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Coupled common fixed point results involving a (φ,ψ)-contractive condition for mixed g-monotone operators in partially ordered metric spaces

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Abstract

In the setting of partially ordered metric spaces, using the notion of compatible mappings, we establish the existence and uniqueness of coupled common fixed points involving a (φ,ψ)-contractive condition for mixed g-monotone operators. Our results extend and generalize the well-known results of Berinde (Nonlinear Anal. TMA 74:7347-7355, 2011; Nonlinear Anal. TMA 75:3218-3228, 2012) and weaken the contractive conditions involved in the results of Alotaibi *et al.* (Fixed Point Theory Appl. 2011:44, 2011), Bhaskar *et al.* (Nonlinear Anal. TMA 65:1379-1393, 2006), and Luong *et al.* (Nonlinear Anal. TMA 74:983-992, 2011). The effectiveness of the presented work is validated with the help of suitable examples.

MSC: 54H10; 54H25

Keywords: partially ordered set; compatible mappings; g-mixed monotone mappings; coupled coincidence point; coupled common fixed point

1 Introduction and preliminaries

Bhaskar and Lakshmikantham [1] introduced the notion of coupled fixed points and proved some coupled fixed point theorems for a mapping with the mixed monotone property in the setting of partially ordered metric spaces. These concepts are defined as follows.

Definition 1.1 [1] Let $(X,\leq)$ be a partially ordered set and $F : X\times X\to X$. The mapping F is said to have the mixed monotone property if $F(x,y)$ is monotone non-decreasing in x and monotone non-increasing in y; that is, for any $x,y\in X$,

$$x_1,x_2\in X,\qquad x_1\leq x_2\quad\text{implies}\quad F(x_1,y)\leq F(x_2,y)$$

and

$$y_1,y_2\in X,\qquad y_1\leq y_2\quad\text{implies}\quad F(x,y_1)\geq F(x,y_2).$$

Definition 1.2 [1] An element $(x,y)\in X\times X$ is called a coupled fixed point of the mapping $F : X\times X\to X$ if $F(x,y)=x$ and $F(y,x)=y$.

Bhaskar and Lakshmikantham [1] proved the following results.

Theorem 1.3 [1] *Let $(X, \leq)$ be a partially ordered set and suppose there exists a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \to X$ be a continuous mapping having the mixed monotone property on X. Assume that there exists a $k \in [0, 1)$ with*

$$d\big(F(x,y), F(u,v)\big) \leq \frac{k}{2}\big[d(x,u) + d(y,v)\big] \tag{1.1}$$

for all $x \geq u$ and $y \leq v$.

If there exist two elements $x_0, y_0 \in X$ with $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$, then there exist $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$.

Theorem 1.4 [1] *Let $(X, \leq)$ be a partially ordered set and suppose there exists a metric d on X such that (X, d) is a complete metric space. Assume that X has the following property:*

(i) *if a non-decreasing sequence $\{x_n\} \to x$, then $x_n \leq x$ for all n,*

(ii) *if a non-increasing sequence $\{y_n\} \to y$, then $y \leq y_n$ for all n.*

Let $F : X \times X \to X$ be a mapping having the mixed monotone property on X. Assume that there exists a $k \in [0, 1)$ with the condition (1.1). *If there exist two elements $x_0, y_0 \in X$ with $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$, then there exist $x, y \in X$ such that $x = F(x, y)$ and $y = F(y, x)$.*

Lakshmikantham and Ćirić [2] extended the notion of mixed monotone property to mixed g-monotone property and generalized the results of Bhaskar and Lakshmikantham [1] by establishing the existence of coupled coincidence point results using a pair of commutative maps.

Definition 1.5 [2] Let $(X, \leq)$ be a partially ordered set and $F : X \times X \to X$ and $g : X \to X$. We say F has the mixed g-monotone property if $F(x, y)$ is monotone g-nondecreasing in its first argument and is monotone g-nonincreasing in its second argument; that is, for any $x, y \in X$,

$$x_1, x_2 \in X, \qquad gx_1 \leq gx_2 \quad \text{implies} \quad F(x_1, y) \leq F(x_2, y)$$

and

$$y_1, y_2 \in X, \qquad gy_1 \leq gy_2 \quad \text{implies} \quad F(x, y_1) \geq F(x, y_2).$$

Definition 1.6 [2] An element $(x, y) \in X \times X$ is called a coupled coincidence point of the mappings $F : X \times X \to X$ and $g : X \to X$ if $F(x, y) = gx$ and $F(y, x) = gy$.

Definition 1.7 [2] An element $(x, y) \in X \times X$ is called a coupled common fixed point of the mappings $F : X \times X \to X$ and $g : X \to X$ if $x = gx = F(x, y)$ and $y = gy = F(y, x)$.

Definition 1.8 [2] Let X be a non-empty set and $F : X \times X \to X$ and $g : X \to X$. We say F and g are commutative if $gF(x, y) = F(gx, gy)$ for all $x, y \in X$.

Later, Choudhury and Kundu [3] introduced the notion of compatibility in the context of coupled coincidence point problems and used the notion to improve the results of Lakshmikantham and Ćirić [2].

Definition 1.9 [3] The mappings $F : X \times X \to X$ and $g : X \to X$ are said to be compatible if

$$\lim_{n\to\infty} d\big(gF(x_n, y_n), F(gx_n, gy_n)\big) = 0 \quad \text{and} \quad \lim_{n\to\infty} d\big(gF(y_n, x_n), F(gy_n, gx_n)\big) = 0$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} gx_n = x$ and $\lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} gy_n = y$ for some $x, y \in X$.

In recent years, following Bhaskar and Lakhsmikantham [1], the existence and uniqueness of coupled fixed points under more general contractive conditions were established by various authors. One can refer to [2, 4–15].

In order to generalize the results of Bhaskar and Lakshmikantham [1], Luong and Thuan [7] considered the following class of control functions.

Definition 1.10 [7] Let Φ denote the class of functions $\varphi : [0,\infty) \to [0,\infty)$ which satisfy

(φ_1) φ is continuous and non-decreasing;
(φ_2) $\varphi(t) = 0$ if and only if $t = 0$;
(φ_3) $\varphi(t + s) \le \varphi(t) + \varphi(s)$, for all $t, s \in [0,\infty)$.

Definition 1.11 [7] Let Ψ denote the class of functions $\psi : [0,\infty) \to [0,\infty)$ which satisfy

(i_ψ) $\lim_{t\to r} \psi(t) > 0$ for all $r > 0$ and $\lim_{t\to 0+} \psi(t) = 0$.

The contractive condition considered by Luong and Thuan [7] is given below:

$$\varphi\big(d\big(F(x, y), F(u, v)\big)\big) \le \frac{1}{2}\varphi\big(d(x, u) + d(y, v)\big) - \psi\left(\frac{d(x, u) + d(y, v)}{2}\right), \tag{1.2}$$

where $\varphi \in \Phi$, $\psi \in \Psi$ and $x \ge u$, $y \le v$.

On the other hand, Alotaibi and Alsulami [16] extended the results of Luong and Thuan [7] for a compatible pair (F, g), where $F : X \times X \to X$ and $g : X \to X$ are the maps satisfying the following contractive condition:

$$\varphi\big(d\big(F(x, y), F(u, v)\big)\big) \le \frac{1}{2}\varphi\big(d(gx, gu) + d(gy, gv)\big) - \psi\left(\frac{d(gx, gu) + d(gy, gv)}{2}\right), \tag{1.3}$$

with $\varphi \in \Phi$, $\psi \in \Psi$ and $gx \ge gu$, $gy \le gv$.

