entropy

MDPI

Article

On the Approximate Solutions of Local Fractional Differential Equations with Local Fractional Operators

Hossein Jafari [1,2,*], Hassan Kamil Jassim [2,3], Fairouz Tchier [4] and Dumitru Baleanu [5,6]

1 Department of Mathematical Sciences, University of South Africa (UNISA), Pretoria 0003, South Africa
2 Department of Mathematics, Faculty of Mathematical Sciences, University of Mazandaran, Babolsar 47416, Iran; hassan.kamil28@yahoo.com
3 Department of Mathematics, Faculty of Education for Pure Sciences, University of Thi-Qar, Nasiriyah 64001, Iraq
4 Department of Mathematics, King Saud University, P.O. Box 22452, Riyadh 11495, Saudi Arabia; ftchier@ksu.edu.sa
5 Department of Mathematics, Çankaya University, Ankara 06530, Turkey; dumitru@cankaya.edu.tr
6 Institute of Space Sciences, P.O. Box MG-23, Magurele-Bucharest RO-76911, Romania
* Correspondence: jafari@umz.ac.ir; Tel.: +98-11-3530-2466; Fax: +98-11-3530-2460

Academic Editors: J. A. Tenreiro Machado and Kevin H. Knuth
Received: 5 March 2016; Accepted: 13 April 2016; Published: 19 April 2016

Abstract: In this paper, we consider the local fractional decomposition method, variational iteration method, and differential transform method for analytic treatment of linear and nonlinear local fractional differential equations, homogeneous or nonhomogeneous. The operators are taken in the local fractional sense. Some examples are given to demonstrate the simplicity and the efficiency of the presented methods.

Keywords: fractional differential equations; local fractional decomposition method; local fractional variational iteration method; local fractional differential transform method

1. Introduction

Fractional calculus is at a stage of rapid development in many areas of science and engineering. Many hidden aspects of real world phenomena from several fields were developed by using this type of fractional calculus. Fractional operators have been applied successfully to describe phenomena with memory effect, although the types of memory types appearing in Nature are still far from being fully understood.

As a result differential equations with arbitrary orders have been subjected to many studies due to their frequent occurrence in different applications in physics, fluid mechanics, physiology, engineering, potential theory and elasticity, among others. Recently, a lot of literature has been published regarding the application of fractional differential equations in nonlinear dynamics [1–7]. Thus, a huge amount of attention has been given to the solution of fractional ordinary and fractional partial differential equations [1–10]. During the last few years local fractional calculus has started to play an important role in describing the complex phenomena which take place on a Cantor set.

Local fractional differential equations are usually difficult to solve analytically, so it is necessary to obtain an efficient approximate solution, and for this the local fractional decomposition method [11–13], local fractional variational iteration method [13,14], local fractional differential transform method [15], and local fractional series expansion method [16,17], have been successfully applied to solve partial differential equations with local fractional operators, while the local fractional Sumudu transform [18], and local fractional Laplace transform [19–21] are used to solve local fractional ordinary differential equations.

We recall that entropy plays an important role in the analysis of anomalous diffusion processes and fractional diffusion equations. These fractional novel entropy indices and fractional operators allowing their implementation in complex dynamical systems [22–28]. Another application is related to local fractional wave equations under fixed entropy arising in fractal hydrodynamics [29].

In this work, we applied the local fractional decomposition method, variational iteration method, and differential transform method to solve local fractional ordinary differential equations. The advantage of these methods respect to other numerical methods is that they don't need discretization.

2. Analysis of the Methods

2.1. Local Fractional Decomposition Method (LFDM)

Let us consider the local fractional differential equation in the following form:

$$L^{(m\vartheta)}\varphi(\gamma)+R_{\vartheta}\varphi(\gamma)=f(\gamma)\ ,\ 0<\vartheta\leqslant 1 \tag{1}$$

where $L^{(m\vartheta)}=\frac{d^{m\vartheta}}{d\gamma^{m\vartheta}}$, $m\in N$ is linear local fractional operator of order $m\vartheta$, R_{ϑ} is linear local fractional operator of order less than $m\vartheta$, and $f(\gamma)$ is the source term. The Equation (1) has a lot of application in physics and engineering.

By defining the $m\vartheta$-fold local fractional integral operator:

$$L^{(-m\vartheta)}\varphi(\gamma)=\frac{1}{\Gamma^{m}(1+\vartheta)}\overbrace{\int_{0}^{\gamma}\cdots\int_{0}^{\gamma}}^{m-time}\varphi(\omega)\ (d\omega)^{m\vartheta} \tag{2}$$

we obtain:

$$L^{(-m\vartheta)}[L^{(m\vartheta)}\varphi(\gamma)+R_{\vartheta}\varphi(\gamma)]=L^{(-m\vartheta)}[f(\gamma)] \tag{3}$$

Hence, we have:

$$\varphi(\gamma)=\rho(\gamma)+L^{(-m\vartheta)}[f(\gamma)]-L^{(-m\vartheta)}[R_{\vartheta}\varphi(\gamma)], \tag{4}$$

where $\rho(\gamma)$ is obtained from the initial conditions.

In the LFDM we express the solution $\varphi(\gamma)$ of local fractional differential Equation (1) in a series form defined by:

$$\varphi(\gamma)=\sum_{n=0}^{\infty}\varphi_n(\gamma) \tag{5}$$

Substituting Equation (5) into both sides of Equation (4) yields:

$$\sum_{n=0}^{\infty}\varphi_n(\gamma)=\rho(\gamma)+L^{(-m\vartheta)}[f(\omega)]-L^{(-m\vartheta)}\left[R_{\vartheta}\left(\sum_{n=0}^{\infty}\varphi_n(\omega)\right)\right] \tag{6}$$

The components $\varphi_n(\gamma)$, $n\geqslant 0$ of the solution $\varphi(\gamma)$ are completely determined in a recursive manner by:

$$\begin{aligned}\varphi_0(\gamma)&=\rho(\gamma)+L^{(-m\vartheta)}[f(\omega)],\\ \varphi_{n+1}(\gamma)&=-L^{(-m\vartheta)}[R_{\vartheta}\ \varphi_n(\omega)],\ n\geqslant 0\end{aligned} \tag{7}$$

2.2. Local Fractional Variational Iteration Method (LFVIM)

We consider the following local fractional differential equation:

$$L^{(m\vartheta)}\varphi(\gamma)+R_{\vartheta}\varphi(\gamma)+N_{\vartheta}\varphi(\gamma)=f(\gamma)\ ,\ 0<\vartheta\leqslant 1 \tag{8}$$

where N_{ϑ} is nonlinear local fractional operator.

