematics [24]. Approximate and numarical methods are used for most of fractional differential equations because these equations do not have exact analytic solutions. Also the transform method was applied to differential equations of fractional order by Erturk, Momani and

Odibat [25]. It is also shown [26, 27] that the Homotopy perturbation method and Adomian decomposition method have been used for solving many problems.

Variational iteration method is applied to Helmholtz equation by Momani and Odibat and the results confirmed that this method is compatible with those obtained by Adomian decomposition method [28, 29, 30, 31]. We would like to indicate that the variational iteration method was proposed by Ji-Huan He [32] and it is a powerful analytical method. As it is known the exact solutions for most fractional differential equations do not exist, therefore, the variational iteration method is needed in order to find approximate solutions (see [33]).

Variational iteration method has been applied to the classical Davey-Stewartson equations by Jafari and Alipour [34] and the results show that this method is suitable for the solutions of the Davey-Stewartson equations. Abdou and Soliman have applied the variational iteration method for solving three species nonlinear partial differential equations (Schrödinger-KdV, generalized KdV and shallow water equations) and the results obtained by variational iteration method show that this method is a proper for solving nonlinear equations [35].

In this thesis, the applications of the variational iteration method are presented in order to provide the approximate solutions for the fractional Davey-Stewartson equations.

The over view of this thesis is as follows:

Briefly, general information about the variational principles and the variational iteration method are given in the Chapter 2. Basic definitions of Caputo fractional derivatives and Riemann-Liouville fractional integrals and derivatives are presented in Chapter 3. Variational iteration method is applied to the fractional Davey-Stewartson equationsand the corresponding numerical solutions and figures are shown in Chapter4. Chapter 5 is devoted to our conclusion.

CHAPTER 2

VARIATIONAL ITERATION METHOD

2.1 Calculus of Variations

The calculus of variations deals with the changes in functionals [36]. The variational principles are very importent in many branches of science and engineering and they have plenty of important applications. "A functional is a correspondence between a function in some class and the set of real numbers" [36]. We consider examples

$$\Phi(y) = \int_a^b f(x, y, y')\ dx, \qquad \Phi(y) = \max_{a \le x \le b} |y(x)|. \tag{2.1.1}$$

The function f is defined as known function of its arguments and $y(x)$ is designated for the functional Φ. Φ is a real number and it can be calculated. The spaces of applicable functions are continuous and having continous first derivatives on the interval a to b. This class is denoted by C^1 (a, b) [36].

The class of functions φ in C^1 is considered and they satisfy $y(a) = y_1$ and $y(b) = y_2$. "For what functions $y(x)$ in φ is the functional $\Phi(x)$ described in (2.1.1) stationary?" [36]. The stationary quality is described as similar to functions and if function is stationary at the point $\bar{x}$, then

$$\left.\frac{dg(x)}{dx}\right|_{x=\bar{x}}=0, \qquad \lim_{\varepsilon\to 0}\frac{g(\bar{x}+\varepsilon)-g(\bar{x})}{\varepsilon}=0. \tag{2.1.2}$$

We consider the function $y+\varepsilon\eta(x)$ and here y is in φ and $\eta\in C^1$ is zero at a and b. In that case the sum $y+\varepsilon\eta$ that is also in φ for all values of ε. We define the derivative of the functional that is called the variation as follows [36]:

$$\varepsilon\lim_{\varepsilon\to 0}\frac{\Phi(y+\varepsilon\eta)-\Phi(y)}{\varepsilon}. \tag{2.1.3}$$

The functions that make (2.1.3) zero are known as extremals. Equation (2.1.3) can be used for finding the conditions extremals satisfy.

If we consider functional

$$\Phi(\varepsilon)=\int_a^b f\left(x,\bar{y}+\varepsilon\eta,\overline{y'}+\varepsilon\eta'\right)dx, \tag{2.1.4}$$

here $\bar{y}$ and η are specified, thus Φ is a function of ε. The function is differentiated with respect to ε and evaluated at $\varepsilon=0$, namely

$$\left.\frac{d\Phi}{d\varepsilon}\right|_{\varepsilon=0}=\int_a^b\left(f_y\eta+f_{y'}\eta'\right)dx, \qquad f_y=\frac{\partial f}{\partial y}, \qquad f_{y'}=\frac{\partial f}{\partial y'}. \tag{2.1.5}$$

The last term is integrated by parts and then following equation is obtained as in [36];

$$\int_a^b f_{y'}\frac{d\eta}{dx}dx=\int_a^b\frac{d}{dx}\left[f_{y'}\eta\right]dx-\int_a^b\eta\frac{d}{dx}f_{y'}\ dx$$

$$=\left[\eta f_{y'}\right]_b^a-\int_a^b\eta\frac{d}{dx}f_{y'}\ dx. \tag{2.1.6}$$

Then the first term becomes zero, but nevertheless, on account of the conditions on η the derivative of (2.1.5) is given by

$$\Phi'(0) = \int_a^b \eta(x)\left[f_y - \frac{d}{dx} f_{y'} \right] dx = 0 . \tag{2.1.7}$$

Provided that $M(x)$ is in C, $\eta(x)$ is in C^1 and becomes zero at a and b and if

$$\int_a^b \eta(x) M(x) dx = 0 \tag{2.1.8}$$

for all possible functions η, then as in [36]

$$M(x) = 0, \ a \le x \le b . \tag{2.1.9}$$

Given that M is different from zero at some small region of x, η could be constructed so that it is zero everywhere excluding near where $M \neq 0$, and it has the same sign as M so the value of integral would be positive, hence the original assumption would become false, ($M \neq 0$). Provided that Φ is stationary for all possible values of variations η, in that case [36]

$$[f]_y \equiv f_y - \frac{d}{dx} f_{y'} = 0 \tag{2.1.10}$$

which is known as Lagrange- Euler equation.

2.2 Variational Iteration Method

Variational iteration method proposed by Ji-Huan He [37, 38, 34] is used for many problems in areas of science and engineering. This method is used for solving nonlinear and linear ordinary differential equations as well as for solving various engineering problems. “Ji -Huan He was the first to apply the variational iteration method to fractional differential equations” [39]. The variational iteration method, restricted variation, correction function and lagrange multiplier are explained in [40]. There are various applications of this method in the references [33, 37, 41, 42, 43]. Ji-Huan He has applied this method for solving autonomous ordinary differential systems [38]. The main speciality of variational iteration method is its flexibility and ability for solving nonlinear equations [33].

We consider the following general nonlinear equation,

$$Ly + Ny = g(t), \tag{2.2.1}$$

here L is a linear operator, N is a nonlinear operator and g (t) is a known analytical function. According to the variational iteration method, the correction functional can be constructed as follows:

$$y_{n+1}(t) = y_n(t) + \int_0^t \lambda\{Ly_n(\xi) + N\tilde{y}_n(\xi) - g(\xi)\}\,d\xi, \tag{2.2.2}$$

where λ is a general Lagrange multiplier and it can be identified optimally via the variational theory, $y_0(x)$ is an initial approximation with possible unknowns, and $\tilde{y}_n$ is considered as a restricted variation i.e. $\delta\tilde{y}_n = 0$ [37]. This shows that variational iteration method is simple and influential. As a result this method is suitable for solving some nonlinear problems [36].

Reference [36] was analyzed and following restricted variation was defined in view of it. The variational principles can characterize engineering aspects of many problems.

As in [36], we start with variational principles of nonlinear problem,

$$N(u) - f = 0 \tag{2.2.3}$$

and we use the “adjoint” equation by making use of the Fréchet differentials. The variational integral is given by

$$I(u,v)=\int\left[vN(u)-vf-ug\right]dV \ , \qquad (2.2.4)$$

and its first variational has the form

$$\delta I=\int\left\{\delta v\left[N(u)-f\right]+\delta u\left[\widetilde{N'_u}v-g\right]\right\}dV=0 \, . \qquad (2.2.5)$$

The Euler equataions are

$\delta v : N(u)-f=0$,

$\delta u : \widetilde{N'_u}v-g\equiv N^*(u,v)-g=0$.

Hence an "adjoint variational principle" can be defined for the equation (2.2.3) and its adjoint becomes

$$\widetilde{N'}_u v-g\equiv N^*(u,v)-g=0 \, . \qquad (2.2.6)$$

The variational equation can be written as follows

$$\delta\bar{J}=\int\left[N(u)-f\right]\delta u \ dV \, . \qquad (2.2.7)$$

In Equation (2.2.7) $\delta\bar{J}$ indicates that J might not exist and (2.2.7) is not necessarily the variational of any functional [36], making (2.2.7) denoting a "quasi-variational principle."

There is another approach by writing (2.2.4) as follows:

$$K\left(u,u^0\right)=\int\left[N\left(u^0\right)-f\right]u \ dV \, . \qquad (2.2.8)$$

By keeping u^0 fixed variations are made with respect to u as,

$$\delta_u I=\int\left[N\left(u^0\right)-f\right]\delta u \ dV \, . \qquad (2.2.9)$$

The notation $\delta_u I$ denotes variations depending only in u. After the variation we substitute $u = u^0$ and obtain (2.2.3) as the “Euler equation”. This type of principle is called a “restricted variational principle” [36]. For in tis case, Euler equation of the adjoint equation (2.2.6), $u = u^0$ is used.

The restricted variational principle is not stationary unless

$$N^*(u,u) - g = 0 \ ,$$

that results from setting $v = u = u^0$ in (2.2.6) [36].

2.3 Lagrange Multipliers

Lagrange multiplier method is applied intensively in the area of calculus of variations. The Variations can be discussed by using the Lagrange multipliers that are useful for variational solutions.

Our aim is to make the functional $\mathrm{J} = \int_a^b F(x, y, y')\,dx$ stationary among all functions in φ subject to the condition that the functional $\mathrm{K} = \int_a^b G(x, y, y')\,dx$ has a prescribed value K_1 [36].

