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Research Article

On the Definitions of Nabla Fractional Operators

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We show that two recent definitions of discrete nabla fractional sum operators are related. Obtaining such a relation between two operators allows one to prove basic properties of the one operator by using the known properties of the other. We illustrate this idea with proving power rule and commutative property of discrete fractional sum operators. We also introduce and prove summation by parts formulas for the right and left fractional sum and difference operators, where we employ the Riemann-Liouville definition of the fractional difference. We formalize initial value problems for nonlinear fractional difference equations as an application of our findings. An alternative definition for the nabla right fractional difference operator is also introduced.

1. Introduction

The following definitions of the backward (nabla) discrete fractional sum operators were given in [1, 2], respectively. For any given positive real number α, we have

$$\nabla_a^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} \left(t-\rho(s)\right)^{\overline{\alpha-1}} f(s), \tag{1.1}$$

where $t \in \{a+1, a+2, \ldots\}$, and

$$\triangledown_a^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t} \left(t-\rho(s)\right)^{\overline{\alpha-1}} f(s), \tag{1.2}$$

where $t \in \{a, a+1, a+2, \ldots\}$.

One significant difference between these two operators is that the sum in (1.1) starts at $a+1$ and the sum in (1.2) starts at a. In this paper we aim to answer the following question:

Do these two definitions lead the development of the theory of the nabla fractional difference equations in two directions?

In order to answer this question, we first obtain a relation between the operators in (1.1) and (1.2). Then we illustrate how such a relation helps one to prove basic properties of the one operator if similar properties of the other are already known.

In recent years, discrete fractional calculus gains a great deal of interest by several mathematicians. First Miller and Ross [3] and then Gray and Zhang [1] introduced discrete versions of the Riemann-Liouville left fractional integrals and derivatives, called the fractional sums and differences with the delta and nabla operators, respectively. For recent developments of the theory, we refer the reader to the papers [2, 4–19]. For further reading in this area, we refer the reader to the books on fractional differential equations [20–23].

The paper is organized as follows. In Section 2, we summarize some of basic notations and definitions in discrete nabla calculus. We employ the Riemann-Liouville definition of the fractional difference. In Section 3, we obtain two relations between the operators $\nabla_a^{-\alpha}$ and $\triangledown_a^{-\alpha}$. So by the use of these relations we prove some properties for $\nabla_a^{-\alpha}$-operator. Section 4 is devoted to summation by parts formulas. In Section 5, we formalize initial value problems and obtain corresponding summation equation with $\nabla_a^{-\alpha}$-operator. This section can be considered as an application of the results in Section 3. Finally, in Section 6, a definition of the nabla right fractional difference resembling the nabla right fractional sum is formulated. This definition can be used to prove continuity of the nabla right fractional differences with respect to the order α.

2. Notations and Basic Definitions

Definition 2.1. (i) For a natural number m, the m rising (ascending) factorial of t is defined by

$$t^{\overline{m}} = \prod_{k=0}^{m-1}(t+k), \quad t^{\overline{0}} = 1. \tag{2.1}$$

(ii) For any real number α, the rising function is defined by

$$t^{\overline{\alpha}} = \frac{\Gamma(t+\alpha)}{\Gamma(t)}, \quad t \in \mathbb{R} - \{\dots,-2,-1,0\},\ 0^{\overline{\alpha}} = 0. \tag{2.2}$$

Throughout this paper, we will use the following notations.

(i) For real numbers a and b, we denote $\mathbb{N}_a = \{a, a+1, \dots\}$ and ${}_b\mathbb{N} = \{\dots, b-1,\ b\}$.

(ii) For $n \in \mathbb{N}$, we define

$$_{\ominus}\Delta^n f(t) := (-1)^n \Delta^n f(t). \tag{2.3}$$

Definition 2.2. Let $\rho(t) = t - 1$ be the backward jump operator. Then

(i) the (nabla) left fractional sum of order $\alpha > 0$ (starting from a) is defined by

$$\nabla_a^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t - \rho(s))^{\overline{\alpha-1}} f(s), \quad t \in \mathbb{N}_{a+1} \tag{2.4}$$

(ii) the (nabla) right fractional sum of order $\alpha > 0$ (ending at b) is defined by

$${}_b\nabla^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=t}^{b-1} (s - \rho(t))^{\overline{\alpha-1}} f(s) = \frac{1}{\Gamma(\alpha)} \sum_{s=t}^{b-1} (\sigma(s) - t)^{\overline{\alpha-1}} f(s), \quad t \in_{b-1} \mathbb{N}. \tag{2.5}$$

We want to point out that the nabla left fractional sum operator has the following characteristics.