According to the theory of local fractional variational iteration algorithm [13,14], we can write the iteration formula as:

$$\varphi_{n+1}(\gamma)=\varphi_n(\gamma)+\frac{1}{\Gamma(1+\vartheta)}\int_0^{\gamma}\frac{\mu^{\vartheta}}{\Gamma(1+\vartheta)}\left[L^{(m\vartheta)}\varphi_n(\omega)+R_{\vartheta}\varphi_n(\omega)+N_{\vartheta}\varphi_n(\omega)-f(\gamma)\right](d\omega)^{\vartheta} \tag{9}$$

where $\frac{\mu^{\vartheta}}{\Gamma(1+\vartheta)}$ is a fractal Lagrange multiplier.

Making the local fractional variation of Equation (9), we have:

$$\delta^{\vartheta}\varphi_{n+1}(\gamma)=\delta^{\vartheta}\varphi_n(\gamma)+\delta^{\vartheta}\frac{1}{\Gamma(1+\vartheta)}\int_0^{\gamma}\frac{\mu^{\vartheta}}{\Gamma(1+\vartheta)}\left[L^{(m\vartheta)}\varphi_n(\omega)+R_{\vartheta}\widetilde{\varphi_n}(\omega)+N_{\vartheta}\widetilde{\varphi_n}(\omega)-f(\gamma)\right](d\omega)^{\vartheta} \tag{10}$$

where $\widetilde{\varphi_n}$ is considered as a restricted local fractional variation; that is $\delta^{\vartheta}\varphi_n=0$ (see [20]).

If $m=2$, we get:

$$\frac{\mu^{\vartheta}}{\Gamma(1+\vartheta)}=\frac{(\omega-\gamma)^{\vartheta}}{\Gamma(1+\vartheta)} \tag{11}$$

so that iteration is expressed as:

$$\varphi_{n+1}(\gamma)=\varphi_n(\gamma)+\frac{1}{\Gamma(1+\vartheta)}\int_0^{\gamma}\frac{(\omega-\gamma)^{\vartheta}}{\Gamma(1+\vartheta)}\left[L^{(2\vartheta)}\varphi_n(\omega)+R_{\vartheta}\varphi_n(\omega)+N_{\vartheta}\varphi_n(\omega)-f(\gamma)\right](d\omega)^{\vartheta} \tag{12}$$

If $m=3$, we get:

$$\frac{\mu^{\vartheta}}{\Gamma(1+\vartheta)}=-\frac{(\omega-\gamma)^{2\vartheta}}{\Gamma(1+2\vartheta)} \tag{13}$$

so that iteration is expressed as:

$$\varphi_{n+1}(\gamma)=\varphi_n(\gamma)-\frac{1}{\Gamma(1+\vartheta)}\int_0^{\gamma}\frac{(\omega-\gamma)^{2\vartheta}}{\Gamma(1+2\vartheta)}\left[L^{(3\vartheta)}\varphi_n(\omega)+R_{\vartheta}\varphi_n(\omega)+N_{\vartheta}\varphi_n(\omega)-f(\gamma)\right](d\omega)^{\vartheta} \tag{14}$$

Finally, the solution is:

$$\varphi(\gamma)=\lim_{n\to\infty}\varphi_n(\gamma) \tag{15}$$

2.3. Local Fractional Differential Transform Method (LFDTM)

In this method, the definition of local fractional differential transform and its theorems are introduced via the general local fractional Taylor theorems [20,21].

Definition 1. The local fractional differential transform of the function $\varphi(\gamma)$ is defined by the following formula:

$$\aleph(k)=\frac{1}{\Gamma(1+k\vartheta)}\left[\frac{d^{k\vartheta}\varphi(\gamma)}{d\gamma^{k\vartheta}}\right]_{\gamma=0} \tag{16}$$

where $k=0,1,\ldots,m$ and $0<\vartheta\leqslant 1$. In this work, the lowercase $\varphi(\gamma)$ represents the original function and the uppercase $\aleph(k)$ stand for the transformed function.

Definition 2. The local fractional differential inverse transform of $\aleph(k)$ is defined as follows:

$$\varphi(\gamma) = \sum_{k=0}^{\infty} \aleph(k)\, \gamma^{k\vartheta} \tag{17}$$

Theorem 1. If $(\gamma) = u(\gamma) + v(\gamma)$, then we have:

$$\aleph(k) = U(k) + V(k) \tag{18}$$

Theorem 2. If $\varphi(\gamma) = u(\gamma) \cdot v(\gamma)$, then we have:

$$\aleph(k) = \sum_{\rho=0}^{k} U(\rho)\, V(k-\rho) \tag{19}$$

Theorem 3. If $\varphi(\gamma) = \frac{d^{n\vartheta}}{d\gamma^{n\vartheta}} u(\gamma)$, where $n \in N$, then we have:

$$\aleph(k) = \frac{\Gamma(1+(k+n)\vartheta)}{\Gamma(1+k\vartheta)} U(k+n) \tag{20}$$

Theorem 4. If $\varphi(\gamma) = \frac{\gamma^{n\vartheta}}{\Gamma(1+n\vartheta)}$, where $n \in N$, then we have:

$$\aleph(k) = \frac{\delta_{\vartheta}(k-n)}{\Gamma(1+k\vartheta)} \tag{21}$$

where the local fractional Dirac delta function is [20]:

$$\delta_{\vartheta}(k-n) = \begin{cases} 1, & k = n \\ 0, & k \neq n \end{cases} \tag{22}$$

Hence, according to the LFDTM and its theorems we can construct the iteration formula for local fractional differential Equation (8) in the form:

$$\frac{\Gamma(1+(k+m)\vartheta)}{\Gamma(1+k\vartheta)} \aleph(k+m) = F(k) - R_{\vartheta}\aleph(k) - N_{\vartheta}\aleph(k) \tag{23}$$

where $R_{\vartheta}\aleph(k)$, $N_{\vartheta}\aleph(k)$ and $F(k)$ are the transformations of the functions $R_{\vartheta}\varphi(\gamma)$, $N_{\vartheta}\varphi(\gamma)$ and $f(\gamma)$ respectively.

