ON MODELING THE GROUNDWATER FLOW WITHIN A CONFINED AQUIFER

ABDON ATANGANA[1], DUMITRU BALEANU[2,3,4]

[1]Institute for Groundwater Studies, Faculty of Natural and Agricultural Sciences, South Africa
E-mail: abdonatangana@yahoo.fr
[2]Department of Chemical and Materials Engineering, Faculty of Engineering,
King Abdulaziz University, P.O. Box
80204, Jeddah, Saudi Arabia
[3]Department of Mathematics and Computer Sciences, Faculty of Arts and Sciences,
Cankaya University, 06530 Ankara, Turkey
[4]Institute of Space Sciences, Magurele-Bucharest, Romania
E-mail: dumitru@cankaya.edu.tr

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The groundwater flow equation is used to simulate the movement of water under the confined aquifer. In this paper we study a modification of the groundwater flow equation within a newly proposed derivative. We numerically solve the generalized groundwater flow equation with the Crank-Nicholson scheme. We also analytically solve the generalized equation *via* the method of separation of variable.

Key words: groundwater flow equation; fractional derivative; β-derivative.

1. INTRODUCTION

In recent years, attention has been paid by several scholars working in the field of groundwater studies to model with accuracy the movement of water under the confined geological formation called aquifer. The first attempt to describe this phenomenon has been done by Theis [1]; see also Ref. [2]. However, in his work, Theis put in place some assumptions, which help him to have a more simplified equation. For instance, Theis [1] assumed that the aquifer is homogeneous, but all the aquifer investigated were not at all homogeneous, but rather they were heterogeneous. In fact we have concluded that it is not possible to find a homogeneous aquifer in a real life situation. Clearly, the model presented by Theis [1] has some limitations and this has been a worry in the circle of groundwater studies.

To the best of our knowledge, the authors of Ref. [3] were the first to generalize the groundwater flow equation to the concept of *fractional derivative*. However the fractional derivative used in their work was the Liouville one [4]. Due to the rewards of the fractional derivative application to the groundwater flow equation, the groundwater flow equation using the Caputo fractional derivative was investigated in [5–7], which allows the use of the standard initial and boundary conditions. The results obtained from this study were much better than the one obtained with Riemann-Liouville fractional derivative. Due to the benefits provided by the fractional calculus while modeling real world problems, several applications of this type of calculus were proposed [8–15]. A new type of derivative was proposed and applied in Refs. [16,17]. Although the conformable derivative displays useful properties than Caputo and Riemann-Liouville fractional derivatives, this last version has very big disadvantage because all differentiable functions has zero as conformable derivative at the point zero. The aim of this paper is to analyze the modification of the groundwater flow equation within this new derivative.

2. THE BETA-DERIVATIVE AND ITS GEOMETRICAL INTERPRETATION

We shall briefly present in this section the β- derivative and its numerical presentation.

Definition 1 [18]: Let f be a function, such that $f:[a,\infty)\to\Re$. Then the $\beta-$derivative is defined as:

$$ {}_0^AD_x^\beta(f(x))=\lim_{\varepsilon\to 0}\frac{f\left(x+\varepsilon\left(x+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}\right)-f(x)}{\varepsilon} \tag{1}$$

for all $x\ge a,\beta\in(0,1]$. Then if the limit of the above exists, f is said to be $\beta-$derivable.

The β- derivative can be written as

$$ {}_0^AD_x^\beta(f(x))=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\left(x+\frac{1}{\Gamma(\beta)}\right)^{1-\beta} \tag{2}$$

We can easily prove this result by making the notation with h as follows

$$h=\left(x+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}.$$

We obtain

$$ {}_0^AD_x^\beta(f(x))=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}\left(x+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}.$$

and this completes the proof. Accordingly, we will propose the numerical approximation of the β- derivative as follows

$$ {}_0^A D_x^{\beta}(f(x_i)) = \lim_{h\to 0} \frac{f(x_{i+1}) - f(x_i)}{h}\left(x_i + \frac{1}{\Gamma(\beta)}\right)^{1-\beta}, \quad 0<\beta\le 1 \tag{3a} $$

$$ {}_0^A D_x^{\beta}(f(x_i)) = \lim_{h\to 0} \frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{h}\left(x_i + \frac{1}{\Gamma(\beta)}\right)^{2-\beta}, \quad 1<\beta\le 2 \tag{3b} $$

