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Research Article

Fractional Sums and Differences with Binomial Coefficients

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In fractional calculus, there are two approaches to obtain fractional derivatives. The first approach is by iterating the integral and then defining a fractional order by using Cauchy formula to obtain Riemann fractional integrals and derivatives. The second approach is by iterating the derivative and then defining a fractional order by making use of the binomial theorem to obtain Grünwald-Letnikov fractional derivatives. In this paper we formulate the delta and nabla discrete versions for left and right fractional integrals and derivatives representing the second approach. Then, we use the discrete version of the Q-operator and some discrete fractional dual identities to prove that the presented fractional differences and sums coincide with the discrete Riemann ones describing the first approach.

1. Introduction and Preliminaries

Fractional calculus (FC) is developing very fast in both theoretical and applied aspects. As a result, FC is used intensively and successfully in the last few decades to describe the anomalous processes which appear in complex systems [1–6]. Very recently, important results in the field of fractional calculus and its applications were reported (see e.g., [7–10] and the references therein). The complexity of the real world phenomena is a great source of inspiration for the researchers to invent new fractional tools which will be able to dig much dipper into the mysteries of the mother nature. Historically the FC passed through different periods of evolutions, and it started to face very recently a new provocation: how to formulate properly its discrete counterpart [11–24]. At this stage, we have to stress on the fact that in the classical discrete equations their roots are based on the functional difference equations, therefore, the natural question is to find the generalization of these equations to the fractional case. In other words, we will end up with generalizations of the basic operators occurring in standard difference equations. As it was expected, there were several attempts to do this generalization as well as to apply this new techniques to investigate the dynamics of some complex processes. In recent years, the discrete counterpart of the fractional Riemann-Liouville, Caputo, was investigated mainly thinking how to apply techniques from the time scales calculus to the expressions of the fractional operators. Despite of the beauty of the obtained results, one simple question arises: can we obtain the same results from a new point of view which is more simpler and more intuitive? Having all above mentioned thinks in mind we are going to use the binomial theorem in order to get Grünwald-Letnikov fractional derivatives. After that, we proved that the results obtained coincide with the ones obtained by the discretization of the Riemann-Liouville operator. In this manner, we believe that it becomes more clear what the fractional difference equations bring new in description of the related complex phenomena described.

For a natural number n, the fractional polynomial is defined by

$$t^{(n)} = \prod_{j=0}^{n-1} (t-j) = \frac{\Gamma(t+1)}{\Gamma(t+1-n)}, \tag{1}$$

where Γ denotes the special gamma function and the product is zero when $t + 1 - j = 0$ for some j. More generally, for arbitrary α, define

$$t^{(\alpha)} = \frac{\Gamma(t+1)}{\Gamma(t+1-\alpha)}, \tag{2}$$

where the convention of that division at pole yields zero. Given that the forward and backward difference operators are defined by

$$\Delta f(t) = f(t+1) - f(t), \qquad \nabla f(t) = f(t) - f(t-1), \tag{3}$$

respectively, we define iteratively the operators $\Delta^m = \Delta(\Delta^{m-1})$ and $\nabla^m = \nabla(\nabla^{m-1})$, where m is a natural number.

Here are some properties of the factorial function.

Lemma 1 (see [13]). *Assume the following factorial functions are well defined.*

(i) $\Delta t^{(\alpha)} = \alpha t^{(\alpha-1)}$.

(ii) $(t-\mu)t^{(\mu)} = t^{(\mu+1)}$, *where* $\mu \in \mathbb{R}$.

(iii) $\mu^{(\mu)} = \Gamma(\mu+1)$.

(iv) *If* $t \le r$, *then* $t^{(\alpha)} \le r^{(\alpha)}$ *for any* $\alpha > r$.

(v) *If* $0 < \alpha < 1$, *then* $t^{(\alpha\nu)} \ge (t^{(\nu)})^{\alpha}$.

(vi) $t^{(\alpha+\beta)} = (t-\beta)^{(\alpha)} t^{(\beta)}$.

