Chandok and Tas *Journal of Inequalities and Applications* 2014, 2014:61
http://www.journalofinequalitiesandapplications.com/content/2014/1/61

Journal of Inequalities and Applications
a SpringerOpen Journal

RESEARCH **Open Access**

An original coupled coincidence point result for a pair of mappings without MMP

Sumit Chandok[1*] and Kenan Tas[2]

*Correspondence:
chansok.s@gmail.com;
chandhok.sumit@gmail.com
[1]Department of Mathematics, Khalsa College of Engineering & Technology, Punjab Technical University, Ranjit Avenue, Amritsar, 143001, India
Full list of author information is available at the end of the article

Abstract

The purpose of this paper is to establish a coupled coincidence point theorem for a pair of mappings without MMP (mixed monotone property) in metric spaces endowed with partial order, which is not an immediate consequence of a well-known theorem in the literature. Also, we present a result on the existence and uniqueness of coupled common fixed points. The results presented in the paper generalize and extend some of the results of Bhaskar and Lakshmikantham (Nonlinear Anal. 65:1379-1393, 2006), Choudhury, Metiya and Kundu (Ann. Univ. Ferrara 57:1-16, 2011), Harjani, Lopez and Sadarangani (Nonlinear Anal. 74:1749-1760, 2011) and of Luong and Thuan (Bull. Math. Anal. Appl. 2:16-24, 2010) for the mappings having no MMP. We introduce an example that there exists a common coupled fixed point of the mappings g and F such that F does not satisfy the g-mixed monotone property, and also g and F do not commute.

MSC: 41A50; 47H10; 54H25

Keywords: coupled fixed point; mixed monotone property; ordered metric spaces

1 Introduction and preliminaries

Fixed point theory is one of the famous and traditional theories in mathematics and has a large number of applications. The Banach contraction mapping is one of the pivotal results of analysis. It is a very popular tool for solving existence problems in many different fields of mathematics. There are a lot of generalizations of the Banach contraction principle in the literature. Ran and Reurings [1] extended the Banach contraction principle in partially ordered sets with some applications to linear and nonlinear matrix equations. While Nieto and Rodŕiguez-López [2] extended the result of Ran and Reurings and applied their main theorems to obtain a unique solution for a first-order ordinary differential equation with periodic boundary conditions. Bhaskar and Lakshmikantham [3] introduced the concept of mixed monotone mappings and obtained some coupled fixed point results. Also, they applied their results on a first-order differential equation with periodic boundary conditions. Recently, many researchers have obtained fixed point, common fixed point, coupled fixed point and coupled common fixed point results in cone metric spaces, fuzzy metric spaces, intuitionistic fuzzy normed spaces, partially ordered metric spaces and others (see [1–25]).

Definition 1.1 Let (X,d) be a metric space and $F : X \times X \to X$ and $g : X \to X$, F and g are said to *commute* if $F(gx, gy) = g(F(x, y))$ for all $x, y \in X$.

Definition 1.2 Let (X,d) be a metric space and let $g: X \to X$, $F: X \times X \to X$. The mappings g and F are said to be *compatible* if $\lim_{n\to\infty} d(gF(x_n,y_n), F(gx_n,gy_n)) = 0$ and $\lim_{n\to\infty} d(gF(y_n,x_n), F(gy_n,gx_n)) = 0$ hold whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\lim_{n\to\infty} F(x_n,y_n) = \lim_{n\to\infty} gx_n$ and $\lim_{n\to\infty} F(y_n,x_n) = \lim_{n\to\infty} gy_n$.

Definition 1.3 Let $(X,\preceq)$ be a partially ordered set and $F: X \times X \to X$. The mapping F is said to be *non-decreasing* if for $x,y \in X$, $x \preceq y$ implies $F(x) \preceq F(y)$ and *non-increasing* if for $x,y \in X$, $x \preceq y$ implies $F(x) \succeq F(y)$.

Definition 1.4 Let $(X,\preceq)$ be a partially ordered set and $F: X \times X \to X$ and $g: X \to X$. The mapping F is said to have the *mixed g-monotone property* if $F(x,y)$ is monotone g-non-decreasing in x and monotone g-non-increasing in y, that is, for any $x,y \in X$,

$$x_1,x_2 \in X, \quad gx_1 \preceq gx_2 \quad \Rightarrow \quad F(x_1,y) \preceq F(x_2,y)$$

and

$$y_1,y_2 \in X, \quad gy_1 \preceq gy_2 \quad \Rightarrow \quad F(x,y_1) \succeq F(x,y_2).$$

If g = identity mapping in Definition 1.4, then the mapping F is said to have the *mixed monotone property*.

Recently, Đoric *et al.* [12] showed that the mixed monotone property in coupled fixed point results for mappings in ordered metric spaces can be replaced by another property which is often easy to check. In particular, it is automatically satisfied in the case of a totally ordered space, the case which is important in applications. Hence, these results can be applied in a much wider class of problems.

If elements x, y of a partially ordered set $(X,\preceq)$ are comparable (*i.e.*, $x \preceq y$ or $y \preceq x$ holds) we will write $x \asymp y$. Let $g: X \to X$ and $F: X \times X \to X$. We will consider the following condition:

$$\text{if } x,y,u,v \in X \text{ are such that } gx \asymp F(x,y) = gu, \text{ then } F(x,y) \asymp F(u,v).$$

If g is an identity mapping, for all x, y, v, if $x \asymp F(x,y)$ then $F(x,y) \asymp F(F(x,y),v)$.

Đoric *et al.* [12] gave some examples that these conditions may be satisfied when F does not have the g-mixed monotone property.