(i) $\nabla_a^{-\alpha}$ maps functions defined on $\mathbb{N}_a$ to functions defined on $\mathbb{N}_a$.

(ii) $\nabla_a^{-n} f(t)$ satisfies the nth order discrete initial value problem

$$\nabla^n y(t) = f(t), \qquad \nabla^i y(a) = 0, \quad i = 0, 1, \ldots, n - 1. \tag{2.6}$$

(iii) The Cauchy function $(t - \rho(s))^{\overline{n-1}} / \Gamma(n)$ satisfies $\nabla^n y(t) = 0$.

In the same manner, it is worth noting that the nabla right fractional sum operator has the following characteristics.

(i) ${}_b\nabla^{-\alpha}$ maps functions defined on ${}_b\mathbb{N}$ to functions defined on ${}_b\mathbb{N}$.

(ii) ${}_b\nabla^{-n} f(t)$ satisfies the nth order discrete initial value problem

$${}_\ominus\Delta^n y(t) = f(t), \qquad {}_\ominus\Delta^i y(b) = 0, \quad i = 0, 1, \ldots, n - 1. \tag{2.7}$$

(iii) The Cauchy function $(s - \rho(t))^{\overline{n-1}} / \Gamma(n)$ satisfies ${}_\ominus\Delta^n y(t) = 0$.

Definition 2.3. (i) The (nabla) left fractional difference of order $\alpha > 0$ is defined by

$$\nabla_a^{\alpha} f(t) = \nabla^n \nabla_a^{-(n-\alpha)} f(t) = \frac{\nabla^n}{\Gamma(n - \alpha)} \sum_{s=a+1}^{t} (t - \rho(s))^{\overline{n-\alpha-1}} f(s), \quad t \in \mathbb{N}_{a+1}. \tag{2.8}$$

(ii) The (nabla) right fractional difference of order $\alpha > 0$ is defined by

$${}_b\nabla^{\alpha} f(t) = {}_\ominus\Delta^n {}_b\nabla^{-(n-\alpha)} f(t) = \frac{{}_\ominus\Delta^n}{\Gamma(n - \alpha)} \sum_{s=t}^{b-1} (s - \rho(t))^{\overline{n-\alpha-1}} f(s), \quad t \in_{b-1} \mathbb{N}. \tag{2.9}$$

Here and throughout the paper $n = [\alpha] + 1$, where $[\alpha]$ is the greatest integer less than or equal α.

Regarding the domains of the fractional difference operators we observe the following.

(i) The nabla left fractional difference ∇_a^{α} maps functions defined on $\mathbb{N}_a$ to functions defined on $\mathbb{N}_{a+n}$ (on $\mathbb{N}_a$ if we think $f = 0$ before a).

(ii) The nabla right fractional difference ${}_b\nabla^{\alpha}$ maps functions defined on ${}_b\mathbb{N}$ to functions defined on ${}_{b-n}\mathbb{N}$ (on ${}_b\mathbb{N}$ if we think $f = 0$ after b).

3. A Relation between the Operators $\nabla_a^{-\alpha}$ and $\triangledown_a^{-\alpha}$

In this section we illustrate how two operators, $\nabla_a^{-\alpha}$ and $\triangledown_a^{-\alpha}$ are related.

Lemma 3.1. *The following holds:*

(i) $\triangledown_{a+1}^{-\alpha} f(t) = \nabla_a^{-\alpha} f(t)$,

(ii) $\triangledown_a^{-\alpha} f(t) = (1/\Gamma(\alpha))(t-a+1)^{\overline{\alpha-1}} f(a) + \nabla_a^{-\alpha} f(t)$.

