MATHEMATICAL COMMUNICATIONS
Math. Commun. **16**(2011), 325–334.

Fixed points of generalized contraction mappings in cone metric spaces

Duran Turkoglu[1,*], Muhib Abuloha[2] and Thabet Abdeljawad[3]

[1] *Department of Mathematics, Faculty of Science and Arts, University of Amasya, TR-05 100, Amasya, Turkey*
[2] *Department of Mathematics, Institute of Science and Technology, Gazi University, TR-06 500, Ankara, Turkey*
[3] *Department of Mathematics and Computer Science, Çankaya University, TR-06 530 Ankara, Turkey*

Received December 1, 2008; accepted October 21, 2010

Abstract. In this paper, we proved a fixed point theorem and a common fixed point theorem in cone metric spaces for generalized contraction mappings where some of the main results of Ćirić in [8, 27] are recovered.

AMS subject classifications: 54H25, 47H10

Key words: Fixed point, generalized contraction mappings, cone metric space, minihedral cone, strongly minihedral cone, common fixed points

1. Introduction

Cone metric spaces were introduced by Huang and Zhang in [14]. The authors described there convergence in cone metric spaces and introduced completeness. Then they proved some fixed point theorems of contractive mappings on cone metric spaces. Some definitions and topological concepts were generalized by Turkoglu and Abuloha in [33] and they proved there that every cone metric space is a topological space. They also generalized the concept of diametrically contractive mappings and proved some fixed point theorems in cone metric spaces. In [1, 15, 34], some common fixed point theorems were proved for maps on cone metric spaces. In [24], some definitions were generalized in cone metric spaces such as c-nonexpansive and (c, λ) uniformly locally contractive functions $f-$closure, $c-$ isometric and some fixed point theorems were proved there. In [2], the authors proved some fixed point theorems in cone metric spaces which generalized those in [14]. In [16], the authors defined the quasi-contraction on cone metric spaces and they proved some fixed point theorems. In [35], the concept of set-valued contractions in cone metric spaces were introduced. In [27], the author gave some results about characterization of best approximations in cone metric spaces. For more recent fixed point theorems in cone metric spaces we refer to (see [5, 6, 13, 18, 23, 29, 30, 36, 37]).

*Corresponding author. *Email addresses:* `dturkoglu@gazi.edu.tr` (D. Turkoglu), `muhib2000@yahoo.com` (M. Abuloha), `thabet@cankaya.edu.tr` (T. Abdeljawad)

Generalized contraction mappings, introduced in [7], are of great importance in fixed point theory. After that and in the last decade, in [12], J. Gornicki, B. E. Rhoades used generalized contraction mappings to obtain common fixed point theorems. Further, this class of generalized contraction mappings was later studied by many authors (see [3, 4, 9, 19, 20, 21, 22, 25, 26, 29, 31, 32]).

In this paper, we proved a fixed point theorem and a common fixed point theorem in cone metric spaces for generalized contraction mappings where some of the main results of Ćirić in [7, 8] are recovered.

2. Preliminaries

Let E be a real Banach space and P a subset of E. Then, P is called a cone if and only if

P1) P is closed, non-empty and $P \neq \{0\}$,

P2) $a, b \in R \ \ a, b \geq 0$; $x, y \in P \Rightarrow ax + by \in P$,

P3) $x \in P$ and $-x \in P \Rightarrow x = 0$.

Given a cone $P \subset E$, we define a partial ordering $\leq$ with respect to P by $x \leq y$ if and only if $y - x \in P$, We write $x < y$ to indicate that $x \leq y$ but $x \neq y$, while $x << y$ will stand for $y - x \in IntP$ ($IntP \cong$ interior of P).

The cone P is called normal if there is a number K, such that for all $x, y \in E$, $0 \leq x \leq y \Rightarrow \| x \| \leq K \| y \|$, where K is called the normal constant of P.