Definition 1.5 An element $(x,y) \in X \times X$ is called a *coupled coincidence point* of the mappings $F: X \times X \to X$ and $g: X \to X$ if $F(x,y) = gx$ and $F(y,x) = gy$.

If g = identity mapping in Definition 1.5, then $(x,y) \in X \times X$ is called a *coupled fixed point*.

The purpose of this paper is to establish some coupled coincidence point results in partially ordered metric spaces for a pair of mappings without mixed monotone property satisfying a contractive condition. Also, we present a result on the existence and uniqueness of coupled common fixed points. Also, we give an example to illustrate the main result in this paper. The results proved generalize some of the results of Bhaskar and Lakshmikantham [3], Choudhury *et al.* [10], Luong and Thuan [17] and Harjani *et al.* [13] for the mappings having no mixed monotone property.

2 Main results

2.1 Coupled common fixed point theorems

In this section, we prove some coupled common fixed point theorems in the context of ordered metric spaces.

We denote by Φ the set of functions $\phi : [0,\infty) \to [0,\infty)$ satisfying:

(i) ϕ is continuous;

(ii) $\phi(t) < t$ for all $t > 0$ and $\phi(t) = 0$ if and only if $t = 0$.

Theorem 2.1 *Let $(X,\preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X,d) is a complete metric space. Suppose that $F : X \times X \to X$ and $g : X \to X$ are self-mappings on X such that the following conditions hold:*

(i) *g is continuous and $g(X)$ is closed;*

(ii) *$F(X \times X) \subseteq g(X)$ and g and F are compatible;*

(iii) *for all $x,y,u,v \in X$, if $g(x) \asymp F(x,y) = gu$, then $F(x,y) \asymp F(u,v)$;*

(iv) *there exist $x_0, y_0 \in X$ such that $gx_0 \asymp F(x_0,y_0)$ and $gy_0 \asymp F(y_0,x_0)$;*

(v) *there exists a non-negative real number L such that*

$$d\bigl(F(x,y),F(u,v)\bigr) \le \phi\bigl(\max\bigl\{d(gx,gu),d(gy,gv)\bigr\}\bigr) + L\min\bigl\{d\bigl(F(x,y),gu\bigr),d\bigl(F(u,v),gx\bigr), d\bigl(F(x,y),gx\bigr),d\bigl(F(u,v),gu\bigr)\bigr\} \tag{2.1}$$

for all $x,y,u,v \in X$ with $gx \succeq gu$ and $gy \preceq gv$, where $\phi \in \Phi$;

(vi) (a) *F is continuous or* (b) *$x_n \to x$, when $n \to \infty$ in X, then $x_n \asymp x$ for sufficiently large n.*

Then there exist $x,y \in X$ such that $F(x,y) = g(x)$ and $F(y,x) = g(y)$, that is, F and g have a coupled coincidence point $(x,y) \in X \times X$.

Proof Using conditions (ii) and (iv), construct sequences $\{x_n\}$ and $\{y_n\}$ in X satisfying $gx_n = F(x_{n-1},y_{n-1})$ and $gy_n = F(y_{n-1},x_{n-1})$ for $n = 1,2,\ldots$.

By (iv), $gx_0 \asymp F(x_0,y_0) = gx_1$ and condition (iii) implies that $gx_1 = F(x_0,y_0) \asymp F(x_1,y_1) = gx_2$. Proceeding by induction, we get that $gx_{n-1} \asymp gx_n$, and similarly, $gy_{n-1} \asymp gy_n$ for each $n \in \mathbb{N}$.

Now from the contractive condition (2.1), we have

$$\begin{aligned} d(gx_{n+1},gx_n) &= d\bigl(F(x_n,y_n),F(x_{n-1},y_{n-1})\bigr) \\ &\le \phi\bigl(\max\bigl\{d(gx_n,gx_{n-1}),d(gy_n,gy_{n-1})\bigr\}\bigr) + L\min\bigl\{d\bigl(F(x_n,y_n),gx_{n-1}\bigr), \\ &\quad d\bigl(F(x_{n-1},y_{n-1}),gx_n\bigr),d\bigl(F(x_n,y_n),gx_n\bigr),d\bigl(F(x_{n-1},y_{n-1}),gx_{n-1}\bigr)\bigr\}, \end{aligned} \tag{2.2}$$

which implies that $d(gx_{n+1},gx_n) \le \phi(\max\{d(gx_n,gx_{n-1}),d(gy_n,gy_{n-1})\})$.

Similarly, we have $d(gy_{n+1},gy_n) \le \phi(\max\{d(gy_n,gy_{n-1}),d(gx_n,gx_{n-1})\})$.

Therefore, from the above two inequalities we have

$$\max\bigl\{d(gx_{n+1},gx_n),d(gy_{n+1},gy_n)\bigr\} \le \phi\bigl(\max\bigl\{d(gx_n,gx_{n-1}),d(gy_n,gy_{n-1})\bigr\}\bigr). \tag{2.3}$$

Since $\phi(t) < t$ for all $t > 0$ and $\phi(t) = 0$ if and only if $t = 0$, from (2.3) we have

$$\max\{d(gx_{n+1}, gx_n), d(gy_{n+1}, gy_n)\} \leq \max\{d(gx_n, gx_{n-1}), d(gy_n, gy_{n-1})\}.$$

Set $\varrho_n := \max\{d(gx_{n+1}, gx_n), d(gy_{n+1}, gy_n)\}$, then $\{\varrho_n\}$ is a non-increasing sequence of positive real numbers. Thus, there is $d \geq 0$ such $\lim_{n\to\infty} \varrho_n = d$.