Proof. The proof of (i) follows immediately from the above definitions (1.1) and (1.2). For the proof of (ii), we have

$$\begin{aligned}
\triangledown_a^{-\alpha} f(t) &= \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t} (t-\rho(s))^{\overline{\alpha-1}} f(s) \\
&= \frac{1}{\Gamma(\alpha)} (t-a+1)^{\overline{\alpha-1}} f(a) + \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{\alpha-1}} f(s) \\
&= \frac{1}{\Gamma(\alpha)} (t-a+1)^{\overline{\alpha-1}} f(a) + \nabla_a^{-\alpha} f(t).
\end{aligned} \tag{3.1}$$

□

Next three lemmas show that the above relations on the operators (1.1) and (1.2) help us to prove some identities and properties for the operator $\nabla_a^{-\alpha}$ by the use of known identities for the operator $\triangledown_a^{-\alpha}$.

Lemma 3.2. *The following holds:*

$$\nabla_a^{-\alpha} \nabla f(t) = \nabla \nabla_a^{-\alpha} f(t) - \frac{(t-a)^{\overline{\alpha-1}}}{\Gamma(\alpha)} f(a). \tag{3.2}$$

Proof. It follows from Lemma 3.1 and Theorem 2.1 in [13]

$$\begin{aligned}
\nabla_a^{-\alpha} \nabla f(t) = \triangledown_{a+1}^{-\alpha} \nabla f(t) &= \nabla \triangledown_a^{-\alpha} f(t) - \frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} f(a) \\
&= \nabla \left\{ \frac{1}{\Gamma(\alpha)} (t-a+1)^{\overline{\alpha-1}} f(a) + \nabla_a^{-\alpha} f(t) \right\} - \frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)} f(a)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{(\alpha-1)}{\Gamma(\alpha)}(t-a+1)^{\overline{\alpha-2}}f(a) + \nabla\nabla_a^{-\alpha}f(t) - \frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}f(a) \\
&= \nabla\nabla_a^{-\alpha}f(t) - \frac{(t-a)^{\overline{\alpha-1}}}{\Gamma(\alpha)}f(a).
\end{aligned}
\tag{3.3}
$$

□

Lemma 3.3. *Let $\alpha > 0$ and $\beta > -1$. Then for $t \in \mathbb{N}_a$, the following equality holds*

$$
\nabla_a^{-\alpha}(t-a)^{\overline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)}(t-a)^{\overline{\alpha+\mu}}. \tag{3.4}
$$

Proof. It follows from Theorem 2.1 in [13]

$$
\nabla_a^{-\alpha}(t-a)^{\overline{\mu}} = \triangledown_{a+1}^{-\alpha}(t-a)^{\overline{\mu}} = \frac{\Gamma(\mu+1)}{\Gamma(\mu+\alpha+1)}(t-a)^{\overline{\alpha+\mu}}. \tag{3.5}
$$

□

Lemma 3.4. *Let f be a real-valued function defined on $\mathbb{N}_a$, and let $\alpha, \beta > 0$. Then*

$$
\nabla_a^{-\alpha}\nabla_a^{-\beta}f(t) = \nabla_a^{-(\alpha+\beta)}f(t) = \nabla_a^{-\beta}\nabla_a^{-\alpha}f(t). \tag{3.6}
$$

Proof. It follows from Lemma 3.1 and Theorem 2.1 in [2]

$$
\nabla_a^{-\alpha}\nabla_a^{-\beta}f(t) = \triangledown_{a+1}^{-\alpha}\triangledown_{a+1}^{-\beta}f(t) = \triangledown_{a+1}^{-(\alpha+\beta)}f(t) = \nabla_a^{-(\alpha+\beta)}f(t). \tag{3.7}
$$

□

Remark 3.5. Let $\alpha > 0$ and $n = [\alpha] + 1$. Then, by Lemma 3.2 we have

$$
\nabla\nabla_a^{\alpha}f(t) = \nabla\nabla^n\left(\nabla_a^{-(n-\alpha)}f(t)\right) = \nabla^n\left(\nabla\nabla_a^{-(n-\alpha)}f(t)\right) \tag{3.8}
$$

or

$$
\nabla\nabla_a^{\alpha}f(t) = \nabla^n\left[\nabla_a^{-(n-\alpha)}\nabla f(t) + \frac{(t-a)^{\overline{n-\alpha-1}}}{\Gamma(n-\alpha)}f(a)\right]. \tag{3.9}
$$

Then, using the identity

$$
\nabla^n\frac{(t-a)^{\overline{n-\alpha-1}}}{\Gamma(n-\alpha)} = \frac{(t-a)^{\overline{-\alpha-1}}}{\Gamma(-\alpha)} \tag{3.10}
$$

we verified that (3.2) is valid for any real α.