For a function of two variables we have the following notation

$$ u(x_i, t_n) = u_i^n $$

Then the approximation of the above expression shall be for the explicit method

$$ {}_0^A D_x^{\beta}(u(x_i,t_n)) = \lim_{h\to 0} \frac{u(x_{i+1},t_n) - 2u(x_i,t_n) + u(x_{i-1},t_n)}{h}\left(x_i + \frac{1}{\Gamma(\beta)}\right)^{2-\beta}, \quad 1<\beta\le 2 \tag{4} $$

Finally if we use the central difference at time $t_{n+1/2}$, then a second-order central difference for the space derivative at position x_j, we have the recurrence equation

$$ {}_0^A D_x^{\beta}(u(x_i,t_n)) = $$

$$ \lim_{h\to 0}\left(\frac{u(x_{i+1},t_{n+1}) - 2u(x_i,t_{n+1}) + u(x_{i-1},t_{n+1})}{2h^2} + \frac{u(x_{i+1},t_n) - 2u(x_i,t_n) + u(x_{i-1},t_n)}{2h^2} \right)\cdot \tag{5} $$

$$ \left(x_i + \frac{1}{\Gamma(\beta)}\right)^{2-\beta}, 1<\beta\le 2. $$

The stability and convergence analysis of the scheme will be verified by solving the well-known Theis groundwater flow equation. This will be presented in the section below. We shall illustrate this technique with an example, see Figs. 1 and 2.

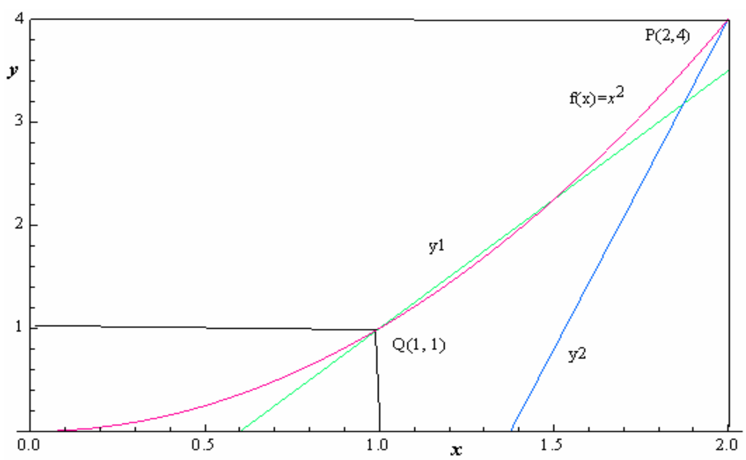

Fig. 1 – The case β = 0.5.

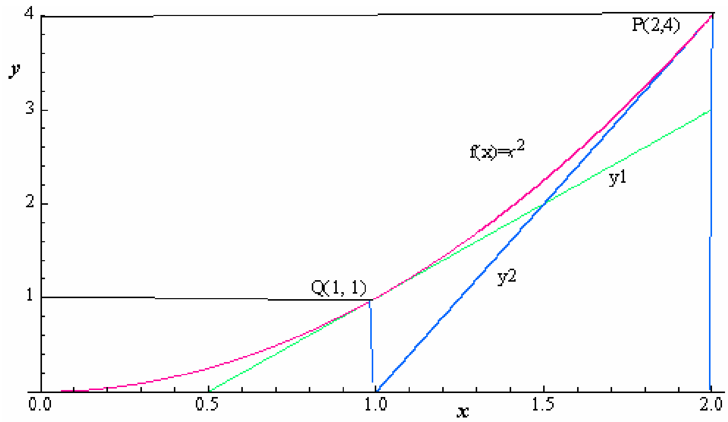

Fig. 2 – The case β = 1.

Let consider the function $f(x)=x^2$, we chose two points $x_0=1$ and $x_1=2$. Since *f* is differentiable and is $\frac{1}{2}$ – differentiable then

$$ {}_{0}^{A}D_x^{\beta}(f(x)) = 2\left(x+\frac{1}{\Gamma(0.5)}\right)^{0.5} x \tag{6}$$

Therefore at the points 1 and 2 we have the following

$$ {}_{0}^{A}D_x^{0.5}(f(1)) = 2\left(1+\frac{1}{\Gamma(0.5)}\right)^{0.5}, \qquad {}_{0}^{A}D_x^{0.5}\left(f(2)\right) = 4\left(2+\frac{1}{\Gamma(0.5)}\right)^{0.5} \tag{7}$$

and

$$y_1-1 = 2\left(1+\frac{1}{\Gamma(0.5)}\right)^{0.5}(x-1), \quad y_2-4 = 4\left(2+\frac{1}{\Gamma(0.5)}\right)^{0.5}(x-2).$$

