ON THE CHRISTENSEN-WANG BOUNDS FOR THE GHOST NUMBER OF A p-GROUP ALGEBRA

FATMA ALTUNBULAK AKSU AND DAVID J. GREEN

Abstract. Christensen and Wang give conjectural upper and lower bounds for the ghost number of the group algebra of a p-group. We apply results of Koshitani and Motose on the nilpotency index of the Jacobson radical to prove the upper bound and most cases of the lower bound.

1. Introduction

Let G be a group and k a field of characteristic p. A map $f\colon M \to N$ in the stable category $\mathrm{stmod}(kG)$ of finitely generated kG-modules is called a *ghost* if it vanishes under Tate cohomology, that is if $f_*\colon \hat{H}^*(G,M) \to \hat{H}^*(G,N)$ is zero. The ghost maps then form an ideal in the stable category; Chebolu, Christensen and Mináč [4] define the *ghost number* of kG to be the nilpotency degree of this ideal.

If G is a p-group, then by [3] the ghost ideal is nontrivial – that is, the ghost number exceeds one – unless G is C_2 or C_3. But the exact value of the ghost number is only known in a few cases; for example, it is not yet known for the quaternion group Q_8.

In [5], Christensen and Wang give conjectural upper and lower bounds for the ghost number of a p-group. Our main result establishes most cases of this conjecture:

Theorem 1.1. *Let G be a p-group of order p^n, and k a field of characteristic p. Then*

(1) ghost number$(kG) \leq$ ghost number(kC_{p^n}).

(2) *If G is neither extraspecial of exponent p for odd p, nor extraspecial of order p^3 and exponent p^2 for $p \in \{3,5\}$, then*

$$\text{ghost number}(k(C_p)^n) \leq \text{ghost number}(kG)\,.$$

We do not know whether the lower bound holds in the excluded cases. The upper bound is only rarely attained:

Proposition 1.2. *Let G be a group of order p^n, and k a field of characteristic p. If G is non-cyclic but has the same ghost number as C_{p^n}, then $p = 2$; and G is one of the groups $C_2 \times C_{2^{n-1}}$, Q_{2^n}, SD_{2^n} or Mod_{2^n}.*

Remark 1.3. By work of Chebolu, Christensen and Mináč – specifically, Theorem 5.4 and Corollary 5.12 of [4] – it follows that

$$\text{ghost number}(k(C_2 \times C_{2^{n-1}})) = 2^{n-1} = \text{ghost number}(kC_{2^n})\,.$$

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We do not know whether the other groups in Proposition 1.2 attain the upper bound.

The *nilpotency index* of the radical $J(kG)$ is the smallest positive integer s such that $J(kG)^s = 0$. Following Wallace [9], we denote the nilpotency index of the radical by $t(G)$. We shall prove Theorem 1.1 using known properties of $t(G)$. The first link between ghost number and nilpotency index is given by the following result:

Theorem 1.4 ([4], Theorem 4.7)**.** *Let k be a field of characteristic p and let G be a finite p-group. Then*

$$\text{ghost number}(kG) < t(G) \leq |G| \, .$$

For most p-groups we can use $t(G)$ to strengthen the lower bound in Theorem 1.1 (2):

Proposition 1.5. *Let k be a field of characteristic p. If G is a p-group of order p^n which is not elementary abelian, and moreover*

- *G is neither an extraspecial 2-group nor an almost extraspecial 2-group;*
- *G is not extraspecial of exponent p for p odd;*
- *G is not p_-^{1+2} for $p \in \{3, 5\}$;*
- *G is neither C_4 nor C_9,*

then $\quad \text{ghost number}(kG) \geq t((C_p)^n)$.

2. The upper bound

Let us recall the ghost number of a cyclic group.

Theorem 2.1 ([4], Theorem 5.4.)**.**

$$\text{ghost number}(kC_{p^n}) = \left\lceil \frac{p^n - 1}{2} \right\rceil = \begin{cases} 2^{n-1} & p = 2 \\ \frac{p^n-1}{2} & p \text{ odd} \end{cases} .$$

Proof of Theorem 1.1 (1). Let G be any p-group of order p^n. Theorem 1.4 tells us that

$$\text{ghost number}(kG) \leq t(G) - 1 \, .$$

Motose and Ninomiya [8, Thm 1] demonstrated that if $t(G) = |G|$ then G is cyclic; and Koshitani [6, Thm 1.6] showed that if $n \geq 2$ then the following three statements are equivalent:

(1) $t(G) = p^{n-1} + p - 1$
(2) $p^{n-1} < t(G) < p^n$
(3) G is not cyclic, but it does have a cyclic subgroup of index p.