Assume that $\bar{y} = \bar{y}(x)$ is the desired extremal. We consider the family of curves $y = \bar{y} + \varepsilon_1 \eta(x) + \varepsilon_2 \zeta(x)$ where η and ζ satisfy the homogeneous boundary conditions so that y is an admissible function [36]. Given,

$$\Phi(\varepsilon_1, \varepsilon_2) = \int_a^b F(x, \bar{y} + \varepsilon_1 \eta + \varepsilon_2 \zeta, \bar{y}' + \varepsilon_1 \eta' + \varepsilon_2 \zeta')\,dx\,, \tag{2.3.1}$$

this function is a stationary for $\varepsilon_1 = \varepsilon_2 = 0$ and dependent on the constraint such that

$$\psi(\varepsilon_1,\varepsilon_2) = \int_a^b G(x, \bar{y} + \varepsilon_1\eta + \varepsilon_2\zeta, \bar{y}' + \varepsilon_1\eta' + \varepsilon_2\zeta')dx = K_1 . \tag{2.3.2}$$

For small enough values of ε_1 and ε_2 we obtain the following

$$\left\{\frac{\partial}{\partial\varepsilon_1}\left[\lambda_0\Phi(\varepsilon_1,\varepsilon_2) + \lambda\psi(\varepsilon_1,\varepsilon_2)\right]\Big|_{\varepsilon_1=\varepsilon_2=0}\right\} = 0$$

and (2.3.3)

$$\left\{\frac{\partial}{\partial\varepsilon_2}\left[\lambda_0\Phi(\varepsilon_1,\varepsilon_2) + \lambda\psi(\varepsilon_1,\varepsilon_2)\right]\Big|_{\varepsilon_1=\varepsilon_2=0}\right\} = 0 .$$

λ_0 and λ are not equal to zero. As a result we obtain the equations given below [36]

$$\int_a^b \left\{\lambda_0\left[F\right]_y + \lambda\left[G\right]_y\right\} \eta dx = 0 , \tag{2.3.4a}$$

$$\int_a^b \left\{\lambda_0\left[F\right]_y + \lambda\left[G\right]_y\right\} \zeta dx = 0 . \tag{2.3.4b}$$

Here $[F]_y$ and $[G]_y$ are the same as the Euler equation functionals F and G. As η is an arbitrary function in the first equation but ζ is not an arbitrary function in the first equation, the ratio of λ_0 to λ does not depend on ζ. But in the second equation, ζ is arbitrary and this gives $\lambda_0[F]_y + \lambda[G]_y = 0$. If $\lambda_0 \neq 0$ or

$$[G]_y = \frac{d}{dx}G_{y'} - G_y \neq 0 . \tag{2.3.5}$$

Then, we might substitute $\lambda_0 = 1$ and (2.3.4a) leads to for arbitrary η [36],

$$\frac{d}{dx}\left[F_{y'} + \lambda G_{y'}\right] - \frac{\partial}{\partial y}\left[F + \lambda G\right] = 0. \tag{2.3.6}$$

The integrant $F^* = F + \lambda G$ is derived for the Euler equation. The solution to (2.3.6) that has two undetermined constants plus the unknown parameter of λ and these are determined via the two boundary conditions and $\mathrm{K} = K_1$ [36]. Provided that the integrand is of the type F (x, y, y', z, z') and the constraint is given by

$$\mathrm{G}\,(\,x, y, z\,) = 0, \tag{2.3.7}$$

then Euler equation can be obtained by using the Lagrange multipliers in the same manner. Thus λ is a function of x rather than a constant [36]. For example the type of constraint used in fluid mechanics problems is usually a differential equation of the form:

$$G\left(x, y, y', z, z'\right) = 0\,. \tag{2.3.8}$$

If this equation is not solvable with respect to $z = f(y)$, then we use of Lagrange multipliers technique to solve it taking $F^* = F + \lambda G$, with $\lambda = \lambda(x)$ [36].

CHAPTER 3

BASIC DEFINITIONS OF FRACTIONAL CALCULUS

3.1 Riemann-Liouville Fractional Integrals and Fractional Derivatives

Let $\Omega = [a,b]$ $(-\infty < a < b < \infty)$ be a finite interval on the real axis of $\mathbb{R}$. The Riemann-Liouville fractional integrals $I_{a+}^{\alpha} f$ and $I_{b-}^{\alpha} f$ of order α $\left(\Re(\alpha) > 0\right)$ are defined by (see [5]):

$$\left(I_{a+}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_{a}^{x} \frac{f(t)\,dt}{(x-t)^{1-\alpha}}, \qquad x > a;\ \Re(\alpha) > 0, \tag{3.1.1}$$

and

$$\left(I_{b-}^{\alpha} f\right)(x) = \frac{1}{\Gamma(\alpha)} \int_{x}^{b} \frac{f(t)\,dt}{(t-x)^{1-\alpha}}, \qquad x < b;\ \Re(\alpha) > 0, \tag{3.1.2}$$

where α is a Complex number (C), $\Re(\alpha)$ is the real part and $\Gamma(\alpha)$ is the Gamma function [5]:

$$\Gamma(\alpha) = \int_{0}^{\infty} t^{\alpha-1} e^{-t}\,dt, \qquad \Re(\alpha) > 0, \tag{3.1.3}$$

these integrals are known as the left-sided and the right-sided fractional integrals respectively.

When $\alpha = n \in \mathbb{N}$ where $\mathbb{N}$ is the set of all positive integers and the definitions (3.1.1) and (3.1.2) transform to the nth integrals of the following forms [5]: $\left(I_{a+}^{n} f\right)(x)=\int_{a}^{x} dt_{1} \int_{a}^{t_{1}} dt_{2} \ldots \int_{a}^{t_{n-1}} f\left(t_{n}\right) dt_{n}$

$$=\frac{1}{(n-1)!} \int_{a}^{x}(x-t)^{n-1} f(t) dt, \qquad n \in \mathbb{N} \tag{3.1.4}$$

and

$$\left(I_{b-}^{n} f\right)(x)=\int_{x}^{b} dt_{1} \int_{t_{1}}^{b} dt_{2} \ldots \int_{t_{n-1}}^{b} f\left(t_{n}\right) dt_{n}$$

$$=\frac{1}{(n-1)!} \int_{x}^{b}(x-t)^{n-1} f(t) dt, \qquad n \in \mathbb{N}. \tag{3.1.5}$$

The Riemann-Liouville fractional derivatives $D_{a+}^{\alpha} y$ and $D_{b-}^{\alpha} y$ of order $\alpha \in C$ $(\Re(\alpha)) \geq 0$ are defined as follows [5]:

$$\left(D_{a+}^{\alpha} y\right)(x)=\left(\frac{d}{dx}\right)^{n}\left(I_{a+}^{n-\alpha} y\right)(x) \tag{3.1.6}$$

$$=\frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^{n} \int_{a}^{x} \frac{y(t) dt}{(x-t)^{\alpha-n+1}} \qquad (n=[\Re(\alpha)]+1;\ x>a)$$

and

$$\left(D_{b-}^{\alpha} y\right)(x)=\left(-\frac{d}{dx}\right)^{n}\left(I_{b-}^{n-\alpha} y\right)(x) \tag{3.1.7}$$

$$=\frac{1}{\Gamma(n-\alpha)}\left(-\frac{d}{dx}\right)^{n} \int_{x}^{b} \frac{y(t) dt}{(t-x)^{\alpha-n+1}} \qquad (n=[\Re(\alpha)]+1;\ x<b),$$

respectively. Here the meaning of $[\Re(\alpha)]$ is the integral part of $\Re(\alpha)$. Especially, when $\alpha=n \in \mathbb{N}_{0}$ then

$\left(D_{a+}^{0} y\right)(x)=\left(D_{b-}^{0} y\right)(x)=y(x);\ \left(D_{a+}^{n} y\right)(x)=y^{(n)}(x),$

and $\left(D_{b-}^{n}y\right)(x)=(-1)^{n}\,y^{(n)}(x), \quad (n\in\mathbb{N})$. (3.1.8)

Here, $y^{(n)}(x)$ denotes the usual derivative of $y(x)$ of order *n*. Provided that $0<\Re(\alpha)<1$,

$$\left(D_{a+}^{\alpha}y\right)(x)=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{a}^{x}\frac{y(t)\,dt}{(x-t)^{\alpha-[\Re(\alpha)]}} \quad \left(0<\Re(\alpha)<1;\ x>a\right), \tag{3.1.9}$$

$$\left(D_{b-}^{\alpha}y\right)(x)=-\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x}^{b}\frac{y(t)\,dt}{(t-x)^{\alpha-[\Re(\alpha)]}} \quad \left(0<\Re(\alpha)<1;\ x<b\right). \tag{3.1.10}$$

Property 3.1.1: In the case of $\Re(\alpha)\geq 0$ and $\beta\in C\ \left(\Re(\beta)>0\right)$,

$$\left(I_{a+}^{\alpha}(t-a)^{\beta-1}\right)(x)=\frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(x-a)^{\beta+\alpha-1} \qquad \left(\Re(\alpha)>0\right), \tag{3.1.11}$$

$$\left(D_{a+}^{\alpha}(t-a)^{\beta-1}\right)(x)=\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(x-a)^{\beta-\alpha-1} \qquad \left(\Re(\alpha)\geq 0\right) \tag{3.1.12}$$

and

$$\left(I_{b-}^{\alpha}(b-t)^{\beta-1}\right)(x)=\frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(b-x)^{\beta+\alpha-1} \qquad \left(\Re(\alpha)>0\right), \tag{3.1.13}$$

$$\left(D_{b-}^{\alpha}(b-t)^{\beta-1}\right)(x)=\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(b-x)^{\beta-\alpha-1} \qquad \left(\Re(\alpha)\geq 0\right). \tag{3.1.14}$$

Especially, Riemann-Liouville fractional derivatives of a constant are not equal to zero, provided that $\beta=1$ and $\Re(\alpha)\geq 0$,

$$\left(D_{a+}^{\alpha}1\right)(x)=\frac{(x-a)^{-\alpha}}{\Gamma(1-\alpha)}, \quad \left(D_{b-}^{\alpha}1\right)(x)=\frac{(b-x)^{-\alpha}}{\Gamma(1-\alpha)} \qquad \left(0<\Re(\alpha)<1\right). \tag{3.1.15}$$

Conversely for $j = 1, 2, \cdots, \ [\Re(\alpha)] + 1$,

$$\left(D_{a+}^{\alpha}(t-a)^{\alpha-j}\right)(x) = 0, \quad \left(D_{b-}^{\alpha}(b-t)^{\alpha-j}\right)(x) = 0. \tag{3.1.16}$$

Lemma 3.1.1: Let $\Re(\alpha) > 0$, $n = [\Re(\alpha)] + 1$ and let $f_{n-\alpha}(x) = \left(I_{a+}^{n-\alpha} f\right)(x)$ is the fractional integral (3.1.1) of order $n-\alpha$ (see [5]).