By using Lemma 3.1, Remark 3.5, and the identity $\nabla(t-a)^{\overline{\alpha-1}} = (\alpha-1)(t-a)^{\overline{\alpha-2}}$, we arrive inductively at the following generalization.

Theorem 3.6. *For any real number α and any positive integer p, the following equality holds:*

$$\nabla_{a+p-1}^{-\alpha}\nabla^p f(t) = \nabla^p \nabla_{a+p-1}^{-\alpha} f(t) - \sum_{k=0}^{p-1} \frac{\left(t-(a+p-1)\right)^{\overline{\alpha-p+k}}}{\Gamma(\alpha+k-p+1)} \nabla^k f(a+p-1), \tag{3.11}$$

where f is defined on $\mathbb{N}_a$.

Lemma 3.7. *For any $\alpha > 0$, the following equality holds:*

$${}_b\nabla_{\ominus}^{-\alpha}\Delta f(t) = {}_{\ominus}\Delta\, {}_b\nabla^{-\alpha} f(t) - \frac{(b-t)^{\overline{\alpha-1}}}{\Gamma(\alpha)} f(b). \tag{3.12}$$

Proof. By using of the following summation by parts formula

$$\begin{aligned} &\Delta_s\left[\left(\rho(s)-\rho(t)\right)^{\overline{\alpha-1}} f(s)\right] \\ &\quad = (\alpha-1)\left(s-\rho(t)\right)^{\overline{\alpha-2}} f(s) + \left(s-\rho(t)\right)^{\overline{\alpha-1}} \Delta f(s) \end{aligned} \tag{3.13}$$

we have

$$\begin{aligned} {}_b\nabla_{\ominus}^{-\alpha}\Delta f(t) &= -\frac{1}{\Gamma(\alpha)}\sum_{s=t}^{b-1}\left(s-\rho(t)\right)^{\overline{\alpha-1}}\Delta f(s) \\ &= \frac{1}{\Gamma(\alpha)}\left[-\sum_{s=t}^{b-1}\Delta_s\left(\left(\rho(s)-\rho(t)\right)^{\overline{\alpha-1}} f(s)\right) + (\alpha-1)\sum_{s=t}^{b-1}\left(s-\rho(t)\right)^{\overline{\alpha-2}} f(s)\right] \\ &= \frac{1}{\Gamma(\alpha-1)}\sum_{s=t}^{b-1}\left(s-\rho(t)\right)^{\overline{\alpha-2}} f(s) - \frac{(b-t)^{\overline{\alpha-1}}}{\Gamma(\alpha)} f(b). \end{aligned} \tag{3.14}$$

On the other hand,

$$\begin{aligned} &{}_{\ominus}\Delta\, {}_b\nabla^{-\alpha} f(t) \\ &\quad = -\frac{1}{\Gamma(\alpha)}\sum_{s=t}^{b-1}\Delta_t\left(s-\rho(t)\right)^{\overline{\alpha-1}} f(s) = \frac{1}{\Gamma(\alpha-1)}\sum_{s=t}^{b-1}\left(s-\rho(t)\right)^{\overline{\alpha-2}} f(s), \end{aligned} \tag{3.15}$$

where the identity

$$\Delta_t\left(s-\rho(t)\right)^{\overline{\alpha-1}} = -(\alpha-1)\left(s-\rho(t)\right)^{\overline{\alpha-2}} \tag{3.16}$$

and the convention that $(0)^{\overline{\alpha-1}} = 0$ are used. □

Remark 3.8. Let $\alpha > 0$ and $n = [\alpha] + 1$. Then, by the help of Lemma 3.7 we have

