FRACTAL BOUNDARY VALUE PROBLEMS FOR INTEGRAL AND DIFFERENTIAL EQUATIONS WITH LOCAL FRACTIONAL OPERATORS

by

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In the present paper we investigate the fractal boundary value problems for the Fredholm and Volterra integral equations, heat conduction and wave equations by using the local fractional decomposition method. The operator is described by the local fractional operators. The four illustrative examples are given to elaborate the accuracy and reliability of the obtained results.

Key words: *local fractional decomposition method, heat conduction equations, integral equations, wave equations, boundary value problems*

Introduction

Fractals are used in many engineering applications such as porous media modelling, nano fluids, fracture mechanics, and many other applications in nanoscale [1, 2] where various transport phenomena cannot be described by smooth continuum approach and need the fractal nature of the objects to be taken into account. For the transport phenomena performed in fractal objects the local temperature depends on the fractal dimensions where adequate physical results can be achieved by the application of local fractional models and relevant solution approaches. In [3] heat diffusion problem is analyzed in fractal geometry cooling surface and in [4] fractal heat conduction problem is solved by using the local fractional variation iteration method.

The decomposition method introduced by Adomian [5, 6] was broadly applied to solve a wide class of linear and nonlinear problems described by ordinary, partial differential equations and integral equations [7-9]. There are exist some other analytical methods widely applied to solve non-linear problems, among them: the variational iteration method [10, 11], the homotopy pertur-

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bation method [12, 13], the heat-balance integral method [14, 15], the complex transform method [16, 17], the homotopy analysis method [18], the fractional sub-equation method [19] and the fractional variational iteration method [20-22] and more details can be seen in [23].

The local fractional derivative is a relatively new tool to investigate local fractal behaviors of differential equations with fractal conditions [24-28]. Several works have been published on efficient methods for solving such local fractional differential equations with non-differential functions [29, 30, 39, 40]. The integral equations via local fractional calculus theory [27, 28] were first proposed in [29] and developed in the case of local fractional Fredholm and Volterra integral equations [31]. For the differential and integral equations with local fractional derivative and integral operators, there are exist some analytical methods, such as: the local fractional variational iteration method [30, 32], the local fractional decomposition method [33, 34], the local fractional Fourier series method [4, 35], the local fractional Fourier transform method [12, 36], the local fractional Laplace transform method [12], the local fractional Z transform method [37], *etc*.

Taking into account the existing background in solution of fractal boundary problems the present communication addresses analytical solutions performed by the local fractional decomposition method to two problems: the local fractional integral equations and the local fractional wave equation.

Analytical method

For seek of clarity of the explanation, the local fractional decomposition method will be briefly outlined. For integral equations a compact recurrence scheme has been developed [7-29, 31]. The initial (zeroth) approximation in this case is:

$$u_0(x) = f(x) \tag{1}$$

and

$$u_{n+1}(x) = \frac{\lambda^\alpha}{\Gamma(1+\alpha)} \int_a^b K(x,t) u_n(x) (\mathrm{d}t)^\alpha, \quad n \in N \tag{2}$$

Local fractional integral of $f(x)$ of order $\alpha (0 < \alpha \le 1)$ in the interval [a, b] is denoted by [23-35]:

$${}_a I_b^{(\alpha)} f(x) = \frac{1}{\Gamma(1+\alpha)} \int_a^b f(t) (\mathrm{d}t)^\alpha = \frac{1}{\Gamma(1+\alpha)} \lim_{\Delta t \to 0} \sum_{j=0}^{j=N-1} f(t_j)(\Delta t_j)^\alpha \tag{3}$$

with a partition of the interval [a, b], which is defined through (t_j, t_{j+1}) with $\Delta t_j = t_{j+1} - t_j$, $\Delta t = \max\{\Delta t_1, \Delta t_2, \Delta t_j, ...\}$, $j = 0, ..., N-1$, $t_0 = a$ and $t_N = b$.

Hence, we can determine a few terms in the series such as $u(x) = \sum_{n=0}^{\infty} u_n(x)$ by truncating the series at certain term. The local fractional Volterra integral equation is written in the form [26, 31]:

$$u(x) = f(x) + \frac{\lambda^\alpha}{\Gamma(1+\alpha)} \int_0^x K(x,t) u(t) (\mathrm{d}t)^\alpha \tag{4}$$

and hence

$$u_0(x) = f(x) \tag{5}$$

Substituting $u(x) = \sum_{n=0}^{\infty} u_n(x)$ into eq. (4) implies:

$$\sum_{n=0}^{\infty} u_n(x) = f(x) + \frac{\lambda^\alpha}{\Gamma(1+\alpha)} \int_0^x K(x,t) \left\{ \sum_{n=0}^{\infty} u_n(x) \right\} (\mathrm{d}t)^\alpha \tag{6}$$