3. Illustrative Examples

In this section, some examples for homogeneous and nonhomogeneous local fractional differential equations within local fractional derivative operator are presented in order to demonstrate the simplicity and the efficiency of the above methods.

3.1. Homogeneous Local Fractional Differential Equations

Example 1. Let us consider the homogeneous local fractional differential equation with local fractional derivative in the form:

$$\varphi^{(2\vartheta)}(\gamma) - \varphi(\gamma) = 0\,,\ 0 < \vartheta \leqslant 1 \tag{24}$$

subject to the initial conditions given as:

$$\varphi(0) = 0\,,\ \varphi^{(\vartheta)}(0) = 1 \tag{25}$$

- By using LFDM:

Adopting the inverse operator $L^{(-2\vartheta)}$ to both sides of Equation (24) and using the initial conditions leads to:

$$L^{(-2\vartheta)}[L^{(2\vartheta)}\varphi(\gamma)] = L^{(-2\vartheta)}[\varphi(\gamma)]$$

Hence, we get:

$$\varphi(\gamma) = \frac{\gamma^{\vartheta}}{\Gamma(1+\vartheta)} + L^{(-2\vartheta)}[\varphi(\gamma)] \tag{26}$$

According to the LFDM we decompose the unknown function $\varphi(\gamma)$ as an in finite series:

$$\varphi(\gamma) = \sum_{n=0}^{\infty} \varphi_n(\gamma) \tag{27}$$

Substituting Equation (27) into Equation (26) yields:

$$\sum_{n=0}^{\infty} \varphi_n(\gamma) = \frac{\gamma^{\vartheta}}{\Gamma(1+\vartheta)} + L^{(-2\vartheta)}\left[\sum_{n=0}^{\infty} \varphi_n(\gamma)\right] \tag{28}$$

The components $\varphi_n(\gamma)$, $n \geqslant 0$ can be completely determined by using the following iterative formula:

$$\varphi_0(\gamma) = \frac{\gamma^{\vartheta}}{\Gamma(1+\vartheta)} \tag{29}$$

$$\varphi_{n+1}(\gamma) = L^{(-2\vartheta)}[\varphi_n(\gamma)] \ , n \geqslant 0 \tag{30}$$

In view of Equations (29) and (30), we have the following approximations:

$$\varphi_0(\gamma) = \frac{\gamma^{\vartheta}}{\Gamma(1+\vartheta)}$$

$$\varphi_1(\gamma) = L^{(-2\vartheta)}[\varphi_0(\gamma)] = L^{(-2\vartheta)}\left[\frac{\gamma^{\vartheta}}{\Gamma(1+\vartheta)}\right] = \frac{\gamma^{3\vartheta}}{\Gamma(1+3\vartheta)}$$

$$\varphi_2(\gamma) = L^{(-2\vartheta)}[\varphi_1(\gamma)] = L^{(-2\vartheta)}\left[\frac{\gamma^{3\vartheta}}{\Gamma(1+3\vartheta)}\right] = \frac{\gamma^{5\vartheta}}{\Gamma(1+5\vartheta)}$$

$$\varphi_3(\gamma) = L^{(-2\vartheta)}[\varphi_2(\gamma)] = L^{(-2\vartheta)}\left[\frac{\gamma^{5\vartheta}}{\Gamma(1+5\vartheta)}\right] = \frac{\gamma^{7\vartheta}}{\Gamma(1+7\vartheta)}$$

and so on. Finally, the solution of Equation (24) in series form is given by:

$$\varphi(\gamma) = \frac{\gamma^{\vartheta}}{\Gamma(1+\vartheta)} + \frac{\gamma^{3\vartheta}}{\Gamma(1+3\vartheta)} + \frac{\gamma^{5\vartheta}}{\Gamma(1+5\vartheta)} + \frac{\gamma^{7\vartheta}}{\Gamma(1+7\vartheta)} + \cdots = sinh_{\vartheta}\left(\gamma^{\vartheta}\right) \tag{31}$$

- By using LFVIM:

Using Equation (12), we write the iterative formula of Equation (24) as:

$$\varphi_{n+1}(\gamma) = \varphi_n(\gamma) + \frac{1}{\Gamma(1+\vartheta)} \int_0^{\gamma} \frac{(\omega-\gamma)^{\vartheta}}{\Gamma(1+\vartheta)} \left[\varphi_n^{(2\vartheta)}(\omega) - \varphi_n(\omega)\right] (d\omega)^{\vartheta} \tag{32}$$

Start with the zeroth approximation:

$$\varphi_0(\gamma) = \frac{\gamma^{\vartheta}}{\Gamma(1+\vartheta)} \tag{33}$$

Utilizing Equations (32) and (33), we obtain the successive approximations:

$$\begin{aligned} \varphi_1(\gamma) &= \varphi_0(\gamma) + \frac{1}{\Gamma(1+\vartheta)} \int_0^{\gamma} \frac{(\omega-\gamma)^{\vartheta}}{\Gamma(1+\vartheta)} \left[\varphi_0^{(2\vartheta)}(\omega) - \varphi_0(\omega)\right] (d\omega)^{\vartheta} \\ &= \frac{\gamma^{\vartheta}}{\Gamma(1+\vartheta)} + \frac{\gamma^{3\vartheta}}{\Gamma(1+3\vartheta)} \end{aligned}$$

$$\begin{aligned}\varphi_2(\gamma) &= \varphi_1(\gamma) + \frac{1}{\Gamma(1+\vartheta)}\int\limits_0^\gamma \frac{(\omega-\gamma)^\vartheta}{\Gamma(1+\vartheta)}\left[\varphi_1^{(2\vartheta)}(\omega) - \varphi_1(\omega)\right](d\omega)^\vartheta \\ &= \frac{\gamma^\vartheta}{\Gamma(1+\vartheta)} + \frac{\gamma^{3\vartheta}}{\Gamma(1+3\vartheta)} + \frac{\gamma^{5\vartheta}}{\Gamma(1+5\vartheta)} \\ \varphi_3(\gamma) &= \varphi_2(\gamma) + \frac{1}{\Gamma(1+\vartheta)}\int\limits_0^\gamma \frac{(\omega-\gamma)^\vartheta}{\Gamma(1+\vartheta)}\left[\varphi_2^{(2\vartheta)}(\omega) - \varphi_2(\omega)\right](d\omega)^\vartheta \\ &= \frac{\gamma^\vartheta}{\Gamma(1+\vartheta)} + \frac{\gamma^{3\vartheta}}{\Gamma(1+3\vartheta)} + \frac{\gamma^{5\vartheta}}{\Gamma(1+5\vartheta)} + \frac{\gamma^{7\vartheta}}{\Gamma(1+7\vartheta)}\end{aligned}$$