3. GENERALIZATION OF THEIS GROUNDWATER EQUATION

The mathematical formula describing the movement of water in a confined aquifer was first proposed by Theis and we will present it as [3]

$$\frac{S}{T}\,{}_{0}^{A}D_t^{\beta}\left(s(r,t)\right) = \partial_{rr}^2\left(s(r,t)\right)+\frac{1}{r}\partial_r\left(s(r,t)\right) \tag{8}$$

The above equation is subjected to the following initial and boundary conditions

$$s(r,0)=s_0, \qquad \lim_{r\to\infty} = s_0 \qquad Q = 2\pi r_b K d\,\partial_r\left(s\left(r_b,t\right)\right), \tag{9}$$

where $s(r,t)$ is the drawdown or change in the level of water; S is the specific storativity of the aquifer, and T is the transmissivity, with K as the hydraulic conductivities of the main and confining layers, d is the thickness of the main and confining layers, and Q is the discharge rate of the pumping. We shall use the Crank-Nicholson scheme [19] to solve numerically equation (8) together with equation (9).

3.1. NUMERICAL SOLUTION AND ITS STABILITY

Before performing the numerical methods, we assume that equation (8) has a unique and sufficiently smooth solution. However in order to establish numerical schemes for equation (9), we let: $x_j = jh,\ 0 \le j \le M,\ Mh = J,\ t_k = k\tau,\ 0 \le k \le N$ and $N\tau = T$ and we let h to be the step size and τ to be the time size, and M and N to be the grid points. We shall introduce the Crank-Nicholson numerical method [19] as follows. At the onset, the discretization of the first- and second-order space derivative are identified as

$$\partial_r s = \frac{1}{2}\left(\left(\frac{s_{j+1}^{k+1} - s_{j-1}^{k+1}}{h}\right) + \left(\frac{s_{j+1}^{k} - s_{j-1}^{k}}{h}\right)\right) + O(h) \tag{10}$$

The second order derivative shall be considered as

$$\partial_{rr} s = \frac{1}{2}\left(\left(\frac{s_{j+1}^{k+1} - 2s_j^{k+1} + s_{j-1}^{k+1}}{h^2}\right) + \left(\frac{s_{j+1}^{k} - 2s_j^{k} + s_{j-1}^{k}}{h^2}\right)\right) + O(h^2), \quad s = \frac{1}{2}\left(s_j^{k+1} + s_j^{k}\right) \tag{11}$$

Finally, the Crank-Nicholson scheme for the fractional derivative is given as

$${}_0^A D_t^{\beta} s = \frac{1}{2}\left(t_k + \frac{1}{\Gamma(\beta)}\right)^{1-\beta}\left(\left(\frac{s_j^{k+1} - s_j^{k-1}}{\tau}\right)\right) + O(\tau) \tag{12}$$

Nonetheless by replacing equations (12), (11) and (10) into equation (8) we will obtain the following

$$\begin{aligned} &\frac{1}{2}\left(t_k + \frac{1}{\Gamma(\beta)}\right)^{1-\beta}\left(\left(\frac{s_{j+1}^{k+1} - s_{j+1}^{k-1}}{\tau}\right) + \left(\frac{s_j^{k+1} - s_j^{k-1}}{\tau}\right)\right)\frac{S}{T} = \\ &\frac{1}{2}\left(\left(\frac{s_{j+1}^{k+1} - 2s_j^{k+1} + s_j^{k+1}}{h^2}\right) + \left(\frac{s_{j+1}^{k} - 2s_j^{k} + s_{j-1}^{k}}{h^2}\right)\right) + \\ &\frac{1}{2}\left(\left(\frac{s_{j+1}^{k+1} - s_{j-1}^{k+1}}{h}\right) + \left(\frac{s_{j+1}^{k} - s_{j-1}^{k}}{h}\right)\right)\frac{1}{r_j}. \end{aligned} \tag{13}$$

For simplicity let us put

$$\frac{1}{2}\left(t_k+\frac{1}{\Gamma(\beta)}\right)^{1-\beta}\frac{S}{T\tau}=\gamma_k,\qquad \mu_j=\frac{1}{2\pi r_j},\qquad a=\frac{1}{2h^2}$$

Such that equation (13) becomes

$$\begin{aligned}&\gamma_k\left(s_j^{k+1}-s_j^{K}\right)=a\left(s_{j+1}^{k+1}-2s_j^{k+1}+s_{j-1}^{k+1}+s_{j+1}^{k}-s_j^{k}+s_{j-1}^{k}\right)+\\&\mu_j\left(s_{j+1}^{k+1}-s_{j-1}^{k+1}+s_{j+1}^{k}-s_{j-1}^{k}\right)\end{aligned}\tag{14}$$

The above discretization can be converted to a matrix. We shall present in detail the stability analysis of the Crank-Nicholson scheme for the modified groundwater equation *via* the $\beta-$derivative.