The cone P is called regular if every increasing sequence which is bounded from above is convergent. That is if $\{x_n\}$ is a sequence such that $x_1 \leq x_2 \leq ... \leq x_n \leq y$ for some $y \in E$, then there is $x \in E$ such that $\| x_n - x \| \to 0$ as $n \to \infty$. Equivalently, the cone P is called regular if every decreasing sequence which is bounded from below is convergent [14]. P is called a minihedral cone if $\sup\{x, y\}$ exists for all $x, y \in E$, and strongly minihedral if every subset of E which is bounded from above has a supremum and hence any subset of E which is bounded from below has an infimum [11]. Throughout this article we assume that the cone P is normal with constant K and P is a cone in E with $intP \neq \varnothing$ and $\leq$ is a partial ordering with respect to P.

Definition 1 (See [14])**.** *A cone metric space is an ordered pair (X, d), where X is any set and $d : X \times X \to E$ is a mapping satisfying:*

d1) $0 < d(x, y)$ *for all* $x, y \in X$, *and* $d(x, y) = 0$ *if and only if* $x = y$,

d2) $d(x, y) = d(y, x)$ *for all* $x, y \in X$,

d3) $d(x, y) \leq d(x, z) + d(z, y)$ *for all* $x, y, z \in X$.

Definition 2 (See [14])**.** *Let (X, d) be a cone metric space, $\{x_n\}$ a sequence in X and $x \in X$. If for any $c \in E$ with $c \gg 0$, there is N such that for all $n > N$, $d(x_n, x) \ll c$, then $\{x_n\}$ is said to be convergent and $\{x_n\}$ converge to x. (i.e. $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$).*

Definition 3 (See [14]). *Let (X, d) be a cone metric space, $\{x_n\}$ a sequence in X, if for any $c \in E$ with $c \gg 0$, there is N such that for all $n, m > N$, $d(x_m, x_n) \ll c$ then $\{x_n\}$ is called a Cauchy sequence in X.*

Lemma 1 (See [14]). *Let (X, d) be a cone metric space, P a normal cone with a normal constant K. Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X and $y_n \to y$, $x_n \to x$ as $(n \to \infty)$, then $d(x_n, y_n) \to d(x, y)$ as $n \to \infty$.*

Lemma 2 (See [14]). *Let (X, d) be a cone metric space, P a normal cone with a normal constant K. Let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ converge to x if and only if $d(x_n, x) \to 0$ as $n \to \infty$.*

Lemma 3 (See [14]). *Let (X, d) be a cone metric space, P a normal cone with a normal constant K. Let $\{x_n\}$ be a sequence in X. Then $\{x_n\}$ is a Cauchy sequence if and only if $d(x_n, x_m) \to 0$ as $m, n \to \infty$.*

If our cone is strongly minihedral, then we can define continuous functions.

Definition 4 (See [33]). *A map $T : (X, d) \to (X, d)$ is called continuous at $x \in X$, if for each $V \in \tau_c$ containing Tx, there exists $U \in \tau_c$ containing x such that $T(U) \subset V$. If T is continuous at each $x \in X$, then it is called continuous, where the metric topology τ_c is*

$$\tau_c = \{U \subset X : \forall x \in U, \exists B \in \beta, x \in B \subset U\},$$
$$\beta = \{B(x, c) : x \in X, c \gg 0\}, B(x, c) = \{y \in X : d(x, y) \ll c\}.$$

In [33], it was proved that T is continuous if and only if it is sequentially continuous. That is, the condition $x_n \in (X, d)$, $x_n \to x \in X$ implies that $Tx_n \to Tx$ in (X, d).

Lemma 4 (See [28]). *There is not normal cone with a normal constant $K < 1$.*

The reader can refer to the proof of this lemma and its related example in [16].

Example 1 (See[29]). *Let $X = \{a_1, a_2, ...\}$ be a countable set of distinct points, $E = (l^2, \|\|_2)$ and $P = \left\{\{x_n\}_{n\geq 1} \in l^2 : x_n \geq 0\, (\forall n \geq 1)\right\}$. Put $x_i = \left\{\frac{3^i}{n}\right\}_{n\geq 1}$ for all $i \geq 1$ and note that $x_i \in l^2\, (i \geq 1)$. Define the map $d : X \times X \to P$ by*

$$d(a_i, a_j) = |x_i - x_j| = \left\{\frac{|3^i - 3^j|}{n}\right\}_{n\geq 1}$$

It is easy to see that (X, d) is a cone metric space, the normal constant of P is $M = 1$ and there is no Cauchy sequence in (X, d). Hence (X, d) is a complete cone metric space.