$$\varphi_n(\gamma) = \frac{\gamma^\vartheta}{\Gamma(1+\vartheta)} + \frac{\gamma^{3\vartheta}}{\Gamma(1+3\vartheta)} + \frac{\gamma^{5\vartheta}}{\Gamma(1+5\vartheta)} + \frac{\gamma^{7\vartheta}}{\Gamma(1+7\vartheta)} + \cdots + \frac{\gamma^{(2n+1)\vartheta}}{\Gamma(1+(2n+1)\vartheta)}$$

Hence, we get the solution of Equation (24) as follows:

$$\varphi(\gamma) = \lim_{n\to\infty}\varphi_n(\gamma) = sinh_\vartheta\left(\gamma^\vartheta\right) \tag{34}$$

- By using LFDTM:

Taking the local fractional differential transform of Equation (24), we obtain the following iteration relation:

$$\frac{\Gamma(1+(k+2)\vartheta)}{\Gamma(1+k\vartheta)}\aleph(k+2) - \aleph(k) = 0 \tag{35}$$

or:

$$\aleph(k+2) = \frac{\Gamma(1+k\vartheta)}{\Gamma(1+(k+2)\vartheta)}\aleph(k) \tag{36}$$

From the initial conditions Equation (25), we get:

$$\aleph(0) = 0,\ \aleph(1) = \frac{1}{\Gamma(1+\vartheta)} \tag{37}$$

Using iteration Equations (36) and (37), we obtain the following values of $\aleph(k)$ successively:

$$\aleph(2) = \frac{1}{\Gamma(1+2\vartheta)}\aleph(0) = 0$$

$$\aleph(3) = \frac{\Gamma(1+\vartheta)}{\Gamma(1+3\vartheta)}\aleph(1) = \frac{1}{\Gamma(1+3\vartheta)}$$

$$\aleph(4) = \frac{\Gamma(1+2\vartheta)}{\Gamma(1+4\vartheta)}\aleph(2) = 0$$

$$\aleph(5) = \frac{\Gamma(1+3\vartheta)}{\Gamma(1+5\vartheta)}\aleph(3) = \frac{1}{\Gamma(1+5\vartheta)}$$

$$\aleph(6) = \frac{\Gamma(1+4\vartheta)}{\Gamma(1+6\vartheta)}\aleph(4) = 0$$

$$\aleph(7) = \frac{\Gamma(1+5\vartheta)}{\Gamma(1+7\vartheta)}\aleph(5) = \frac{1}{\Gamma(1+7\vartheta)}$$

Finally, the local fractional differential inverse transform leads to:

$$\begin{aligned}\varphi(\gamma) &= \sum_{k=0}^{\infty}\aleph(k)\gamma^{k\vartheta} \\ &= \frac{\gamma^\vartheta}{\Gamma(1+\vartheta)} + \frac{\gamma^{3\vartheta}}{\Gamma(1+3\vartheta)} + \frac{\gamma^{5\vartheta}}{\Gamma(1+5\vartheta)} + \frac{\gamma^{7\vartheta}}{\Gamma(1+7\vartheta)} + \cdots\end{aligned}$$

that gives the exact solution by:

$$\varphi(\gamma) = sinh_{\vartheta}\left(\gamma^{\vartheta}\right) \tag{38}$$

From Equations (31), (34) and (38), the approximate solution of the given problem Equation (24) by using local fractional decomposition method is the same results as that obtained by the local fractional variational iteration method and by the local fractional differential transformation method.

3.2. Nonhomogeneous Local Fractional Differential Equation

Example 2. Consider the nonhomogeneous local fractional differential equation with local fractional derivative in the form:

$$\varphi^{(3\vartheta)}(\gamma) - \varphi(\gamma) = \frac{\gamma^{2\vartheta}}{\Gamma(1+2\vartheta)}\,,\ 0 < \vartheta \leqslant 1 \tag{39}$$

subject to initial conditions:

$$\varphi(0) = 1\,,\ \varphi^{(\vartheta)}(0) = 1,\ \varphi^{(2\vartheta)}(0) = 0 \tag{40}$$

- By using LFDM:

Taking the inverse operator $L^{(-3\vartheta)}$ to both sides of Equation (39) and using the initial conditions leads to:

$$L^{(-3\vartheta)}\left[L^{(3\vartheta)}\varphi(\gamma)\right] = L^{(-3\vartheta)}\left[\frac{\gamma^{2\vartheta}}{\Gamma(1+2\vartheta)} + \varphi(\gamma)\right] \tag{41}$$

Hence, we get:

$$\varphi(\gamma) = 1 + \frac{\gamma^{\vartheta}}{\Gamma(1+\vartheta)} + \frac{\gamma^{5\vartheta}}{\Gamma(1+5\vartheta)} + L^{(-3\vartheta)}\left[\varphi(\gamma)\right] \tag{42}$$

According to the LFDM we decompose the unknown function $\varphi(\gamma)$ as an in finite series:

$$\varphi(\gamma) = \sum_{n=0}^{\infty}\varphi_n(\gamma) \tag{43}$$

Substituting Equation (43) into Equation (42) yields:

$$\sum_{n=0}^{\infty}\varphi_n(\gamma) = 1 + \frac{\gamma^{\vartheta}}{\Gamma(1+\vartheta)} + \frac{\gamma^{5\vartheta}}{\Gamma(1+5\vartheta)} + L^{(-3\vartheta)}\left[\sum_{n=0}^{\infty}\varphi_n(\gamma)\right] \tag{44}$$

The components $\varphi_n(\gamma)\,,\ n \geqslant\ 0$ can be completely determined by using the cursive relationship:

$$\varphi_0(\gamma) = 1 + \frac{\gamma^{\vartheta}}{\Gamma(1+\vartheta)} + \frac{\gamma^{5\vartheta}}{\Gamma(1+5\vartheta)}, \varphi_{n+1}(\gamma) = L^{(-3\vartheta)}\left[\varphi_n(\gamma)\right]\,,\ n \geqslant 0 \tag{45}$$