Also, for our purposes we list down the following two properties, the proofs of which are straightforward:

$$\begin{aligned} \nabla_s (s-t)^{(\alpha-1)} &= (\alpha-1)\left(\rho(s)-t\right)^{(\alpha-2)} \\ \nabla_t \left(\rho(s)-t\right)^{(\alpha-1)} &= -(\alpha-1)\left(\rho(s)-t\right)^{(\alpha-2)}. \end{aligned} \tag{4}$$

For the sake of the nabla fractional calculus, we have the following definition.

Definition 2 (see [25–28]). (i) For a natural number m, the m rising (ascending) factorial of t is defined by

$$t^{\overline{m}} = \prod_{k=0}^{m-1} (t+k), \qquad t^{\overline{0}} = 1. \tag{5}$$

(ii) For any real number, the α rising function is defined by

$$t^{\overline{\alpha}} = \frac{\Gamma(t+\alpha)}{\Gamma(t)}, \quad t \in \mathbb{R} - \{\ldots,-2,-1,0\}, \quad 0^{\alpha} = 0. \tag{6}$$

Regarding the rising factorial function, we observe the following:

(i)

$$\nabla\left(t^{\overline{\alpha}}\right) = \alpha t^{\overline{\alpha-1}}, \tag{7}$$

(ii)

$$\left(t^{\overline{\alpha}}\right) = (t+\alpha-1)^{(\alpha)}, \tag{8}$$

(iii)

$$\Delta_t \left(s-\rho(t)\right)^{\overline{\alpha}} = -\alpha\left(s-\rho(t)\right)^{\overline{\alpha-1}}. \tag{9}$$

Notation:

(i) For a real $\alpha > 0$, we set $n = [\alpha] + 1$, where $[\alpha]$ is the greatest integer less than α.

(ii) For real numbers a and b, we denote $\mathbb{N}_a = \{a, a+1, \ldots\}$ and ${}_b\mathbb{N} = \{b, b-1, \ldots\}$.

(iii) For $n \in \mathbb{N}$ and real a, we denote

$${}_{\ominus}\Delta^n f(t) \triangleq (-1)^n \Delta^n f(t). \tag{10}$$

(iv) For $n \in \mathbb{N}$ and real b, we denote

$$\nabla_{\ominus}^n f(t) \triangleq (-1)^n \nabla^n f(t). \tag{11}$$

The following definition and the properties followed can be found in [29] and the references therein.

Definition 3 (see [29]). Let $\sigma(t) = t+1$ and $\rho(t) = t-1$ be the forward and backward jumping operators, respectively. Then

(i) the (delta) left fractional sum of order $\alpha > 0$ (starting from a) is defined by

$$\Delta_a^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a}^{t-\alpha} (t-\sigma(s))^{(\alpha-1)} f(s), \quad t \in \mathbb{N}_{a+\alpha}. \tag{12}$$

(ii) The (delta) right fractional sum of order $\alpha > 0$ (ending at b) is defined by

$$\begin{aligned} {}_b\Delta^{-\alpha} f(t) &= \frac{1}{\Gamma(\alpha)} \sum_{s=t+\alpha}^{b} (s-\sigma(t))^{(\alpha-1)} f(s) \\ &= \frac{1}{\Gamma(\alpha)} \sum_{s=t+\alpha}^{b} \left(\rho(s)-t\right)^{(\alpha-1)} f(s), \quad t \in {}_{b-\alpha}\mathbb{N}. \end{aligned} \tag{13}$$

(iii) The (nabla) left fractional sum of order $\alpha > 0$ (starting from a) is defined by

$$\nabla_a^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=a+1}^{t} \left(t-\rho(s)\right)^{\overline{\alpha-1}} f(s), \quad t \in \mathbb{N}_{a+1}. \tag{14}$$

(iv) The (nabla) right fractional sum of order $\alpha > 0$ (ending at b) is defined by

$$\begin{aligned} {}_b\nabla^{-\alpha} f(t) &= \frac{1}{\Gamma(\alpha)} \sum_{s=t}^{b-1} \left(s-\rho(t)\right)^{\overline{\alpha-1}} f(s) \\ &= \frac{1}{\Gamma(\alpha)} \sum_{s=t}^{b-1} (\sigma(s)-t)^{\overline{\alpha-1}} f(s), \quad t \in {}_{b-1}\mathbb{N}. \end{aligned} \tag{15}$$

Regarding the delta left fractional sum, we observe the following:

(i) $\Delta_a^{-\alpha}$ maps functions defined on $\mathbb{N}_a$ to functions defined on $\mathbb{N}_{a+\alpha}$.