3. Fixed point of generalized contraction mappings

For $x_1, x_2 \in X$ the scalar distant $d_c(x_1, x_2)$ between x_1 and x_2 is defined by $d_c(x_1, x_2) = \|d(x_1, x_2)\|$.

Theorem 1. *Let (X, d) be a complete cone metric space with a normal constant $K \geq 1$ and $T : X \to X$ a selfmapping on X such that for each $x, y \in X$:*

$$d_c(Tx, Ty) \leq \alpha(x, y) d_c(x, y) + \beta(x, y) d_c(x, Tx) + \gamma(x, y) d_c(y, Ty) + \delta(x, y)[d_c(x, Ty) + d_c(y, Tx)], \quad (1)$$

where $\alpha, \beta, \gamma, \delta$ are functions from $X \times X$ into $[0, 1)$ such that

$$\lambda = \sup\{\alpha(x, y) + \beta(x, y) + \gamma(x, y) + 2K\delta(x, y) : x, y \in X\} < 1, \quad (2)$$

then

(i) T has a unique fixed point, say $u \in X$,

(ii) $T^n x \to u$ as $n \to \infty$, for each $x \in X$,

(iii) $d_c(T^n x, u) \leq \frac{\lambda^n}{1-\lambda} d_c(x, Tx)$.

Proof. Fix $x \in X$. Let $\{x_n\}$ be defined by $x_0 = x$, $x_1 = Tx_0$, $x_2 = Tx_1$, ... $x_{n+1} = Tx_n, \ldots$. From (1),

$$d_c(x_n, x_{n+1}) = d_c(Tx_{n-1}, Tx_n) \leq \alpha d_c(x_{n-1}, x_n) + \beta d_c(x_{n-1}, x_n) + \gamma d_c(x_n, x_{n+1}) + \delta[d_c(x_{n-1}, x_{n+1}) + d_c(x_n, x_n)].$$

Or

$$d_c(x_n, x_{n+1}) = d_c(Tx_{n-1}, Tx_n) \leq \alpha d_c(x_{n-1}, x_n) + \beta d_c(x_{n-1}, x_n) + \gamma d_c(x_n, x_{n+1}) + \delta d_c(x_{n-1}, x_{n+1}), \quad (3)$$

where α, β, γ and δ evaluated at (x_{n-1}, x_n). By the triangle inequality we have

$$d(x_{n-1}, x_{n+1}) \leq d(x_{n-1}, x_n) + d(x_n, x_{n+1}).$$

Hence,

$$\begin{aligned} d_c(x_{n-1}, x_{n+1}) &\leq K\|d(x_{n-1}, x_n) + d(x_n, x_{n+1})\| \\ &\leq K(d_c(x_{n-1}, x_n) + d_c(x_n, x_{n+1})) \\ &\leq 2K \max\{d_c(x_{n-1}, x_n), d_c(x_n, x_{n+1})\}. \end{aligned} \quad (4)$$

By (4) equation (3) turns to be

$$d_c(x_n, x_{n+1}) \leq (\alpha + \beta + \gamma) \max\{d_c(x_{n-1}, x_n), d_c(x_n, x_{n+1})\} + 2K\delta \max\{d_c(x_{n-1}, x_n), d_c(x_n, x_{n+1})\}.$$

Then

$$d_c(x_n, x_{n+1}) \leq \lambda \max\{d_c(x_{n-1}, x_n), d_c(x_n, x_{n+1})\}.$$

Since $\lambda < 1$, then

$$d_c(x_n, x_{n+1}) \leq \lambda d_c(x_{n-1}, x_n). \quad (5)$$

By inductivity, we obtain

$$d_c(x_n,x_{n+1}) \le \lambda d_c(x_{n-1},x_n) \le \lambda.\lambda d_c(x_{n-2},x_{n-1}) \le ... \le \lambda^n d_c(x,Tx). \quad (6)$$