We consider the class Φ redefined by Berinde [5] as follows.

Definition 1.12 [5] Let Φ denote the class of functions $\varphi : [0,\infty) \to [0,\infty)$ which satisfy

(i_φ) φ is continuous and (strictly) increasing;
(ii_φ) $\varphi(t) < t$ for all $t > 0$;
(iii_φ) $\varphi(t + s) \le \varphi(t) + \varphi(s)$ for all $t, s \in [0,\infty)$.

Note that by (i_φ) and (ii_φ), we have $\varphi(t) = 0$ if and only if $t = 0$.

Berinde [5] weakened the contractive conditions (1.1) and (1.2) by considering the more general one

$$\varphi\left(\frac{d(F(x,y),F(u,v)) + d(F(y,x),F(v,u))}{2}\right) \le \varphi\left(\frac{d(x,u) + d(y,v)}{2}\right) - \psi\left(\frac{d(x,u) + d(y,v)}{2}\right) \qquad (1.4)$$

for a mixed monotone mapping $F : X \times X \to X$, $x \ge u$, $y \le v$, where $\varphi \in \Phi$ and $\psi \in \Psi$.

The present work extends and generalizes several results presented in the literature of fixed point theory. Our theorems directly derive the main results of Berinde [4, 5]. We give suitable examples to show how our results extend the well-known results of Alotaibi *et al.* [16], Bhaskar *et al.* [1] and Luong *et al.* [7] by significantly weakening the involved contractive condition.

2 Main results

Theorem 2.1 *Let $(X, \le)$ be a partially ordered set and suppose there exists a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \to X$, $g : X \to X$ be two maps with F having the mixed g-monotone property on X such that there exist two elements $x_0, y_0 \in X$ with $gx_0 \le F(x_0, y_0)$ and $gy_0 \ge F(y_0, x_0)$. Suppose there exist $\varphi \in \Phi$ and $\psi \in \Psi$ such that*

$$\varphi\left(\frac{d(F(x,y),F(u,v)) + d(F(y,x),F(v,u))}{2}\right) \le \varphi\left(\frac{d(gx,gu) + d(gy,gv)}{2}\right) - \psi\left(\frac{d(gx,gu) + d(gy,gv)}{2}\right) \qquad (2.1)$$

for all $x, y, u, v \in X$ with $gx \ge gu$ and $gy \le gv$.

Suppose that $F(X \times X) \subseteq g(X)$, g is continuous and the pair (F, g) is compatible.

Also suppose either

(a) *F is continuous, or*

(b) *X has the following property:*

  (i) *if a non-decreasing sequence $\{x_n\} \to x$, then $gx_n \le gx$ for all n;*

  (ii) *if a non-increasing sequence $\{y_n\} \to y$, then $gy \le gy_n$ for all n.*

Then there exist $x, y \in X$ such that $gx = F(x, y)$ and $gy = F(y, x)$; that is, F and g have a coupled coincidence point in X.

Proof Let $x_0, y_0 \in X$ such that $gx_0 \le F(x_0, y_0)$ and $gy_0 \ge F(y_0, x_0)$. Since $F(X \times X) \subseteq g(X)$, we can choose $x_1, y_1 \in X$ such that $gx_1 = F(x_0, y_0)$, $gy_1 = F(y_0, x_0)$. Again, we can choose $x_2, y_2 \in X$ such that $gx_2 = F(x_1, y_1)$, $gy_2 = F(y_1, x_1)$.

Continuing this process, we can construct sequences $\{gx_n\}$ and $\{gy_n\}$ in X such that

$$gx_{n+1} = F(x_n, y_n), \qquad gy_{n+1} = F(y_n, x_n) \quad \text{for all } n \ge 0. \qquad (2.2)$$

We shall prove, for all $n \ge 0$, that

$$gx_n \le gx_{n+1}, \qquad (2.3)$$

$$gy_n \ge gy_{n+1}. \qquad (2.4)$$

Since $gx_0 \leq F(x_0, y_0)$ and $gy_0 \geq F(y_0, x_0)$, $gx_1 = F(x_0, y_0)$, $gy_1 = F(y_0, x_0)$, we have $gx_0 \leq gx_1$, $gy_0 \geq gy_1$; that is, (2.3) and (2.4) hold for $n = 0$.

Suppose that (2.3) and (2.4) hold for some $n > 0$, *i.e.*, $gx_n \leq gx_{n+1}$, $gy_n \geq gy_{n+1}$. As F has the mixed g-monotone property, by (2.2), we have

$$gx_{n+1} = F(x_n, y_n) \leq F(x_{n+1}, y_n) \leq F(x_{n+1}, y_{n+1}) = gx_{n+2},$$

and

$$gy_{n+1} = F(y_n, x_n) \geq F(y_{n+1}, x_n) \geq F(y_{n+1}, x_{n+1}) = gy_{n+2};$$

that is,

$$gx_{n+1} \leq gx_{n+2} \quad \text{and} \quad gy_{n+1} \geq gy_{n+2}.$$

Then, by mathematical induction, it follows that (2.3) and (2.4) hold for all $n \geq 0$.

If, for some $n \geq 0$, we have $(gx_{n+1}, gy_{n+1}) = (gx_n, gy_n)$, then $F(x_n, y_n) = gx_n$ and $F(y_n, x_n) = gy_n$; that is, F and g have a coincidence point. So, now onwards, we suppose $(gx_{n+1}, gy_{n+1}) \neq (gx_n, gy_n)$ for all $n \geq 0$; that is, we suppose that either $gx_{n+1} = F(x_n, y_n) \neq gx_n$ or $gy_{n+1} = F(y_n, x_n) \neq gy_n$.

Since $gx_n \geq gx_{n-1}$ and $gy_n \leq gy_{n-1}$, by (2.1) and (2.2), we have, for all $n \geq 0$, that

$$\begin{aligned}
&\varphi\left(\frac{d(gx_{n+1}, gx_n) + d(gy_{n+1}, gy_n)}{2}\right) \\
&\quad = \varphi\left(\frac{d(F(x_n, y_n), F(x_{n-1}, y_{n-1})) + d(F(y_n, x_n), F(y_{n-1}, x_{n-1}))}{2}\right) \\
&\quad \leq \varphi\left(\frac{d(gx_n, gx_{n-1}) + d(gy_n, gy_{n-1})}{2}\right) - \psi\left(\frac{d(gx_n, gx_{n-1}) + d(gy_n, gy_{n-1})}{2}\right).
\end{aligned} \tag{2.5}$$

Since ψ is non-negative, we have

$$\varphi\left(\frac{d(gx_{n+1}, gx_n) + d(gy_{n+1}, gy_n)}{2}\right) \leq \varphi\left(\frac{d(gx_n, gx_{n-1}) + d(gy_n, gy_{n-1})}{2}\right).$$

By the monotonicity of φ, we have

$$\frac{d(gx_{n+1}, gx_n) + d(gy_{n+1}, gy_n)}{2} \leq \frac{d(gx_n, gx_{n-1}) + d(gy_n, gy_{n-1})}{2}.$$

Let $R_n = \frac{d(gx_{n+1}, gx_n) + d(gy_{n+1}, gy_n)}{2}$, then $\{R_n\}$ is a monotone decreasing sequence of non-negative real numbers. Therefore, there exists some $R \geq 0$ such that

$$\lim_{n \to \infty} R_n = \lim_{n \to \infty} \left[\frac{d(gx_{n+1}, gx_n) + d(gy_{n+1}, gy_n)}{2}\right] = R. \tag{2.6}$$

We claim that $R = 0$.