Suppose that $d > 0$, letting $n \to \infty$ two sides of (2.3) and using the properties of ϕ, we have

$$d = \lim_{n\to\infty} d(gy_n, gy_{n+1}) \leq \lim_{n\to\infty} \phi\bigl(\max\{d(gy_{n+1}, gy_n), d(gx_{n+1}, gx_n)\}\bigr) = \phi(d) < d, \tag{2.4}$$

which is a contradiction. Hence $d = 0$, *i.e.*,

$$\lim_{n\to\infty} \varrho_n = \lim_{n\to\infty} \max\{d(gx_{n+1}, gx_n), d(gy_{n+1}, gy_n)\} = 0. \tag{2.5}$$

Now, we shall prove that $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences. Suppose, to the contrary, that at least one of $\{gx_n\}$ or $\{gy_n\}$ is not a Cauchy sequence. This means that there exists an $\epsilon > 0$ for which we can find subsequences $\{gx_{n(k)}\}$, $\{gx_{m(k)}\}$ of $\{gx_n\}$ and $\{gy_{n(k)}, gy_{m(k)}\}$ of $\{gy_n\}$ with $n(k) > m(k) \geq k$ such that

$$\max\{d(gx_{n(k)}, gx_{m(k)}), d\bigl(gy_{n(k)}, gy_m(k)\bigr)\} \geq \epsilon. \tag{2.6}$$

Further, corresponding to $m(k)$, we can choose $n(k)$ in such a way that it is the smallest integer with $n(k) > m(k) \geq k$ and satisfies (2.6). Then

$$\max\{d(gx_{n(k)-1}, gx_{m(k)}), d(gy_{n(k)-1}, gy_{m(k)})\} < \epsilon. \tag{2.7}$$

Using the triangle inequality and (2.7), we have

$$d(gx_{n(k)}, gx_{m(k)}) \leq d(gx_{n(k)}, gx_{n(k)-1}) + d(gx_{n(k)-1}, gx_{m(k)}) < d(gx_{n(k)}, gx_{n(k)-1}) + \epsilon \tag{2.8}$$

and

$$d(gy_{n(k)}, gy_{m(k)}) \leq d(gy_{n(k)}, gy_{n(k)-1}) + d(gy_{n(k)-1}, gy_{m(k)}) < d(gy_{n(k)}, gy_{n(k)-1}) + \epsilon. \tag{2.9}$$

From (2.6), (2.8) and (2.9), we have

$$\begin{aligned}\epsilon &\leq \max\{d(gx_{n(k)}, gx_{m(k)}), d(gy_{n(k)}, gy_{m(k)})\}\\ &< \max\{d(gx_{n(k)}, gx_{n(k)-1}), d(gy_{n(k)}, gy_{n(k)-1})\} + \epsilon.\end{aligned}$$

Letting $k \to \infty$ in the inequalities above and using (2.5), we get

$$\lim_{k\to\infty} \max\{d(gx_{n(k)}, gx_{m(k)}), d(gy_{n(k)}, gy_{m(k)})\} = \epsilon. \tag{2.10}$$

By the triangle inequality,

$$d(gx_{n(k)}, gx_{m(k)}) \leq d(gx_{n(k)}, gx_{n(k)-1}) + d(gx_{n(k)-1}, gx_{n(k)-1}) + d(gx_{m(k)-1}, gx_{m(k)})$$

and

$$d(gy_{n(k)}, gy_{m(k)}) \leq d(gy_{n(k)}, gy_{n(k)-1}) + d(gy_{n(k)-1}, gy_{m(k)-1}) + d(gy_{m(k)-1}, gy_{m(k)}).$$

From the last two inequalities and (2.6), we have

$$\begin{aligned}\epsilon &\leq \max\{d(gx_{n(k)}, gx_{m(k)}), d(gy_{n(k)}, gy_{m(k)})\} \\ &\leq \max\{d(gx_{n(k)}, gx_{n(k)-1}), d(gy_{n(k)}, gy_{n(k)-1})\} \\ &\quad + \max\{d(gx_{m(k)-1}, gx_{m(k)}), d(gy_{m(k)-1}, gy_{m(k)})\} \\ &\quad + \max\{d(gx_{n(k)-1}, gx_{m(k)-1}), d(gy_{n(k)-1}, gy_{m(k)-1})\}. \end{aligned} \tag{2.11}$$

Again, by the triangle inequality,

$$\begin{aligned} d(gx_{n(k)-1}, gx_{m(k)-1}) &\leq d(gx_{n(k)-1}, gx_{m(k)}) + d(gx_{m(k)}, gx_{m(k)-1}) \\ &< d(gx_{m(k)}, gx_{m(k)-1}) + \epsilon, \end{aligned}$$

and

$$\begin{aligned} d(gy_{n(k)-1}, gy_{m(k)-1}) &\leq d(gy_{n(k)-1}, gy_{m(k)}) + d(gy_{m(k)}, gy_{m(k)-1}) \\ &< d(gy_{m(k)}, gy_{m(k)-1}) + \epsilon. \end{aligned}$$

Therefore,

$$\begin{aligned} &\max\{d(gx_{n(k)-1}, gx_{m(k)-1}), d(gy_{n(k)-1}, gy_{m(k)-1})\} \\ &\quad < \max\{d(gx_{m(k)}, gx_{m(k)-1}), d(gy_{m(k)}, gy_{m(k)-1})\} + \epsilon. \end{aligned} \tag{2.12}$$

Taking $k \to \infty$ in (2.11) and (2.12) and using (2.5), (2.10), we have

$$\lim_{k\to\infty} \max\{d(gx_{n(k)-1}, gx_{m(k)-1}), d(gy_{n(k)-1}, gy_{m(k)-1})\} = \epsilon. \tag{2.13}$$