(ii) $u(t) = \Delta_a^{-n} f(t)$, $n \in \mathbb{N}$, satisfies the initial value problem:

$$\begin{gathered} \Delta^n u(t) = f(t), \quad t \in N_a, \\ u(a + j - 1) = 0, \quad j = 1, 2, \ldots, n. \end{gathered} \tag{16}$$

(iii) The Cauchy function $(t - \sigma(s))^{(n-1)}/(n-1)!$ vanishes at $s = t - (n-1), \ldots, t-1$.

Regarding the delta right fractional sum, we observe the following:

(i) ${}_b\Delta^{-\alpha}$ maps functions defined on ${}_b\mathbb{N}$ to functions defined on ${}_{b-\alpha}\mathbb{N}$.

(ii) $u(t) = {}_b\Delta^{-n} f(t)$, $n \in \mathbb{N}$, satisfies the initial value problem:

$$\begin{gathered} \nabla_{\ominus}^n u(t) = f(t), \quad t \in {}_bN, \\ u(b - j + 1) = 0, \quad j = 1, 2, \ldots, n. \end{gathered} \tag{17}$$

(iii) The Cauchy function $(\rho(s) - t)^{(n-1)}/(n-1)!$ vanishes at $s = t+1, t+2, \ldots, t+(n-1)$.

Regarding the nabla left fractional sum, we observe the following:

(i) $\nabla_a^{-\alpha}$ maps functions defined on $\mathbb{N}_a$ to functions defined on $\mathbb{N}_a$.

(ii) $\nabla_a^{-n} f(t)$ satisfies the nth-order discrete initial value problem:

$$\nabla^n y(t) = f(t), \qquad \nabla^i y(a) = 0, \quad i = 0, 1, \ldots, n-1. \tag{18}$$

(iii) The Cauchy function $(t - \rho(s))^{\overline{n-1}}/\Gamma(n)$ satisfies $\nabla^n y(t) = 0$.

Regarding the nabla right fractional sum we observe the following:

(i) ${}_b\nabla^{-\alpha}$ maps functions defined on ${}_b\mathbb{N}$ to functions defined on ${}_b\mathbb{N}$.

(ii) ${}_b\nabla^{-n} f(t)$ satisfies the nth-order discrete initial value problem:

$$_{\ominus}\Delta^n y(t) = f(t), \qquad _{\ominus}\Delta^i y(b) = 0, \quad i = 0, 1, \ldots, n-1. \tag{19}$$

The proof can be done inductively. Namely, assuming it is true for n, we have

$$_{\ominus}\Delta^{n+1}\, {}_b\nabla^{-(n+1)} f(t) = {}_{\ominus}\Delta^n \left[-\Delta\, {}_b\nabla^{-(n+1)} f(t)\right]. \tag{20}$$

By the help of (9), it follows that

$$_{\ominus}\Delta^{n+1}\, {}_b\nabla^{-(n+1)} f(t) = {}_{\ominus}\Delta^n\, {}_b\nabla^{-n} f(t) = f(t). \tag{21}$$

The other part is clear by using the convention that $\sum_{k=s+1}^{s} = 0$.

(iii) The Cauchy function $(s - \rho(t))^{\overline{n-1}}/\Gamma(n)$ satisfies $_{\ominus}\Delta^n y(t) = 0$.

Definition 4. (i) [12] The (delta) left fractional difference of order $\alpha > 0$ (starting from a) is defined by

$$\begin{aligned} \Delta_a^{\alpha} f(t) &= \Delta^n \Delta_a^{-(n-\alpha)} f(t) \\ &= \frac{\Delta^n}{\Gamma(n-\alpha)} \sum_{s=a}^{t-(n-\alpha)} (t - \sigma(s))^{(n-\alpha-1)} f(s), \\ &\qquad t \in \mathbb{N}_{a+(n-\alpha)}. \end{aligned} \tag{22}$$