On the contrary, suppose that $R > 0$.

Taking limit as $n \to \infty$ on both sides of (2.5) and using the properties of φ and ψ, we have

$$\begin{aligned}\varphi(R) &= \lim_{n\to\infty}\varphi(R_n) \le \lim_{n\to\infty}\bigl[\varphi(R_{n-1}) - \psi(R_{n-1})\bigr]\\ &= \varphi(R) - \lim_{R_{n-1}\to R}\psi(R_{n-1}) < \varphi(R),\end{aligned}$$

a contradiction.

Thus, $R = 0$; that is,

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty}\left[\frac{d(gx_{n+1}, gx_n) + d(gy_{n+1}, gy_n)}{2}\right] = 0. \tag{2.7}$$

Next, we shall show that $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences.

If possible, suppose that at least one of $\{gx_n\}$ and $\{gy_n\}$ is not a Cauchy sequence. Then there exists an $\varepsilon > 0$ for which we can find subsequences $\{gx_{n(k)}\}$, $\{gx_{m(k)}\}$ of $\{gx_n\}$ and $\{gy_{n(k)}\}$, $\{gy_{m(k)}\}$ of $\{gy_n\}$ with $n(k) > m(k) \ge k$ such that

$$r_k = \frac{d(gx_{n(k)}, gx_{m(k)}) + d(gy_{n(k)}, gy_{m(k)})}{2} \ge \varepsilon. \tag{2.8}$$

Further, corresponding to $m(k)$, we can choose $n(k)$ in such a way that it is the smallest integer with $n(k) > m(k) \ge k$ and satisfies (2.8). Then

$$\frac{d(gx_{n(k)-1}, gx_{m(k)}) + d(gy_{n(k)-1}, gy_{m(k)})}{2} < \varepsilon. \tag{2.9}$$

By (2.8), (2.9) and the triangle inequality, we have

$$\begin{aligned}\varepsilon \le r_k &= \frac{d(gx_{n(k)}, gx_{m(k)}) + d(gy_{n(k)}, gy_{m(k)})}{2}\\ &\le \frac{\{d(gx_{n(k)}, gx_{n(k)-1}) + d(gx_{n(k)-1}, gx_{m(k)}) + d(gy_{n(k)}, gy_{n(k)-1}) + d(gy_{n(k)-1}, gy_{m(k)})\}}{2}\\ &< \frac{d(gx_{n(k)}, gx_{n(k)-1}) + d(gy_{n(k)}, gy_{n(k)-1})}{2} + \varepsilon.\end{aligned}$$

Letting $k \to \infty$ and using (2.7) in the last inequality, we have

$$\lim_{k\to\infty} r_k = \lim_{k\to\infty}\left[\frac{d(gx_{n(k)}, gx_{m(k)}) + d(gy_{n(k)}, gy_{m(k)})}{2}\right] = \varepsilon. \tag{2.10}$$

Again, by the triangle inequality

$$\begin{aligned} r_k &= \frac{d(gx_{n(k)}, gx_{m(k)}) + d(gy_{n(k)}, gy_{m(k)})}{2}\\ &\le \frac{\left\{\begin{array}{l} d(gx_{n(k)}, gx_{n(k)+1}) + d(gx_{n(k)+1}, gx_{m(k)+1}) + d(gx_{m(k)+1}, gx_{m(k)})\\ + d(gy_{n(k)}, gy_{n(k)+1}) + d(gy_{n(k)+1}, gy_{m(k)+1}) + d(gy_{m(k)+1}, gy_{m(k)})\end{array}\right\}}{2}\\ &= R_{n(k)} + R_{m(k)} + \frac{d(gx_{n(k)+1}, gx_{m(k)+1}) + d(gy_{n(k)+1}, gy_{m(k)+1})}{2}.\end{aligned}$$

By the monotonicity of φ and the property (iii$_\varphi$), we have

$$\varphi(r_k) \leq \varphi(R_{n(k)}) + \varphi(R_{m(k)}) + \varphi\left(\frac{d(gx_{n(k)+1}, gx_{m(k)+1}) + d(gy_{n(k)+1}, gy_{m(k)+1})}{2}\right). \tag{2.11}$$

Since $n(k) > m(k)$, $gx_{n(k)} \geq gx_{m(k)}$ and $gy_{n(k)} \leq gy_{m(k)}$.

Then by (2.1) and (2.2), we have

$$\begin{aligned}
&\varphi\left(\frac{d(gx_{n(k)+1}, gx_{m(k)+1}) + d(gy_{n(k)+1}, gy_{m(k)+1})}{2}\right)\\
&\quad = \varphi\left(\frac{d(F(x_{n(k)}, y_{n(k)}), F(x_{m(k)}, y_{m(k)})) + d(F(y_{n(k)}, x_{n(k)}), F(y_{m(k)}, x_{m(k)}))}{2}\right)\\
&\quad \leq \varphi\left(\frac{d(gx_{n(k)}, gx_{m(k)}) + d(gy_{n(k)}, gy_{m(k)})}{2}\right)\\
&\qquad - \psi\left(\frac{d(gx_{n(k)}, gx_{m(k)}) + d(gy_{n(k)}, gy_{m(k)})}{2}\right)\\
&\quad = \varphi(r_k) - \psi(r_k).
\end{aligned} \tag{2.12}$$

By (2.11) and (2.12), we have

$$\varphi(r_k) \leq \varphi(R_{n(k)}) + \varphi(R_{m(k)}) + \varphi(r_k) - \psi(r_k).$$

Letting $k \to \infty$, using (2.7), (2.10) and the properties of φ and ψ in the last inequality, we have

$$\begin{aligned}
\varphi(\varepsilon) &\leq \varphi(0) + \varphi(0) + \varphi(\varepsilon) - \lim_{k\to\infty} \psi(r_k)\\
&= \varphi(\varepsilon) - \lim_{r_k\to\varepsilon} \psi(r_k) < \varphi(\varepsilon),
\end{aligned}$$

a contradiction.

Therefore, both $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences in X. By the completeness of X, there exist $x, y \in X$ such that

$$\lim_{n\to\infty} F(x_n, y_n) = \lim_{n\to\infty} gx_n = x \quad \text{and} \quad \lim_{n\to\infty} F(y_n, x_n) = \lim_{n\to\infty} gy_n = y. \tag{2.13}$$

Since F and g are compatible mappings, we have from (2.13)

$$\lim_{n\to\infty} d\bigl(gF(x_n, y_n), F(gx_n, gy_n)\bigr) = 0, \tag{2.14}$$

$$\lim_{n\to\infty} d\bigl(gF(y_n, x_n), F(gy_n, gx_n)\bigr) = 0. \tag{2.15}$$

Let the condition (a) hold.