Since $n(k) > m(k)$, $gx_{n(k)-1} \succeq gx_{m(k)-1}$ and $gy_{n(k)-1} \preceq gy_{m(k)-1}$. Then from (2.1) we have

$$\begin{aligned} d(gx_{n(k)}, gx_{m(k)}) &= d(F(x_{n(k)-1}, y_{n(k)-1}), F(x_{m(k)-1}, y_{m(k)-1})) \\ &\leq \phi(\max\{d(gx_{n(k)-1}, gx_{m(k)-1}), d(gy_{n(k)-1}, gy_{m(k)-1})\}) \\ &\quad + L\min\{d(F(x_{n(k)-1}, y_{n(k)-1}), gx_{m(k)-1}), d(F(x_{m(k)-1}, y_{m(k)-1}), gx_{n(k)-1}), \\ &\quad d(F(x_{n(k)-1}, y_{n(k)-1}), gx_{n(k)-1}), d(F(x_{m(k)-1}, y_{m(k)-1}), gx_{m(k)-1})\} \\ &\leq \phi(\max\{d(gx_{n(k)-1}, gx_{m(k)-1}), d(gy_{n(k)-1}, gy_{m(k)-1})\}) \\ &\quad + L\min\{d(gx_{n(k)}, gx_{n(k)-1}), d(gx_{m(k)}, gx_{m(k)-1})\}. \end{aligned} \tag{2.14}$$

Similarly,

$$\begin{aligned} d(gy_{m(k)}, gy_{n(k)}) &\leq \phi(\max\{d(gx_{n(k)-1}, gx_{m(k)-1}), d(gy_{n(k)-1}, gy_{m(k)-1})\}) \\ &\quad + L\min\{d(gy_{m(k)}, gy_{m(k)-1}), d(gy_{n(k)}, gy_{n(k)-1})\}. \end{aligned} \tag{2.15}$$

From (2.14) and (2.15), we have

$$\begin{aligned}
&\max\{d(gx_{n(k)}, gx_{m(k)}), d(gy_{n(k)}, gy_{m(k)})\} \\
&\quad \le \phi(\max\{d(gx_{n(k)-1}, gx_{m(k)-1}), d(gy_{n(k)-1}, gy_{m(k)-1})\}) \\
&\qquad + L\min\{d(gx_{n(k)}, gx_{n(k)-1}), d(gx_{m(k)}, gx_{m(k)-1})\} \\
&\qquad + L\min\{d(gy_{m(k)}, gy_{m(k)-1}), d(gy_{n(k)}, gy_{n(k)-1})\}.
\end{aligned}$$

Letting $n \to \infty$ in the above inequality and using (2.5), (2.10), (2.13) and the properties of ϕ, we have

$$\epsilon \le \phi(\epsilon) + 2L\min\{0,0\} < \epsilon,$$

which is a contradiction. Therefore, $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences and since $g(X)$ is closed in a complete metric space (condition (i)), there exist $x, y \in g(X)$ such that $\lim_{n\to\infty} gx_n = \lim_{n\to\infty} F(x_{n-1}, y_{n-1}) = x$ and $\lim_{n\to\infty} gy_n = \lim_{n\to\infty} F(y_{n-1}, x_{n-1}) = y$.

Compatibility of F and g (condition (ii)) implies that

$$\lim_{n\to\infty} d(g(F(x_n, y_n)), F(gx_n, gy_n)) = 0$$

and

$$\lim_{n\to\infty} d(g(F(y_n, x_n)), F(gy_n, gx_n)) = 0.$$

Consider the two possibilities given in condition (vi).

(a) Suppose that F is continuous. Using the triangle inequality, we get that

$$d(gx, F(gx_n, gy_n)) \le d(gx, g(F(x_n, y_n))) + d(g(F(x_n, y_n)), F(gx_n, gy_n)).$$

By taking limit $n \to \infty$ and using the continuity of F and g, we have $d(gx, F(x,y)) = 0$, *i.e.*, $gx = F(x,y)$ and, in a similar way, we have $gy = F(y,x)$. Thus F and g have a coupled coincidence point.

(b) In this case $gx_n \asymp u = gx$ and $gy_n \asymp v = gy$ for some $x, y \in X$ and n sufficiently large. For such n, using (2.1) we get

$$\begin{aligned}
d(F(x,y), gx) &\le d(F(x,y), gx_{n+1}) + d(gx_{n+1}, gx) \\
&= d(F(x,y), F(x_n, y_n)) + d(gx_{n+1}, gx) \\
&\le \phi(\max\{d(gx, gx_n), d(gy, gy_n)\}) \\
&\quad + L\min\{d(F(x,y), gx_n), d(F(x_n, y_n), gx_n), \\
&\quad d(F(x,y), gx), d(F(x_n, y_n), gx_n)\} + d(gx_{n+1}, gx).
\end{aligned}$$

Taking $n \to \infty$ in the above inequality and using the compatibility of F and g and the properties of ϕ, we have $d(F(x,y), gx) \le \phi(\max\{0,0\}) + 0 + 0 = 0$. Hence $F(x,y) = gx$. Similarly, one can show that $F(y,x) = gy$. Hence the result. □

Remark 2.1 Very recently, using the equivalence of the three basic metrics, Samet *et al.* [19] show that many of the coupled fixed point theorems are immediate consequences of well-known fixed point theorems in the literature.

In our Theorem 2.1, it is easy to see that if $L \neq 0$ there is no equivalence and this theorem is not a consequence of a known fixed point theorem.