Consequently, we obtain:

$$\varphi_0(\gamma) = 1 + \frac{\gamma^{\vartheta}}{\Gamma(1+\vartheta)} + \frac{\gamma^{5\vartheta}}{\Gamma(1+5\vartheta)}$$

$$\begin{aligned}
\varphi_1(\gamma) = L^{(-3\vartheta)}\left[\varphi_0(\gamma)\right] &= L^{(-3\vartheta)}\left[1 + \tfrac{\gamma^{\vartheta}}{\Gamma(1+\vartheta)} + \tfrac{\gamma^{5\vartheta}}{\Gamma(1+5\vartheta)}\right] \\
&= \tfrac{\gamma^{3\vartheta}}{\Gamma(1+3\vartheta)} + \tfrac{\gamma^{4\vartheta}}{\Gamma(1+4\vartheta)} + \tfrac{\gamma^{8\vartheta}}{\Gamma(1+8\vartheta)} \\
\varphi_2(\gamma) = L^{(-3\vartheta)}\left[\varphi_1(\gamma)\right] &= L^{(-3\vartheta)}\left[\tfrac{\gamma^{3\vartheta}}{\Gamma(1+3\vartheta)} + \tfrac{\gamma^{4\vartheta}}{\Gamma(1+4\vartheta)} + \tfrac{\gamma^{8\vartheta}}{\Gamma(1+8\vartheta)}\right] \\
&= \tfrac{\gamma^{6\vartheta}}{\Gamma(1+6\vartheta)} + \tfrac{\gamma^{7\vartheta}}{\Gamma(1+7\vartheta)} + \tfrac{\gamma^{11\vartheta}}{\Gamma(1+11\vartheta)}
\end{aligned}$$

$$\begin{aligned}\varphi_3(\gamma) = L^{(-3\vartheta)}[\varphi_2(\gamma)] &= L^{(-3\vartheta)}\left[\frac{\gamma^{6\vartheta}}{\Gamma(1+6\vartheta)} + \frac{\gamma^{7\vartheta}}{\Gamma(1+7\vartheta)} + \frac{\gamma^{11\vartheta}}{\Gamma(1+11\vartheta)}\right] \\ &= \frac{\gamma^{9\vartheta}}{\Gamma(1+6\vartheta)} + \frac{\gamma^{10\vartheta}}{\Gamma(1+7\vartheta)} + \frac{\gamma^{14\vartheta}}{\Gamma(1+11\vartheta)}\end{aligned}$$

and so on. The solution in series form:

$$\varphi(\gamma) = 1 + \frac{\gamma^{\vartheta}}{\Gamma(1+\vartheta)} + \frac{\gamma^{3\vartheta}}{\Gamma(1+3\vartheta)} + \frac{\gamma^{4\vartheta}}{\Gamma(1+4\vartheta)} + \frac{\gamma^{5\vartheta}}{\Gamma(1+5\vartheta)} + \cdots \tag{46}$$

is readily obtained. Therefore, the exact solution can be written as:

$$\varphi(\gamma) = E_{\vartheta}\left(\gamma^{\vartheta}\right) - \frac{\gamma^{2\vartheta}}{\Gamma(1+2\vartheta)} \tag{47}$$

- By using LFVIM:

Using Equation (14), we write the iteration formula of Equation (39) as:

$$\varphi_{n+1}(\gamma) = \varphi_n(\gamma) - \frac{1}{\Gamma(1+\vartheta)}\int_0^{\gamma}\frac{(\omega-\gamma)^{2\vartheta}}{\Gamma(1+2\vartheta)}\left[\varphi_n^{(3\vartheta)}(\omega) - \varphi_n(\omega) - \frac{\omega^{2\vartheta}}{\Gamma(1+2\vartheta)}\right](d\omega)^{\vartheta} \tag{48}$$

Start with the zeroth approximation:

$$\varphi_0(\gamma) = 1 + \frac{\gamma^{\vartheta}}{\Gamma(1+\vartheta)} + \frac{\gamma^{5\vartheta}}{\Gamma(1+5\vartheta)} \tag{49}$$

Substituting Equation (49) into Equation (48), we obtain the successive approximations:

$$\begin{aligned}\varphi_1(\gamma) &= \varphi_0(\gamma) - \frac{1}{\Gamma(1+\vartheta)}\int_0^{\gamma}\frac{(\omega-\gamma)^{2\vartheta}}{\Gamma(1+2\vartheta)}\left[\varphi_0^{(3\vartheta)}(\omega) - \varphi_0(\omega) - \frac{\omega^{2\vartheta}}{\Gamma(1+2\vartheta)}\right](d\omega)^{\vartheta} \\ &= \frac{\gamma^{\vartheta}}{\Gamma(1+\vartheta)} - \frac{1}{\Gamma(1+\vartheta)}\int_0^{\gamma}\frac{(\omega-\gamma)^{2\vartheta}}{\Gamma(1+2\vartheta)}\left[-1 - \frac{\omega^{\vartheta}}{\Gamma(1+\vartheta)} - \frac{\omega^{5\vartheta}}{\Gamma(1+5\vartheta)}\right](d\omega)^{\vartheta} \\ &= 1 + \frac{\gamma^{\vartheta}}{\Gamma(1+\vartheta)} + \frac{\gamma^{3\vartheta}}{\Gamma(1+3\vartheta)} + \frac{\gamma^{4\vartheta}}{\Gamma(1+4\vartheta)} + \frac{\gamma^{5\vartheta}}{\Gamma(1+5\vartheta)} + \frac{\gamma^{8\vartheta}}{\Gamma(1+8\vartheta)}\end{aligned}$$