$L_p(a,b)$ $\{(1 \le p \le \infty)\}$ is denoted the Lebesgue space of comlex-valued measurable functions f.

a) Provided that $1 \le p \le \infty$ and $f(x) \in I_{a+}^{\alpha}(L_p)$, then

$$\left(I_{a+}^{\alpha} D_{a+}^{\alpha} f\right)(x) = f(x). \tag{3.1.17}$$

b) Provided that $f(x) \in L_1(a,b)$ and $f_{n-\alpha}(x) \in AC^n[a,b]$, the equality

$$\left(I_{a+}^{\alpha} D_{a+}^{\alpha} f\right)(x) = f(x) - \sum_{j=1}^{n} \frac{f_{n-\alpha}^{(n-j)}(a)}{(\alpha - j + 1)} (x-a)^{\alpha-j} \tag{3.1.18}$$

is true nearly everywhere on $[a,b]$. In [5] $AC^n[a,b]$ was denoted to the space of complex- valued functions $f(x)$ that have continuous derivatives up to order $n-1$ on $[a,b]$ such that $f^{(n-1)}(x) \in AC\ \ [a,b]$ for $n \in \mathbb{N} = \{1,2,3...\}$, $AC^n[a,b] = \left\{ f : [a,b] \to C \text{ and } (D^{n-1} f)(x) \in AC\ \ [a,b]\ (D = \frac{d}{dx}) \right\}$. C is the set of complex numbers. Especially, $AC^1\ \ [a,b] = AC\ \ [a,b]$.

Especially, on the condition that $0 < \Re(\alpha) < 1$, then

$$\left(I_{a+}^{\alpha} D_{a+}^{\alpha} f\right)(x) = f(x) - \frac{f_{1-\alpha}(a)}{\Gamma(\alpha)} (x-a)^{\alpha-1}, \tag{3.1.19}$$

here $f_{1-\alpha}(x) = \left(I_{a+}^{1-\alpha} f\right)(x)$ and for $\alpha = n \in \mathbb{N}$, the following form is valid:

$$\left(I_{a+}^{n}D_{a+}^{n}f\right)(x)=f(x)-\sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(x-a)^{k}. \tag{3.1.20}$$

Property 3.1.2: Let $\alpha>0$ and $\beta>0$ be such that $n-1<\alpha\le n$, $m-1<\beta\le m$ $(n,m\in\mathbb{N})$ and $\alpha+\beta<n$, let $f\in L_1(a,b)$ and $f_{m-\alpha}\in AC^{m}([a,b])$. Then

$$\left(D_{a+}^{\alpha}D_{a+}^{\beta}f\right)(x)=\left(D_{a+}^{\alpha+\beta}f\right)(x)-\sum_{j=1}^{m}\left(D_{a+}^{\beta-j}f\right)(a+)\frac{(x-a)^{-j-\alpha}}{\Gamma(1-j-\alpha)}. \tag{3.1.21}$$

Proof. Since $n>\alpha+\beta$, using (3.1.6) and the semigroup property as in reference [5]

$$\left(I_{a+}^{\alpha}I_{a+}^{\beta}f\right)(x)=\left(I_{a+}^{\alpha+\beta}f\right)(x), \text{ and } \left(I_{b-}^{\alpha}I_{b-}^{\beta}f\right)(x)=\left(I_{b-}^{\alpha+\beta}f\right)(x) \tag{3.1.22}$$

and

$$\left(D_{a+}^{\alpha}D_{a+}^{\beta}f\right)(x)=\left(\frac{d}{dx}\right)^{n}\left(I_{a+}^{n-\alpha}D_{a+}^{\beta}f\right)(x)=\left(\frac{d}{dx}\right)^{n}\left[I_{a+}^{n-\alpha-\beta}\left(I_{a+}^{\beta}D_{a+}^{\beta}f\right)\right](x). \tag{3.1.23}$$

As $f\in L_1(a,b)$ and $f_{m-\alpha}\in AC^{m}([a,b])$ through the following explanation (3.1.25) is obtained where α is replaced by β in (3.1.24).

As long as $\Re(\alpha)>0$ and $f(x)\in L_p(a,b)$ $(1\le p\le\infty)$, then

$$\left(D_{a+}^{\alpha}I_{a+}^{\alpha}f\right)(x)=f(x) \qquad \text{and} \qquad \left(D_{b-}^{\alpha}I_{b-}^{\alpha}f\right)(x)=f(x) \qquad (\Re(\alpha)>0)$$

(3.1.24) Hence nearly everywhere on $[a,b]$,

$$\left(I_{a+}^{\beta}D_{a+}^{\beta}f\right)(t)=f(t)-\sum_{j=1}^{m}\frac{\left(I_{a+}^{m-\beta}f\right)^{(m-j)}(a+)}{\Gamma(\beta-j+1)}(x-a)^{\beta-j}. \tag{3.1.25}$$

Lemma3.1.2: Let $\Re(\alpha)>0$ and $n=[\Re(\alpha)]+1$. In addition let $g_{n-\alpha}(x)=\left(I_{b-}^{n-\alpha}g\right)(x)$ be the fractional integral (3.1.2) of order $n-\alpha$ (see [5]).

a) If $1\le p\le\infty$ and $g(x)\in I_{b-}^{\alpha}(L_p)$, then

$$\left(I_{b-}^{\alpha}D_{b-}^{\alpha}g\right)(x)=g(x). \tag{3.1.26}$$

b) If g(x)$\in L_1(a,b)$ and $g_{n-\alpha}(x)\in AC^n[a,b]$, then

$$\left(I_{b-}^{\alpha}D_{b-}^{\alpha}g\right)(x)=g(x)-\sum_{j=1}^{n}\frac{(-1)^{n-j}\,g_{n-\alpha}^{n-j}(a)}{\Gamma(\alpha-j+1)}(b-x)^{\alpha-j} \tag{3.1.27}$$

exist nearly everywhere on $[a,b]$.

If $\left(0<\Re(\alpha)<1\right)$ then, the form is obtained

$$\left(I_{b-}^{\alpha}D_{b-}^{\alpha}g\right)(x)=g(x)-\frac{g_{1-\alpha}(a)}{\Gamma(a)}(b-x)^{\alpha-1}, \tag{3.1.28}$$

the following equality holds for $g_{1-\alpha}(x)=\left(I_{b-}^{1-\alpha}g\right)(x)$ and $\alpha=n\in\mathbb{N}$;

$$\left(I_{b-}^{n}D_{b-}^{n}g\right)(x)=g(x)-\sum_{k=0}^{n-1}\frac{(-1)^{k}\,g^{(k)}(b)}{k!}(b-x)^{k}. \tag{3.1.29}$$

3.2 Caputo Fractional Derivative

Let $[a,b]$ be closed interval in $\mathbb{R}$ [5]. Let $D_{a+}^{\alpha}\left[y(t)\right](x)\equiv\left(D_{a+}^{\alpha}y\right)(x)$ and $D_{b-}^{\alpha}\left[y(t)\right](x)\equiv\left(D_{b-}^{\alpha}y\right)(x)$ be the Riemann-Liouville fractional derivatives of order $\alpha\in C$ $\left(\Re(\alpha)\ge 0\right)$ where these are defined through (3.1.6) and (3.1.7).

The fractional derivatives are $({}^{C}D_{a+}^{\alpha}y)(x)$ and $({}^{C}D_{b-}^{\alpha}y)(x)$ of order $\alpha\in C$ $\left(\Re(\alpha)>0\right)$ on $[a,b]$ and these are defined as follows:

$$\left({}^{C}D_{a+}^{\alpha}y\right)(x)=\left[D_{a+}^{\alpha}\left(y(t)-\sum_{k=0}^{n-1}\frac{y^{(k)}(a)}{k!}(t-a)^{k}\right)\right](x) \tag{3.2.1}$$

and

$$\left({}^{C}D_{b-}^{\alpha}y\right)(x)=\left[D_{b-}^{\alpha}\left(y(t)-\sum_{k=0}^{n-1}\frac{y^{(k)}(b)}{k!}(b-t)^{k}\right)\right](x), \tag{3.2.2}$$

respectively, here n is defined by [5]:

$$n=[\Re(a)]+1 \text{ for } \alpha\notin\mathbb{N}_0 \; n=\alpha \text{ for } \alpha\in\mathbb{N}_0. \tag{3.2.3}$$

In that case these derivatives are called the left-sided and the right-sided Caputo fractional derivatives of order α.

When $0<\Re(\alpha)<1$, holds (3.2.1) and (3.2.2) are formed

$$\left({}^{C}D_{a+}^{\alpha}y\right)(x)=\left(D_{a+}^{\alpha}\left[y(t)-y(a)\right]\right)(x), \tag{3.2.4}$$

$$\left({}^{C}D_{b-}^{\alpha}y\right)(x)=\left(D_{b-}^{\alpha}\left[y(t)-y(b)\right]\right)(x). \tag{3.2.5}$$

Riemann-Liouville fractional derivatives $(D_{a+}^{\alpha}y)(x)$ and $(D_{b-}^{\alpha}y)(x)$ and the Caputo derivatives are connected

$$\left({}^{C}D_{a+}^{\alpha}y\right)(x)=\left(D_{a+}^{\alpha}y\right)(x)-\sum_{k=0}^{n-1}\frac{y^{(k)}(a)}{\Gamma(k-\alpha+1)}(x-a)^{k-\alpha} \quad \left(n=[\Re(\alpha)]+1\right) \tag{3.2.6}$$

and

$$\left({}^{C}D_{b-}^{\alpha}y\right)(x)=\left(D_{b-}^{\alpha}y\right)(x)-\sum_{k=0}^{n-1}\frac{y^{(k)}(b)}{\Gamma(k-\alpha+1)}(b-x)^{k-\alpha}, \quad \left(n=[\Re(\alpha)]+1\right). \tag{3.2.7}$$

Especially, when $0<\Re(\alpha)<1$ the forms are given by;

$$\left({}^{C}D_{a+}^{\alpha}y\right)(x)=\left(D_{a+}^{\alpha}y\right)(x)-\frac{y(a)}{\Gamma(1-\alpha)}(x-a)^{-\alpha} \tag{3.2.8}$$

and

$$\left({}^{C}D_{b-}^{\alpha}y\right)(x)=\left(D_{b-}^{\alpha}y\right)(x)-\frac{y(b)}{\Gamma(1-\alpha)}(b-x)^{-\alpha}. \tag{3.2.9}$$