For all $n \geq 0$, we have

$$d\bigl(gx, F(gx_n, gy_n)\bigr) \leq d\bigl(gx, gF(x_n, y_n)\bigr) + d\bigl(gF(x_n, y_n), F(gx_n, gy_n)\bigr).$$

Taking $n \to \infty$ in the last inequality, using the inequalities (2.13), (2.14) and the continuities of F and g, we have $d(gx, F(x,y)) = 0$; that is, $gx = F(x,y)$. Again, for all $n \geq 0$,

$$d\big(gy, F(gy_n, gx_n)\big) \leq d\big(gy, gF(y_n, x_n)\big) + d\big(gF(y_n, x_n), F(gy_n, gx_n)\big).$$

Taking $n \to \infty$ in the last inequality, using the inequalities (2.13), (2.15) and the continuities of F and g, we have $d\big(gy, F(y,x)\big) = 0$; that is, $gy = F(y,x)$. Hence, the element $(x,y) \in X \times X$ is a coupled coincidence point of the mappings $F : X \times X \to X$ and $g : X \to X$.

Next, we suppose that the condition (b) holds.

By (2.3), (2.4) and (2.13), we have $\{gx_n\}$ is a non-decreasing sequence, $gx_n \to x$ and $\{gy_n\}$ is a non-increasing sequence, $gy_n \to y$ as $n \to \infty$. Hence, by the assumption (b), we have for all $n \geq 0$,

$$ggx_n \leq gx \quad \text{and} \quad ggy_n \geq gy. \tag{2.16}$$

Since F and g are compatible mappings and g is continuous, by inequalities (2.13)-(2.15), we have

$$\lim_{n\to\infty} ggx_n = gx = \lim_{n\to\infty} g\big(F(x_n, y_n)\big) = \lim_{n\to\infty} F(gx_n, gy_n), \tag{2.17}$$

and

$$\lim_{n\to\infty} ggy_n = gy = \lim_{n\to\infty} g\big(F(y_n, x_n)\big) = \lim_{n\to\infty} F(gy_n, gx_n). \tag{2.18}$$

Now,

$$d\big(F(x,y), gx\big) \leq d\big(F(x,y), ggx_{n+1}\big) + d(ggx_{n+1}, gx);$$

that is,

$$d\big(F(x,y), gx\big) \leq d\big(F(x,y), gF(x_n, y_n)\big) + d(ggx_{n+1}, gx).$$

Taking $n \to \infty$ in the last inequality and using (2.17), we have

$$\begin{aligned} d\big(F(x,y), gx\big) &\leq \lim_{n\to\infty} d\big(F(x,y), gF(x_n, y_n)\big) + \lim_{n\to\infty} d(ggx_{n+1}, gx) \\ &\leq \lim_{n\to\infty} d\big(F(x,y), F(gx_n, gy_n)\big). \end{aligned} \tag{2.19}$$

Similarly,

$$d\big(F(y,x), gy\big) \leq \lim_{n\to\infty} d\big(F(y,x), F(gy_n, gx_n)\big). \tag{2.20}$$

By (2.19), (2.20) and the property (i_φ), we have

$$\begin{aligned} &\varphi\left(\frac{d(F(x,y), gx) + d(F(y,x), gy)}{2}\right) \\ &\quad \leq \lim_{n\to\infty} \varphi\left(\frac{d(F(x,y), F(gx_n, gy_n)) + d(F(y,x), F(gy_n, gx_n))}{2}\right). \end{aligned} \tag{2.21}$$

By (2.1) and (2.16), we have

$$\varphi\left(\frac{d(F(x,y),F(gx_n,gy_n))+d(F(y,x),F(gy_n,gx_n))}{2}\right)$$
$$\leq \varphi\left(\frac{d(gx,ggx_n)+d(gy,ggy_n)}{2}\right)-\psi\left(\frac{d(gx,ggx_n)+d(gy,ggy_n)}{2}\right). \tag{2.22}$$

Inserting (2.22) in (2.21), we have

$$\begin{aligned}
&\varphi\left(\frac{d(F(x,y),gx)+d(F(y,x),gy)}{2}\right)\\
&\quad\leq \lim_{n\to\infty}\left[\varphi\left(\frac{d(gx,ggx_n)+d(gy,ggy_n)}{2}\right)-\psi\left(\frac{d(gx,ggx_n)+d(gy,ggy_n)}{2}\right)\right]\\
&\quad= \lim_{n\to\infty}\varphi\left(\frac{d(gx,ggx_n)+d(gy,ggy_n)}{2}\right)-\lim_{n\to\infty}\psi\left(\frac{d(gx,ggx_n)+d(gy,ggy_n)}{2}\right).
\end{aligned}$$

By (2.17), (2.18), the continuity of φ and $\lim_{t\to 0+}\psi(t)=0$, we get

$$\begin{aligned}
\varphi\left(\frac{d(F(x,y),gx)+d(F(y,x),gy)}{2}\right) &\leq \lim_{n\to\infty}\varphi\left(\frac{d(gx,ggx_n)+d(gy,ggy_n)}{2}\right)\\
&= \varphi(0)=0.
\end{aligned}$$

Since φ is non-negative and $\varphi(0)=0$, we have

$$d\big(F(x,y),gx\big)=0 \quad\text{and}\quad d\big(F(y,x),gy\big)=0;$$

that is,

$$F(x,y)=gx \quad\text{and}\quad F(y,x)=gy.$$

Hence, the element $(x,y)\in X\times X$ is a coupled coincidence point of the mappings $F:X\times X\to X$ and $g:X\to X$. □

Now, we give an example in support of Theorem 2.1.

Example 2.1 Let $X=[0,1]$. Then $(X,\leq)$ is a partially ordered set with the natural ordering of real numbers.

Let $d(x,y)=|x-y|$ for $x,y\in X$.

Then (X,d) is a complete metric space.

Let $:X\to X$ be defined as

$$g(x)=x^2,\quad \text{for all } x\in X.$$

Let $F:X\times X\to X$ be defined as

$$F(x,y)=\begin{cases}\frac{x^2-y^2}{4}, & \text{if } x,y\in[0,1], x\geq y,\\ 0, & \text{if } x<y.\end{cases}$$

Let $\{x_n\}$ and $\{y_n\}$ be two sequences in X such that

$$\lim_{n\to\infty} F(x_n, y_n) = a, \qquad \lim_{n\to\infty} g(x_n) = a,$$
$$\lim_{n\to\infty} F(y_n, x_n) = b \quad \text{and} \quad \lim_{n\to\infty} g(y_n) = b.$$

Now, for all $n \geq 0$,

$$g(x_n) = x_n^2, \qquad g(y_n) = y_n^2,$$
$$F(x_n, y_n) = \begin{cases} \frac{x_n^2 - y_n^2}{4}, & \text{if } x, y \in [0,1], x_n \geq y_n, \\ 0, & \text{if } x_n < y_n, \end{cases}$$

and

$$F(y_n, x_n) = \begin{cases} \frac{y_n^2 - x_n^2}{4}, & \text{if } x, y \in [0,1], y_n \geq x_n, \\ 0, & \text{if } y_n < x_n. \end{cases}$$

Obviously, $a = 0$ and $b = 0$.

Then it follows that

$$d\big(gF(x_n, y_n), F(gx_n, gy_n)\big) \to 0 \quad \text{as } n \to \infty,$$

and

$$d\big(gF(y_n, x_n), F(gy_n, gx_n)\big) \to 0 \quad \text{as } n \to \infty.$$

Hence, the mappings F and g are compatible in X. Clearly, F obeys the mixed g-monotone property. Also, $F(X \times X) \subseteq g(X)$.