$$\begin{aligned}\varphi_2(\gamma) &= \varphi_1(\gamma) - \frac{1}{\Gamma(1+\vartheta)}\int_0^{\gamma}\frac{(\omega-\gamma)^{2\vartheta}}{\Gamma(1+2\vartheta)}\left[\varphi_1^{(3\vartheta)}(\omega) - \varphi_1(\omega) - \frac{\omega^{2\vartheta}}{\Gamma(1+2\vartheta)}\right](d\omega)^{\vartheta} \\ &= \frac{\gamma^{\vartheta}}{\Gamma(1+\vartheta)} - \frac{1}{\Gamma(1+\vartheta)}\int_0^{\gamma}\frac{(\omega-\gamma)^{2\vartheta}}{\Gamma(1+2\vartheta)}\left[-\frac{\omega^{3\vartheta}}{\Gamma(1+3\vartheta)} - \frac{\omega^{4\vartheta}}{\Gamma(1+4\vartheta)} - \frac{\omega^{8\vartheta}}{\Gamma(1+8\vartheta)}\right](d\omega)^{\vartheta} \\ &= 1 + \frac{\gamma^{\vartheta}}{\Gamma(1+\vartheta)} + \frac{\gamma^{3\vartheta}}{\Gamma(1+3\vartheta)} + \frac{\gamma^{4\vartheta}}{\Gamma(1+4\vartheta)} + \frac{\gamma^{5\vartheta}}{\Gamma(1+5\vartheta)} + \frac{\gamma^{6\vartheta}}{\Gamma(1+6\vartheta)} + \frac{\gamma^{7\vartheta}}{\Gamma(1+7\vartheta)} \\ &\quad + \frac{\gamma^{8\vartheta}}{\Gamma(1+8\vartheta)} + \frac{\gamma^{11\vartheta}}{\Gamma(1+11\vartheta)}\end{aligned}$$

$$\varphi_n(\gamma) = \sum_{k=0}^{n}\frac{\gamma^{k\vartheta}}{\Gamma(1+2\vartheta)} - \frac{\gamma^{2\vartheta}}{\Gamma(1+2\vartheta)}$$

The LFVIM admits the use of:

$$\varphi(\gamma) = \lim_{n\to\infty}\varphi_n(\gamma), \tag{50}$$

that gives the solution by:

$$\varphi(\gamma) = E_{\vartheta}\left(\gamma^{\vartheta}\right) - \frac{\gamma^{2\vartheta}}{\Gamma(1+2\vartheta)} \tag{51}$$

- By using LFDTM:

Applying the local fractional differential transform of Equation (39), we obtain the following iteration relation:

$$\frac{\Gamma(1+(k+3)\vartheta)}{\Gamma(1+k\vartheta)}\aleph(k+3)-\aleph(k)=\frac{\delta_\vartheta(k-2)}{\Gamma(1+2\vartheta)} \tag{52}$$

or

$$\aleph(k+3)=\frac{\Gamma(1+k\vartheta)}{\Gamma(1+(k+3)\vartheta)}\left[\aleph(k)+\frac{\delta_\vartheta(k-2)}{\Gamma(1+2\vartheta)}\right] \tag{53}$$

From the initial conditions Equation (40), we get:

$$\aleph(0)=1,\ \aleph(1)=\frac{1}{\Gamma(1+\vartheta)},\ \aleph(2)=0 \tag{54}$$

Using iteration Equation (53), we obtain the following values of $\aleph(k)$ successively:

$$\aleph(3)=\frac{1}{\Gamma(1+3\vartheta)}\left[\aleph(0)+\frac{\delta_\vartheta(k-2)}{\Gamma(1+2\vartheta)}\right]=\frac{1}{\Gamma(1+3\vartheta)}\cdot 1=\frac{1}{\Gamma(1+3\vartheta)}$$

$$\aleph(4)=\frac{\Gamma(1+\vartheta)}{\Gamma(1+4\vartheta)}\left[\aleph(1)+\frac{\delta_\vartheta(k-2)}{\Gamma(1+2\vartheta)}\right]=\frac{\Gamma(1+\vartheta)}{\Gamma(1+4\vartheta)}\cdot\frac{1}{\Gamma(1+\vartheta)}=\frac{1}{\Gamma(1+4\vartheta)}$$

$$\aleph(5)=\frac{\Gamma(1+2\vartheta)}{\Gamma(1+5\vartheta)}\left[\aleph(2)+\frac{\delta_\vartheta(k-2)}{\Gamma(1+2\vartheta)}\right]=\frac{\Gamma(1+2\vartheta)}{\Gamma(1+5\vartheta)}\cdot\frac{1}{\Gamma(1+2\vartheta)}=\frac{1}{\Gamma(1+5\vartheta)}$$

$$\aleph(6)=\frac{\Gamma(1+3\vartheta)}{\Gamma(1+6\vartheta)}\left[\aleph(3)+\frac{\delta_\vartheta(k-2)}{\Gamma(1+2\vartheta)}\right]=\frac{\Gamma(1+3\vartheta)}{\Gamma(1+6\vartheta)}\cdot\frac{1}{\Gamma(1+3\vartheta)}=\frac{1}{\Gamma(1+6\vartheta)}$$

Finally, the local fractional differential inverse transform leads to:

$$\begin{aligned}\varphi(\gamma) &= \sum_{k=0}^{\infty}\aleph(k)\gamma^{k\vartheta}\\ &= 1+\tfrac{\gamma^{\vartheta}}{\Gamma(1+\vartheta)}+\tfrac{\gamma^{3\vartheta}}{\Gamma(1+3\vartheta)}+\tfrac{\gamma^{4\vartheta}}{\Gamma(1+4\vartheta)}+\tfrac{\gamma^{5\vartheta}}{\Gamma(1+5\vartheta)}+\ldots\end{aligned} \tag{55}$$

that gives the solution by:

$$\varphi(\gamma)=E_\vartheta\left(\gamma^{\vartheta}\right)-\frac{\gamma^{2\vartheta}}{\Gamma(1+2\vartheta)} \tag{56}$$

From Equations (47), (51) and (56), the approximate solution of the given problem Equation (39) by using local fractional decomposition method is the same results as that obtained by the local fractional variational iteration method and by the local fractional differential transformation method.