In the case of $\alpha \notin \mathbb{N}_0$ [5], the Caputo fractional derivatives (3.2.1) and (3.2.2) become the Riemann-Liouville fractional derivatives (3.1.6) and (3.1.7),

$$\left({}^{C}D_{a+}^{\alpha}y\right)(x)=\left(D_{a+}^{\alpha}y\right)(x) \tag{3.2.10}$$

when $y(a)=y'(a)=\ldots=y^{(n-1)}(a)=0,\left(n=\left[\Re(\alpha)\right]+1\right)$, and

$$\left({}^{C}D_{b-}^{\alpha}y\right)(x)=\left(D_{b-}^{\alpha}y\right)(x), \tag{3.2.11}$$

if $y(b)=y'(b)=\ldots=y^{(n-1)}(b)=0$, $\left(n=[\Re(\alpha)]+1\right)$

Especially, when $0<\Re(\alpha)<1$ the equalities are obtained,

$$\left({}^{C}D_{a+}^{\alpha}y\right)(x)=\left(D_{a+}^{\alpha}y\right)(x), \quad \text{when } y(a)=0 \tag{3.2.12}$$

and

$$\left({}^{C}D_{b-}^{\alpha}y\right)(x)=\left(D_{b-}^{\alpha}y\right)(x), \text{when } y(b)=0. \tag{3.2.13}$$

Provide that $\alpha=n\in\mathbb{N}$ and the common derivative $y^{(n)}(x)$ of order n exists, in that case $\left({}^{(C)}D_{a+}^{n}y\right)(x)$ become $y^{(n)}(x)$, meanwhile $\left({}^{C}D_{b-}^{n}y\right)(x)$ becoming $y^{(n)}(x)$ with exactness to the fixed multiplier $(-1)^n$ (see [5]):

$$\left({}^{C}D_{a+}^{n}y\right)(x)=y^{(n)}(x) \text{ and } \left({}^{C}D_{b-}^{n}y\right)(x)=(-1)^{n}y^{(n)}(x) \quad (n\in\mathbb{N}). \tag{3.2.14}$$

Theorem 3.2.1: Let $\Re(\alpha)\geq 0$ and let n be given by (3.2.3). Provided that $y(x)\in AC^{n}[a,b]$, in that case the Caputo fractional derivatives $\left({}^{C}D_{a+}^{\alpha}y\right)(x)$ and $\left({}^{C}D_{b-}^{\alpha}y\right)(x)$ hold nearly everywhere on $[a,b]$.

If $\alpha \notin \mathbb{N}_0$, $\left({}^C D_{a+}^{\alpha} y\right)(x)$ and $\left({}^C D_{b-}^{\alpha} y\right)(x)$ are denoted by

$$\left({}^C D_{a+}^{\alpha} y\right)(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x \frac{y^{(n)}(t)\,dt}{(x-t)^{\alpha-n+1}} = \left(I_{a+}^{n-\alpha} D^n y\right)(x) \tag{3.2.15}$$

and

$$\left({}^C D_{b-}^{\alpha} y\right)(x) = \frac{1}{\Gamma(n-\alpha)} \int_x^b \frac{y^{(n)}(t)\,dt}{(t-x)^{\alpha-n+1}} = (-1)^n \left(I_{b-}^{n-\alpha} D^n y\right) \tag{3.2.16}$$

respectively, where $D = \dfrac{d}{dx}$ and $n = [\Re(\alpha)] + 1$.

Especially, when $0 < \Re(\alpha) < 1$ and $y(x) \in AC[a,b]$,

$$\left({}^C D_{a+}^{\alpha} y\right)(x) = \frac{1}{\Gamma(1-\alpha)} \int_a^x \frac{y'(t)\,dt}{(x-t)^{\alpha}} = \left(I_{a+}^{1-a} Dy\right)(x) \tag{3.2.17}$$

and

$$\left({}^C D_{b-}^{\alpha} y\right)(x) = -\frac{1}{\Gamma(1-\alpha)} \int_x^b \frac{y'(t)\,dt}{(t-x)^{\alpha}} = -\left(I_{b-}^{1-\alpha} Dy\right)(x) \tag{3.2.18}$$

Proof: Let $\alpha \notin \mathbb{N}_0$. When equations (3.2.1) and (3.1.6) which integrate by parts the inner integral and differentiating. (It is possible via the conditions of the theorem), (see [5]):

$$\left({}^C D_{a+}^{\alpha} y\right)(x) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dx^n} \left\{ -\frac{(x-t)^{n-\alpha}}{n-\alpha} \left[y(t) - \sum_{k=0}^{n-1} \frac{y^{(k)}(a)}{k!} (t-a)^k \right] \Big|_{t=a}^{x} \right.$$

$$\left. + \int_a^x \frac{(x-t)^{n-\alpha}}{n-\alpha} \frac{d}{dt} \left[y(t) - \sum_{k=0}^{n-1} \frac{y^{(k)}(a)}{k!} (t-a)^k \right] dt \right\}$$

$$= \frac{1}{\Gamma(n-\alpha)} \frac{d^{n-1}}{dx^{n-1}} \int_a^x (x-t)^{n-\alpha-1} \left[y'(t) - \sum_{k=1}^{n-1} \frac{y^{(k)}(a)}{(k-1)!} (t-a)^{k-1} \right] dt$$

$$= \cdots = \frac{1}{\Gamma(n-\alpha)} \frac{d}{dx} \int_a^x (x-t)^{n-\alpha-1} \left[y^{(n-1)}(t) - y^{(n-1)}(a) \right] dt .$$

If the above argument is used again,

$$\left({}^C D_{a+}^{\alpha} y \right)(x) = \frac{1}{\Gamma(n-\alpha)} \int_a^x (x-t)^{n-\alpha-1} y^{(n)}(t)\, dt$$

the result of (3.2.15) is obtained.

Theorem 3.2.2: Let $\Re(\alpha) \geq 0$ and let n be given by (3.2.3). Moreover let $y(x) \in C^n[a,b]$.

In that case the Caputo fractional derivatives are $\left({}^C D_{a+}^{\alpha} y \right)(x)$ and $\left({}^C D_{b-}^{\alpha} y \right)(x)$ and these are continuous on $[a,b]$: $\left({}^C D_{a+}^{\alpha} y \right)(x) \in C[a,b]$ and $\left({}^C D_{b-}^{\alpha} y \right)(x) \in C[a,b]$.

a) If $\alpha \notin \mathbb{N}_0$, then $\left({}^C D_{a+}^{\alpha} y \right)(x)$ and $\left({}^C D_{b-}^{\alpha} y \right)(x)$ are represented by (3.2.15) and (3.2.16), respectively. Furthermore,

$$\left({}^C D_{a+}^{\alpha} y \right)(a) = \left({}^C D_{b-}^{\alpha} y \right)(b) = 0. \tag{3.2.19}$$

Especially, they have the forms (3.2.17) and (3.2.18) for $0 < \Re(\alpha) < 1$.

b) If $\alpha = n \in \mathbb{N}_0$, in that case the fractional derivatives $\left({}^C D_{a+}^{\alpha} y \right)(x)$ and $\left({}^C D_{b-}^{\alpha} y \right)(x)$ have representations given by (3.2.14).

Property 3.2.1: Let $\Re(\alpha) > 0$ and let n be given via (3.2.3) and let $\Re(\beta) > 0$. Then [5]:

$$\left({}^{C}D_{a+}^{\alpha}(t-a)^{\beta-1}\right)(x)=\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(x-a)^{\beta-1} \qquad (\Re(\beta)>n) \tag{3.2.20}$$

$$\left({}^{C}D_{b-}^{\alpha}(b-t)^{\beta-1}\right)(x)=\frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(b-x)^{\beta-1} \qquad (\Re(\beta)>n) \tag{3.2.21}$$

and

$$\left({}^{C}D_{a+}^{\alpha}(t-a)^{k}\right)(x)=0 \text{ and } \left({}^{C}D_{b-}^{\alpha}(t-a)^{k}\right)(x)=0,\ k=0,1,\ldots,n-1. \tag{3.2.22}$$

Especially, the equalities are given

$$\left({}^{C}D_{a+}^{\alpha}1\right)(x)=0 \text{ and } \left({}^{C}D_{b-}^{\alpha}1\right)(x)=0. \tag{3.2.23}$$

Lemma 3.2.1: Let $\Re(\alpha)>0$ and let $y(x)\in L_{\infty}(a,b)$ or $y(x)\in C[a,b]$.

Provided that $\Re(\alpha)\notin\mathbb{N}$ or $\alpha\in\mathbb{N}_0$, in that case,

$$\left({}^{C}D_{a+}^{\alpha}I_{a+}^{\alpha}y\right)(x)=y(x) \text{ and } \left({}^{C}D_{b-}^{\alpha}I_{b-}^{\alpha}y\right)(x)=y(x). \tag{3.2.24}$$

Proof: Let $y(x)\in L_{\infty}(a,b)$ $\left(y(x)\in C[a,b]\right)$, and let $\Re(\alpha)\notin\mathbb{N}$ $n=\left[\Re(\alpha)\right]+1$ or $n\in\mathbb{N}$ and $k=0,1,\cdots,n-1$.

$$\left(I_{a+}^{\alpha}y\right)^{(k)}(x)=\left(I_{a+}^{\alpha-k}y\right)(x), \qquad k=0,1,\cdots n-1. \tag{3.2.25}$$

Seeing that $y(x)\in L_{\infty}(a,b)$, $\left(y(x)\in C[a,b]\right)$, in that case for any $x\in[a,b]$,

$$\left|\left(I_{a+}^{\alpha-k}y\right)(x)\right|\le\frac{K}{\left|\Gamma(\alpha-k)\right|\left[\Re(\alpha)-k\right]}(x-a)^{\Re(\alpha)-k} \quad \left(K=\|y\|L_{\infty}=\left(\|y\|_{C}\right)\right), \tag{3.2.26}$$

for any $k=0,1,\cdots,n-1=\left[\Re(\alpha)\right]$, such that

$$\left(I_{a+}^{\alpha}y\right)^{k}(a+)=0 \qquad (k=0,1,\cdots,n-1). \tag{3.2.27}$$

Consequently, (3.2.10) is used for $\Re(\alpha)\notin\mathbb{N}$,

$$\left({}^{C}D_{a+}^{n}y\right)(x)=\frac{d^{n}}{dx^{n}}\left[y(x)-\sum_{k=0}^{n-1}\frac{y^{(k)}(a)}{k!}(x-a)^{k}\right] \tag{3.2.28}$$

for $(n\in\mathbb{N})$ and $y(x)$ is replaced by $\left(I_{a+}^{\alpha}y\right)(x)$.

$$\left({}^{C}D_{a+}^{\alpha}I_{a+}^{\alpha}y\right)(x)=\left(D_{a+}^{\alpha}I_{a+}^{\alpha}y\right)(x)=y(x), \tag{3.2.29}$$

and so the formula (3.2.24) is obtained.