Remark 2.2 In the above theorem, condition (iii) is a substitution for the mixed monotone property that has been used in most of the coupled fixed point results so far. Note that this condition is trivially satisfied if the order $\preceq$ on X is total, which is the case in most of the examples in articles mentioned in the references.

If g is an identity mapping in the above theorem, we have the following result.

Corollary 2.2 *Let $(X, d, \leq)$ be a complete partially ordered metric space and let $F : X \times X \to X$. Suppose that the following hold*:

(i) *for all $x, y, v \in X$, if $x \asymp F(x, y)$, then $F(x, y) \asymp F(F(x, y), v)$*;
(ii) *there exist $x_0, y_0 \in X$ such that $x_0 \asymp F(x_0, y_0)$ and $y_0 \asymp F(y_0, x_0)$*;
(iii) *there exists a non-negative real number L such that*

$$\begin{aligned} d\bigl(F(x,y),F(u,v)\bigr) \leq{}& \phi\bigl(\max\bigl\{d(x,u),d(y,v)\bigr\}\bigr) \\ &+ L\min\bigl\{d\bigl(F(x,y),u\bigr),d\bigl(F(u,v),x\bigr), \\ &d\bigl(F(x,y),x\bigr),d\bigl(F(u,v),u\bigr)\bigr\} \end{aligned}$$

for all $x, y, u, v \in X$ with $x \succeq u$ and $y \preceq v$, where $\phi \in \Phi$;

(iv) (a) *F is continuous or* (b) *$x_n \to x$, when $n \to \infty$ in X, then $x_n \asymp x$ for sufficiently large n.*

Then there exist $x, y \in X$ such that $F(x, y) = x$ and $y = F(y, x)$, that is, F has a coupled fixed point $(x, y) \in X \times X$.

Remark 2.3 Letting $L = 0$, in inequality (2.1), for all $x, y, u, v \in X$, $\alpha, \beta \geq 0$, $\alpha + \beta < 1$, we have

$$\alpha d(x,u) + \beta d(y,v) \leq (\alpha + \beta)\max\bigl\{d(x,u),d(y,v)\bigr\} = \phi\bigl(\max\bigl\{d(x,u),d(y,v)\bigr\}\bigr),$$

where $\phi(t) = (\alpha + \beta)(t)$ for all $t \geq 0$ is in Φ. Hence Theorem 2.1 generalizes the corresponding coupled fixed point results of Bhaskar and Lakshmikantham [3], Choudhury *et al.* [10], Luong and Thuan [17] and Harjani *et al.* [13] for the mappings having no mixed monotone property.

Taking $L = 0$, we have the following result.

Corollary 2.3 *Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a metric d on X such that (X, d) is a complete metric space. Suppose that $F : X \times X \to X$ and $g : X \to X$ are self-mappings on X such that the following conditions hold*:

(i) *g is continuous and $g(X)$ is closed*;
(ii) *$F(X \times X) \subseteq g(X)$ and g and F are compatible*;

(iii) *for all* $x, y, u, v \in X$, *if* $g(x) \asymp F(x,y) = gu$, *then* $F(x,y) \asymp F(u,v)$;
(iv) *there exist* $x_0, y_0 \in X$ *such that* $gx_0 \asymp F(x_0,y_0)$ *and* $gy_0 \asymp F(y_0,x_0)$;
(v) *F and g satisfy*

$$d\bigl(F(x,y),F(u,v)\bigr) \leq \phi\bigl(\max\bigl\{d(gx,gu),d(gy,gv)\bigr\}\bigr) \tag{2.16}$$

for all $x, y, u, v \in X$ *with* $gx \succeq gu$ *and* $gy \preceq gv$, *where* $\phi \in \Phi$;
(vi) (a) *F is continuous or* (b) $x_n \to x$, *when* $n \to \infty$ *in* X, *then* $x_n \asymp x$ *for sufficiently large n*.

Then there exist $x, y \in X$ *such that* $F(x,y) = g(x)$ *and* $gy = F(y,x)$, *that is, F and g have a coupled coincidence point* $(x,y) \in X \times X$.

Corollary 2.4 *Let* $(X, \preceq)$ *be a partially ordered set and suppose that there exists a metric d on X such that* (X,d) *is a complete metric space. Suppose that* $F : X \times X \to X$ *and* $g : X \to X$ *are self-mappings on X such that following conditions hold*:
(i) *g is continuous and* $g(X)$ *is closed*;
(ii) $F(X \times X) \subseteq g(X)$ *and g and F are compatible*;
(iii) *for all* $x, y, u, v \in X$, *if* $g(x) \asymp F(x,y) = gu$, *then* $F(x,y) \asymp F(u,v)$;
(iv) *there exist* $x_0, y_0 \in X$ *such that* $gx_0 \asymp F(x_0,y_0)$ *and* $gy_0 \asymp F(y_0,x_0)$;
(v) *there exist non-negative real numbers* α, β *with* $\alpha + \beta < 1$ *such that*

$$d\bigl(F(x,y),F(u,v)\bigr) \leq \alpha d(gx,gu) + \beta d(gy,gv) \tag{2.17}$$

for all $x, y, u, v \in X$ *with* $gx \succeq gu$ *and* $gy \preceq gv$;
(vi) (a) *F is continuous or* (b) $x_n \to x$, *when* $n \to \infty$ *in* X, *then* $x_n \asymp x$ *for sufficiently large n*.

Then there exist $x, y \in X$ *such that* $F(x,y) = g(x)$ *and* $gy = F(y,x)$, *that is, F and g have a coupled coincidence point* $(x,y) \in X \times X$.

Taking $\alpha = \beta = k \in [0,1)$, we have the following result.