$$\begin{aligned} {}_{\ominus}\Delta\, {}_{b}\nabla^{\alpha} f(t) &= {}_{\ominus}\Delta_{\ominus}\Delta^{n}\left({}_{b}\nabla^{-(n-\alpha)} f(t)\right) \\ &= {}_{\ominus}\Delta^{n}\left({}_{\ominus}\Delta\, {}_{b}\nabla^{-(n-\alpha)} f(t)\right) \end{aligned} \tag{3.17}$$

or

$${}_{\ominus}\Delta\, {}_{b}\nabla^{\alpha} f(t) = {}_{\ominus}\Delta^{n}\left[{}_{b}\nabla_{\ominus}^{-(n-\alpha)} \Delta f(t) + \frac{(b-t)^{\overline{n-\alpha-1}}}{\Gamma(n-\alpha)} f(b)\right]. \tag{3.18}$$

Then, using the identity

$${}_{\ominus}\Delta^{n} \frac{(b-t)^{\overline{n-\alpha-1}}}{\Gamma(n-\alpha)} = \frac{(b-t)^{\overline{-\alpha-1}}}{\Gamma(-\alpha)} \tag{3.19}$$

we verified that (3.12) is valid for any real α.

By using Lemma 3.7, Remark 3.8, and the identity $\Delta(b-t)^{\overline{\alpha-1}} = -(\alpha-1)(b-t)^{\overline{\alpha-2}}$, we arrive inductively at the following generalization.

Theorem 3.9. *For any real number α and any positive integer p, the following equality holds:*

$${}_{b-p+1}\nabla_{\ominus}^{-\alpha} \Delta^{p} f(t) =_{\ominus} \Delta^{p}\, {}_{b-p+1}\nabla^{-\alpha} f(t) - \sum_{k=0}^{p-1} \frac{(b-p+1-t)^{\overline{\alpha-p+k}}}{\Gamma(\alpha+k-p+1)}\, {}_{\ominus}\Delta^{k} f(b-p+1), \tag{3.20}$$

where f is defined on ${}_{b}N$.

We finish this section by stating the commutative property for the right fractional sum operators without giving its proof.

Lemma 3.10. *Let f be a real valued function defined on ${}_{b}\mathbb{N}$, and let $\alpha, \beta > 0$. Then*

$${}_{b}\nabla^{-\alpha}\left[{}_{b}\nabla^{-\beta} f(t)\right] =_{b} \nabla^{-(\alpha+\beta)} f(t) = {}_{b}\nabla^{-\beta}\left[{}_{b}\nabla^{-\alpha} f(t)\right]. \tag{3.21}$$

4. Summation by Parts Formulas for Fractional Sums and Differences

We first state summation by parts formula for nabla fractional sum operators.

Theorem 4.1. *For $\alpha > 0$, $a, b \in \mathbb{R}$, f defined on $\mathbb{N}_a$ and g defined on ${}_{b}\mathbb{N}$, the following equality holds*

$$\sum_{s=a+1}^{b-1} g(s) \nabla_{a}^{-\alpha} f(s) = \sum_{s=a+1}^{b-1} f(s)_{b}\nabla^{-\alpha} g(s). \tag{4.1}$$

Proof. By the definition of the nabla left fractional sum we have

$$\sum_{s=a+1}^{b-1} g(s)\nabla_a^{-\alpha} f(s) = \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{b-1} g(s) \sum_{r=a+1}^{s} (s-\rho(r))^{\overline{\alpha-1}} f(r). \tag{4.2}$$

If we interchange the order of summation we reach (4.1). □

By using Theorem 3.6, Lemma 3.4, and $\nabla_a^{-(n-\alpha)} f(a) = 0$, we prove the following result.