Let $\varphi, \psi : [0,\infty) \to [0,\infty)$ be defined as $\varphi(t) = \frac{t}{2}$, $\psi(t) = \frac{t}{4}$, for $t \in [0,\infty)$.

Also, $x_0 = 0$ and $y_0 = c$ (>0) are two points in X such that $g(x_0) = g(0) = 0 = F(0, c) = F(x_0, y_0)$ and $g(y_0) = g(c) = c^2 \geq \frac{c^2}{4} = F(c, 0) = F(y_0, x_0)$.

Next, we verify inequality (2.1) of Theorem 2.1. We take $x, y, u, v \in X$ such that $gx \geq gu$ and $gy \leq gv$; that is, $x^2 \geq u^2$ and $y^2 \leq v^2$. We discuss the following cases.

Case 1: $x \geq y$, $u \geq v$.

Then

$$\begin{aligned}
&\varphi\left(\frac{d(F(x,y), F(u,v)) + d(F(y,x), F(v,u))}{2}\right) \\
&\quad = \frac{1}{2}\left(\frac{d(F(x,y), F(u,v)) + d(0,0)}{2}\right) = \frac{1}{4} d\left(\frac{x^2 - y^2}{4}, \frac{u^2 - v^2}{4}\right) \\
&\quad = \frac{1}{4}\left|\frac{x^2 - y^2}{4} - \frac{u^2 - v^2}{4}\right| = \frac{1}{4}\left|\frac{(x^2 - u^2) + (v^2 - y^2)}{4}\right| = \frac{1}{4}\left\{\frac{(x^2 - u^2)}{4} + \frac{(v^2 - y^2)}{4}\right\} \\
&\quad \leq \frac{1}{4}\left\{\frac{(x^2 - u^2) + (v^2 - y^2)}{2}\right\} = \frac{1}{4}\left\{\frac{d(gx, gu) + d(gv, gy)}{2}\right\} \\
&\quad = \varphi\left(\frac{d(gx, gu) + d(gv, gy)}{2}\right) - \psi\left(\frac{d(gx, gu) + d(gv, gy)}{2}\right).
\end{aligned}$$

Case 2: $x \geq y$, $u < v$.

Then

$$\begin{aligned}
&\varphi\left(\frac{d(F(x,y),F(u,v)) + d(F(y,x),F(v,u))}{2}\right)\\
&\quad= \frac{1}{2}\left(\frac{d(F(x,y),F(u,v)) + d(F(y,x),F(v,u))}{2}\right)\\
&\quad= \frac{1}{4}\left\{d\left(\frac{x^2-y^2}{4},0\right) + d\left(0,\frac{v^2-u^2}{4}\right)\right\}\\
&\quad= \frac{1}{4}\left\{\left(\frac{x^2-y^2}{4}\right) + \left(\frac{v^2-u^2}{4}\right)\right\}\\
&\quad= \frac{1}{4}\left\{\left(\frac{x^2-u^2}{4}\right) + \left(\frac{v^2-y^2}{4}\right)\right\}\\
&\quad\leq \frac{1}{4}\left\{\left(\frac{x^2-u^2}{2}\right) + \left(\frac{v^2-y^2}{2}\right)\right\} = \frac{1}{4}\left\{\frac{(x^2-u^2)+(v^2-y^2)}{2}\right\}\\
&\quad= \frac{1}{4}\left\{\frac{d(gx,gu)+d(gv,gy)}{2}\right\} = \varphi\left(\frac{d(gx,gu)+d(gv,gy)}{2}\right) - \psi\left(\frac{d(gx,gu)+d(gv,gy)}{2}\right).
\end{aligned}$$

Case 3: $x < y$, $u \geq v$.

Then

$$\begin{aligned}
&\varphi\left(\frac{d(F(x,y),F(u,v)) + d(F(y,x),F(v,u))}{2}\right)\\
&\quad= \frac{1}{2}\left(\frac{d(F(x,y),F(u,v)) + d(F(y,x),F(v,u))}{2}\right)\\
&\quad= \frac{1}{4}\left\{d\left(0,\frac{u^2-v^2}{4}\right) + d\left(\frac{y^2-x^2}{4},0\right)\right\} = \frac{1}{4}\left\{\left(\frac{u^2-v^2}{4}\right) + \left(\frac{y^2-x^2}{4}\right)\right\}\\
&\quad= \frac{1}{4}\left\{\frac{-(x^2-u^2)-(v^2-y^2)}{4}\right\} \leq \frac{1}{4}\left\{\frac{(x^2-u^2)+(v^2-y^2)}{4}\right\}\\
&\quad\leq \frac{1}{4}\left\{\frac{(x^2-u^2)+(v^2-y^2)}{2}\right\} = \frac{1}{4}\left\{\frac{d(gx,gu)+d(gv,gy)}{2}\right\}\\
&\quad= \varphi\left(\frac{d(gx,gu)+d(gv,gy)}{2}\right) - \psi\left(\frac{d(gx,gu)+d(gv,gy)}{2}\right).
\end{aligned}$$

Case 4: $x < y$, $u < v$.

Then

$$\begin{aligned}
&\varphi\left(\frac{d(F(x,y),F(u,v)) + d(F(y,x),F(v,u))}{2}\right)\\
&\quad= \frac{1}{2}\left(\frac{d(0,0) + d(F(y,x),F(v,u))}{2}\right) = \frac{1}{4}d\left(\frac{y^2-x^2}{4},\frac{v^2-u^2}{4}\right)\\
&\quad= \frac{1}{4}\left|\frac{y^2-x^2}{4} - \frac{v^2-u^2}{4}\right| = \frac{1}{4}\left|\frac{-(x^2-u^2)-(v^2-y^2)}{4}\right| = \frac{1}{4}\left\{\frac{|(x^2-u^2)+(v^2-y^2)|}{4}\right\}\\
&\quad= \frac{1}{4}\left\{\frac{(x^2-u^2)+(v^2-y^2)}{4}\right\} \leq \frac{1}{4}\left\{\frac{(x^2-u^2)+(v^2-y^2)}{2}\right\}
\end{aligned}$$

$$= \frac{1}{4}\left\{\frac{d(gx,gu)+d(gv,gy)}{2}\right\} = \varphi\left(\frac{d(gx,gu)+d(gv,gy)}{2}\right) - \psi\left(\frac{d(gx,gu)+d(gv,gy)}{2}\right).$$

Hence, the inequality (2.1) of Theorem 2.1 is satisfied.

Thus, all the conditions of Theorem 2.1 are satisfied, and it can be easily seen that $(0,0)$ is the required coupled coincidence point of F and g in X.

Remark 2.1 If we choose the functions $\varphi(t) = t/2$ and $\psi(t) = t/4$, for $t \in [0,\infty)$, then with this choice of functions, we can obtain the already existing contractive condition. Since φ and ψ are actually contractions, this will be cleared in Corollary 2.3. But if we choose $\varphi(t) = t/(t+1)$ and $\psi(t) = t/3$, for $t \in [0,\infty)$, then with this choice of φ and ψ, the contractive condition (2.1) does not turn to the existing contractive condition.

The next example shows that Theorem 2.1 is more general than Theorem 3.1 in [16] since the contractive condition (2.1) is more general than (1.3).

