OBSERVING DIFFUSION PROBLEMS DEFINED ON CANTOR SETS IN DIFFERENT CO-ORDINATE SYSTEMS

by

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In this paper, the 2-D and 3-D diffusions defined on Cantor sets with local fractional differential operator were discussed in different co-ordinate systems. The 2-D diffusion in Cantorian co-ordinate system can be converted into the symmetric diffusion defined on Cantor sets. The 3-D diffusions in Cantorian co-ordinate system can be observed in the Cantor-type cylindrical and spherical co-ordinate methods.

Key words: *diffusion, Cantor-type circle-co-ordinate method, Cantor-type cylindrical-co-ordinate method, Cantor-type spherical-co-ordinate method, local fractional derivative*

Introduction

Fractional diffusion phenomena [1-8] have stimulated a growing interest in complex systems from fractional calculus view of point. In space and time with different kernels of differentiability, diffusion systems are considered in the non-local fractional derivatives [9]. Povstenko [10, 11] observed the fractional radial diffusion in cylindrical and spherical space. In space and time with non-differentiability [12-17], the local diffusion models were discussed in [18-20]. The diffusions in 3-D and 2-D Cantorian co-ordinate systems were described as [20]:

$$\frac{\mathrm{d}^{\alpha}\Phi(x,y,z,\tau)}{\mathrm{d}\tau^{\alpha}}=D\left[\frac{\partial^{2\alpha}\Phi(x,y,z,\tau)}{\partial x^{2\alpha}}+\frac{\partial^{2\alpha}\Phi(x,y,z,\tau)}{\partial y^{2\alpha}}+\frac{\partial^{2\alpha}\Phi(x,y,z,\tau)}{\partial z^{2\alpha}}\right] \tag{1}$$

$$\frac{\mathrm{d}^{\alpha}\Phi(x,y,\tau)}{\mathrm{d}\tau^{\alpha}}=D\left[\frac{\partial^{2\alpha}\Phi(x,y,\tau)}{\partial x^{2\alpha}}+\frac{\partial^{2\alpha}\Phi(x,y,\tau)}{\partial y^{2\alpha}}\right] \tag{2}$$

where the local fractional partial derivative of $\Phi(x,y,\tau)$ of α $(0<\alpha<1)$ order with respect to τ denotes [12, 13]:

$$\left.\frac{\partial^{\alpha}\Phi(x,y,\tau)}{\partial\tau^{\alpha}}\right|_{\tau=\tau_0}=\frac{\Delta^{\alpha}[\Phi(x,y,\tau)-\Phi(x,y,\tau)]}{(\tau-\tau_0)^{\alpha}} \tag{3}$$

where

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$$\Delta^{\alpha}[\Phi(x,y,\tau)-\Phi(x,y,\tau_0)]\cong\Gamma(1+\nu)[\Phi(x,y,\tau)-\Phi(x,y,\tau_0)] \tag{4}$$

In [18] the applications of the Cantor-type-cylindrical co-ordinates (CCyCs) was observed. Also, the Cantor-type-circle co-ordinates (CCiCs) and Cantor-type-spherical co-ordinates (CSCs) were suggested in references [12] and [13], respectively. The target of this paper is to propose the symmetric-CCyCs (SCCyCs) and -CSCs (SCSCs), and to discuss eqs. (1) in CSC, and (2) in symmetric-Cantor-type- cylindrical, -sphere and -circle forms, respectively.

Basic theories of Cantor-type cylindrical, -spherical, and -circle co-ordinates

The CCyC system

For, $R\in(0,+\infty)$, $z\in(-\infty,+\infty)$, $\theta\in(0,\pi]$, and $x^{2\alpha}+y^{2\alpha}=R^{2\alpha}$, the CCyC system takes the form [12, 18, 20]:

$$\begin{cases} x^{\alpha}=R^{\alpha}\cos_{\alpha}\theta^{\alpha} \\ y^{\alpha}=R^{\alpha}\sin_{\alpha}\theta^{\alpha} \\ z^{\alpha}=z^{\alpha} \end{cases} \tag{5}$$

A local fractional vector (LFV) is defined as [12, 18]:

$$\vec{r}=R^{\alpha}\cos_{\alpha}\theta^{\alpha}\vec{e}_1^{\alpha}+R^{\alpha}\sin_{\alpha}\theta^{\alpha}\vec{e}_2^{\alpha}+z^{\alpha}\vec{e}_3^{\alpha}=r_R\vec{e}_R^{\alpha}+r_{\theta}\vec{e}_{\theta}^{\alpha}+r_z\vec{e}_z^{\alpha} \tag{6}$$