By triangle inequality of a cone metric, for $m>n$ we get

$$d(x_n,x_m) \le d(x_n,x_{n+1}) + d(x_{n+1},x_{n+2}) + ... + d(x_{m-1},x_m).$$

By normality of the cone, equation (6) and that $\|.\|$ satisfies the triangle inequality we obtain

$$\begin{aligned} d_c(x_n,x_m) &\le K(d_c(x_n,x_{n+1}) + d_c(x_{n+1},x_{n+2}) + ... + d_c(x_{m-1},x_m)) \\ &\le K\left(\lambda^n d_c(x,Tx) + \lambda^{n+1} d_c(x,Tx) + ... + \lambda^{m-1} d_c(x,Tx)\right) \\ &\le \frac{\lambda^n}{1-\lambda} K d_c(x,Tx), \end{aligned}$$

or

$$d_c(x_n,x_m) \le \frac{\lambda^n}{1-\lambda} K d_c(x,Tx). \quad (7)$$

Letting $m,n\to\infty$, in (7) Lemma 2 implies that $\{x_n\}$ is a Cauchy sequence. Since (X,d) is a complete cone metric space, then there exists $u\in X$ such that

$$\lim_{n\to\infty} x_n = u. \quad (8)$$

Now, we show that u is a fixed point of T. From (1) and (2),

$$\begin{aligned} d_c(Tu,Tx_n) &\le \alpha d_c(u,x_n) + \beta d_c(u,Tu) + \gamma d_c(x_n,Tx_n) \\ &\quad +\delta[d_c(u,Tx_n) + d_c(x_n,Tu)] \\ &\le (\alpha+\beta+\gamma+2\delta)\max\left\{ \begin{array}{c} d_c(u,x_n), d_c(u,Tu), \\ d_c(x_n,x_{n+1}), d_c(u,x_{n+1}), d_c(x_n,Tu) \end{array} \right\} \\ &\le \lambda \max\{d_c(u,x_n), d_c(u,Tu), d_c(x_n,x_{n+1}), d_c(u,x_{n+1}), d_c(x_n,Tu)\}. \end{aligned}$$

Take the limit as $n\to\infty$, then by (8) and Lemma 1, we obtain

$$d_c(Tu,u) \le \lambda d_c(u,Tu) \quad (9)$$

Since $\lambda<1$, then $d_c(Tu,u)=0$. Hence, $\|d(Tu,u)\|=0$, then $d(Tu,u)=0$ which implies $Tu=u$.

For uniqueness, assume $x,y\in X$ and $x\neq y$ are two fixed points of T. Then, (1) leads to

$$\begin{aligned} d_c(x,y) &= d_c(Tx,Ty) \\ &\le \alpha d_c(x,y) + \beta d_c(x,Tx) + \gamma d_c(y,Ty) + \delta d_c(x,Ty) + \delta d_c(y,Tx) \\ &\le (\alpha+2\delta) d_c(x,y) \le \lambda d_c(x,y), \end{aligned}$$

since $\lambda<1$, then $d_c(x,y)=0$, which implies $x=y$. Since $x\in X$ was arbitrary, then from (8) we conclude that (ii) holds.

To show (iii), taking the limit in (7) as $m\to\infty$ and making use of Lemma 1, we get $d_c(T^n x,u) \le \frac{\lambda^n}{1-\lambda} d_c(x,Tx)$ for each n. The proof is complete. □

Mappings which satisfy (1) and (2) called generalized contractions. From the proof of 1, it clear that if T is a generalized contraction, then it satisfies

$$d_c\left(Tx,Ty\right) \leq \lambda \max\left\{d_c\left(x,y\right), d_c\left(x,Tx\right), d_c\left(y,Ty\right), \frac{1}{2}\left[d_c\left(x,Ty\right)+d_c\left(y,Tx\right)\right]\right\}, \tag{10}$$

where $\lambda \in \langle 0,1\rangle$ and $x,y \in X$.

In [17], Kannan introduced the following contractive condition in the metric space (X,ρ)

$$\rho\left(Tx,Ty\right) \leq \alpha\left[\rho\left(x,Tx\right)+\rho\left(y,Ty\right)\right],\ 0<\alpha<\frac{1}{2} \tag{11}$$

In [14], Kannan's contractive condition (11) was carried to cone metric spaces to prove a fixed point theorem. Clearly, mappings satisfying (11) are a type of generalized contractive mappings.

We know that Banach contraction mappings are continuous. However, generalized contraction mappings are not continuous in general. The following example shows that Kannan's mappings are not necessarily continuous. This example was given in complete metric spaces which are complete cone metric spaces with a normal constant $K=1$.