(ii) [19] The (delta) right fractional difference of order $\alpha > 0$ (ending at b) is defined by

$$\begin{aligned} {}_b\Delta^{\alpha} f(t) &= \nabla_{\ominus}^n\, {}_b\Delta^{-(n-\alpha)} f(t) \\ &= \frac{(-1)^n \nabla^n}{\Gamma(n-\alpha)} \sum_{s=t+(n-\alpha)}^{b} (s - \sigma(t))^{(n-\alpha-1)} f(s), \\ &\qquad t \in {}_{b-(n-\alpha)}\mathbb{N}. \end{aligned} \tag{23}$$

(iii) [20] The (nabla) left fractional difference of order $\alpha > 0$ (starting from a) is defined by

$$\begin{aligned} \nabla_a^{\alpha} f(t) &= \nabla^n \nabla_a^{-(n-\alpha)} f(t) \\ &= \frac{\nabla^n}{\Gamma(n-\alpha)} \sum_{s=a+1}^{t} (t - \rho(s))^{\overline{n-\alpha-1}} f(s), \\ &\qquad t \in \mathbb{N}_{a+1}. \end{aligned} \tag{24}$$

(iv) [29, 30] The (nabla) right fractional difference of order $\alpha > 0$ (ending at b) is defined by

$$\begin{aligned} {}_b\nabla^{\alpha} f(t) &= {}_{\ominus}\Delta^n\, {}_b\nabla^{-(n-\alpha)} f(t) \\ &= \frac{(-1)^n \Delta^n}{\Gamma(n-\alpha)} \sum_{s=t}^{b-1} (s - \rho(t))^{\overline{n-\alpha-1}} f(s), \\ &\qquad t \in {}_{b-1}\mathbb{N}. \end{aligned} \tag{25}$$

Regarding the domains of the fractional type differences we observe the following.

(i) The delta left fractional difference Δ_a^{α} maps functions defined on $\mathbb{N}_a$ to functions defined on $\mathbb{N}_{a+(n-\alpha)}$.

(ii) The delta right fractional difference ${}_b\Delta^{\alpha}$ maps functions defined on ${}_b\mathbb{N}$ to functions defined on ${}_{b-(n-\alpha)}\mathbb{N}$.

(iii) The nabla left fractional difference ∇_a^{α} maps functions defined on $\mathbb{N}_a$ to functions defined on $\mathbb{N}_{a+n}$.

(iv) The nabla right fractional difference ${}_b\nabla^{\alpha}$ maps functions defined on ${}_b\mathbb{N}$ to functions defined on ${}_{b-n}\mathbb{N}$.

Lemma 5 (see [15]). *Let $0 \leq n-1 < \alpha \leq n$, and let $y(t)$ be defined on $\mathbb{N}_a$. Then the following statements are valid:*

(i) $(\Delta_a^{\alpha})y(t-\alpha) = \nabla_{a-1}^{\alpha} y(t)$ *for* $t \in \mathbb{N}_{n+a}$,

(ii) $(\Delta_a^{-\alpha})y(t+\alpha) = \nabla_{a-1}^{-\alpha} y(t)$ *for* $t \in \mathbb{N}_a$.

Lemma 6 (see [29]). *Let $y(t)$ be defined on ${}_{b+1}\mathbb{N}$. Then the following statements are valid:*

(i) $({}_b\Delta^{\alpha})y(t+\alpha) = {}_{b+1}\nabla^{\alpha} y(t)$ *for* $t \in {}_{b-n}\mathbb{N}$,

(ii) $({}_b\Delta^{-\alpha})y(t-\alpha) = {}_{b+1}\nabla^{-\alpha} y(t)$ *for* $t \in {}_b\mathbb{N}$.

If $f(s)$ is defined on $N_a \cap {}_bN$ and $a \equiv b \pmod 1$ then $(Qf)(s) = f(a+b-s)$. The Q-operator generates a dual identity by which the left type and the right type fractional sums and differences are related. Using the change of variable $u = a+b-s$, in [18] it was shown that

$$\Delta_a^{-\alpha} Qf(t) = Q\,{}_b\Delta^{-\alpha} f(t), \tag{26}$$

and, hence,

$$\Delta_a^{\alpha} Qf(t) = \left(Q\,{}_b\Delta^{\alpha} f\right)(t). \tag{27}$$

The proof of (27) follows by (26) and by noting that

$$-Q\nabla f(t) = \Delta Qf(t). \tag{28}$$

Similarly, in the nabla case we have

$$\nabla_a^{-\alpha} Qf(t) = Q\,{}_b\nabla^{-\alpha} f(t), \tag{29}$$

and, hence,

$$\nabla_a^{\alpha} Qf(t) = \left(Q\,{}_b\nabla^{\alpha} f\right)(t). \tag{30}$$

The proof of (30) follows by (29) and that

$$-Q\Delta f(t) = \nabla Qf(t). \tag{31}$$

For more details about the discrete version of the Q-operator we refer to [29].