Corollary 2.5 *Let* $(X, \preceq)$ *be a partially ordered set and suppose that there exists a metric d on X such that* (X,d) *is a complete metric space. Suppose that* $F : X \times X \to X$ *and* $g : X \to X$ *are self-mappings on X such that following conditions hold*:
(i) *g is continuous and* $g(X)$ *is closed*;
(ii) $F(X \times X) \subseteq g(X)$ *and g and F are compatible*;
(iii) *for all* $x, y, u, v \in X$, *if* $g(x) \asymp F(x,y) = gu$, *then* $F(x,y) \asymp F(u,v)$;
(iv) *there exist* $x_0, y_0 \in X$ *such that* $gx_0 \asymp F(x_0,y_0)$ *and* $gy_0 \asymp F(y_0,x_0)$;
(v) *there exists* $k \in [0,1)$ *such that*

$$d\bigl(F(x,y),F(u,v)\bigr) \leq k\,\bigl[d(gx,gu) + d(gy,gv)\bigr] \tag{2.18}$$

for all $x, y, u, v \in X$ *with* $gx \succeq gu$ *and* $gy \preceq gv$;
(vi) (a) *F is continuous or* (b) $x_n \to x$, *when* $n \to \infty$ *in* X, *then* $x_n \asymp x$ *for sufficiently large n*.

Then there exist $x, y \in X$ *such that* $F(x,y) = g(x)$ *and* $gy = F(y,x)$, *that is, F and g have a coupled coincidence point* $(x,y) \in X \times X$.

Now, we shall prove the existence and uniqueness of a coupled common fixed point. Note that if $(X,\preceq)$ is a partially ordered set, then we endow the product space $X\times X$ with the following partial order relation:

$$\text{for } (x,y),(u,v)\in X\times X,\quad (u,v)\preceq(x,y)\quad\Leftrightarrow\quad x\preceq u,\qquad y\succeq v.$$

Theorem 2.6 *In addition to hypotheses of Theorem* 2.1, *suppose that*

(vii) *for every* $(x,y),(u,v)\in X\times X$, *there exists* $(w,z)\in X\times X$ *such that* $(F(w,z),F(z,w))$ *is comparable to* $(F(x,y),F(y,x))$ *and* $(F(u,v),F(v,u))$.

Then F *and* g *have a unique coupled common fixed point, that is, there exists a unique* $(p,q)\in X\times X$ *such that* $p=gp=F(p,q)$ *and* $q=gq=F(q,p)$.

Proof From Theorem 2.1, there exists $(x,y)\in X\times X$ such that $gx=F(x,y)$ and $gy=F(y,x)$. Suppose that there is also $(u,v)\in X\times X$ such that $gu=F(u,v)$ and $gv=F(v,u)$. We will prove that $gx=gu$ and $gy=gv$. Condition (vii) implies that there exists $(w,z)\in X\times X$ such that $(F(w,z),F(z,w))$ is comparable to both $(F(x,y),F(y,x))$ and $(F(u,v),F(v,u))$. Put $w_0=w$, $z_0=z$ and, analogously to the proof of Theorem 2.1, choose sequences $\{w_n\}$, $\{z_n\}$ satisfying

$$gw_n=F(w_{n-1},z_{n-1})\quad\text{and}\quad gz_n=F(z_{n-1},w_{n-1})$$

for $n\in\mathbb{N}$. Starting from $x_0=x$, $y_0=y$ and $u_0=u$, $v_0=v$, choose sequences $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$, $\{v_n\}$, satisfying $gx_n=F(x_{n-1},y_{n-1})$, $gy_n=F(y_{n-1},x_{n-1})$ and $gu_n=F(u_{n-1},v_{n-1})$, $gv_n=F(v_{n-1},u_{n-1})$ for $n\in\mathbb{N}$, taking into account properties of coincidence points, it is easy to see that this can be done so that $x_n=x$, $y_n=y$ and $u_n=u$, $v_n=v$, *i.e.*,

$$gx_n=F(x,y),\qquad gy_n=F(y,x)\quad\text{and}\quad gu_n=F(u,v),\qquad gv_n=F(v,u)\quad\text{for } n\in\mathbb{N}.$$

Since $(F(x,y),F(y,x))=(gx,gy)$ and $(F(w,z),F(z,w))=(gw_1,gz_1)$ are comparable, then $gx\asymp gw_1$ and $gy\asymp gz_1$, and, in a similar way, we have $gx\asymp gw_n$ and $gy\asymp gz_n$. Thus from (2.1) we have

$$\begin{aligned} d(gx,gw_{n+1}) &= d\bigl(F(x,y),F(w_n,z_n)\bigr)\\ &\le \phi\bigl(\max\bigl\{d(gx,gw_n),d(gy,gz_n)\bigr\}\bigr)\\ &\quad + L\min\bigl\{d\bigl(F(x,y),gw_n\bigr),d\bigl(F(w_n,z_n),gx\bigr),\\ &\quad d\bigl(F(x,y),gx\bigr),d\bigl(F(w_n,z_n),gw_n\bigr)\bigr\}, \end{aligned}\tag{2.19}$$

which implies that $d(gx,gw_{n+1})\le\phi(\max\{d(gx,gw_n),d(gy,gz_n)\})$.

Similarly, we can prove that $d(gy,gz_{n+1})\le\phi(\max\{d(gx,gw_n),d(gy,gz_n)\})$.