Example 2 (See [10]). *Let $X=[0,4]$ be the set of real numbers with the usual metric $\rho(x,y)=|y-x|$. Define $T:X\to X$ by*

$$T(x)=\begin{cases}\dfrac{x}{3}, & if \quad x\leq 3,\\ \dfrac{x}{4}, & if \quad 3<x\leq 4\end{cases}. \tag{12}$$

For $x,y\in[0,3]$, we have

$$\begin{aligned}\rho(Tx,Ty) &= \frac{1}{3}\left|x-Tx+Tx-Ty+Ty-y\right| \\ &\leq \frac{1}{3}\left[\rho\left(x,Tx\right)+\rho\left(Tx,Ty\right)+\rho\left(y,Ty\right)\right].\end{aligned} \tag{13}$$

Hence,

$$\rho\left(Tx,Ty\right) \leq \frac{1}{2}\left[\rho\left(x,Tx\right)+\rho\left(y,Ty\right)\right]. \tag{14}$$

Similarly, we obtain that the same inequality holds for $x,y\in\langle 3,4]$.

Now, let $x\in[0,3]$ and $y\in\langle 3,4]$. Then, we have

$$\rho\left(Tx,Ty\right)=\left|\frac{x}{3}-\frac{y}{4}\right| \leq 1 < 1.125 \leq \frac{1}{2}\rho\left(y,Ty\right). \tag{15}$$

Clearly, the inequality holds for $x\in\langle 3,4]$ and $y\in[0,3]$. Thus, T satisfies (11), but T is discontinuous.

Proposition 1. *A generalized contraction mapping T on a complete cone metric space (X,d) has a unique fixed point and at this point it is continuous.*

Proof. From Theorem 1 we know that T has a unique fixed point, say $z \in X$. Let $\{y_n\} \subset X$ be such that $y_n \to z$ as $n \to \infty$. We shall show that $Ty_n \to Tz = z$ as $n \to \infty$. From (10) and Lemma 1 we have

$$\begin{aligned} d_c\left(Ty_n, Tz\right) &\leq \lambda \max\left\{d_c\left(y_n, z\right), d_c\left(y_n, Ty_n\right), d_c\left(y, Ty\right), \frac{1}{2}[d_c\left(y_n, z\right) + d_c\left(z, Ty_n\right)]\right\} \\ &\leq \lambda d_c\left(y_n, z\right) + \lambda d_c\left(Tz, Ty_n\right), \end{aligned} \tag{16}$$

or

$$\begin{aligned} d_c\left(Ty_n, Tz\right) - \lambda d_c\left(Tz, Ty_n\right) &\leq \lambda d_c\left(y_n, z\right) \\ (1-\lambda) d_c\left(Tz, Ty_n\right) &\leq \lambda d_c\left(y_n, z\right) \\ d_c\left(Tz, Ty_n\right) &\leq \frac{\lambda}{1-\lambda} d_c\left(y_n, z\right) \end{aligned} \tag{17}$$

Let $n \to \infty$, then (17) and Lemma 1 imply that $Ty_n \to Tz = z$ in (X, d). Thus, T is continuous at a fixed point. Then the proof is complete. □

4. Common fixed points of generalized contraction mappings

Let S be a non-empty set and let $\{T\alpha\}_{\alpha \in J}$ be a family of selfmappings on S and J an indexing set. A point $u \in S$ is called a common fixed point for a family $\{T\alpha\}_{\alpha \in J}$ if and only if $u = T_\alpha u$ for each $T\alpha$.

Theorem 2. *Let (X, d) be a complete cone metric space with a normal constant $K \geq 1$ and $\{T\alpha\}_{\alpha \in J}$ a family of selfmappings of X. If there exists a fixed $\beta \in J$ such that for each $\alpha \in J$:*

$$d_c\left(T_\alpha x, T_\beta y\right) \leq \lambda \max\left\{d_c\left(x, y\right), d_c\left(x, T_\alpha x\right), d_c\left(y, T_\beta y\right), \frac{1}{2}\left[d_c\left(x, T_\beta y\right) + d_c\left(y, T_\alpha x\right)\right]\right\} \tag{18}$$

for some $\lambda = \lambda(\alpha) \in (0, 1)$ with $\lambda K < 1$ and all $x, y \in X$, then all T_α have a unique common fixed point, which is a unique fixed point of each T_α, $\alpha \in J$.