From the difference calculus or time scale calculus, for a natural n and a sequence f, we recall

$$\begin{aligned} \Delta^n f(t) &= \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(t+n-k), \\ \nabla^n f(t) &= \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(t-n+k). \end{aligned} \tag{32}$$

2. The Fractional Differences and Sums with Binomial Coefficients

We first give the definition of fractional order of (32) in the left and right sense.

Definition 7. The (binomial) delta left fractional difference and sum of order $\alpha > 0$ for a function f defined on $\mathbb{N}_a$ are defined by

(a)

$$B\Delta_a^{\alpha} = \sum_{k=0}^{\alpha+t-a} (-1)^k \binom{\alpha}{k} f(t+\alpha-k), \quad t \in \mathbb{N}_{a+n-\alpha}, \tag{33}$$

(b)

$$B\Delta_a^{-\alpha} = \sum_{k=0}^{\alpha+t-a} (-1)^k \binom{-\alpha}{k} f(t-\alpha-k), \quad t \in \mathbb{N}_{a+\alpha}, \tag{34}$$

where $(-1)^k \binom{-\alpha}{k} = \binom{\alpha+k-1}{k}$.

Definition 8. The (binomial) nabla left fractional difference and sum of order $\alpha > 0$ for a function f defined on $\mathbb{N}_a$, are defined by

(a)

$$B\nabla_a^{\alpha} = \sum_{k=0}^{t-a-1} (-1)^k \binom{\alpha}{k} f(t-k), \quad t \in \mathbb{N}_{a+n}, \tag{35}$$

(b)

$$B\nabla_a^{-\alpha} = \sum_{k=0}^{t-a-1} (-1)^k \binom{-\alpha}{k} f(t-k), \quad t \in \mathbb{N}_a. \tag{36}$$

Analogously, in the right case we can define the following.

Definition 9. The (binomial) delta right fractional difference and sum of order $\alpha > 0$ for a function f defined on ${}_b\mathbb{N}$ are defined by

(a)

$${}_b\Delta B^{\alpha} = \sum_{k=0}^{\alpha+b-t} (-1)^k \binom{\alpha}{k} f(t-\alpha+k), \quad t \in b-n+\alpha\mathbb{N}, \tag{37}$$

(b)

$${}_b\Delta B^{-\alpha} = \sum_{k=0}^{-\alpha+b-t} (-1)^k \binom{-\alpha}{k} f(t+\alpha+k), \quad t \in b-\alpha\mathbb{N}. \tag{38}$$

Definition 10. The (binomial) nabla right fractional difference and sum of order $\alpha > 0$ for a function f defined on ${}_b\mathbb{N}$ are defined by

(a)

$${}_b\nabla B^{\alpha} = \sum_{k=0}^{b-t-1} (-1)^k \binom{\alpha}{k} f(t-k), \quad t \in {}_{b-n}\mathbb{N}, \tag{39}$$

(b)

$${}_b\nabla B^{-\alpha} = \sum_{k=0}^{b-t-1} (-1)^k \binom{-\alpha}{k} f(t-k), \quad t \in {}_b\mathbb{N}. \tag{40}$$

We next proceed to show that the Riemann fractional differences and sums coincide with the binomial ones defined above. We will use the dual identities in Lemma 5 and Lemma 6, and the action of the discrete version of the Q-operator to follow easy proofs and verifications. In [20], the author used a delta Leibniz's rule to obtain the following alternative definition for Riemann delta left fractional differences:

$$\Delta_a^{\alpha} f(t) = \frac{1}{\Gamma(-\alpha)} \sum_{s=a}^{t+\alpha} (t-\sigma(s))^{(-\alpha-1)} f(s), \quad \alpha \notin \mathbb{N},\ t \in \mathbb{N}_{a+n-\alpha}, \tag{41}$$

then proceeded with long calculations and showed, actually, that

$$\Delta_a^{\alpha} f(t) = B\Delta_a^{\alpha} f(t), \qquad \Delta_a^{-\alpha} f(t) = B\Delta_a^{-\alpha} f(t). \tag{42}$$