Hence, we have the local fractional gradient operator (LFGO) and local fractional Laplace operator (LFLO) in the CCyC system [18]:

$$\nabla^{\alpha}\phi(R,\theta,z)=\vec{e}_R^{\alpha}\frac{\partial^{\alpha}}{\partial R^{\alpha}}\phi+\vec{e}_{\theta}^{\alpha}\frac{1}{R^{\alpha}}\frac{\partial^{\alpha}}{\partial\theta^{\alpha}}\phi+\vec{e}_z^{\alpha}\frac{\partial^{\alpha}}{\partial z^{\alpha}}\phi \tag{7}$$

$$\nabla^{2\alpha}\phi(R,\theta,z)=\frac{\partial^{2\alpha}}{\partial R^{2\alpha}}\phi+\frac{1}{R^{2\alpha}}\frac{\partial^{2\alpha}}{\partial\theta^{2\alpha}}\phi+\frac{1}{R^{\alpha}}\frac{\partial^{\alpha}}{\partial R^{\alpha}}\phi+\frac{\partial^{2\alpha}}{\partial z^{2\alpha}}\phi \tag{8}$$

where a LFV is given as [18]:

$$\begin{cases} \vec{e}_R^{\alpha}=\cos_{\alpha}\theta^{\alpha}\vec{e}_1^{\alpha}+\sin_{\alpha}\theta^{\alpha}\vec{e}_2^{\alpha} \\ \vec{e}_{\theta}^{\alpha}=-\sin_{\alpha}\theta^{\alpha}\vec{e}_1^{\alpha}+\cos_{\alpha}\theta^{\alpha}\vec{e}_2^{\alpha} \\ \vec{e}_z^{\alpha}=\vec{e}_3^{\alpha} \end{cases} \tag{9}$$

The CSC system

For, $R\in(0,+\infty)$, $\eta\in(0,\pi)$, $\theta\in(0,2\pi)$, and $x^{2\alpha}+y^{2\alpha}+z^{2\alpha}=R^{2\alpha}$ the CSC system [12, 13] is suggested as:

$$\begin{cases} x^{\alpha}=R^{\alpha}\cos_{\alpha}\eta^{\alpha}\cos_{\alpha}\theta^{\alpha} \\ y^{\alpha}=R^{\alpha}\cos_{\alpha}\eta^{\alpha}\sin_{\alpha}\theta^{\alpha} \\ z^{\alpha}=R^{\alpha}\sin_{\alpha}\eta^{\alpha} \end{cases} \tag{10}$$

A LFV is written [13] as:

$$\begin{aligned} \vec{r}&=R^{\alpha}\cos_{\alpha}\eta^{\alpha}\cos_{\alpha}\theta^{\alpha}\vec{e}_1^{\alpha}+R^{\alpha}\cos_{\alpha}\eta^{\alpha}\sin_{\alpha}\theta^{\alpha}\vec{e}_2^{\alpha}+R^{\alpha}\sin_{\alpha}\eta^{\alpha}\vec{e}_3^{\alpha}= \\ &=\vec{r}_R\vec{e}_R^{\alpha}+\vec{r}_{\eta}\vec{e}_{\eta}^{\alpha}+\vec{r}_{\theta}\vec{e}_{\theta}^{\alpha} \end{aligned} \tag{11}$$

Hence, we get the LFGO and LFLO in the CSC system [13]:

$$\nabla^{\alpha}\phi(R,\eta,\theta)=\vec{e}_R^{\alpha}\frac{\partial^{\alpha}}{\partial R^{\alpha}}\phi+\vec{e}_{\eta}^{\alpha}\frac{1}{R^{\alpha}}\frac{\partial^{\alpha}}{\partial \eta^{\alpha}}\phi+\vec{e}_{\theta}^{\alpha}\frac{1}{R^{\alpha}}\frac{1}{\sin_{\alpha}\eta^{\alpha}}\frac{\partial^{\alpha}}{\partial\theta^{\alpha}}\phi \tag{12}$$

$$\nabla^{2\alpha}\phi(R,\eta,\theta)=\frac{\partial^{2\alpha}\phi}{\partial R^{2\alpha}}+\frac{1}{R^{2\alpha}}\frac{1}{\sin_{\alpha}\eta^{\alpha}}\frac{\partial^{\alpha}}{\partial\eta^{\alpha}}\left(\sin_{\alpha}\eta^{\alpha}\frac{\partial^{\alpha}\phi}{\partial\eta^{\alpha}}\right)+$$