Proof. Let $\alpha \in J$ and $x \in X$ be arbitrary. Consider a sequence defined by $x_0 = x$, $x_{2n+1} = T_\alpha x_{2n}$, $x_{2n+2} = T_\beta x_{2n+1}$, $n \geq 0$. From (18) we get

$$\begin{aligned} d_c\left(x_{2n+1}, x_{2n+2}\right) &= d_c\left(T_\alpha x_{2n}, T_\beta x_{2n+1}\right) \\ &\leq \lambda \max\left\{\begin{array}{c} d_c\left(x_{2n}, x_{2n+1}\right), d_c\left(x_{2n}, x_{2n+1}\right), d_c\left(x_{2n+1}, x_{2n+2}\right), \\ \frac{1}{2}\left[d_c\left(x_{2n}, x_{2n+2}\right) + d_c\left(x_{2n+1}, x_{2n+1}\right)\right] \end{array}\right\}. \end{aligned}$$

Since

$$\begin{aligned} d(x_{2n}, x_{2n+2}) &\leq d\left(x_{2n}, x_{2n+1}\right) + d\left(x_{2n+1}, x_{2n+2}\right) \\ \|d(x_{2n}, x_{2n+2})\| &\leq K\left[\|d\left(x_{2n}, x_{2n+1}\right)\| + \|d\left(x_{2n+1}, x_{2n+2}\right)\|\right] \\ d_c\left(x_{2n}, x_{2n+2}\right) &\leq K\left[d_c\left(x_{2n}, x_{2n+1}\right) + d_c\left(x_{2n+1}, x_{2n+2}\right)\right] \end{aligned} \tag{19}$$

So,

$$\begin{aligned}\frac{1}{2}d_c(x_{2n},x_{2n+2}) &\leq \frac{1}{2}K\left[d_c(x_{2n},x_{2n+1})+d_c(x_{2n+1},x_{2n+2})\right]\\ &\leq K\max\left\{d_c(x_{2n},x_{2n+1}),d_c(x_{2n+1},x_{2n+2})\right\},\end{aligned}$$

we have

$$d_c(x_{2n+1},x_{2n+2}) \leq \lambda K\max\left\{d_c(x_{2n},x_{2n+1}),d_c(x_{2n+1},x_{2n+2})\right\}.$$

Hence, as $\lambda K<1$,

$$d_c(x_{2n+1},x_{2n+2}) \leq \lambda K d_c(x_{2n},x_{2n+1}).$$

Similarly, we get that $d_c(x_{2n},x_{2n+1}) \leq \lambda K d_c(x_{2n-1},x_{2n})$. Thus, for any $n\geq 1$ we have

$$d_c(x_n,x_{n+1}) \leq \lambda K d_c(x_{n-1},x_n) \leq (\lambda K)^2 d_c(x_{n-2},x_{n-1}) \leq ... \leq (\lambda K)^n d_c(x_0,x_1). \tag{20}$$

From (20) and by the triangle inequality of the cone metric and $\|.\|$, for $m>n$ we get

$$\begin{aligned}d_c(x_n,x_m) &\leq K\left(d_c(x_n,x_{n+1})+d_c(x_{n+1},x_{n+2})+...+d_c(x_{m-1},x_m)\right)\\ &\leq K\left((\lambda K)^n d_c(x_0,x_1)+(\lambda K)^{n+1}d_c(x_0,x_1)+...+(\lambda K)^{m-1}d_c(x_0,x_1)\right)\\ &\leq K\left[(\lambda K)^n+(\lambda K)^{n+1}+...+(\lambda K)^{m-1}\right]d_c(x_0,x_1)\\ &\leq K\frac{(\lambda K)^n}{1-(\lambda K)}d_c(x_0,x_1) \qquad (21)\end{aligned}$$