Theorem 11. *Let f be defined on suitable domains and $\alpha > 0$. Then,*

(1)

$$\Delta_a^{\alpha} f(t) = B\Delta_a^{\alpha} f(t), \qquad \Delta_a^{-\alpha} f(t) = B\Delta_a^{-\alpha} f(t), \tag{43}$$

(2)

$${}_b\Delta^{\alpha} f(t) = {}_b\Delta B^{\alpha} f(t), \qquad {}_b\Delta^{-\alpha} f(t) = \Delta B_a^{-\alpha} f(t), \tag{44}$$

(3)

$$\nabla_a^{\alpha} f(t) = B\nabla_a^{\alpha} f(t), \qquad \nabla_a^{-\alpha} f(t) = B\nabla_a^{-\alpha} f(t), \tag{45}$$

(4)

$${}_b\nabla^{\alpha} f(t) = {}_b\nabla B^{\alpha} f(t), \qquad {}_b\nabla^{-\alpha} f(t) = {}_b\nabla B^{-\alpha} f(t). \tag{46}$$

Proof. (1) follows by (42).

(2) By the discrete Q-operator action we have

$$\begin{aligned} {}_b\Delta^{\alpha} f(t) &= Q\Delta_a (Qf)(t) \\ &= Q\sum_{k=0}^{\alpha+t-a} (-1)^k \binom{\alpha}{k} (Qf)(t+\alpha-k) \\ &= {}_b\Delta B^{\alpha} f(t). \end{aligned} \tag{47}$$

The fractional sum part is also done in a similar way by using the Q-operator.

(3) By the dual identity in Lemma 5 (i) and (42), we have

$$\nabla_a^{\alpha} f(t) = \Delta_{a+1}^{\alpha} f(t+\alpha) = B\Delta_{a+1}^{\alpha} f(t+\alpha) = B\nabla_a^{\alpha} f(t). \tag{48}$$

The fractional sum part can be proved similarly by using Lemma 5 (ii) and (42).

(4) The proof can be achieved by either (2) and Lemma 6 or, alternatively, by (3) and the discrete Q-operator. □

Remark 12. In analogous to (41), the authors in [31] used a nabla Leibniz's rule to prove that

$$\nabla_a^{\alpha} f(t) = \frac{1}{\Gamma(-\alpha)} \sum_{s=a+1}^{t} (t-\rho(s))^{\overline{-\alpha-1}} f(s). \tag{49}$$

In [30], the authors used a delta Leibniz's Rule to prove the following formula for nabla right fractional differences:

$${}_b\nabla^{\alpha} f(t) = \frac{1}{\Gamma(-\alpha)} \sum_{s=t}^{b-1} (s-\rho(t))^{\overline{-\alpha-1}} f(s). \tag{50}$$

Similarly, we can use a nabla Leibniz's rule to prove the following formula for the delta right fractional differences:

$${}_b\Delta^{\alpha} f(t) = \frac{1}{\Gamma(-\alpha)} \sum_{s=t-\alpha}^{b} (s-\sigma(t))^{(-\alpha-1)} f(s). \tag{51}$$

We here remark that the proofs of the last three parts of Theorem 11 can be done alternatively by proceeding as in [20] starting from (49), (50), and (51). Also, it is worth mentioning that mixing both delta and nabla operators in defining delta and nabla right Riemann fractional differences was essential in proceeding, through the dual identities and the discrete Q-operator or delta and nabla type Leibniz's rules, to obtain the main results in this paper [29].

3. Conclusion

The impact of fractional calculus in both pure and applied branches of science and engineering started to increase substantially. The main idea of iterating an operator and then generalizing to any order (real or complex) started to be used in the last decade to obtain appropriate discretization for the fractional operators. We mention, from the theory of time scales view point, that how to obtain the fractional operators was a natural question and it was not correlated to the well-known Grünwald-Letnikov approach. We believe that the discretizations obtained recently in the literature for the fractional operators are different from the one reported within Grünwald-Letnikov method. Bearing all of these thinks in mind we proved that the discrete operators via binomial theorem will lead to the same results as the ones by using the discretization of the Riemann-Liouville operators via time scales techniques. The discrete version of the impressive dual tool Q-operator has been used to prove the equivalency.