Therefore, from the above two inequalities we have

$$\max\bigl\{d(gx,gw_{n+1}),d(gy,gz_{n+1})\bigr\}\le\phi\bigl(\max\bigl\{d(gx,gw_n),d(gy,gz_n)\bigr\}\bigr).\tag{2.20}$$

Since $\phi(t)\le t$ for all $t\ge 0$, from (2.20) we have

$$\max\bigl\{d(gx,gw_{n+1}),d(gy,gz_{n+1})\bigr\}\le\max\bigl\{d(gx,gw_n),d(gy,gz_n)\bigr\}.$$

Hence the sequence $\{\delta_n\}$ defined by $\delta_n := \max\{d(gx, gw_{n+1}), d(gy, gz_{n+1})\}$ is non-negative and decreasing and so $\lim_{n\to\infty} \delta_n = \delta$ for some $\delta \geq 0$.

Now, we show that $\delta = 0$. Assume that $\delta > 0$, letting $n \to \infty$ two sides of (2.20) and using the properties of ϕ, we have

$$\delta = \lim_{n\to\infty} d(gy, gz_{n+1}) \leq \lim_{n\to\infty} \phi\bigl(\max\bigl\{d(gx, gw_n), d(gy, gz_n)\bigr\}\bigr) = \phi(\delta) < \delta, \tag{2.21}$$

which is a contradiction. Hence $d = 0$, *i.e.*,

$$\lim_{n\to\infty} d(gx, gw_{n+1}) = 0 = \lim_{n\to\infty} d(gy, gz_{n+1}) = 0. \tag{2.22}$$

Similarly, we can prove that

$$\lim_{n\to\infty} d(gu, gw_{n+1}) = 0 = \lim_{n\to\infty} d(gv, gz_{n+1}). \tag{2.23}$$

Using relations (2.22) and (2.23), together with the triangle inequality, we have $d(gx, gu) = 0$ and $d(gy, gv) = 0$ and so $gx = gu$ and $gy = gv$.

Denote $gx = p$ and $gy = q$. So, we have that

$$gp = g(gx) = gF(x, y) \quad \text{and} \quad gq = g(gy) = gF(y, x). \tag{2.24}$$

By definition of the sequences $\{x_n\}$ and $\{y_n\}$ we have

$$gx_n = F(x, y) = F(x_{n-1}, y_{n-1}) \quad \text{and} \quad gy_n = F(y, x) = F(y_{n-1}, x_{n-1}),$$

and so

$$F(x_{n-1}, y_{n-1}) \to F(x, y) \quad \text{and} \quad gx_n \to F(x, y),$$

as well as

$$F(y_{n-1}, x_{n-1}) \to F(y, x) \quad \text{and} \quad gy_n \to F(y, x).$$

Compatibility of g and F implies that

$$d\bigl(gF(x_n, y_n), F(gx_n, gy_n)\bigr) \to 0, \quad n \to \infty,$$

i.e., $gF(x, y) = F(gx, gy)$. This together with (2.24) implies that $gp = F(p, q)$ and, in a similar way, $gq = F(q, p)$. Thus, we have another coincidence, and by the property we have just proved, it follows that $gp = gx = p$ and $gq = gy = q$. In other words, $p = gp = F(p, q)$ and $q = gq = F(q, p)$, and (p, q) is a common coupled fixed point of g and F.

To prove the uniqueness, assume that (r, s) is another coupled common fixed point. Then by (2.24) we have $r = gr = gp = p$ and $s = gs = gq = q$. Hence we get the result. $\square$

Example 2.7 Let $X = [0, 1]$. Then $(X, \leq)$ is a partially ordered set with the natural ordering of real numbers. Let $d(x, y) = |x - y|$ for all $x, y \in X$. Define a mapping $g : X \to X$ by $g(x) = x^2$

and a mapping $F : X \times X \to X$ by

$$F(x,y) = \begin{cases} \frac{x^2-2y^2}{4} & x \geq y, \\ 0 & x < y. \end{cases}$$

Then it is easy to check all the conditions of Theorems 2.1 and 2.6. In particular, we will check that g and F are compatible.

Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X such that

$$\lim_{n\to\infty} gx_n = \lim_{n\to\infty} F(x_n, y_n) = a \quad \text{and} \quad \lim_{n\to\infty} gy_n = \lim_{n\to\infty} F(y_n, x_n) = b.$$

Then $\frac{a-2b}{4} = a$ and $\frac{b-2a}{4} = b$, where from it follows that $a = b = 0$. Then

$$d\big(gF(x_n, y_n), F(gx_n, gy_n)\big) = \left| \left(\frac{x_n^2 - 2y_n^2}{4}\right)^2 - \frac{x_n^4 - 2y_n^4}{4} \right| \to 0 \quad (n \to \infty),$$

and similarly $d(gF(y_n, x_n), F(gy_n, gx_n)) \to 0$.

Now, we verify inequality (2.1) of Theorem 2.1 for $\phi(t) = \frac{3}{4}t$, $t > 0$ and $L = [0, \infty)$ for all $x, y, u, v \in X$ with $gx \succeq gu$ and $gy \preceq gv$.

$$\begin{aligned} d\big(F(x,y), F(u,v)\big) &= \left| \frac{x^2 - 2y^2}{4} - \frac{u^2 - 2v^2}{4} \right| \\ &\leq \frac{1}{4}\left|x^2 - u^2\right| + \frac{2}{4}\left|y^2 - v^2\right| \\ &\leq \frac{3}{4} \max\left\{\left|x^2 - u^2\right|, \left|y^2 - v^2\right|\right\} \\ &\leq \phi\big(\max\big\{d(gx, gu), d(gy, gv)\big\}\big) \\ &\quad + L \min\big\{d\big(F(x,y), gu\big), d\big(F(u,v), gx\big), \\ &\quad d\big(F(x,y), gx\big), d\big(F(u,v), gu\big)\big\}. \end{aligned}$$

Thus there exists a common coupled fixed point $(0, 0)$ of the mappings g and F. Note that F does not satisfy the g-mixed monotone property. Also, g and F do not commute.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

Author details
[1]Department of Mathematics, Khalsa College of Engineering & Technology, Punjab Technical University, Ranjit Avenue, Amritsar, 143001, India. [2]Department of Mathematics and Computer Science, Cankaya University, Ankara, Turkey.