Lemma 3.2.2: In the case of $\alpha>0$, $a\in\mathbb{R}$ and $\lambda\in C$, $E_{\alpha}(z)$ is the Mittag-Leffler function [5].

$$\left[{}^{C}D_{a+}^{\alpha}E_{\alpha}\left(\lambda(t-a)^{\alpha}\right)\right](x)=\lambda E_{\alpha}\left[\lambda(x-a)^{\alpha}\right] \tag{3.2.30}$$

and

$$\left({}^{C}D_{-}^{\alpha}t^{\alpha-1}E_{\alpha}(\lambda t^{-\alpha})\right)(x)=\frac{1}{x}E_{\alpha,1-\alpha}\left(\lambda x^{-\alpha}\right). \tag{3.2.31}$$

Especially, for $\alpha=n\in\mathbb{N}$,

$$D^{n}E_{n}\left[\lambda(x-a)^{n}\right]=E_{n}\left[\lambda(x-a)^{n}\right] \tag{3.2.32}$$

and

$$D^{n}\left[t^{n-1}E_{n}\left(\lambda t^{-n}\right)\right](x)=\frac{1}{x}E_{n,1-n}\left(\lambda x^{-n}\right)=\frac{\lambda}{x^{n+1}}E_{n}\left(\lambda x^{-n}\right). \tag{3.2.33}$$

Lemma 3.2.3: Let $\alpha>0,$ $n-1<\alpha\le n$ $(n\in\mathbb{N})$ be such that $y(x)\in C^{n}(\mathbb{R}^{+})$ $y^{(n)}(x)\in L_{1}(0,b)$ for any $b>0,$ the estimate is the following form;

$$\left|y(x)\right|\le Be^{q_{0}x} \qquad (x>b>0) \tag{3.2.34}$$

the estimate holds for $y^{(n)}(x)$, the Laplace transforms $(\mathcal{L}y)(p)$ and $\mathcal{L}\left[D^{n}y(t)\right]$

exist and $\lim_{x\to+\infty}\left(D^k y\right)(x)=0$ for $k=0,1,\cdots,n-1$. Consequently the following correlation holds [5]:

$$\left(\mathcal{L}\,{}^C D_{0+}^{\alpha} y\right)(s)=s^{\alpha}\left(\mathcal{L}y\right)(s)-\sum_{k=0}^{n-1} s^{\alpha-k-1}\left(D^k y\right)(0). \tag{3.2.35}$$

Especially, in the case of $0<\alpha\le 1$,

$$\left(\mathcal{L}\,{}^C D_{0+}^{\alpha} y\right)(s)=s^{\alpha}\left(\mathcal{L}y\right)(s)-s^{\alpha-1}y(0). \tag{3.2.36}$$

CHAPTER 4

CLASSICAL AND FRACTIONAL DAVEY-STEWARTSON EQUATIONS

In this chapter, we introduce the Davey-Stewartson equations. Davey-Stewartson (DS) equations were used for various applications. Davey-Stewartson equations were proposed initially for the evolution of weakly nonlinear pockets of water waves in the finite depth by Davey and Stewartson [44]. Jafari and Alipour applied the Homotopy analysis method for solving Davey-Stewartson equations [45]. Wang and Huang used the variable separation approach for solving general (2+1) dimensional DS equations and some important results were obtained by variable separation approach [46]. Babaoglu, Eden and Erbay used generalized Davey-Stewartson equations in reference [47]. Variational iteration method is used for solving Davey-Stewartson equations by Jafari and Alipour [34]. Li used system of the degenerate Davey-Stewartson equations and the global existence of weak solutions and blow up solutions for initial condition are proved [48].

Davey-Stewartson equations are reduced to the (1+1) dimensional nonlinear Schrödinger equation. These equations are

$$iu_t + c_0 u_{xx} + u_{yy} = c_1 |u|^2 u + c_2 u \varphi_x \ , \quad (4.1)$$

$$\varphi_{xx} + c_3 \varphi_{yy} = \left(|u|^2\right)_x$$

Davey and Stewartson initially, derived the above equations (4.1) in the context of water waves [44].

A complex field $u = u(x, y, t)$ and real field $\varphi = \varphi(x, y, t)$ are involved in the Davey-Stewartson equations in two dimensions. $c_0, c_{1,} c_2$ and c_3 are real parameters [49]. For $c_3 > 0$ the field φ is based elliptically on u and so initial data is not specified for φ, but usually initial data is specified only for u.

$$i\frac{\partial u}{\partial t} + c_0 \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = c_1 |u|^2 u + c_2 u \frac{\partial \varphi}{\partial x}, \quad (4.2)$$

$$\frac{\partial^2 \varphi}{\partial x} + c_3 \frac{\partial^2 \varphi}{\partial y^2} = \frac{\partial\left(|q|^2\right)}{\partial x} = 0.$$

These equations are completely integrable for following conditions. In this case of $c_0 = -1,\ c_1 = 1,\ c_2 = -2$ and $c_3 = 1$, Davey-Stewartson equations are known as DS-I. For this case $c_0 = 1, c_1 = -1, c_2 = 2$ and $c_3 = -1$, these equations are known as DS-II. c_0 and c_3 are physical parameters that play a determining role in the classification of these equations.

These equations are categorized as elliptic-elliptic, elliptic-hyperbolic, hyperbolic-elliptic, hyperbolic-hyperbolic according to (c_0, c_3) signs. The signs are respectively (+, +), (+, -), (-, +) and (-, -) [50].

In this thesis we use the variational iteration method to obtain approximate solutions for Davey-Stewartson equations and their approximate solutions are provided. We use fractional Davey-Stewartson equations in the following form [39]:

$$\frac{1}{2}\sigma^4\frac{\partial^\alpha q}{\partial y^\alpha}+\frac{1}{2}\sigma^2\frac{\partial^2 q}{\partial x^2}+i\frac{\partial q}{\partial t}+\lambda|q|^2 q-\frac{\partial\phi}{\partial x}q=0, \qquad 1<\alpha\leq 2$$

$$\frac{\partial^2\phi}{\partial x^2}-\sigma^2\frac{\partial^\alpha\phi}{\partial y^\alpha}-2\lambda\frac{\partial(|q|^2)}{\partial x}=0\,.$$

Here, $\frac{\partial^\alpha}{\partial y^\alpha}$ is Caputo fractional derivative.

In case of $\alpha=2$ and $\sigma=1$, this condition is especial and it is known as classical Davey-Stewartson-I equation. In this case of $\alpha=2$ and $\sigma=i$, this condition is known as classical Davey-Stewartson-II equation. The parameter λ features the focusing or defocusing state [34]. The most known cases of integrable equations of two examples arise as higher dimensional generalizations of the nonlinear Schrödinger Equation for the classical Davey-Stewartson-I and II [44].

We apply the variational iteration method to approximete solutions of the fractional Davey-Stewartson equations. Moreover we obtain numerical results and these results show the nature of the surfaces/curves as the fractional derivative parameters changed.

4.1 Variational Iteration Method Applied to the Fractional Davey- Stewartson Equations

In this section the variational iteration method is applied to fractional Davey-Stewartson equations.

$$\frac{1}{2}\sigma^4\frac{\partial^\alpha q}{\partial y^\alpha}+\frac{1}{2}\sigma^2\frac{\partial^2 q}{\partial x^2}+i\frac{\partial q}{\partial t}+\lambda|q|^2 q-\frac{\partial\phi}{\partial x}q=0\ , \tag{4.1.1}$$

$$\frac{\partial^2\phi}{\partial x^2}-\sigma^2\frac{\partial^\alpha\phi}{\partial y^\alpha}-2\lambda\frac{\partial\left(|q|^2\right)}{\partial x}=0.$$

Initially we separate the amplitude of surface wave packet q in real and imaginary parts, namely, $q=u+iv$. Using the above DS equations become [39]

$$\frac{\partial^\alpha u}{\partial y^\alpha}+\frac{1}{\sigma^2}\frac{\partial^2 u}{\partial x^2}-\frac{2}{\sigma^4}\frac{\partial v}{\partial t}+\frac{2\lambda}{\sigma^4}\left(u^3+v^2u\right)-\frac{2}{\sigma^4}\left(\frac{\partial\phi}{\partial x}u\right)=0, \tag{4.1.2}$$

$$\frac{\partial^\alpha v}{\partial y^\alpha}+\frac{1}{\sigma^2}\frac{\partial^2 v}{\partial x^2}-\frac{2}{\sigma^4}\frac{\partial u}{\partial t}+\frac{2\lambda}{\sigma^4}\left(v^3+u^2v\right)-\frac{2}{\sigma^4}\left(\frac{\partial\phi}{\partial x}v\right)=0,$$

$$\frac{\partial^\alpha\phi}{\partial y^\alpha}-\frac{1}{\sigma^2}\frac{\partial^2\phi}{\partial x^2}+\frac{2\lambda}{\sigma^2}\frac{\partial\left(u^2+v^2\right)}{\partial x}=0.$$

Using the variational iteration method, the correction functional can be taken as [39]:

$$u_{n+1}(x,y,t)=u_n(x,y,t)+I_y^\beta\left[\lambda_1\left(\frac{\partial^\alpha u_n(x,y,t)}{\partial y^\alpha}+\frac{1}{\sigma^2}\frac{\partial^2 u_n(x,y,t)}{\partial x^2}-\frac{2}{\sigma^4}\frac{\partial v_n(x,y,t)}{\partial t}\right.\right.$$

$$\left.\left.+\frac{2\lambda}{\sigma^4}\left(u_n(x,y,t)^3+v_n(x,y,t)^2u_n(x,y,t)\right)-\frac{2}{\sigma^4}\left(\frac{\partial\phi_n(x,y,t)}{\partial x}u_n(x,y,t)\right)\right)\right]$$

$$=u_n(x,y,t)+\frac{1}{\Gamma(\beta)}\int_0^y(y-\zeta)^{(\beta-1)}\lambda_1(\zeta)\left[\frac{\partial^\alpha u_n(x,\zeta,t)}{\partial\zeta^\alpha}+\frac{1}{\sigma^2}\frac{\partial^2 u_n(x,\zeta,t)}{\partial x^2}\right.$$

$$-\frac{2}{\sigma^4}\frac{\partial v_n(x,\zeta,t)}{\partial t}+\frac{2\lambda}{\sigma^4}\left(u_n(x,\zeta,t)^3+v_n(x,\zeta,t)^2u_n(x,\zeta,t)\right)$$
$$-\frac{2}{\sigma^4}\left(\frac{\partial\phi_n(x,\zeta,t)}{\partial x}u_n(x,\zeta,t)\right)\Bigg], \tag{4.1.3}$$