Theorem 4.2. *For $\alpha > 0$, and f defined in a suitable domain $\mathbb{N}_a$, the following are valid*

$$\nabla_a^{\alpha}\nabla_a^{-\alpha} f(t) = f(t), \tag{4.3}$$

$$\nabla_a^{-\alpha}\nabla_a^{\alpha} f(t) = f(t), \quad \text{when } \alpha \notin \mathbb{N}, \tag{4.4}$$

$$\nabla_a^{-\alpha}\nabla_a^{\alpha} f(t) = f(t) - \sum_{k=0}^{n-1} \frac{(t-a)^{\overline{k}}}{k!} \nabla^k f(a), \quad \text{when } \alpha = n \in \mathbb{N}. \tag{4.5}$$

We recall that $D^{-\alpha}D^{\alpha} f(t) = f(t)$, where $D^{-\alpha}$ is the Riemann-Liouville fractional integral, is valid for sufficiently smooth functions such as continuous functions. As a result of this it is possible to obtain integration by parts formula for a certain class of functions (see [23] page 76, and for more details see [22]). Since discrete functions are continuous we see that the term $\nabla_a^{-(1-\alpha)} f(t)|_{t=a}$, for $0 < \alpha < 1$ disappears in (4.4), with the application of the convention that $\sum_{s=a+1}^{a} f(s) = 0$.

By using Theorem 3.9, Lemma 3.10, and ${}_b\nabla^{-(n-\alpha)} f(b) = 0$, we obtain the following.

Theorem 4.3. *For $\alpha > 0$, and f defined in a suitable domain ${}_b\mathbb{N}$, we have*

$$ {}_b\nabla^{-\alpha}\ {}_b\nabla^{-\alpha} f(t) = f(t), \tag{4.6}$$

$$ {}_b\nabla_b^{-\alpha}\nabla^{\alpha} f(t) = f(t), \quad \text{when } \alpha \notin \mathbb{N}, \tag{4.7}$$

$$ {}_b\nabla^{-\alpha}{}_b\nabla^{\alpha} f(t) = f(t) - \sum_{k=0}^{n-1} \frac{(b-t)^{\overline{k}}}{k!}\, {}_\ominus\Delta^k f(b), \quad \text{when } \alpha = n \in \mathbb{N}. \tag{4.8}$$

Theorem 4.4. *Let $\alpha > 0$ be noninteger. If f is defined on ${}_bN$ and g is defined on N_a, then*

$$\sum_{s=a+1}^{b-1} f(s)\nabla_a^{\alpha} g(s) = \sum_{s=a+1}^{b-1} g(s)_b\nabla^{\alpha} f(s). \tag{4.9}$$

Proof. Equation (4.7) implies that

$$\sum_{s=a+1}^{b-1} f(s)\nabla_a^{\alpha} g(s) = \sum_{s=a+1}^{b-1} {}_b\nabla^{-\alpha}\left({}_b\nabla^{\alpha} f(s)\right)\nabla_a^{\alpha} g(s). \tag{4.10}$$

And by Theorem 4.1 we have

$$\sum_{s=a+1}^{b-1} f(s)\nabla_a^{\alpha} g(s) = \sum_{s=a+1}^{b-1} {}_b\nabla^{\alpha} f(s)\nabla_a^{-\alpha}\nabla_a^{\alpha} g(s). \tag{4.11}$$

Then the result follows by (4.4). □

5. Initial Value Problems

Let us consider the following initial value problem for a nonlinear fractional difference equation

$$\nabla_{a-1}^{\alpha} y(t) = f\big(t, y(t)\big) \quad \text{for} \quad t = a+1, a+2, \dots, \tag{5.1}$$

$$\nabla_{a-1}^{-(1-\alpha)} y(t)|_{t=a} = y(a) = c, \tag{5.2}$$

where $0 < \alpha < 1$ and a is any real number.

Apply the operator $\nabla_a^{-\alpha}$ to each side of (5.1) to obtain

$$\nabla_a^{-\alpha}\nabla_{a-1}^{\alpha} y(t) = \nabla_a^{-\alpha} f\big(t, y(t)\big). \tag{5.3}$$