$$+\frac{2}{R^{\alpha}}\frac{\partial^{\alpha}\phi}{\partial R^{\alpha}}+\frac{1}{R^{2\alpha}}\frac{1}{\sin_{\alpha}^{2}\eta^{\alpha}}\frac{\partial^{2\alpha}\phi}{\partial\theta^{2\alpha}} \tag{13}$$

where a LFV [13] is determined by:

$$\begin{cases}\vec{e}_R^{\alpha}=\sin_{\alpha}\eta^{\alpha}\cos_{\alpha}\theta^{\alpha}\vec{e}_1^{\alpha}+\sin_{\alpha}\eta^{\alpha}\sin_{\alpha}\theta^{\alpha}\vec{e}_2^{\alpha}+\cos_{\alpha}\eta^{\alpha}\vec{e}_3^{\alpha}\\ \vec{e}_{\eta}^{\alpha}=\cos_{\alpha}\eta^{\alpha}\cos_{\alpha}\theta^{\alpha}\vec{e}_1^{\alpha}+\cos_{\alpha}\eta^{\alpha}\sin_{\alpha}\theta^{\alpha}\vec{e}-\sin_{\alpha}\eta^{\alpha}\vec{e}_3^{\alpha}\\ \vec{e}_{\theta}^{\alpha}=-\sin_{\alpha}\theta^{\alpha}\vec{e}_1^{\alpha}+\cos_{\alpha}\theta^{\alpha}\vec{e}_2^{\alpha}\end{cases} \tag{14}$$

The CCiC system

For $R\in(0,+\infty)$, and $\theta\in(0,2\pi)$ the CCiC system [12] is considered as:

$$\begin{cases}x^{\alpha}=R^{\alpha}\cos_{\alpha}\theta^{\alpha}\\ y^{\alpha}=R^{\alpha}\sin_{\alpha}\theta^{\alpha}\end{cases} \tag{15}$$

where $R>0$ and $0<\theta<2\pi$.

A LFV is expressed as [12]:

$$\vec{r}=R^{\alpha}\cos_{\alpha}\theta^{\alpha}\vec{e}_1^{\alpha}+R^{\alpha}\sin_{\alpha}\theta^{\alpha}\vec{e}_2^{\alpha}=\vec{r}_R\vec{e}_R^{\alpha}+\vec{r}_{\theta}\vec{e}_{\theta}^{\alpha} \tag{16}$$

Hence, we have a LFV [12]:

$$\begin{cases}\vec{e}_R^{\alpha}=\cos_{\alpha}\theta^{\alpha}\vec{e}_1^{\alpha}+\sin_{\alpha}\theta^{\alpha}\vec{e}_2^{\alpha}\\ \vec{e}_{\theta}^{\alpha}=-\sin_{\alpha}\theta^{\alpha}\vec{e}_1^{\alpha}+\cos_{\alpha}\theta^{\alpha}\vec{e}_2^{\alpha}\end{cases} \tag{17}$$

such that the LFGO and LFLO in the CCiC system reads as [12]:

$$\nabla^{\alpha}\phi(R,\theta)=\vec{e}_R^{\alpha}\frac{\partial^{\alpha}}{\partial R^{\alpha}}\phi+\vec{e}_{\theta}^{\alpha}\frac{1}{R^{\alpha}}\frac{\partial^{\alpha}}{\partial\theta^{\alpha}}\phi \tag{18}$$

$$\nabla^{2\alpha}\phi(R,\theta)=\frac{\partial^{2\alpha}}{\partial R^{2\alpha}}\phi+\frac{1}{R^{2\alpha}}\frac{\partial^{2\alpha}}{\partial\theta^{2\alpha}}\phi+\frac{1}{R^{\alpha}}\frac{\partial^{\alpha}}{\partial R^{\alpha}}\phi \tag{19}$$

The Cantor-type-cylindrical and -spherical symmetries

The Cantor-type-cylindrical symmetry (SCCy)

Adopting eq. (7), the LFGO in the SCCy form is presented as:

$$\nabla^{\alpha}\phi(R)=\vec{e}_R^{\alpha}\frac{\partial^{\alpha}}{\partial R^{\alpha}}\phi \tag{20}$$

Similarly, from eq. (8), the LFLO in the SCCy form can be written as:

$$\nabla^{2\alpha}\phi(R)=\frac{\partial^{2\alpha}}{\partial R^{2\alpha}}\phi+\frac{1}{R^{\alpha}}\frac{\partial^{\alpha}}{\partial R^{\alpha}}\phi \tag{21}$$