Example 2.2 Let $X = \mathbb{R}$. Then $(X,\le)$ is a partially ordered set with the natural ordering of real numbers. Let $d : X \times X \to R^+$ be defined by

$$d(x,y) = |x-y| \quad \text{for } x,y \in X.$$

Then (X,d) is a complete metric space.

Define $F : X \times X \to X$ by $F(x,y) = \frac{x-5y}{20}$, $(x,y) \in X \times X$ and $g : X \to X$ by $g(x) = \frac{x}{2}$, $x \in X$.

Clearly, $F(X \times X) \subseteq g(X)$, F is continuous and has the mixed g-monotone property, the pair (F,g) is compatible and satisfies the condition (2.1) but does not satisfy the condition (1.3). Assume, to the contrary, that there exist $\varphi \in \Phi$ (in accordance with Definition 1.10) and $\psi \in \Psi$ such that (1.3) holds. Then we must have

$$\varphi\left(\left|\frac{x-5y}{20} - \frac{u-5v}{20}\right|\right) \le \frac{1}{2}\varphi\left(\left|\frac{x}{2}-\frac{u}{2}\right| + \left|\frac{y}{2}-\frac{v}{2}\right|\right) - \psi\left(\frac{|\frac{x}{2}-\frac{u}{2}| + |\frac{y}{2}-\frac{v}{2}|}{2}\right)$$
$$= \frac{1}{2}\varphi\left(\frac{|x-u|+|y-v|}{2}\right) - \psi\left(\frac{|x-u|+|y-v|}{4}\right)$$

for all $x \ge u$ and $y \le v$. Take $x = u$, $y \ne v$ in the last inequality and let $\rho = \frac{|y-v|}{4}$, we obtain

$$\varphi(\rho) \le \frac{1}{2}\varphi(2\rho) - \psi(\rho), \quad \rho > 0.$$

But by (φ_3) we have $\frac{1}{2}\varphi(2\rho) \le \varphi(\rho)$ and hence we deduce that, for all $\rho > 0$, $\psi(\rho) \le 0$, that is, $\psi(\rho) = 0$, which contradicts (i_ψ). This shows that F does not satisfy (1.3).

Now, we prove that (2.1) holds. Indeed, for $x \ge u$ and $y \le v$, we have

$$\left|\frac{x-5y}{20} - \frac{u-5v}{20}\right| \le \frac{1}{20}|x-u| + \frac{1}{4}|y-v|,$$

and

$$\left|\frac{y-5x}{20}-\frac{v-5u}{20}\right|\leq\frac{1}{20}|y-v|+\frac{1}{4}|x-u|.$$

By summing up the last two inequalities, we get exactly (2.1) with $\varphi(t)=\frac{1}{2}t$, $\psi(t)=\frac{1}{5}t$. Also, $x_0=-1$, $y_0=1$ are the two points in X such that $gx_0\leq F(x_0,y_0)$ and $gy_0\geq F(y_0,x_0)$. F, g, φ, ψ satisfy all the conditions of Theorem 2.1. So, by Theorem 2.1, we obtain that F and g have a coupled coincidence point $(0,0)$, but Theorem 3.1 in [16] cannot be applied to F and g in this example.

The following Corollary 2.1 is Theorem 2 in [5].

Corollary 2.1 [5] *Let $(X,\leq)$ be a partially ordered set and suppose there exists a metric d on X such that (X,d) is a complete metric space. Let $F:X\times X\to X$ be a mapping having the mixed monotone property on X such that there exist two elements $x_0,y_0\in X$ with $x_0\leq F(x_0,y_0)$ and $y_0\geq F(y_0,x_0)$. Suppose there exist $\varphi\in\Phi$ and $\psi\in\Psi$ such that*

$$\varphi\left(\frac{d(F(x,y),F(u,v))+d(F(y,x),F(v,u))}{2}\right)\leq\varphi\left(\frac{d(x,u)+d(y,v)}{2}\right)-\psi\left(\frac{d(x,u)+d(y,v)}{2}\right)\qquad(2.23)$$

for all $x,y,u,v\in X$ with $x\geq u$ and $y\leq v$. Suppose either

(a) *F is continuous, or*

(b) *X has the following property:*

  (i) *if a non-decreasing sequence $\{x_n\}\to x$, then $x_n\leq x$ for all n;*

  (ii) *if a non-increasing sequence $\{y_n\}\to y$, then $y\leq y_n$ for all n.*

Then there exist $x,y\in X$ such that $x=F(x,y)$ and $y=F(y,x)$.

Proof Taking g to be an identity mapping in Theorem 2.1, we obtain Corollary 2.1. □

The following example shows that Corollary 2.1 is more general than Theorem 1.3 (*i.e.*, Theorem 2.1 in [1]) and Theorem 2.1 in [7], since the contractive condition (2.23) is more general than (1.1) and (1.2).

Example 2.3 Let $X=\mathbb{R}$. Then $(X,\leq)$ is a partially ordered set with the natural ordering of real numbers. Let $d:X\times X\to R^+$ be defined by

$$d(x,y)=|x-y|\quad\text{for } x,y\in X.$$

Then (X,d) is a complete metric space.

Define $F:X\times X\to X$ by $F(x,y)=\frac{x-3y}{6}$, $(x,y)\in X\times X$.

Then F is continuous, has the mixed monotone property and satisfies the condition (2.23) but does not satisfy either the condition (1.1) or the condition (1.2). Indeed, assume there exists $k\in[0,1)$ such that (1.1) holds. Then we must have

$$\left|\frac{x-3y}{6}-\frac{u-3v}{6}\right|\leq\frac{k}{2}\{|x-u|+|y-v|\},\quad x\geq u\text{ and }y\leq v,$$

by which, for $x = u$, we get

$$|y - v| \leq k|y - v|, \quad y \leq v,$$

which for $y < v$ implies $1 \leq k$, a contradiction, since $k \in [0,1)$. Hence, F does not satisfy (1.1).

Further, (1.2) is also not satisfied. Assume, to the contrary, that there exist $\varphi \in \Phi$ (in accordance with Definition 1.10) and $\psi \in \Psi$ such that (1.2) holds. Then we must have

$$\varphi\left(\left|\frac{x-3y}{6} - \frac{u-3v}{6}\right|\right) \leq \frac{1}{2}\varphi\big(|x-u| + |y-v|\big) - \psi\left(\frac{|x-u| + |y-v|}{2}\right),$$

for all $x \geq u$ and $y \leq v$. Take $x = u$, $y \neq v$ in the last inequality and let $\alpha = \frac{|y-v|}{2}$, we obtain

$$\varphi(\alpha) \leq \frac{1}{2}\varphi(2\alpha) - \psi(\alpha), \quad \alpha > 0.$$

But by (φ_3), we have $\frac{1}{2}\varphi(2\alpha) \leq \varphi(\alpha)$ and hence we deduce that, for all $\alpha > 0$, $\psi(\alpha) \leq 0$, that is, $\psi(\alpha) = 0$, which contradicts (i_ψ). This shows that F does not satisfy (1.2).

Now, we prove that (2.23) holds. Indeed, for $x \geq u$ and $y \leq v$, we have

$$\left|\frac{x-3y}{6} - \frac{u-3v}{6}\right| \leq \frac{1}{6}|x-u| + \frac{1}{2}|y-v|,$$

and

$$\left|\frac{y-3x}{6} - \frac{v-3u}{6}\right| \leq \frac{1}{6}|y-v| + \frac{1}{2}|x-u|.$$

By summing up the last two inequalities, we get exactly (2.23) with $\varphi(t) = \frac{1}{2}t$, $\psi(t) = \frac{1}{6}t$. Also, $x_0 = -1$, $y_0 = 1$ are the two points in X such that $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$.