$$v_{n+1}(x,y,t)=v_n(x,y,t)+I_y^{\beta}\left[\lambda_2\left(\frac{\partial^{\alpha}v_n(x,y,t)}{\partial y^{\alpha}}+\frac{1}{\sigma^2}\frac{\partial^2 v_n(x,y,t)}{\partial x^2}+\frac{2}{\sigma^4}\frac{\partial u_n(x,y,t)}{\partial t}\right.\right.$$
$$\left.\left.+\frac{2\lambda}{\sigma^4}\left(v_n(x,y,t)^3+u_n(x,y,t)^2v_n(x,y,t)\right)-\frac{2}{\sigma^4}\left(\frac{\partial\phi_n(x,y,t)}{\partial x}v_n(x,y,t)\right)\right)\right]$$
$$=v_n\left(x,y,t\right)+\frac{1}{\Gamma(\beta)}\int_0^y\left(y-\zeta\right)^{(\beta-1)}\lambda_2(\zeta)\left[\frac{\partial^{\alpha}v_n\left(x,\zeta,t\right)}{\partial\zeta^{\alpha}}+\frac{1}{\sigma^2}\frac{\partial^2 v_n\left(x,\zeta,t\right)}{\partial x^2}\right.$$
$$+\frac{2}{\sigma^4}\frac{\partial u_n(x,\zeta,t)}{\partial t}+\frac{2\lambda}{\sigma^4}\left(v_n(x,\zeta,t)^3+u_n(x,\zeta,t)^2v_n(x,\zeta,t)\right)$$
$$\left.-\frac{2}{\sigma^4}\left(\frac{\partial\phi_n(x,\zeta,t)}{\partial x}v_n(x,\zeta,t)\right)\right], \tag{4.1.4}$$

$$\phi_{n+1}(x,y,t)=\phi_n(x,y,t)+I_y^{\beta}\left[\lambda_3\left(\frac{\partial^{\alpha}\phi_n(x,y,t)}{\partial y^{\alpha}}-\frac{1}{\sigma^2}\frac{\partial^2\phi_n(x,y,t)}{\partial x^2}\right.\right.$$
$$\left.\left.+\frac{2\lambda}{\sigma^2}\frac{\partial}{\partial x}\left(v_n(x,y,t)^2+u_n(x,y,t)^2\right)\right)\right]$$
$$=\phi_n\left(x,y,t\right)+\frac{1}{\Gamma(\beta)}\int_0^y\left(y-\zeta\right)^{(\beta-1)}\lambda_3(\zeta)\left[\frac{\partial^{\alpha}\phi_n\left(x,\zeta,t\right)}{\partial\zeta^{\alpha}}-\frac{1}{\sigma^2}\frac{\partial^2\phi_n\left(x,\zeta,t\right)}{\partial x^2}\right.$$
$$\left.+\frac{2\lambda}{\sigma^2}\frac{\partial}{\partial x}\left(v_n(x,\zeta,t)^2+u_n(x,\zeta,t)^2\right)\right] \tag{4.1.5}$$

Here I_y^{β} is the Riemann-Liouville fractional integral operator of order $\beta=\alpha-\text{floor}(\alpha)$ which is $\beta=\alpha-1$ $\left(\text{see [31]}\right)$ with respect to the variable y

and λ_i , $i = 1,2,3$ the general Lagrange multipliers and $\Gamma(\alpha)$ is the Gamma function [39]. Some approximation must be made for determining approximately Lagrange multipliers.

We can write the correction functional (4.1.3), (4.1.4) and (4.1.5) that can be approximately denotes

$$u_{n+1}(x,y,t) = u_n(x,y,t) + \int_0^y \lambda_1(\zeta)\left[\frac{\partial^2 u_n(x,\zeta,t)}{\partial \zeta^2} + \frac{1}{\sigma^2}\frac{\partial^2 \tilde{u}_n(x,\zeta,t)}{\partial x^2}\right. \tag{4.1.6}$$

$$-\frac{2}{\sigma^4}\frac{\partial \tilde{v}_n(x,\zeta,t)}{\partial t} + \frac{2\lambda}{\sigma^4}\left(\tilde{u}_n(x,\zeta,t)^3 + \tilde{v}_n(x,\zeta,t)^2\tilde{u}_n(x,\zeta,t)\right)$$

$$\left. -\frac{2}{\sigma^4}\left(\frac{\partial \tilde{\phi}_n(x,\zeta,t)}{\partial x}\tilde{u}_n(x,\zeta,t)\right)\right],$$

$$v_{n+1}(x,y,t) = v_n(x,y,t) + \int_0^y \lambda_2(\zeta)\left[\frac{\partial^2 v_n(x,\zeta,t)}{\partial \zeta^2} + \frac{1}{\sigma^2}\frac{\partial^2 \tilde{v}_n(x,\zeta,t)}{\partial x^2}\right. \tag{4.1.7}$$

$$+\frac{2}{\sigma^4}\frac{\partial \tilde{u}_n(x,\zeta,t)}{\partial t} + \frac{2\lambda}{\sigma^4}\left(\tilde{v}_n(x,\zeta,t)^3 + \tilde{u}_n(x,\zeta,t)^2\tilde{v}_n(x,\zeta,t)\right)$$

$$\left. -\frac{2}{\sigma^4}\left(\frac{\partial \tilde{\phi}_n(x,\zeta,t)}{\partial x}\tilde{v}_n(x,\zeta,t)\right)\right],$$

$$\phi_{n+1}(x,y,t) = \phi_n(x,y,t) + \int_0^y \lambda_3(\zeta)\left[\frac{\partial^2 \phi_n(x,\zeta,t)}{\partial \zeta^2} - \frac{1}{\sigma^2}\frac{\partial \tilde{\phi}_n(x,\zeta,t)}{\partial x^2}\right. \tag{4.1.8}$$

$$\left. +\frac{2\lambda}{\sigma^2}\frac{\partial}{\partial x}\left(\tilde{v}_n(x,\zeta,t)^2 + \tilde{u}_n(x,\zeta,t)^2\right)\right].$$

Then $\tilde{u}_n$, $\tilde{v}_n$ and $\tilde{\phi}_n$ are considered as restricted variations, where $\delta\tilde{u}_n = \delta\tilde{v}_n = \delta\tilde{\phi}_n = 0$ in these functions.

Both sides of the equations (4.1.6), (4.1.7) and (4.1.8) are multiplied by restricted variations then these equations are equalized to zero. For finding the optimal λ_1, λ_2 and λ_3 we continue [39]:

$$\delta u_{n+1}(x,y,t)=\delta u_n(x,y,t)+\delta\int_0^y \lambda_1(\zeta)\left[\frac{\partial^2 u_n(x,\zeta,t)}{\partial\zeta^2}+\frac{1}{\sigma^2}\frac{\partial^2\tilde{u}_n(x,\zeta,t)}{\partial x^2}\right.$$
$$-\frac{2}{\sigma^4}\frac{\partial\tilde{v}_n(x,\zeta,t)}{\partial t}+\frac{2\lambda}{\sigma^4}\left(\tilde{u}_n(x,\zeta,t)^3+\tilde{v}_n(x,\zeta,t)^2\tilde{u}_n(x,\zeta,t)\right)$$
$$\left.-\frac{2}{\sigma^4}\left(\frac{\partial\tilde{\phi}_n(x,\zeta,t)}{\partial x}\tilde{u}_n(x,\zeta,t)\right)\right]=0,$$

$$\delta v_{n+1}(x,y,t)=\delta v_n(x,y,t)+\delta\int_0^y \lambda_2(\zeta)\left[\frac{\partial^2 v_n(x,\zeta,t)}{\partial\zeta^2}+\frac{1}{\sigma^2}\frac{\partial^2\tilde{v}_n(x,\zeta,t)}{\partial x^2}\right.$$
$$+\frac{2}{\sigma^4}\frac{\partial\tilde{u}_n(x,\zeta,t)}{\partial t}+\frac{2\lambda}{\sigma^4}\left(\tilde{v}_n(x,\zeta,t)^3+\tilde{u}_n(x,\zeta,t)^2\tilde{v}_n(x,\zeta,t)\right)$$
$$\left.-\frac{2}{\sigma^4}\left(\frac{\partial\tilde{\phi}_n(x,\zeta,t)}{\partial x}\tilde{v}_n(x,\zeta,t)\right)\right]=0,$$

$$\delta\phi_{n+1}(x,y,t)=\delta\phi_n(x,y,t)+\delta\int_0^y \lambda_3(\zeta)\left[\frac{\partial^2\phi_n(x,\zeta,t)}{\partial\zeta^2}-\frac{1}{\sigma^2}\frac{\partial^2\tilde{\phi}_n(x,\zeta,t)}{\partial x^2}\right.$$
$$\left.+\frac{2\lambda}{\sigma^4}\frac{\partial}{\partial x}\left(\tilde{v}_n(x,\zeta,t)^2+\tilde{u}_n(x,\zeta,t)^2\right)\right]=0.$$

We obtain the stationary conditions for $i=1,\ 2,\ 3$ as:

$\lambda_i''(\zeta)\big|_{\zeta=y}=0$,

$1-\lambda_i'(\zeta)\big|_{\zeta=y}=0$,

$\lambda_i(\zeta)\big|_{\zeta=y}=0 \qquad\qquad \Leftrightarrow \qquad \lambda_i(\zeta)=\zeta-y.$

Instead of inserting $\lambda_i(\zeta)=\zeta-y$, ($i=1,\ 2,\ 3$) into the functional of equations (4.1.3), (4.1.4) and (4.1.5), the iteration formulas are obtained [39]:

$$u_{n+1}(x,y,t)=u_n(x,y,t)-(\alpha-1)\boldsymbol{I}_y^{\alpha}\left[\frac{\partial^{\alpha}u_n(x,y,t)}{\partial y^{\alpha}}+\frac{1}{\sigma^2}\frac{\partial^2 u_n(x,y,t)}{\partial x^2}-\frac{2}{\sigma^4}\frac{\partial v_n(x,y,t)}{\partial t}\right.$$
$$\left.+\frac{2\lambda}{\sigma^4}\left(u_n(x,y,t)^3+v_n(x,y,t)^2u_n(x,y,t)\right)-\frac{2}{\sigma^4}\left(\frac{\partial\phi_n(x,y,t)}{\partial x}u_n(x,y,t)\right)\right],\quad (4.1.9)$$

$$v_{n+1}(x,y,t)=v_n(x,y,t)-(\alpha-1)\boldsymbol{I}_y^{\alpha}\left[\frac{\partial^{\alpha}v_n(x,y,t)}{\partial y^{\alpha}}+\frac{1}{\sigma^2}\frac{\partial^2 v_n(x,y,t)}{\partial x^2}+\frac{2}{\sigma^4}\frac{\partial u_n(x,y,t)}{\partial t}\right.$$
$$\left.+\frac{2\lambda}{\sigma^4}\left(v_n(x,y,t)^3+u_n(x,y,t)^2v_n(x,y,t)\right)-\frac{2}{\sigma^4}\left(\frac{\partial\phi_n(x,y,t)}{\partial x}v_n(x,y,t)\right)\right],$$