Then using the definition of the fractional difference and sum operators we obtain

$$\begin{gathered} \nabla_a^{-\alpha}\left\{\nabla\nabla_{a-1}^{-(1-\alpha)} y(t)\right\} = \nabla_a^{-\alpha} f\big(t, y(t)\big) \\ \nabla_a^{-\alpha}\left\{\nabla\nabla_a^{-(1-\alpha)} y(t) + \nabla\left\{\frac{(t-a+1)^{\overline{-\alpha}}}{\Gamma(1-\alpha)} y(a)\right\}\right\} = \nabla_a^{-\alpha} f\big(t, y(t)\big), \end{gathered} \tag{5.4}$$

$$\nabla_a^{-\alpha}\nabla_a^{\alpha} y(t) + \nabla_a^{-\alpha}\nabla\left\{\frac{(t-a+1)^{\overline{-\alpha}}}{\Gamma(1-\alpha)} y(a)\right\} = \nabla_a^{-\alpha} f\big(t, y(t)\big). \tag{5.5}$$

It follows from Lemma 3.2 that

$$\nabla_a^{-\alpha}\nabla\left\{\frac{(t-a+1)^{\overline{-\alpha}}}{\Gamma(1-\alpha)} y(a)\right\} = \nabla\nabla_a^{-\alpha}\left\{\frac{(t-a+1)^{\overline{-\alpha}}}{\Gamma(1-\alpha)} y(a)\right\} - \frac{(t-a)^{\overline{\alpha-1}}}{\Gamma(\alpha)} y(a). \tag{5.6}$$

Note that

$$\nabla_a^{-\alpha}\left\{\frac{(t-a+1)^{\overline{-\alpha}}}{\Gamma(1-\alpha)} y(a)\right\} = \nabla_{a-1}^{-\alpha}\frac{(t-(a-1))^{\overline{-\alpha}}}{\Gamma(1-\alpha)} y(a) - \frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(1-\alpha)} y(a). \tag{5.7}$$

Hence we obtain

$$
\begin{aligned}
&\nabla_a^{-\alpha}\nabla\left\{\frac{(t-a+1)^{\overline{-\alpha}}}{\Gamma(1-\alpha)}y(a)\right\}\\
&=\nabla\left\{\nabla_{a-1}^{-\alpha}\frac{(t-(a-1))^{\overline{-\alpha}}}{\Gamma(1-\alpha)}y(a)-\frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(1-\alpha)}y(a)\right\}-\frac{(t-a)^{\overline{\alpha-1}}}{\Gamma(\alpha)}y(a)\\
&=y(a)\nabla\left\{\Gamma(1-\alpha)(t-(a-1))^{\overline{0}}\right\}-(\alpha-1)y(a)\frac{(t-a+1)^{\overline{\alpha-2}}}{\Gamma(\alpha)}-\frac{(t-a)^{\overline{\alpha-1}}}{\Gamma(\alpha)}y(a)\\
&=-\frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}y(a),
\end{aligned}
\tag{5.8}
$$

which follows from the power rule in Lemma 3.3.

Let us put this expression back in (5.5) and use (4.4), we have

$$y(t)=\frac{(t-a+1)^{\overline{\alpha-1}}}{\Gamma(\alpha)}y(a)+\nabla_a^{-\alpha}f(t,y(t)). \tag{5.9}$$

Thus, we have proved the following lemma.

Lemma 5.1. *y is a solution of the initial value problem,* (5.1), (5.2), *if, and only if, y has the representation* (5.9).

Remark 5.2. A similar result has been obtained in the paper [13] with the operator $\triangledown_a^{\alpha}$. And the initial value problem has been defined in the following form

$$\triangledown_a^{\alpha}y(t)=f(t,y(t)) \quad \text{for } t=a+1,\ a+2,\ldots, \tag{5.10}$$

$$\triangledown_a^{-(1-\alpha)}y(t)|_{t=a}=y(a)=c, \tag{5.11}$$

where $0<\alpha\le 1$ and a is any real number. The subscript a of the term $\triangledown_a^{\alpha}y(t)$ on the left hand side of (5.10) indicates directly that the solution has a domain starts at a. The nature of this notation helps us to use the nabla transform easily as one can see in the papers [2, 12]. On the other hand, the subscript $a-1$ in (5.1) indicates that the solution has a domain starts at a.

6. An Alternative Definition of Nabla Fractional Differences

Recently, the authors in [18], by the help of a nabla Leibniz's Rule, have rewritten the nabla left fractional difference in a form similar to the definition of the nabla left fractional sum. In this section, we do this for the nabla right fractional differences.