So, by Corollary 2.1, we obtain that F has a coupled fixed point $(0,0)$ but neither Theorem 2.1 in [1] nor Theorem 2.1 in [7] can be applied to F in this example.

The following Corollary 2.2 is Corollary 1 in [5].

Corollary 2.2 [5] *Let $(X, \leq)$ be a partially ordered set and suppose there exists a metric d on X such that (X, d) is a complete metric space. Let $F : X \times X \to X$ be a mapping having the mixed monotone property on X such that there exist two elements $x_0, y_0 \in X$ with $x_0 \leq F(x_0, y_0)$ and $y_0 \geq F(y_0, x_0)$. Suppose there exists $\psi \in \Psi$ such that*

$$\begin{aligned} &d\big(F(x,y), F(u,v)\big) + d\big(F(y,x), F(v,u)\big) \\ &\quad \leq d(x,u) + d(y,v) - 2\psi\left(\frac{d(x,u) + d(y,v)}{2}\right) \end{aligned} \tag{2.24}$$

for all $x, y, u, v \in X$ with $x \geq u$ and $y \leq v$. Suppose either

(a) *F is continuous, or*

(b) *X has the following property:*

(i) *if a non-decreasing sequence* $\{x_n\} \to x$, *then* $x_n \leq x$ *for all* n;
(ii) *if a non-increasing sequence* $\{y_n\} \to y$, *then* $y \leq y_n$ *for all* n.

Then F has a coupled fixed point in X.

Proof Note that if $\psi \in \Psi$, then for all $r > 0$, $r\psi \in \Psi$. Now divide (2.24) by 4 and take $\varphi(t) = \frac{1}{2}t$, $t \in [0,\infty)$, then the condition (2.24) reduces to (2.1) with $\psi_1 = \frac{1}{2}\psi$ and $g(x) = x$; and hence by Theorem 2.1, we obtain Corollary 2.2. □

Corollary 2.3 *Let* $(X,\leq)$ *be a partially ordered set and suppose there exists a metric d on X such that* (X,d) *is a complete metric space. Let* $F : X \times X \to X$, $g : X \to X$ *be two maps with F having the mixed g-monotone property on X such that there exist two elements* $x_0, y_0 \in X$ *with* $gx_0 \leq F(x_0,y_0)$ *and* $gy_0 \geq F(y_0,x_0)$. *Suppose there exists a real number* $k \in [0,1)$ *such that*

$$d\big(F(x,y),F(u,v)\big) + d\big(F(y,x),F(v,u)\big) \leq k\big[d(gx,gu) + d(gy,gv)\big] \tag{2.25}$$

for all $x,y,u,v \in X$ *with* $x \geq u$, $y \leq v$. *Suppose either*

(a) *F is continuous, or*
(b) *X has the following property*:
(i) *if a non-decreasing sequence* $\{x_n\} \to x$, *then* $gx_n \leq gx$ *for all* n;
(ii) *if a non-increasing sequence* $\{y_n\} \to y$, *then* $gy \leq gy_n$ *for all* n.

Suppose that $F(X \times X) \subseteq g(X)$, *g is continuous and the pair* (F,g) *is compatible, then there exist* $x,y \in X$ *such that* $gx = F(x,y)$ *and* $gy = F(y,x)$.

Proof Taking $\varphi(t) = \frac{t}{2}$ and $\psi(t) = (1-k)\frac{t}{2}$, $0 \leq k < 1$, in Theorem 2.1, we obtain Corollary 2.3. □

Remark 2.2 (i) Corollary 2.3 is an extension of the recent coupled fixed point result of Berinde (Theorem 3 in [4]) to a coupled coincidence point theorem for a pair of compatible mappings having the mixed g-monotone property.

(ii) Again, the choice of functions F and g in Example 2.2 shows that Corollary 2.3 is more general than Theorem 3.1 in [16], since the contractive condition (2.23) is more general than (1.3). Indeed, the contractive condition (1.3) does not hold for the choice of functions F and g, but (2.25) holds exactly for $k = \frac{3}{5}$ with $x_0 = -1$ and $y_0 = 1$ and yields $(0,0)$ as the coupled coincidence point of F and g.

Corollary 2.4 *Let* $(X,\leq)$ *be a partially ordered set and suppose there exists a metric d on X such that* (X,d) *is a complete metric space. Let* $F : X \times X \to X$, *be a mapping having the mixed monotone property on X such that there exist two elements* $x_0, y_0 \in X$ *with* $x_0 \leq F(x_0,y_0)$ *and* $y_0 \geq F(y_0,x_0)$. *Suppose there exists a real number* $k \in [0,1)$ *such that*

$$d\big(F(x,y),F(u,v)\big) + d\big(F(y,x),F(v,u)\big) \leq k\big[d(x,u) + d(y,v)\big] \tag{2.26}$$

for all $x,y,u,v \in X$ *with* $x \geq u$, $y \leq v$. *Suppose either*

(a) *F is continuous, or*
(b) *X has the following property*:

(i) *if a non-decreasing sequence* $\{x_n\} \to x$, *then* $x_n \leq x$ *for all* n;

(ii) *If a non-increasing sequence* $\{y_n\} \to y$, *then* $y \leq y_n$ *for all* n.

Then F *has a coupled fixed point in* X.

Proof Taking g to be the identity mapping in Corollary 2.3, we obtain Corollary 2.4. □

Remark 2.3 (i) By considering the condition of continuity of F in Corollary 2.4, we obtain Theorem 3 in [4].

(ii) Again, the choice of the function F in Example 2.3 shows that Corollary 2.4 is more general than Theorem 1.3 (*i.e.*, Theorem 2.1 in [1]) and Theorem 2.1 in [7], since the contractive condition (2.26) is more general than (1.1) and (1.2). Indeed, the contractive conditions (1.1) and (1.2) do not hold for the choice of the function F, but (2.26) holds exactly for $k = \frac{2}{3}$ with $x_0 = -1$ and $y_0 = 1$ and yields $(0,0)$ as the coupled fixed point of F.

Now, in order to prove the existence and uniqueness of the coupled common fixed point for our main results, we need the following lemma.

Lemma 2.1 *Let* $F : X \times X \to X$ *and* $g : X \to X$ *be compatible maps and let an element* $(x,y) \in X \times X$ *such that* $gx = F(x,y)$ *and* $gy = F(y,x)$ *exist, then* $gF(x,y) = F(gx,gy)$ *and* $gF(y,x) = F(gy,gx)$.