(4.1.10)

$$\phi_{n+1}(x,y,t)=\phi_n(x,y,t)-(\alpha-1)\boldsymbol{I}_y^{\alpha}\left[\frac{\partial^{\alpha}\phi_n(x,y,t)}{\partial y^{\alpha}}-\frac{1}{\sigma^2}\frac{\partial^2\phi_n(x,y,t)}{\partial x^2}\right.$$
$$\left.+\frac{2\lambda}{\sigma^2}\frac{\partial}{\partial x}\left(v_n(x,y,t)^2+u_n(x,y,t)^2\right)\right].\quad (4.1.11)$$

These are the necessary equations for numerical results. In addition, we freely chose for n =0 the initial approximations $u_0(x,y,t)$, $v_0(x,y,t)$ and $\phi_0(x,y,t)$. We approximate the solutions $u(x,y,t)=\lim_{n\to\infty}u_n(x,y,t)$, $v(x,y,t)=\lim_{n\to\infty}v_n(x,y,t)$ and $\phi(x,y,t)=\lim_{n\to\infty}\phi_n(x,y,t)$ through N^{th} terms $x_{\mathrm{N}}(x,y,t)$, $y_{\mathrm{N}}(x,y,t)$ and $\phi_{\mathrm{N}}(x,y,t)$ [39].

4.2 Numerical Results

We produce the numerical results to support our study in this part and the following initial conditions are considered [39]:

$$u(x,0,t) = r\sec h\left[s(x-ct)\right]\cos\left[(k_1x+k_3t)\right],$$

$$v(x,0,t) = r\sec h\left[s(x-ct)\right]\sin\left[(k_1x+k_3t)\right], \qquad (4.2.1)$$

$$\phi(x,0,t) = f\ \tanh\left[s(x-ct)\right],$$

Here we use these parematers,

$c = k_2 + \sigma^2 k_1$, $r = \sqrt{-(2k_3 + k_1^2\sigma^2 + k_2^2)/\lambda}$, $s = \sqrt{(2k_3 + k_1^2\sigma^2 + k_2^2)/\sigma^2}$,

$f = (2\sigma\sqrt{-\lambda})/(1-\sigma^2)$ and k_i $(i = 1;\ 2;3)$ are arbitrary constants.

The exact solutions, for the special case $\alpha = 2$ is given (see [45]):

$$u(x,y,t) = r\sec h[s(x+y-ct)]\cos[(k_1x+k_2y+k_3t)],$$

$$v(x,y,t) = r\sec h[s(x+y-ct)]\sin[(k_1x+k_2y+k_3t)]\ , \qquad (4.2.2)$$

$$\phi(x,y,t) = f\ \tanh[s(x+y-ct)].$$

They are initial equations,

$$u_0(x,y,t) = r\sec h[s(x-ct)]\cos[(k_1x+k_3t)],$$

$$v_0(x,y,t) = r\sec h[s(x-ct)]\sin[(k_1x+k_3t)],$$

$$\phi_0(x,y,t) = f\ \tanh[s(x-ct)].$$

According to variational iteration method (4.1.9), (4.1.10), (4.1.11), we use initial equations to obtain the first approximations for u, v and ϕ as follows:

$$u_1 = r\cos\left[xk_1+tk_3\right]\sec h\left[s(-ct+x)\right]+\frac{2frsy^{\alpha}\sec h\left[s(-ct+x)\right]^3\cos\left[xk_1+tk_3\right]}{\sigma^4\Gamma(\alpha)}$$

$$-\frac{2frsy^{\alpha}\sec h\left[s(-ct+x)\right]^3\cos\left[xk_1+tk_3\right]}{\sigma^4\Gamma(\alpha+1)}+\frac{rs^2y^{\alpha}\cos\left[k_1x+k_3t\right]\sec h\left[s(-ct+x)\right]^3}{\sigma^2\Gamma(\alpha)}$$

$$-\frac{rs^2y^{\alpha}\cos\left[xk_1+tk_3\right]\sec h\left[s(-ct+x)\right]^3}{\sigma^2\Gamma(\alpha+1)}-\frac{2r^3y^{\alpha}\lambda\cos\left[xk_1+tk_3\right]^3\sec h\left[s(-ct+x)\right]^3}{\sigma^4\Gamma(\alpha)}$$

$$+\cdots\cdots],$$

$$v_1 = r\sin\left[xk_1+tk_3\right]\sec h\left[s(-ct+x)\right]+\frac{2frsy^{\alpha}\sec h\left[s(-ct+x)\right]^3\sin\left[xk_1+tk_3\right]}{\sigma^4\Gamma(\alpha)}$$

$$-\frac{2frsy^{\alpha}\sec h\left[s(-ct+x)\right]^3\sin\left[xk_1+tk_3\right]}{\sigma^4\Gamma(\alpha+1)}+\frac{rs^2y^{\alpha}\sin\left[k_1x+k_3t\right]\sec h\left[s(-ct+x)\right]^3}{\sigma^2\Gamma(\alpha)}$$

$$-\frac{rs^2y^{\alpha}\sin\left[xk_1+tk_3\right]\sec h\left[s(-ct+x)\right]^3}{\sigma^2\Gamma(\alpha+1)}-\frac{2r^3y^{\alpha}\lambda\sin\left[xk_1+tk_3\right]^3\sec h\left[s(-ct+x)\right]^3}{\sigma^4\Gamma(\alpha)}$$

$$+\cdots\cdots]$$

and

$$\phi_1 = f\tanh\left[s(-ct+x)\right]-\frac{2fs^2y^{\alpha}\sec h\left[s(-ct+x)\right]^2\tanh\left[s(-ct+x)\right]}{\sigma^2\Gamma(\alpha)}$$

$$+\frac{2fs^2y^{\alpha}\tanh\left[s(-ct+x)\right]\sec h\left[s(-ct+x)\right]^2}{\sigma^2\Gamma(\alpha+1)}$$

$$+\frac{4r^2sy^{\alpha}\lambda\sec h\left[s(-ct+x)\right]^2\tanh\left[s(-ct+x)\right]}{\sigma^2\Gamma(\alpha)}+\cdots\cdots]$$

Similarly, we may apply this iteration procedure to obtain the sequences $\left\{u_n\left(x,y,t\right)\right\}_{n=0}^{\infty}$, $\left\{v_n\left(x,y,t\right)\right\}_{n=0}^{\infty}$, $\left\{\phi_n\left(x,y,t\right)\right\}_{n=0}^{\infty}$ which are convergent to

$u(x, y, t)$, $v(x, y, t)$ and $\phi(x, y, t)$ respectively.

In addition we obtained the absolute errors between the approximate solutions for value of $\alpha = 1.98$ using the variational iteration method and the exact solutions and Tables 1-3 show these absolute errors.

Then we obtained Figures 1-3 and these Figures show the approximate solutions (4.1.2) obtained for values of $\alpha = 1.98$ and $\alpha = 1.8$ via the variational iteration method and exact solutions. Figures 1-3 show that the solutions obtained via the existing method is almost identical with the exact solutions.

Finally, here it should be indicated that only the two-order term of the variational iteration solution for the especial condition $y = 0.2$, $k_1 = 0.1$, $k_2 = 0.03$, $k_3 = -0.3$, $\sigma = I$, $\lambda = 1$ used in evaluating the approximate solutions and the results are presented on Tables 1-3 and Figures. 1-3 (see[39]).

Table 1: Absolute Errors of $u(x,y,t)$ Equation

$\left|u(x,y,t)-u_n(x,y,t)\right|$ for values of $0.1 \le t \le 0.5$ and $7 \le x \le 20$.
$\left\{u(x,y,t)\right.$ is exact solution and $u_n(x,y,t)$ is approximate solution$\left.\right\}$

x / t	0.1	0.2	0.3	0.4	0.5
20	1.28402×10^{-8}	1.17181×10^{-8}	1.05973×10^{-8}	9.47895×10^{-9}	8.36397×10^{-9}
17	1.97597×10^{-8}	8.07466×10^{-9}	3.49088×10^{-9}	1.49275×10^{-8}	2.6226×10^{-8}
14	9.95054×10^{-7}	1.10588×10^{-6}	1.21449×10^{-6}	1.32079×10^{-6}	1.42471×10^{-6}
11	2.18989×10^{-5}	2.28375×10^{-5}	2.37448×10^{-5}	2.46202×10^{-5}	2.54634×10^{-5}
7	7.73714×10^{-4}	7.8569×10^{-4}	7.96811×10^{-4}	8.07073×10^{-4}	8.16478×10^{-4}

Table 2: Absolute Errors of $v(x,y,t)$ Equation

$\left|v(x,y,t)-v_n(x,y,t)\right|$ for values of $0.1 \le t \le 0.5$ and $7 \le x \le 20$.
$\left\{v(x,y,t)\right.$ is exact solution and $\mathrm{v}_n(x,y,t)$ is approximate solution$\left.\right\}$

x / t	0.1	0.2	0.3	0.4	0.5
20	4.73679×10^{-8}	4.75531×10^{-8}	4.76923×10^{-8}	4.77859×10^{-8}	4.78342×10^{-8}
17	5.18236×10^{-7}	5.15453×10^{-7}	5.12222×10^{-7}	5.08554×10^{-7}	5.04455×10^{-7}
14	5.17454×10^{-6}	5.09924×10^{-6}	5.02004×10^{-6}	4.93706×10^{-6}	4.85041×10^{-6}
11	4.67751×10^{-5}	4.55803×10^{-5}	4.43562×10^{-5}	4.31045×10^{-5}	4.18265×10^{-5}
7	7.05161×10^{-4}	6.69475×10^{-4}	6.33556×10^{-4}	5.97442×10^{-4}	5.61168×10^{-4}

Table 3: Absolute Errors of $\phi(x, y, t)$ Equation

$\left|\phi(x, y, t)-\phi_n(x, y, t)\right|$ for value of $0.1 \le t \le 0.5$ and $7 \le x \le 20$.