The Cantor-type-spherical symmetry (SCS)

Using eq. (12), the LFGO in the SCS form is:

$$\nabla^{\alpha}\phi(R)=\vec{e}_R^{\alpha}\frac{\partial^{\alpha}}{\partial R^{\alpha}}\phi \tag{22}$$

Similarly, from eq. (13), the LFLO takes in the SCS form:

$$\nabla^{2\alpha}\phi(R)=\frac{\partial^{2\alpha}\phi}{\partial R^{2\alpha}}+\frac{2}{R^{\alpha}}\frac{\partial^{\alpha}\phi}{\partial R^{\alpha}} \tag{23}$$

Applications to diffusion problems

In view of eq. (13), we change eq. (1) into the 3-D diffusion in the CSC system:

$$\frac{\mathrm{d}^{\alpha}\Phi}{\mathrm{d}\tau^{\alpha}}=$$

$$D\left(\frac{\partial^{2\alpha}\Phi}{\partial R^{2\alpha}}+\frac{1}{R^{2\alpha}}\frac{1}{\sin_{\alpha}\eta^{\alpha}}\frac{\partial^{\alpha}}{\partial\eta^{\alpha}}\sin_{\alpha}\eta^{\alpha}\frac{\partial^{\alpha}\Phi}{\partial\eta^{\alpha}}+\frac{2}{R^{\alpha}}\frac{\partial^{\alpha}\Phi}{\partial R^{\alpha}}+\frac{1}{R^{2\alpha}}\frac{1}{\sin_{\alpha}^{2}\eta^{\alpha}}\frac{\partial^{2\alpha}\Phi}{\partial\theta^{2\alpha}}\right) \tag{24}$$

where $\Phi=\Phi(R,\eta,\theta,\tau)$.

From eq. (21) the 2-D diffusion in SCCy is:

$$\frac{\mathrm{d}^{\alpha}\Phi}{\mathrm{d}\tau^{\alpha}}=D\left(\frac{\partial^{2\alpha}\Phi}{\partial R^{2\alpha}}+\frac{1}{R^{\alpha}}\frac{\partial^{\alpha}\Phi}{\partial R^{\alpha}}+\frac{\partial^{2\alpha}\Phi}{\partial z^{2\alpha}}\right) \tag{25}$$

where $\Phi=\Phi(R,z,\tau)$.

From eq. (19), eq. (2) becomes the 2-D diffusion in CCiC system, namely:

$$\frac{\mathrm{d}^{\alpha}\Phi}{\mathrm{d}\tau^{\alpha}}=D\left(\frac{\partial^{2\alpha}\Phi}{\partial R^{2\alpha}}+\frac{1}{R^{2\alpha}}\frac{\partial^{2\alpha}\Phi}{\partial\theta^{2\alpha}}+\frac{1}{R^{\alpha}}\frac{\partial^{\alpha}\Phi}{\partial R^{\alpha}}\right) \tag{26}$$

where $\Phi=\Phi(R,\theta,\tau)$.

Making use of eq. (21), the diffusion in SCCy form reads:

$$\frac{\mathrm{d}^{\alpha}\Phi}{\mathrm{d}\tau^{\alpha}}=D\left(\frac{\partial^{2\alpha}\Phi}{\partial R^{2\alpha}}+\frac{1}{R^{\alpha}}\frac{\partial^{\alpha}\Phi}{\partial R^{\alpha}}\right) \tag{27}$$

where $\Phi=\Phi(R,\tau)$.

Adopting eq. (23), the diffusion in SCCy form can be written as:

$$\frac{\mathrm{d}^{\alpha}\Phi}{\mathrm{d}\tau^{\alpha}} = D\left(\frac{\partial^{2\alpha}\Phi}{\partial R^{2\alpha}} + \frac{2}{R^{\alpha}}\frac{\partial^{\alpha}\Phi}{\partial R^{\alpha}}\right) \tag{28}$$

where $\Phi = \Phi(R, \tau)$.

Conclusions

In this paper, we suggest the SCCy and SCS and presented the diffusion problems in Cantor-type co-ordinates. The diffusions in SCCy and SCS forms have obtained. The diffusions defined on Cantor sets are easily observed each other by different Cantor-type co-ordinate systems.

Nomenclature

D – a diffusive constant, [–]
x, y, z – space co-ordinates, [m]
$\Phi(x, y, z, \tau)$ –concentration, [–]
$\Phi(x, y, \tau)$ – concentration , [–]

Greek symbols

α – time fractal dimensional order, [–]
τ – time, [s]

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