The components $u_0(x)$, $u_1(x)$, $u_2(x)$, ... , $u_n(x)$, ... of the function $u(x)$ can be completely determined if we set:

$$
\begin{aligned}
&u_0(x) = f(x) \\
&u_1(x) = \frac{\lambda^\alpha}{\Gamma(1+\alpha)} \int_0^x K(x,t) u_0(x) (\mathrm{d}t)^\alpha \\
&u_2(x) = \frac{\lambda^\alpha}{\Gamma(1+\alpha)} \int_0^x K(x,t) u_1(x) (\mathrm{d}t)^\alpha \\
&\dots \\
&u_n(x) = \frac{\lambda^\alpha}{\Gamma(1+\alpha)} \int_0^x K(x,t) u_{n-1}(x) (\mathrm{d}t)^\alpha
\end{aligned} \tag{7}
$$

and so on. This set of equations can be written in compact recurrence scheme as

$$u_0(x) = f(x) \tag{8}$$

and

$$u_{n+1}(x) = \frac{\lambda^\alpha}{\Gamma(1+\alpha)} \int_0^x K(x,t) u_n(x) (\mathrm{d}t)^\alpha, \quad n \in N \tag{9}$$

Hence, we give the local fractional series solution $u(x) = \sum_{n=0}^{4} u_n(x)$. The above processes were discussed in [37, 38].

Let us consider the local fractional Volterra integral equation given through:

$$u(x) = f(x) + \frac{\lambda^\alpha}{\Gamma(1+\alpha)} \int_0^a K(x,t) u(t) (\mathrm{d}t)^\alpha \tag{10}$$

where a is a constant. We can set up on an iteration formula based on the decomposition method:

$$u_{n+1}(x) = \frac{\lambda^\alpha}{\Gamma(1+\alpha)} \int_a^b K(x,t) u_n(t) (\mathrm{d}t)^\alpha, \quad n \in N \tag{11}$$

An initial condition can be expressed as:

$$u_0(x) = f(x) \tag{12}$$

Thus, we arrive at a local fractional series solution, namely:

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \tag{13}$$

Moreover, we can write the general local fractional differential equation in a local fractional differential operator form:

$$L_x^{(2\alpha)} u(x) + R_x^\alpha u(x) = f(x) \tag{14}$$

In eq. (14) $L_x^{(2\alpha)}$ is the local fractional $2\alpha^{\text{th}}$ order differential operator [27, 28], which by the definition reads:

$$L_s^{(s\alpha)} s(x) = \frac{\mathrm{d}^\alpha}{\mathrm{d}x^\alpha}\left[\frac{\mathrm{d}^\alpha}{\mathrm{d}x^\alpha} x(x)\right] \tag{15}$$

and

$$R_x^\alpha s(x) = \left.\frac{\mathrm{d}^\alpha s(x)}{\mathrm{d}x^\alpha}\right|_{x=x_0} = \lim_{x \to x_0} \frac{\mathrm{d}[s(x) - s(x_0)]}{(x - x_0)^\alpha} \tag{16}$$

is the local fractional α^{th} order differential operator ($0 < \alpha \le 1$) and $s(x)$ is the local fractional continuous [27-33].

Applying the inverse operator $L_x^{(-2\alpha)}$ to both sides of eq. (14) yields:

$$L_x^{(-2\alpha)} L_x^{(2\alpha)} u(x) = -L_x^{(-2\alpha)} R_x^\alpha u(x) + L_x^{(-2\alpha)} f(x) \tag{17}$$

If the inverse differential operator $L_x^{(-2\alpha)}$ exists, according to the local fractional decomposition method mentioned, we have:

$$\begin{cases} u_{n+1}(x) = -L_x^{(-2\alpha)} R_x^{\alpha} u_n(x) \\ u_0(x) = r(x) \end{cases} \tag{18}$$

where $r(x) = L_x^{(-2\alpha)} f(x)$ and

$$L_x^{(-2\alpha)} u(x) = \left[\frac{1}{\Gamma(1+\alpha)}\right]^2 \int_0^x \int_0^{t_2} u(t_1)(\mathrm{d}t_1)^{\alpha} (\mathrm{d}t_2)^{\alpha} \tag{19}$$

Proof of the existence of the inverse differential operator $L_x^{(-2\alpha)}$ as local fractional integral operator one can find in [28]. Finally, we can find a solution in the form:

$$u(x) = \sum_{n=0}^{\infty} u_n(x) \tag{20}$$

Hence, we can obtain that [27-28, 34] the following condition is obeyed:

$$\left| f(x) - f(x_0) \right| < \varepsilon^{\alpha} \tag{21}$$

where fractal dimension of $f(x)$ is equal to α for any $x \in$(a, b).