Received: 6 September 2013 Accepted: 5 January 2014 Published: 10 Feb 2014

References

1. Ran, ACM, Reurings, MCB: A fixed point theorem in partially ordered sets and some applications to matrix equations. Proc. Am. Math. Soc. **132**(5), 1435-1443 (2004)
2. Nieto, JJ, López, RR: Contractive mapping theorems in partially ordered sets and applications to ordinary differential equations. Order **22**, 223-239 (2005)

3. Bhaskar, TG, Lakshmikantham, V: Fixed point theorems in partially ordered metric spaces and applications. Nonlinear Anal. **65**, 1379-1393 (2006)
4. Agarwal, RP, El-Gebeily, MA, O'Regan, D: Generalized contractions in partially ordered metric spaces. Appl. Anal. **87**, 1-8 (2008)
5. Chandok, S: Some common fixed point theorems for generalized f-weakly contractive mappings. J. Appl. Math. Inform. **29**, 257-265 (2011)
6. Chandok, S, Cho, YJ: Coupled common fixed point theorems for mixed g-monotone mappings in partially ordered metric spaces. An. Univ. Oradea, Fasc. Mat. 21(1) (2014, in press)
7. Chandok, S, Dinu, S: Common fixed points for weak ψ-contractive mappings in ordered metric spaces with applications. Abstr. Appl. Anal. **2013**, Article ID 879084 (2013)
8. Chandok, S, Khan, MS, Abbas, M: Common fixed point theorems for nonlinear weakly contractive mappings. Ukr. Math. J. (2013, in press)
9. Chandok, S, Khan, MS, Rao, KPR: Some coupled common fixed point theorems for a pair of mappings satisfying a contractive condition of rational type without monotonicity. Int. J. Math. Anal. **7**(9), 433-440 (2013)
10. Choudhury, BS, Metiya, N, Kundu, A: Coupled coincidence point theorems in ordered metric spaces. Ann. Univ. Ferrara **57**, 1-16 (2011)
11. Choudhury, BS, Kundu, A: A coupled coincidence point result in partially ordered metric spaces for compatible mappings. Nonlinear Anal. **73**, 2524-2531 (2010)
12. Đoric, D, Kadelburg, Z, Radenović, S: Coupled fixed point results for mappings without mixed monotone property. Appl. Math. Lett. (2012). doi:10.1016/j.aml.2012.02.022
13. Harjani, J, Lopez, B, Sadarangani, K: Fixed point theorems for mixed monotone operators and applications to integral equations. Nonlinear Anal. **74**, 1749-1760 (2011)
14. Jachymski, J: Equivalent conditions for generalized contractions on (ordered) metric spaces. Nonlinear Anal. **74**, 768-774 (2011)
15. Kim, JK, Chandok, S: Coupled common fixed point theorems for generalized nonlinear contraction mappings with the mixed monotone property in partially ordered metric spaces. Fixed Point Theory Appl. **2013**, Article ID 307 (2013)
16. Lakshmikantham, V, Ćirić, L: Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces. Nonlinear Anal. **70**, 4341-4349 (2009)
17. Luong, NV, Thuan, NX: Coupled fixed point theorems in partially ordered metric spaces. Bull. Math. Anal. Appl. **2**, 16-24 (2010)
18. Samet, B: Coupled fixed point theorems for a generalized Meir-Keeler contraction in partially ordered metric spaces. Nonlinear Anal. **72**, 4508-4517 (2010)
19. Samet, B, Karapınar, E, Aydi, H, Rajic, VC: Discussion on some coupled fixed point theorems. Fixed Point Theory Appl. **2013**, Article ID 50 (2013)
20. Karapınar, E, Luong, NV, Thuan, NX: Coupled coincidence points for mixed monotone operators in partially ordered metric spaces. Arab. J. Math. **1**, 329-339 (2012)
21. Kutbi, MA, Azam, A, Ahmad, J, Di Bari, C: Some common coupled fixed point results for generalized contraction in complex-valued metric spaces. J. Appl. Math. **2013**, Article ID 352927 (2013)
22. Sintunavarat, W, Cho, YJ, Kumam, P: Coupled coincidence point theorems for contractions without commutative condition in intuitionistic fuzzy normed spaces. Fixed Point Theory Appl. **2011**, Article ID 81 (2011)
23. Sintunavarat, W, Petruşel, A, Kumam, P: Common coupled fixed point theorems for w-compatible mappings without mixed monotone property. Rend. Circ. Mat. Palermo **61**, 361-383 (2012)
24. Sintunavarat, W, Kumam, P, Cho, YJ: Coupled fixed point theorems for nonlinear contractions without mixed monotone property. Fixed Point Theory Appl. **2012**, Article ID 170 (2012)
25. Agarwal, RP, Sintunavarat, W, Kumam, P: Coupled coincidence point and common coupled fixed point theorems lacking the mixed monotone property. Fixed Point Theory Appl. **2013**, Article ID 22 (2013)

10.1186/1029-242X-2014-61
Cite this article as: Chandok and Tas: An original coupled coincidence point result for a pair of mappings without MMP. *Journal of Inequalities and Applications* 2014, 2014:61