3.3. Nonlinear Local Fractional Differential Equations

Example 3. Let us consider the nonlinear local fractional differential equation with local fractional derivative operator:

$$\varphi^{(2\vartheta)}(\gamma)+\varphi^{(\vartheta)}(\gamma)-2\varphi^{2}(\gamma)=0\,,\ 0<\vartheta<1 \tag{57}$$

subject to initial conditions:

$$\varphi(0)=0\,,\ \varphi^{(\vartheta)}(0)=1 \tag{58}$$

- By using LFVIM:

Using Equation (14), we write the iteration formula of Equation (57) as:

$$\varphi_{n+1}(\gamma) = \varphi_n(\gamma) + \frac{1}{\Gamma(1+\vartheta)} \int_0^{\gamma} \frac{(\omega-\gamma)^{\vartheta}}{\Gamma(1+\vartheta)} \left[\varphi_n^{(2\vartheta)}(\omega) + \varphi_n^{(\vartheta)}(\omega) - 2\varphi_n^2(\omega)\right] (d\omega)^{\vartheta} \tag{59}$$

Start with the zeroth approximation:

$$\varphi_0(\gamma) = \frac{\gamma^{\vartheta}}{\Gamma(1+\vartheta)} \tag{60}$$

Substituting Equation (60) into Equation (59), we obtain the successive approximations:

$$\begin{aligned} \varphi_1(\gamma) &= \varphi_0(\gamma) - \tfrac{1}{\Gamma(1+\vartheta)} \int_0^{\gamma} \tfrac{(\omega-\gamma)^{2\vartheta}}{\Gamma(1+2\vartheta)} \left[\varphi_0^{(2\vartheta)}(\omega) + \varphi_0^{(\vartheta)}(\omega) - 2\varphi_0^2(\omega)\right] (d\omega)^{\vartheta} \\ &= \tfrac{\gamma^{\vartheta}}{\Gamma(1+\vartheta)} - \tfrac{\gamma^{2\vartheta}}{\Gamma(1+2\vartheta)} + \tfrac{4\gamma^{4\vartheta}}{\Gamma(1+4\vartheta)} \\ \varphi_2(\gamma) &= \varphi_1(\gamma) - \tfrac{1}{\Gamma(1+\vartheta)} \int_0^{\gamma} \tfrac{(\omega-\gamma)^{2\vartheta}}{\Gamma(1+2\vartheta)} \left[\varphi_1^{(2\vartheta)}(\omega) + \varphi_1^{(\vartheta)}(\omega) - 2\varphi_1^2(\omega)\right] (d\omega)^{\vartheta} \\ &= \tfrac{\gamma^{\vartheta}}{\Gamma(1+\vartheta)} - \tfrac{\gamma^{2\vartheta}}{\Gamma(1+2\vartheta)} + \tfrac{\gamma^{3\vartheta}}{\Gamma(1+3\vartheta)} + \tfrac{4\gamma^{4\vartheta}}{\Gamma(1+4\vartheta)} + \tfrac{16\gamma^{5\vartheta}}{\Gamma(1+5\vartheta)} \end{aligned}$$

Therefore, the approximation solution is

$$\varphi(\gamma) = \frac{\gamma^{\vartheta}}{\Gamma(1+\vartheta)} - \frac{\gamma^{2\vartheta}}{\Gamma(1+2\vartheta)} + \frac{\gamma^{3\vartheta}}{\Gamma(1+3\vartheta)} + \frac{4\gamma^{4\vartheta}}{\Gamma(1+4\vartheta)} + \frac{16\gamma^{5\vartheta}}{\Gamma(1+5\vartheta)} + \ldots \tag{61}$$

- By using LFDTM:

Taking the local fractional differential transform of Equation (57), we obtain the following iteration relation:

$$\frac{\Gamma(1+(k+2)\vartheta)}{\Gamma(1+k\vartheta)} \aleph(k+2) + \frac{\Gamma(1+(k+1)\vartheta)}{\Gamma(1+k\vartheta)} \aleph(k+1) - 2\sum_{\rho=0}^{k} \aleph(\rho)\aleph(k-\rho) = 0 \tag{62}$$

or:

$$\aleph(k+2) = \frac{\Gamma(1+k\vartheta)}{\Gamma(1+(k+2)\vartheta)} \left[2\sum_{\rho=0}^{k} \aleph(\rho)\aleph(k-\rho) - \frac{\Gamma(1+(k+1)\vartheta)}{\Gamma(1+k\vartheta)} \aleph(k+1)\right] \tag{63}$$

From the initial conditions Equation (58), we get:

$$\aleph(0) = 0,\ \aleph(1) = \frac{1}{\Gamma(1+\vartheta)} \tag{64}$$

Using iteration Equation (64), we obtain the following values of $\aleph(k)$ successively:

$$\aleph(2) = \frac{1}{\Gamma(1+2\vartheta)} \left[2\aleph(0)\aleph(0) - \frac{\Gamma(1+\vartheta)}{1} \aleph(1)\right] = -\frac{1}{\Gamma(1+2\vartheta)}$$

$$\aleph(3) = \frac{\Gamma(1+\vartheta)}{\Gamma(1+3\vartheta)} \left[4\aleph(0)\aleph(1) - \frac{\Gamma(1+2\vartheta)}{\Gamma(1+\vartheta)} \aleph(2)\right] = \frac{1}{\Gamma(1+3\vartheta)}$$

Finally, the local fractional differential inverse transform leads to:

$$\begin{aligned} \varphi(\gamma) &= \sum_{k=0}^{\infty} \aleph(k)\gamma^{k\vartheta} \\ &= \tfrac{\gamma^{\vartheta}}{\Gamma(1+\vartheta)} - \tfrac{\gamma^{2\vartheta}}{\Gamma(1+\vartheta)} + \tfrac{\gamma^{3\vartheta}}{\Gamma(1+3\vartheta)} + \ldots \end{aligned} \tag{65}$$

4. Conclusions

In this work, LFDM, LFVIM and LFDTM have been successfully used to find the approximate solutions of homogeneous and nonhomogeneous local fractional ordinary differential equations. The results show that the three methods are powerful and efficient for solving linear and nonlinear local fractional differential equations, and therefore, can be widely applied in other problems. Furthermore, the local fractional variational iteration method (LFVIM) requires the evaluation of the Lagrange multiplier, while the local fractional differential transformation method (LFDTM), which is based on the local fractional Taylor theorem, constructs solutions in the form of polynomial series by means of an iterative procedure.

Acknowledgments: This research project was supported by a grant from the "Research Center of the Center for Female Scientific and Medical Colleges", Deanship of Scientific Research, King Saud University.

Author Contributions: Hassan Kamil Jassim and Hossein Jafari wrote some sections of the manuscript; Fairouz Tchier and Dumitru Baleanu prepared some other sections of the paper and analyzed. All authors have read and approved the final manuscript.