Proof Since the pair (F,g) is compatible, it follows that

$$\lim_{n\to\infty} d\big(gF(x_n,y_n),F\big(g(x_n),g(y_n)\big)\big) = 0,$$
$$\lim_{n\to\infty} d\big(gF(y_n,x_n),F\big(g(y_n),g(x_n)\big)\big) = 0,$$

whenever $\{x_n\}$ and $\{y_n\}$ are sequences in X such that $\lim_{n\to\infty} F(x_n,y_n) = \lim_{n\to\infty} g(x_n) = a$, $\lim_{n\to\infty} F(y_n,x_n) = \lim_{n\to\infty} g(y_n) = b$ for some $a,b \in X$. Taking $x_n = x$, $y_n = y$ and using $gx = F(x,y)$, $gy = F(y,x)$, it follows that

$$d\big(gF(x,y),F(gx,gy)\big) = 0 \quad \text{and} \quad d\big(gF(y,x),F(gy,gx)\big) = 0.$$

Hence, $gF(x,y) = F(gx,gy)$ and $gF(y,x) = F(gy,gx)$. □

Theorem 2.2 *In addition to the hypothesis of Theorem* 2.1, *suppose that for every* $(x,y),(x^*,y^*) \in X \times X$, *there exists a* $(u,v) \in X \times X$ *such that* $(F(u,v),F(v,u))$ *is comparable to* $(F(x,y),F(y,x))$ *and* $(F(x^*,y^*),F(y^*,x^*))$. *Then* F *and* g *have a unique coupled common fixed point*; *that is, there exists a unique* $(x,y) \in X \times X$ *such that* $x = g(x) = F(x,y)$ *and* $y = g(y) = F(y,x)$.

Proof By Theorem 2.1, the set of coupled coincidences is non-empty. In order to prove the theorem, we shall first show that if (x,y) and (x^*,y^*) are coupled coincidence points, that is, if $gx = F(x,y)$, $gy = F(y,x)$ and $gx^* = F(x^*,y^*)$, $gy^* = F(y^*,x^*)$, then

$$gx = gx^* \quad \text{and} \quad gy = gy^*. \tag{2.27}$$

By assumption, there is $(u,v) \in X \times X$ such that $(F(u,v),F(v,u))$ is comparable with $(F(x,y),F(y,x))$ and $(F(x^*,y^*),F(y^*,x^*))$. Put $u_0 = u$, $v_0 = v$ and choose $u_1,v_1 \in X$ so that $gu_1 = F(u_0,v_0)$, $gv_1 = F(v_0,u_0)$.

Then, similarly as in the proof of Theorem 2.1, we can inductively define sequences $\{gu_n\}$ and $\{gv_n\}$ such that $gu_{n+1} = F(u_n,v_n)$ and $gv_{n+1} = F(v_n,u_n)$.

Further, set $x_0 = x$, $y_0 = y$, $x_0^* = x^*$, $y_0^* = y^*$ and, in the same way, define the sequences $\{gx_n\}$, $\{gy_n\}$ and $\{gx_n^*\}$, $\{gy_n^*\}$. Then it is easy to show that

$$gx_{n+1} = F(x_n,y_n), \qquad gy_{n+1} = F(y_n,x_n)$$

and

$$gx_{n+1}^* = F(x_n^*,y_n^*), \qquad gy_{n+1}^* = F(y_n^*,x_n^*) \quad \text{for all } n \geq 0.$$

Since $(F(u,v),F(v,u)) = (gu_1,gv_1)$ and $(F(x,y),F(y,x)) = (gx_1,gy_1) = (gx,gy)$ are comparable, then $gu_1 \geq gx$ and $gv_1 \leq gy$. It is easy to show that (gu_n,gv_n) and (gx,gy) are comparable, that is, $gu_n \geq gx$ and $gv_n \leq gy$ for all $n \geq 1$. Thus by (2.1),

$$\begin{aligned} &\varphi\left(\frac{d(gu_{n+1},gx) + d(gv_{n+1},gy)}{2}\right) \\ &\quad = \varphi\left(\frac{d(F(u_n,v_n),F(x,\ y)) + d(F(v_n,u_n),F(y,x))}{2}\right) \\ &\quad \leq \varphi\left(\frac{d(gu_n,gx) + d(gv_n,gy)}{2}\right) - \psi\left(\frac{d(gu_n,gx) + d(gv_n,gy)}{2}\right). \end{aligned} \tag{2.28}$$

Since ψ is non-negative, we have

$$\varphi\left(\frac{d(gu_{n+1},gx) + d(gv_{n+1},gy)}{2}\right) \leq \varphi\left(\frac{d(gu_n,gx) + d(gv_n,gy)}{2}\right).$$

By the monotonicity of φ, we have

$$\frac{d(gu_{n+1},gx) + d(gv_{n+1},gy)}{2} \leq \frac{d(gu_n,gx) + d(gv_n,gy)}{2}. \tag{2.29}$$

Thus, the sequence $\{d_n\}$ defined by $d_n = \frac{d(gu_n,gx)+d(gv_n,gy)}{2}$, is a monotonically decreasing sequence of non-negative real numbers, so there exists some $d \geq 0$ such that $\lim_{n\to\infty} d_n = d$.

We shall show that $d = 0$. Suppose, to the contrary, that $d > 0$. Then taking limit as $n \to \infty$, in (2.28) and using the continuity of φ, we have

$$\varphi(d) \leq \varphi(d) - \lim_{d_n \to d} \psi(d_n) < \varphi(d),$$

a contradiction. Thus, $d = 0$; that is, $\lim_{n\to\infty} d_n = 0$.

Hence, it follows that $gu_n \to gx$, $gv_n \to gy$.

Similarly, one can show that $gu_n \to gx^*$, $gv_n \to gy^*$.

By the uniqueness of the limit, it follows that $gx = gx^*$ and $gy = gy^*$. Thus, we proved (2.27).

Since $gx = F(x,y)$, $gy = F(y,x)$ and the pair (F,g) is compatible, then by Lemma 2.1, it follows that

$$ggx = gF(x,y) = F(gx,gy) \quad \text{and} \quad ggy = gF(y,x) = F(gy,gx). \tag{2.30}$$

Denote $gx = z$, $gy = w$. Then by (2.30),

$$gz = F(z,w) \quad \text{and} \quad gw = F(w,z). \tag{2.31}$$

Thus, (z,w) is a coupled coincidence point.

Then by (2.27) with $x^* = z$ and $y^* = w$, it follows that $gz = gx$ and $gw = gy$; that is,

$$gz = z, \qquad gw = w. \tag{2.32}$$

By (2.31) and (2.32),

$$z = gz = F(z,w) \quad \text{and} \quad w = gw = F(w,z).$$

Therefore, (z,w) is the coupled common fixed point of F and g.

To prove the uniqueness, assume that (p,q) is another coupled common fixed point of F and g. Then by (2.27), we have $p = gp = gz = z$ and $q = gq = gw = w$. □

Corollary 2.5 *In addition to the hypothesis of Corollary* 2.3, *suppose that for every* $(x,y),(x^*,y^*) \in X \times X$, *there exists a* $(u,v) \in X \times X$ *such that* $(F(u,v),F(v,u))$ *is comparable to* $(F(x,y),F(y,x))$ *and* $(F(x^*,y^*),F(y^*,x^*))$. *Then* F *and* g *have a unique coupled common fixed point*; *that is, there exists a unique* $(x,y) \in X \times X$ *such that* $x = g(x) = F(x,y)$ *and* $y = g(y) = F(y,x)$.

Proof Taking $\varphi(t) = \frac{t}{2}$ and $\psi(t) = (1-k)\frac{t}{2}$, $0 \leq k < 1$ in Theorem 2.2, we obtain Corollary 2.5. □

Remark 2.4 Indeed, $(0,0)$ is the unique coupled common fixed point of the maps F and g in Example 2.1 in view of Theorem 2.2 and Corollary 2.5.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors read and approved the final manuscript.

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