The following delta Leibniz's Rule will be used:

$$\Delta_t\sum_{s=t}^{b-1}g(s,t)=\sum_{s=t}^{b-1}\Delta_t g(s,t)-g(t,\ t+1). \tag{6.1}$$

Using the following identity

$$\Delta_t (s-\rho(t))^{\overline{\alpha}} = -\alpha (s-\rho(t))^{\overline{\alpha-1}}, \tag{6.2}$$

and the definition of the nabla right fractional difference (ii) of Definition 2.3, for $\alpha > 0$, $\alpha \notin \mathbb{N}$ we have

$$\begin{aligned}
{}_b\nabla^{\alpha} f(t) &= (-1)^n \Delta^n {}_b\nabla^{-(n-\alpha)} f(t) \\
&= \frac{(-1)^n \Delta^n}{\Gamma(n-\alpha)} \sum_{s=t}^{b-1} (s-\rho(t))^{\overline{n-\alpha-1}} f(s) \\
&= \frac{(-1)^n \Delta^{n-1}}{\Gamma(n-\alpha)} \Delta_t \sum_{s=t}^{b-1} (s-\rho(t))^{\overline{n-\alpha-1}} f(s) \\
&= \frac{(-1)^n \Delta^{n-1}}{\Gamma(n-\alpha)} \left[-(n-\alpha-1) \sum_{s=t}^{b-1} (s-\rho(t))^{\overline{n-\alpha-2}} f(s) - (t-t)^{\overline{n-\alpha-1}} \right] \\
&= \frac{-(-1)^n \Delta^{n-1}}{\Gamma(n-\alpha-1)} \sum_{s=t}^{b-1} (s-\rho(t))^{\overline{n-\alpha-2}} f(s).
\end{aligned} \tag{6.3}$$

By applying the Leibniz's Rule (6.1), $n-1$ number of times we get

$${}_b\nabla^{\alpha} f(t) = \frac{1}{\Gamma(-\alpha)} \sum_{s=t}^{b-1} (s-\rho(t))^{\overline{-\alpha-1}} f(s). \tag{6.4}$$

In the above, it is to be insisted that $\alpha \notin \mathbb{N}$ is required due the fact that the term $1/\Gamma(-\alpha)$ is undefined for negative integers. Therefore we can proceed and unify the definitions of nabla right fractional sums and differences similar to Definition 5.3 in [18]. Also, the alternative formula (6.4) can be employed, similar to Theorem 5.4 in [18], to show that the nabla right fractional difference ${}_b\nabla^{\alpha} f$ is continuous with respect to $\alpha \geq 0$.

7. Conclusions

In fractional calculus there are two approaches to obtain fractional derivatives. The first approach is by iterating the integral and then defining a fractional order by using Cauchy formula to obtain Riemann fractional integrals and derivatives. The second approach is by iterating the derivative and then defining a fractional order by making use of the binomial theorem to obtain Grünwald-Letnikov fractional derivatives. In this paper we followed the discrete form of the first approach via the nabla difference operator. However, we noticed that in the right fractional difference case we used both the nabla and delta difference operators. This setting enables us to obtain reasonable summation by parts formulas for nabla fractional sums and differences in Section 4 and to obtain an alternative definition for nabla right fractional differences through the delta Leibniz's Rule in Section 6.

While following the discrete form of the first approach, two types of fractional sums and hence fractional differences appeared; one type by starting from a and the other type, which obeys the general theory of nabla time scale calculus, by starting from $a+1$ in the left case and ending at $b-1$ in the right case. Section 3 discussed the relation between the two types of operators, where certain properties of one operator are obtained by using the second operator.

An initial value problem discussed in Section 5 is an important application exposing the derived properties of the two types of operators discussed throughout the paper, where the solution representation was obtained explicitly for order $0 < \alpha < 1$. Regarding this example we remark the following. In fractional calculus, Initial value problems usually make sense for functions not necessarily continuous at a (left case) so that the initial conditions are given by means of a^+. Since sequences are nice continuous functions then in Theorem 4.2, which is the tool in solving our example, the identity (4.4) appears without any initial condition. To create an initial condition in our example we shifted the fractional difference operator so that it started at $a-1$.

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