If $p = 2$ and G is not cyclic then by Koshitani's result and Theorem 2.1

$$t(G) - 1 \leq 2^{n-1} = \text{ghost number}(kC_{p^n}) \, .$$

If p is odd and G not cyclic, then $t(G) - 1 \leq p^{n-1} + p - 2$. This is strictly smaller than ghost number$(kC_{p^n}) = \frac{p^n-1}{2}$, except for the one case $p^n = 3^2$. But the cyclic group of order 9 has ghost number 4, whereas $C_3 \times C_3$ has ghost number 3 by [5, Thm 1.1]. □

Proof of Proposition 1.2. Inspecting the proof of Theorem 1.1 (1) we see that $p = 2$, and that G has a cyclic subgroup of index p. By the classification of such groups (see e.g. [1, 23.4]) it follows that G is either D_{2^n} or one of the stated groups. But D_{2^n} has ghost number $2^{n-2} + 1$ by [5, Cor 1.1]. □

3. Nilpotency index and a lower bound

The following result is a special case of [5, Thm 4.3]:

Theorem 3.1 (Christensen–Wang)**.** *Let G be a finite p-group and k a field of characteristic p. Suppose that $C \leq Z(G)$ is cyclic of order p. Then*

$$\text{ghost number}(kG) \geq t(G/C)\,.$$

Proof. In [5, Thm 4.3], take M_n to be the trivial kC-module. Then the induced kG-module $k(G/C)$ has ghost length equal to its radical length. But its radical length is $t(G/C)$, and by definition the ghost number is the largest ghost length. □

One immediate corollary generalizes [4, Corollary 5.12]:

Corollary 3.2. *Let H be a 2-group and $G = H \times (C_2)^r$ for $r \geq 1$. Then*

$$\text{ghost number}(kG) = t(G) - 1 = t(H) + r - 1\,.$$

Proof. The Jennings series of G is given by

$$\Gamma_s(G) = \begin{cases} \Gamma_1(H) \times C_2^r & s = 1 \\ \Gamma_s(H) & \text{otherwise} \end{cases}.$$

By Jennings' Theorem (Theorem 3.14.6 in [2]) it follows that $t(G) = t(H) + r$. For the first inequality it suffices to consider the case $r = 1$; so $G = H \times C$ with $C \cong C_2$. Theorem 3.1 tells us that

$$\text{ghost number}(kG) \geq t(G/C) = t(H) = t(G) - 1\,.$$

Now apply Theorem 1.4. □

4. Proposition 1.5: the (almost) extraspecial case

Recall that a p-group G is *extraspecial* if $\Phi(G)$, $[G,G]$ and $Z(G)$ coincide and have order p; and *almost* extraspecial if $\Phi(G) = [G,G]$ has order p, but $Z(G)$ is cyclic of order p^2. That is, an almost extraspecial group is a central product of the form $H * C_{p^2}$, with H extraspecial. The following lemma is presumably well known.

Lemma 4.1. *Suppose that G is a p-group of order p^n whose Frattini subgroup has order p. Then*

$$t(G) = \begin{cases} (n+1)(p-1)+1 & \text{if } G \text{ has exponent } p \\ (p+n-1)(p-1)+1 & \text{if } G \text{ has exponent } p^2 \end{cases}.$$

In particular, if $p = 2$ then $t(G) = n + 2$.

Proof. Consider the Jennings series $\Gamma_r(G)$. We certainly have $\Gamma_1(G) = G$ and $\Gamma_2(G) = \Phi(G)$. If the exponent is p then $\Gamma_3(G) = 1$, whereas if the exponent is p^2 then $\Gamma_p(G) = \Gamma_2(G)$ and $\Gamma_{p+1}(G) = 1$. The result follows by Jennings' Theorem (Theorem 3.14.6 in [2]). □

Recall from [8, Thm 1] that $t(C_p^n) = n(p-1) + 1$.

Proposition 4.2. *Let G be a 2-group of order 2^n whose Frattini subgroup has order 2. If G is neither C_4 nor extraspecial nor almost extraspecial then*

$$\text{ghost number}(kG) = n + 1 = t(C_2^n)\,.$$

Proof. By assumption, G has the form $G = H \times C$ with $C \cong C_2$, and $\Phi(H)$ cyclic of order 2. So ghost number$(kG) = t(H)$ by Corollary 3.2, and $t(H) = n + 1$ by Lemma 4.1. □

Proposition 4.3. *Let p be an odd prime and G be a p-group of order p^n whose Frattini subgroup has order p. Then*

$$\text{ghost number}(kG) \geq n(p-1) + 1 = t(C_p^n)$$

except possibly in the following cases:

- *G is extraspecial of exponent p, for any odd p;*
- *G is extraspecial of order p^3 and exponent p^2 for $p \in \{3, 5\}$.*
- *$G = C_9$, with ghost number 4 and $t(C_3 \times C_3) = 5$.*

Remark 4.4. In the proof we use the Proposition 4.9 from [4]: If H is a subgroup of a finite p-group G, then

$$\text{ghost number}(kH) \leq \text{ghost number}(kG)\,.$$

Proof. By assumption we have $\Phi(G) \leq \Omega_1(Z(G))$. Since $\Phi(G) \neq 1$ we have $n \geq 2$.
Step 1: Reduction to the case $\Phi(G) = [G, G] = \Omega_1(Z(G))$.
If $\Phi(G) \lneq \Omega_1(Z(G))$ then there is $C \leq Z(G)$ with $|C| = p$ and $C \cap \Phi(G) = 1$, hence $|\Phi(G/C)| = p$ and so by Theorem 3.1 and Lemma 4.1

$$\text{ghost number}(G) \geq t(G/C) \geq n(p-1) + 1\,.$$

So we may assume that $\Phi(G) = \Omega_1(Z(G))$. If $[G, G] \neq \Phi(G)$ then G is abelian; and therefore cyclic of order p^2, since $\Omega_1(G) = \Phi(G)$. By Theorem 2.1 the ghost number is $\frac{p^2-1}{2}$; for $p > 3$ this is at least $2p - 1$.
Step 2: Reduction to the case G extraspecial.
Extraspecial means that $\Phi(G) = Z(G) = [G, G]$. So if G is not extraspecial then $\Phi(G) \lneq Z(G)$, so $Z(G) \cong C_{p^2}$ and there is a maximal subgroup $E < G$ with $G = EZ(G)$ and $E \cap Z(G) = \Phi(G)$. It follows that E is extraspecial, with $\Phi(E) = \Phi(G)$. As E is extraspecial it has a maximal subgroup of the form $H \times C_p$, with $\Phi(G) \leq H$. Then $L := HZ(G) \times C_p$ is maximal in G, and by Theorem 3.1 and Remark 4.4

$$\text{ghost number}(kG) \geq \text{ghost number}(L) \geq t(HZ(G))\,.$$

As $HZ(G)$ has order p^{n-2} and exponent p^2, Lemma 4.1 says that $t(HZ(G)) = (p + n - 3)(p-1) + 1$.
Step 3: Reduction to the case $G \cong p_-^{1+2}$
We may asume that G has exponent p^2, so $G \cong p_+^{1+2r}$. If $r \geq 2$ then G has a maximal subgroup of the form $H \times C_p$, where H has order p^{n-2} and exponent p^2. The result now follows by the argument of the previous step.
Step 4: The case $G \cong p_-^{1+2}$
G has a subgroup of the form C_{p^2}, so ghost number$(kG) \geq$ ghost number(C_{p^2}) by Remark 4.4. But C_{p^2} has ghost number $\frac{p^2-1}{2}$, which exceeds $3p - 2$ for $p \geq 7$. □

5. The lower bound

Lemma 5.1. *Let G be a p-group of order p^n. If $|\Phi(G)| > p$, then*

$$\text{ghost number}(kG) > n(p-1) + 1 = t(C_p^n)\,.$$

Proof. Let $C \leq \Phi(G) \cap Z(G)$ be cyclic of order p. Then ghost number$(kG) \geq t(G/C)$ by Theorem 3.1. Since G/C has order p^{n-1} and is not elementary abelian, we have $t(G/C) \geq n(p-1)+1$ by [7, Thm 6]. But $t(C_p^n) = n(p-1)+1$ by [8, Thm 1]. □

Proof of Proposition 1.5. Follows from Propositions 4.2 and 4.3, and Lemma 5.1. □

Proof of Theorem 1.1 (2). Let G have order p^n.

First suppose that $p = 2$. Let $C \leq Z(G)$ have order 2, then ghost number$(kG) \geq t(G/C)$ by Theorem 3.1. Since G/C has order 2^{n-1} we have $t(G/C) \geq (n-1)+1 = n = t(C_2^n) - 1$. The result follows since ghost number$(kH) \leq t(H) - 1$ by Theorem 1.4.

Now suppose that p is odd. By Proposition 1.5 we only need consider the case $G = C_9$, with ghost number 4. But $C_3 \times C_3$ has ghost number 3 by [5, Thm 1.1]. □

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Dept of Mathematics and Computer Science, Çankaya University, 06790 Ankara, Turkey

E-mail address: `altunbulak@cankaya.edu.tr`

Dept of Mathematics, Friedrich-Schiller-Universität Jena, 07737 Jena, Germany

E-mail address: `david.green@uni-jena.de`