Illustrative examples

Several illustrative examples demonstrating the efficiency of the suggested local fractional decomposition method are given as follows,

Example 1.

First, we solve the local fractional Fredholm integral equation:

$$u(x) = \frac{x^{\alpha}}{\Gamma(1+\alpha)} + \frac{1}{\Gamma(1+\alpha)} \int_0^1 \frac{(t-1)^{\alpha}}{\Gamma(1+\alpha)} u(t)(\mathrm{d}t)^{\alpha} \tag{22}$$

The zeroth approximation is as:

$$u_0(x) = \frac{x^{\alpha}}{\Gamma(1+\alpha)} \tag{23}$$

The first approximation is expressed through

$$u_1(x) = \frac{1}{\Gamma(1+\alpha)} \int_0^1 \frac{(t-1)^{\alpha}}{\Gamma(1+\alpha)} \frac{t^{\alpha}}{\Gamma(1+\alpha)} (\mathrm{d}t)^{\alpha} = -\frac{1}{\Gamma(1+3\alpha)} \tag{24}$$

Proceeding in this manner, we can find the second approximation as:

$$u_2(x) = \frac{1}{\Gamma(1+\alpha)} \int_0^1 \frac{(t-1)^{\alpha}}{\Gamma(1+\alpha)} u_1(t)(\mathrm{d}t)^{\alpha} = -\frac{1}{\Gamma(1+\alpha)} \int_0^1 \frac{(t-1)^{\alpha}}{\Gamma(1+\alpha)} \frac{1}{\Gamma(1+3\alpha)} (\mathrm{d}t)^{\alpha} = \frac{1}{\Gamma(1+3\alpha)} \frac{1}{\Gamma(1+2\alpha)} \tag{25}$$

Finally, we get:

$$u_n = -\frac{1}{\Gamma(1+3\alpha)} \left[\frac{1}{\Gamma(1+2\alpha)}\right]^{n-1}, \quad n \in N \tag{26}$$

Hence, we arrive at the solution:

$$u(x) = \lim_{n\to\infty} \sum_{n=0}^{\infty} u_n(x) = \frac{x^{\alpha}}{\Gamma(1+\alpha)} + \lim_{n\to 4} \left[-\frac{1}{\Gamma(1+3\alpha)} \sum_{n=1}^{\infty} \left(\frac{1}{\Gamma(1+2\alpha)} \right)^{n-1} \right] =$$

$$= \frac{x^{\alpha}}{\Gamma(1+\alpha)} - \frac{1}{\Gamma(1+3\alpha)} \frac{\dfrac{1}{\Gamma(1+2\alpha)}}{1 - \dfrac{1}{\Gamma(1+2\alpha)}} = \frac{x^{\alpha}}{\Gamma(1+\alpha)} - \frac{1}{\Gamma(1+3\alpha)[\Gamma(1+2\alpha)-1]} \tag{27}$$

if $n \to \infty$.

Example 2.

Similarly, for the solution of the local fractional Volterra integral equation:

$$u(x)=1+\frac{1}{\Gamma(1+\alpha)}\int_0^x\frac{(x-t)^\alpha}{\Gamma(1+\alpha)}u(t)(\mathrm{d}t)^\alpha \tag{28}$$

the zeroth approximation is suggested as:

$$u_0(x)=1 \tag{29}$$

Then, the second approximation is:

$$u_1(x)=\frac{1}{\Gamma(1+\alpha)}\int_0^x\frac{(x-t)^\alpha}{\Gamma(1+\alpha)}(\mathrm{d}t)^\alpha=\frac{x^{2\alpha}}{\Gamma(1+2\alpha)} \tag{30}$$

Proceeding in this manner, the third approximation is:

$$u_2(x)=\frac{1}{\Gamma(1+\alpha)}\int_0^x\frac{(x-t)^\alpha}{\Gamma(1+\alpha)}u_1(t)(\mathrm{d}t)^\alpha=\frac{1}{\Gamma(1+\alpha)}\int_0^x\frac{(x-t)^\alpha}{\Gamma(1+\alpha)}\left(\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}\right)(\mathrm{d}t)^\alpha=\frac{x^{4\alpha}}{\Gamma(1+\alpha)} \tag{31}$$

Therefore, we have:

$$u_n=\frac{x^{2n\alpha}}{\Gamma(1+2n\alpha)},\quad n\in N \tag{32}$$

In conclusion, when $n\to\infty$ we arrive at:

$$u(x)=\lim_{n\to\infty}\sum_{n=0}^{\infty}u_n(x)=\lim_{n\to\infty}\left(\sum_{n=0}^{\infty}\frac{x^{2n\alpha}}{\Gamma(1+2n\alpha)}\right)=\cosh_\alpha(x^\alpha) \tag{33}$$

where $\cosh_\alpha x_\alpha=\sum_{k=0}^{\infty}x^{2\alpha k}/\Gamma(1+2\alpha k)$ is a hyperbolic cosine function defined on a Cantor set [27, 28, 34].