$\{\phi(x, y, t)$ is exact solution and $\phi_n(x, y, t)$ is approximate solution$\}$

x/t	0.1	0.2	0.3	0.4	0.5
20	9.992×10^{-16}	1.1102×10^{-15}	1.1102×10^{-4}	9.992×10^{-16}	9.992×10^{-16}
17	1.169×10^{-13}	1.157×10^{-13}	1.1457×10^{-13}	1.1324×10^{-13}	1.1224×10^{-13}
14	1.2696×10^{-11}	1.256×10^{-11}	1.2439×10^{-11}	1.231×10^{-11}	1.219×10^{-11}
11	1.3837×10^{-9}	1.370×10^{-9}	1.3565×10^{-9}	1.3431×10^{-9}	1.3299×10^{-9}
7	7.2190×10^{-7}	6.6947×10^{-4}	7.07465×10^{-7}	7.0031×10^{-7}	6.9321×10^{-7}

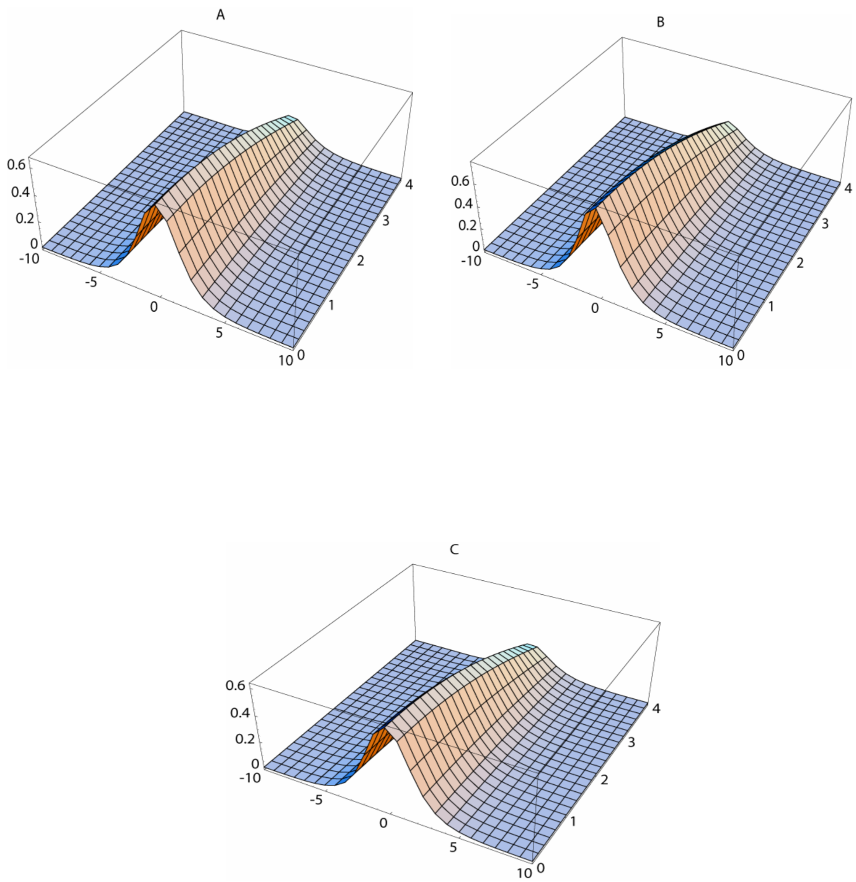

Figure 1: The Surface Shows the Solution $u(x, y, t)$ for Equation (4.1.2)

Here A shows approximate solution when $\alpha = 1.98$; B is an exact solution; C is an approximate solution for $\alpha = 1.8$.

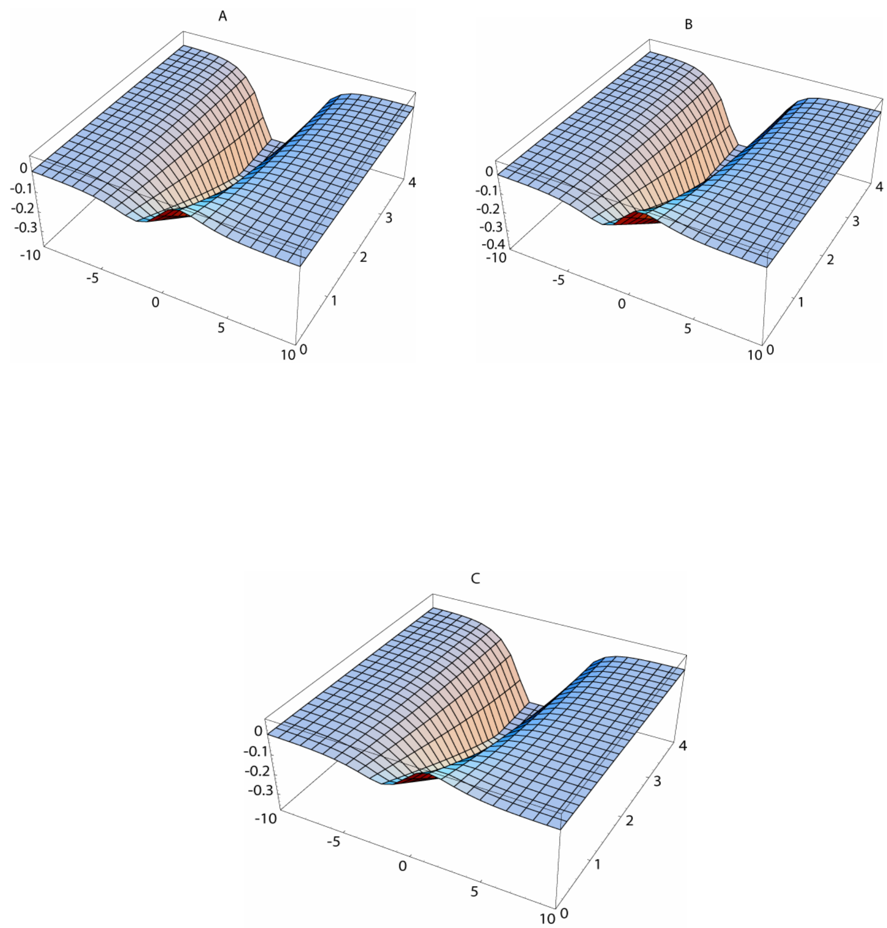

Figure 2: The Surface Shows the Solution $v(x, y, t)$ for Equation (4.1.2)

Here A is an approximate solution for $\alpha = 1.98$; B is an exact solution; C shows an approximate solution when $\alpha = 1.8$.

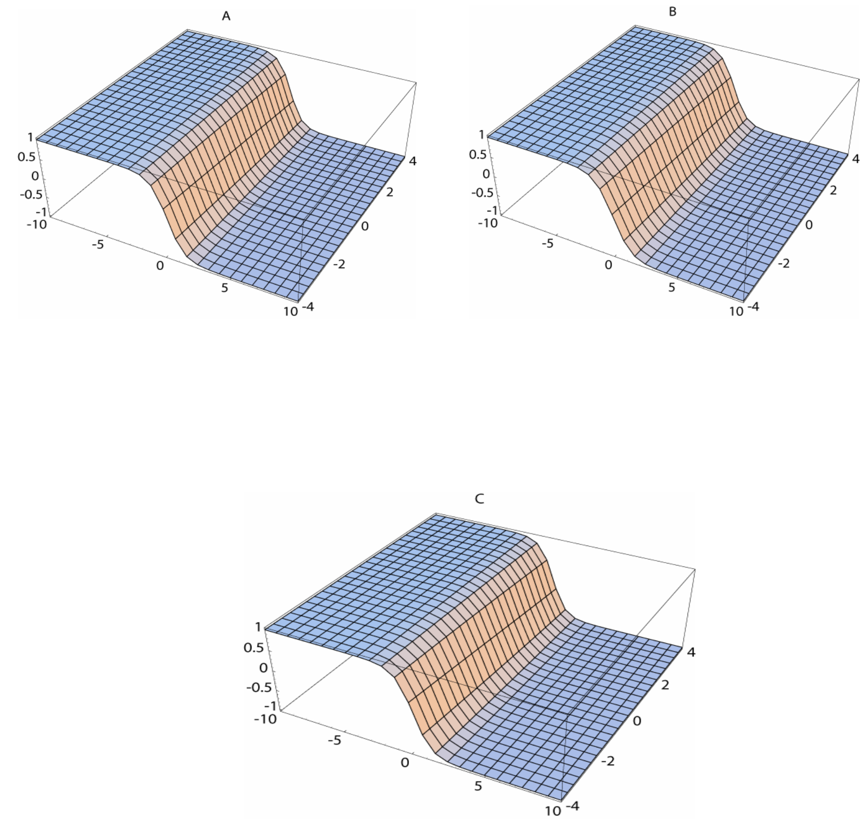

Figure 3: The Surface Shows the Solution $\phi(x, y, t)$ for Equation (4.1.2)

Here A represents an approximate solution for $\alpha = 1.98$; B is an exact solution; C shows an approximate solution when $\alpha = 1.8$.

CHAPTER 5

CONCLUSIONS

The use of the fractional differential equations in several areas of science and engineering requires new methods and techniques in order to solve these types of equations. Among several methods proposed during the last decades to solve this issue we have chosen the Variational iteration method which has been used previously in different areas and which represents a powerful method suitable for handling both linear and nonlinear fractional differential equations.

In this study we have concentrated on the numerical study of the fractional Davey-Stewartson differential equations within Caputo's derivative by applying the variational iteration method. We have fractionalized the classical Davey-Stewartson equations within Caputo derivative and we solve the obtained equations numerically. We have obtained the approximate solution and the exact solution for $u(x,y,t)$, $y(x,y,t)$ and $\phi(x,y,t)$ respectively. The obtained figures show the approximate solution for $\alpha = 1.98$ and $\alpha = 1.8$ and the exact solution. It was observed that these figures are close to each other.

When the results obtained via the variational iteration method have been compared with exact solutions, these results show that the proposed approach is a promising tool for solving many nonlinear and linear fractional differential equations.

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CURRICULUM VITAE

PERSONAL INFORMATION

Surname, Name: Yılmaz, Tuğba
Nationality: Turkish (TC)
Date and Place of Birth: 10.03.1987, Ankara
Marital Status: Single
Phone: +09 537 441 32 03
Email: tugba87yilmaz@yahoo.com.tr

EDUCATION

Degree	Institution	Year of Graduation
BS	Ankara Universty Astronomy and Space Sciences	2009
High School	Ankara Cumhuriyet High School	2005

FOREIGN LANGUAGES

Upper Intermediate English