If in (21) we let $m,n\to\infty$, then by Lemma 2, we conclude that $\{x_n\}$ is a Cauchy sequence. Since X is complete, there is a $z\in X$ such that

$$\lim_{n\to\infty} x_n = z. \tag{22}$$

From (18) we have

$$\begin{aligned}d_c(T_\beta z,x_{2n+1}) &= d_c(T_\beta z,T_\alpha x_{2n})\\ &\leq \lambda\max\left\{\begin{array}{c} d_c(z,x_{2n}),d_c(z,T_\beta z),d_c(x_{2n},x_{2n+1}),\\ \frac{1}{2}\left[d_c(z,x_{2n+1})+d_c(x_{2n},T_\beta z)\right]\end{array}\right\}.\end{aligned}$$

Taking the limit as $n\to\infty$, then by (22) and Lemma 1 we get $d_c(T_\beta z,z)\leq \lambda d_c(z,T_\beta z)$. Therefore, $d_c(T_\beta z,z)=0$ and so $T_\beta z=z$. To show that z is a fixed point of all $\{T_\alpha\}_{\alpha\in J}$, let $\alpha\in J$ be arbitrary. Then from (18) with $x=y=z=T_\beta z$ we have

$$d_c(z,T_\alpha z)=d_c(T_\beta z,T_\alpha z)\leq \lambda(\alpha)\max\left\{d_c(z,T_\alpha z),\frac{1}{2}d_c(z,T_\alpha z)\right\}$$

and hence $T_\alpha z=z$. Thus, all T_α have a common fixed point. Suppose that w is another fixed point of T_β. Then it follows as above, that w is a common fixed point of all $\{T_\alpha\}_{\alpha\in J}$. Thus, from (18) we have $d_c(z,w)=d_c(T_\beta z,T_\alpha w)\leq \lambda d_c(z,w)$ and so $z=w$. Thus, z is a unique common fixed point of all $\{T_\alpha\}_{\alpha\in J}$. The proof is complete. □

Acknowledgement

The research has been supported by The Scientific and Technological Research Council of Turkey (TUBITAK-Turkey). We would also like to thank the referees for their valuable comments.

References

[1] M. ABBAS, G. JUNGCK, *Common Fixed Point Results for Non Commuting Mappings Without Continuity in Cone Metric Spaces*, J. Math. Anal. Appl. **341**(2008), 416–420.

[2] M. ABBAS, B. E. RHOADES, *Fixed and Periodic Point Results in Cone Metric Spaces*, Appl. Math. Lett. **9**(2009), 1364–1368.

[3] J. ACHARI, *Ćirić's Generalized Contractions in L-Spaces*, Glas Mat. Ser. III **13**(1978), 307–310.

[4] J. ACHARI, *Generalized Multi-valued Contractions and Fixed Points*, Rev. Roum. Math. Pures Appl. **24**(1979), 179–182.

[5] M. ARSHAD, A. AZAM, P. VETRO, *Some common fixed point results in cone uniform spaces*, Fixed Point Theory and Appl. **2009**(2009), Article ID 493965.

[6] A. AZAM, M. ARSHAD, *Common fixed points of generalized contractive maps in cone metric spaces*, Bull. Iranian Math. Soc. **35**(2009), 255–264.

[7] LJ. B. ĆIRIĆ, *Generalized Contractions and Fixed Point Theorems*, Publ. Inst. Math. **12**(1971), 19–26.

[8] LJ. B. ĆIRIĆ, *On a Family of Contractive Maps and Fixed Points*, Publ. Inst. Math. **17**(1974), 45–51.

[9] LJ. B. ĆIRIĆ, *On Common Fixed Points in Uniform Spaces*, Publ. Inst. Math. **24**(1978), 39–43.

[10] LJ. B. ĆIRIĆ, *Fixed Point Theory: Contraction Mapping Principle*, Faculty of Mechanical Engineering, University of Belgrade, Beograd, 27. marta 80, 2003.

[11] K. DEIMLING, *Nonlinear Functional Analysis*, Springer-Verlage, Berlin, Heidelberg and New York, 1985.

[12] J. GORNICKI, B. E. RHOADES, *A General Fixed Point Theorem for Involutions*, Indian J. Pure. Appl. Math. **27**(1996), 13-23.

[13] R. H. HAGHI, SH. REZAPOUR, *Fixed points of multifunctions on regular cone metric spaces*, Expo. Math. **28**(2010), 71–77.

[14] L. G. HUANG, X. ZHANG, *Cone Metric Spaces and Fixed Point Theorems of Contractive Mappings*, J. Math. Anal. Appl. **332**(2007), 1468–1476.