Example 3.

Let us consider the local fractional heat conduction equation with no heat generation in fractal media and dimensionless variables [4], which reads:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}=\frac{\partial^{2\alpha}u(x,t)}{\partial x^{2\alpha}} \tag{34}$$

subject to the following fractal initial boundary conditions:

$$u_x^{(\alpha)}(x,0)=0,\quad u(x,0)=E_\alpha(x^\alpha),\quad (0\le x\le l) \tag{35}$$

where in eq. (34) $u(x, t) = T(x, t)$ is the temperature field.

Hence, the recurrence formula takes the form

$$\begin{cases}u_{n+1}(x,t)=-L_t^{(-\alpha)}L_{xx}^{(2\alpha)}u_n(x,t)\\ u_0(x,t)=E_\alpha(x^\alpha)\end{cases} \tag{36}$$

We can develop a solution in a form of local fractional series, namely:

$$u(x,t)=\lim_{i\to\infty}\sum_{i=0}^{\infty}u_i(x,t)=E_\alpha(x^\alpha)\lim_{n\to\infty}\left(\sum_{i=0}^{\infty}\frac{t^{(2i+1)\alpha}}{\Gamma(1+2i+1)\alpha)}\right)=E_\alpha(x^\alpha)\sinh_\alpha(t^\alpha) \tag{37}$$

where $\sinh_\alpha t^\alpha=\sum_{k=0}^{\infty}t^{\alpha(2k+1)}/\Gamma[1+(2k+1)\alpha]$ is a hyperbolic cosine function defined on a Cantor set [27, 28, 34], and $E_\alpha(x^\alpha)=\sum_{k=0}^{\infty}x^{\alpha k}/\Gamma(1+k\alpha)$ is the Mittag-Leffler function defined on a Cantor set [27, 28, 34, 35].

Example 4.

Consider the local fractional wave equation [41]:

$$\frac{\partial^{2\alpha} u(x,t)}{\partial t^{2\alpha}} = a^{2\alpha} \frac{\partial^{2\alpha} u(x,t)}{\partial x^{2\alpha}} + \frac{\partial^{\alpha} u(x,t)}{\partial x^{\alpha}} \tag{38}$$

with fractal initial boundary conditions:

$$u_t^{(\alpha)}(x,0) = 0, \quad u(x,0) = \frac{x^{2\alpha}}{\Gamma(1+2\alpha)} \ (0 \le x \le l) \tag{39}$$

Hence, the structure of the recurrence formula is:

$$\begin{cases} u_{n+1}(x,t) = a^{2\alpha} L_{tt}^{(-2\alpha)} L_{xx}^{(2\alpha)} u_n(x,t) + L_{tt}^{(-2\alpha)} L_x^{(\alpha)} u_n(x,t) \\ u_0(x,t) = \dfrac{x^{2\alpha}}{\Gamma(1+2\alpha)} \end{cases} \tag{40}$$

Then, the solution can be expressed as:

$$u(x,t) = \frac{x^{2\alpha}}{\Gamma(1+2\alpha)} + \frac{x^{\alpha}}{\Gamma(1+2\alpha)} \frac{t^{\alpha}}{\Gamma(1+2\alpha)} + a^{2\alpha} \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} \tag{41}$$

The examples of using Adomian decomposition method in solving the integral and differential equations and in heat transfer problems can be seen in [6, 7, 9]. For examples presented there, one can easily implement local fractional calculus when fractal boundary value problems are considered.

Conclusions

Presented integral and differential equations play a very important role in heat conduction problems. Some initial boundary problems for certain partial differential equations in physics are reducible to the above integral equation. The development of the investigation for dynamics of the complex systems requires new methods and techniques to be developed. On the other hand many of the classical methods were generalized within the local fractional calculus environment but still some hidden behaviors cannot be revealed properly. In this work, we investigated the fractal initial boundary value problems for local fractional equations with local fractional operators, which are set up on fractals. For these reasons, based on the local fractional operators, the local fractional decomposition method to solve local fractional equations has been applied. The method focuses especially on the approximation methodology for processing local fractional equations. The methodology has been exemplified by four illustrative problems demonstrating the accuracy and the reliabilities of the local fractional decomposition method.

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Nomenclature

E_α – Mittag-Leffler function defined on fractal set
${}_aI_b^{(\alpha)}$ – local fractional integral
$L_x^{(\alpha)}$ – local fractional order differential operator
t – time, [s]
x – space co-ordinate, [m]

Greek symbols

α – fractal dimension
Γ – gamma function

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