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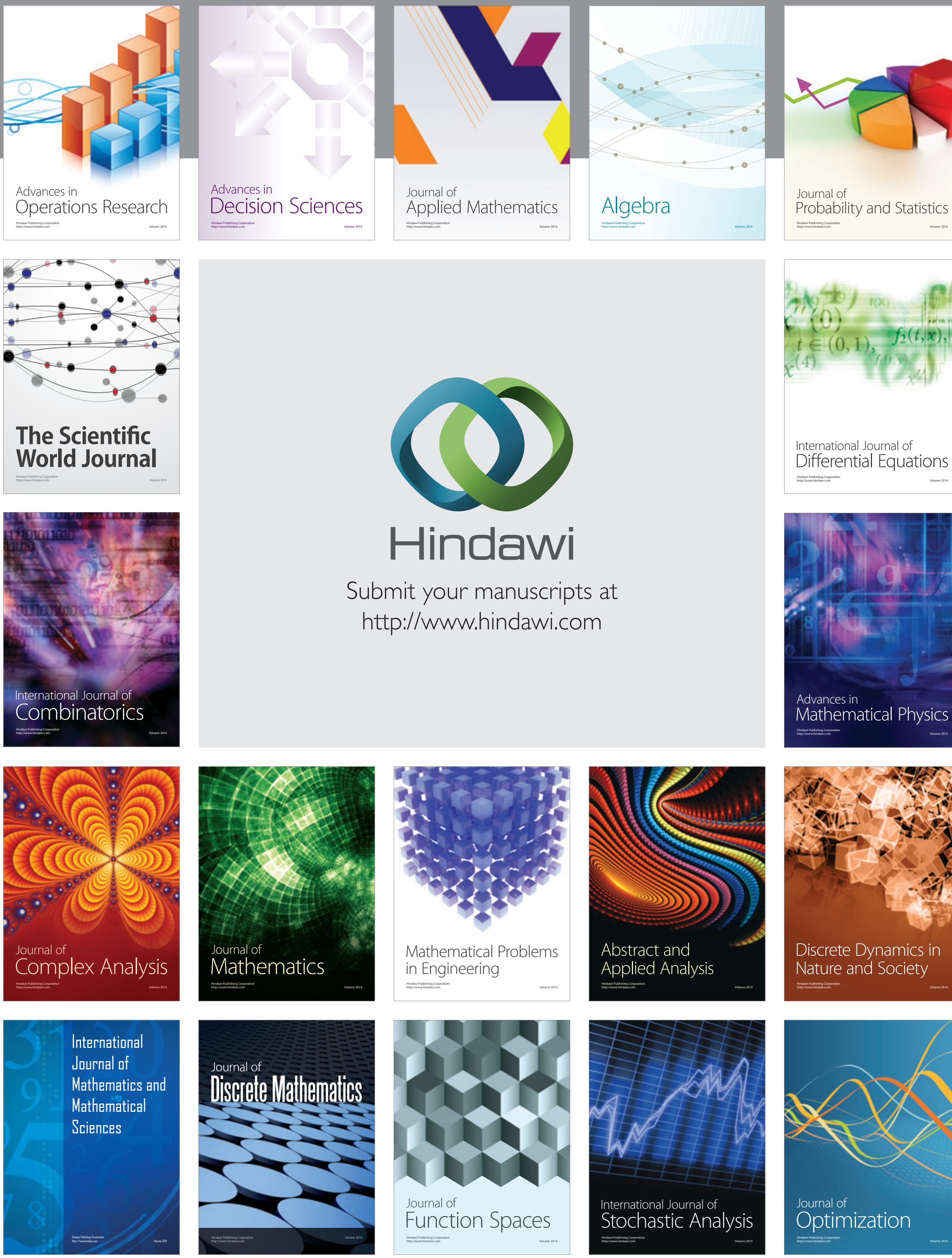
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