Conflicts of Interest: The authors declare no conflict of interest.

References

1. Kilbas, A.A.; Srivastava, H.M.; Trujillo, J.J. *Theory and Applications of Fractional Differential Equations*; Elsevier: San Diego, CA, USA, 2006.
2. He, J.H. Approximate analytical solution for seepage flow with fractional derivatives in porous media. *Comput. Methods Appl. Mech. Eng.* **1998**, *167*, 57–68. [CrossRef]
3. Podlubny, I. *Fractional Differential Equations*; Academic Press: New York, NY, USA, 1999.
4. Hilfer, R. *Applications of Fractional Calculus in Physics*; World Scientific: Singapore, Singapore, 2000.
5. Gao, X.; Yu, J. Synchronization of two coupled fractional order chaotic oscillators. *Chaos Solitons Fractals* **2005**, *26*, 141–145. [CrossRef]
6. Lu, J.G. Chaotic dynamics and synchronization of fractional order Arneodo's systems. *Chaos Solitons Fractals* **2005**, *26*, 1125–1133. [CrossRef]
7. Lu, J.G.; Chen, G. A note on the fractional-order Chen system. *Chaos Solitons Fractals* **2006**, *27*, 685–688. [CrossRef]
8. Chalishajar, D.N.; Malar, K.; Karthikeyan, K. Approximate controllability of abstract impulsive fractional neutral evolution equations with infinite delay in Banach spaces. *Electron. J. Differ. Equ.* **2013**, *2013*, 1–21.
9. Chalishajar, D.N.; Karthikeyan, K. Boundary value problems for impulsive fractional evolution integrodifferential equations with Gronwall's inequality in Banach spaces. *Discontin. Nonlinearity Complex.* **2014**, *3*, 33–48. [CrossRef]
10. Chalishajar, D.N.; Karthikeyan, K. Existence and uniqueness results for boundary value problems of higher order fractional integro-differential equations involving Gronwall's inequality in Banach spaces. *Acta Math. Sci.* **2013**, *33*, 758–772. [CrossRef]
11. Yan, S.P.; Jafari, H.; Jassim, H.K. Local fractional Adomian decomposition and function decomposition methods for solving Laplace equation within local fractional operators. *Adv. Math. Phys.* **2014**, *2014*, 161580. [CrossRef]
12. Jafari, H.; Jassim, H.K. Local fractional Adomian decomposition method for solving two dimensional heat conduction equations within local fractional operators. *J. Adv. Math.* **2014**, *9*, 2574–2582.
13. Baleanu, D.; Machado, J.A.T.; Cattani, C.; Baleanu, C.M.; Yang, X.J. Local fractional variational iteration and decomposition methods for wave equation on Cantor sets within local fractional operators. *Abstr. Appl. Anal.* **2014**, *2014*, 535048. [CrossRef]
14. Jafari, H.; Jassim, H.K. Local fractional variational iteration method for nonlinear partial differential equations within local fractional operators. *Appl. Appl. Math.* **2015**, *10*, 1055–1065.
15. Yang, X.J.; Machado, J.A.T.; Srivastava, H.M. A new numerical technique for solving the local fractional diffusion equation: Two-dimensional extended differential transform approach. *Appl. Math. Comput.* **2016**, *274*, 143–151. [CrossRef]

16. Yang, A.M.; Yang, X.J.; Li, Z.B. Local fractional series expansion method for solving wave and diffusion equations Cantor sets. *Abstr. Appl. Anal.* **2013**, *2013*, 351057. [CrossRef]
17. Jafari, H.; Jassim, H.K. Local fractional series expansion method for solving Laplace and Schrodinger equations on Cantor sets within local fractional operators. *Int. J. Math. Comput. Res.* **2014**, *2*, 736–744.
18. Srivastava, H.M.; Golmankhaneh, A.K.; Baleanu, D.; Yang, X.J. Local fractional Sumudu transform with application to IVPs on Cantor set. *Abstr. Appl. Anal.* **2014**, *2014*, 620529. [CrossRef]
19. Zhao, C.G.; Yang, A.M.; Jafari, H.; Haghbin, A. The Yang-Laplace transform for solving the IVPs with local fractional derivative. *Abstr. Appl. Anal.* **2014**, *2014*, 386459. [CrossRef]
20. Yang, X.J. *Advanced Local Fractional Calculus and Its Applications*; World Science Publisher: New York, NY, USA, 2012.
21. Yang, Y.J.; Hua, L.Q. Variational iteration transform method for fractional differential equations with local fractional derivative. *Abstr. Appl. Anal.* **2014**, *2014*, 760957. [CrossRef]
22. Hristov, J. Transient heat diffusion with a non-singular fading memory: From the Cattaneo constitutive equation with Jeffrey's kernel to the Caputo-Fabrizio time-fractional derivative. *Therm. Sci.* **2016**, *20*, 19. [CrossRef]
23. Golmankhaneh, A.K.; Baleanu, D. New derivatives on the fractal subset of real-line. *Entropy* **2016**, *18*, 1. [CrossRef]
24. Povstenko, Y. Generalized boundary conditions for the time-fractional advection diffusion equation. *Entropy* **2015**, *17*, 4028–4039. [CrossRef]
25. Machado, J.A.T. Fractional order generalized information. *Entropy* **2014**, *16*, 2350–2361. [CrossRef]
26. Machado, J.A.T. Entropy analysis of fractional derivatives and their approximation. *J. Appl. Nonlinear Dyn.* **2012**, *1*, 109–112. [CrossRef]
27. Magin, R.L.; Ingo, C.; Colon-Perez, L.; Triplett, W.; Mareci, T.H. Characterization of anomalous diffusion in porous biological tissues using fractional order derivatives and entropy. *Microporous Mesoporous Mater.* **2013**, *178*, 39–43. [CrossRef] [PubMed]
28. Ingo, C.; Magin, R.L.; Parrish, T.B. New insights into the fractional order diffusion equation using entropy and kurtosis. *Entropy* **2014**, *16*, 5838–5852. [CrossRef]
29. Zhang, Y.D.; Baleanu, D.; Yang, X.J. On a local fractional wave equation under fixed entropy arising in fractal hydrodynamics. *Entropy* **2014**, *16*, 6254–6262. [CrossRef]