[15] D. ILIĆ, V. RAKOČEVIĆ, *Common Fixed Points for Maps on Cone Metric Space*, J. Math. Anal. Appl. **341**(2008), 876–882.

[16] D. ILIĆ, V. RAKOČEVIĆ, *Quasi-Contraction on Cone Metric Space*, Appl. Math. Lett. **22**(2009), 728–731.

[17] R. KANNAN, *Some Results on Fixed Points*, Bull. Calcutta Math. Soc. **60**(1968), 71–76.

[18] D. KLIM, D. WARDOWSKI, *Dynamic processes and fixed points of set valued nonlinear contractions in cone metric spaces*, Nonlinear Anal. **71**(2009), 5170–5175.

[19] B. G. PACHPATTE, *Extension of Ćirić's Maps and Fixed Point Theorem*, Chung Yuan J. **8**(1979), 13-16. .

[20] T. K. PAL, M. MAITI, J. ACHARI, *Extension of Ćirić's Generalized Contractions*, Mat. Vesnik **28**(1976), 313–317.

[21] T. K. PAL, M. MAITI, *Extensions of Fixed Point Theorems of Rhoades and Ćirić*, Proc. Amer. Math. Soc. **64**(1977), 283–286.

[22] T. K. Pal, M. Maiti, *On Orbital Contractions and Ćirić's Generalized Contractions*, Glas Mat. Ser. III **13**(1978), 81–90.

[23] S. Radonevic, *Common fixed points under contractive conditions in cone metric spaces*, Comput. Math. Appl. **58**(2009), 1273–1278.

[24] P. Raja, S. M. Vaezpour, *Some Extensions of Banach's Contraction Principle in Complete Cone Metric Spaces*, Fixed Point Theory Appl. **2008**(2008), Article ID 768294.

[25] B. K. Ray, *On Ćirić's Fixed Point Theorem*, Fund. Math. **94**(1977), 221-229.

[26] B. K. Ray, *On Some Theorems of Ćirić*, Mat. Vesnik **30**(1978), 215–218.

[27] Sh. Rezapour, *Best Approximations in Cone Metric Spaces*, Math. Morav. **11**(2007), 85–88.

[28] Sh. Rezapour, R. Hamlbarani, *Some Notes on the Paper "Cone Metric Spaces and Fixed Point Theorems of Contractive Mappings"*, J. Math. Anal. Appl. **345**(2008), 719–724.

[29] Sh. Rezapour, R. H. Haghi, *Fixed point of multifunctions on cone metric spaces*, Numer. Funct. Anal. Optim. **30**(2009), 825–832.

[30] Sh. Rezapour, R. H. Haghi, *Two results about fixed point of multifunctions*, Math. Proc. R. Ir. Azad. **109**(2009), 155–161.

[31] B. E. Rhoades, *A Fixed Point Theorem for Some Non-self-mappings*, Math. Japonica **23**(1978), 457-459.

[32] K. L. Singh, *A Remark on a Paper: "Extensions of Fixed Point Theorems of Rhoades and Ćirić*, (by T. K. Pal, M. Maiti in Proc. Amer. Math. Soc. **64**(1977), 283–286), Bull. Math. Soc. Math. R. S. Roumanie (Bucharest) **72**(1980).

[33] D. Turkoglu, M. Abuloha, *Cone Metric Spaces and Fixed Point Theorems in Diametrically Contractive Mappings*, Acta Sinica Math. **26**(2010), 489–496.

[34] P. Vetro, *Common fixed points in cone metric spaces*, Rend. Circ. Mat. Palermo **56**(2007), 464–468.

[35] D. Wardowski, *Endpoint and Fixed Points of Set-valued Contractions in Cone Metric Spaces*, Nonlinear Anal. **71**(2009), 512–516.

[36] Wei-Shih Du, *A note on cone metric fixed point theory and its equivalence*, Nonlinear Anal. **72**(2010), 2259–2261.

[37] K. Wlodarczyk, R. Pleabaniak, C. Obczynski, *Convergence theorems, best approximation and best proximity for set valued dynamic systems of relatively quasi-asyptotic contractions in cone uniform spaces*, Nonlinear Anal. **71**(2009), 5022–5031.