Let $\xi_j^k=s_j^k-S_j^k$, where S_j^k is the approximate solution at the point $(x_l,t_k),(k=1,2,...,N,\ l=1,2,...,M)$ and in addition

$$\xi^k=\left[\xi_1^k,\xi_2^k,...,\xi_M^k\right]^T\tag{15}$$

We shall reformulate equation (14) as follows

$$\begin{aligned}&\xi_j^{k+1}\left(\gamma_k+2a\right)=\xi_j^k\left(\gamma_k-2a\right)+\xi_{j+1}^{k+1}\left(\mu_j+a\right)+\\&\xi_{j-1}^{k+1}\left(a-\mu_j\right)+\xi_{j+1}^{k}\left(\mu_j+a\right)+\xi_{j-1}^{k}\left(a-\mu_j\right)\end{aligned}\tag{16}$$

The above equation is a recurrence relation for the error. Equations (14) and (16) show, that both the error and numerical solution have the same growth or decay behavior with respect to time. Since the equation under study is linear with periodic boundary condition, the spatial variation of error may be expended in a finite Fourier [19] series, in the interval *J*, which is the length of the aquifer, as

$$\xi(r)=\sum_{m=1}^{M}B_m e^{jk_m r}\tag{17}$$

We shall recall that $k_m=\dfrac{\pi m}{J}$ with $m=1,2,...,M$ and $M/\Delta r$. Time requirement of the error is encompassed by supposing that the amplitude of error B_m is a function of time [19–20]. Since the error develops an exponentially decay with time, it is wise to undertake that the amplitude alters exponentially with time; consequently

$$\xi(r,t)=\sum_{m=1}^{M}e^{bt}e^{jk_m r},\ b\text{ is constant}\tag{18}$$

The difference equation for error is linear, it follows that the growth of error of a typical term is

$$\xi_m(r,t)=e^{bt}e^{jk_m r}\tag{19}$$

The stability features can be predicted exploiting just this form for error with no loss in oversimplification. Nonetheless to investigate the variation of error in steps of time, we shall replace equation (19) in (16), but this will happen only after we write down

$$\begin{aligned} &\xi_j^k = e^{bt} e^{jk_m r}, \xi_j^{k+1} = e^{b(t+\Delta t)} e^{jk_m r}, \\ &\xi_{j+1}^{k+1} = e^{b(t+\Delta t)} e^{jk_m(r+\Delta r)}, \xi_{j-1}^{k+1} = e^{b(t+\Delta t)} e^{jk_m(r-\Delta r)}, \xi_{j+1}^{k-1} = e^{b(t-\Delta t)} e^{jk_m(r+\Delta r)}, \\ &\xi_{j+1}^k = e^{bt} e^{jk_m(r+\Delta r)}, \xi_{j-1}^k = e^{bt} e^{jk_m(r-\Delta r)}. \end{aligned} \tag{20}$$

This shall yield

$$\begin{aligned} &e^{b(t+\Delta t)} e^{jk_m r} (\gamma_k + 2a) = e^{bt} e^{jk_m r} (\gamma_k - 2a) + \\ &e^{b(t+\Delta t)} e^{jk_m(r+\Delta r)} (\mu_j + a) + e^{b(t+\Delta t)} e^{jk_m(r-\Delta r)} (a - \mu_j) \\ &+ e^{bt} e^{jk_m(r+\Delta r)} (\mu_j + a) + e^{bt} e^{jk_m(r-\Delta r)} (a - \mu_j) \end{aligned} \tag{21}$$

Now after re-arranging and simplification we obtain

$$e^{b\Delta t} = \frac{\gamma_k - 2a + (a + \mu_j) e^{j\Delta r k_m} + (a - \mu_j) e^{-j\Delta r k_m}}{\gamma_k + 2a + (a + \mu_j) e^{j\Delta r k_m} + (a - \mu_j) e^{-j\Delta r k_m}} \tag{22}$$

Note that the stability is achieved if and only if

$$\left| \frac{\xi_j^{k+1}}{\xi_j^k} \right| < 1. \tag{23}$$

But note that

$$\frac{\xi_j^{k+1}}{\xi_j^k} = e^{b\Delta t} \tag{24}$$

Thus in our case of study we have then

$$\frac{\xi_j^{k+1}}{\xi_j^k} = \frac{\gamma_k - 2a + (a + \mu_j) e^{j\Delta r k_m} + (a - \mu_j) e^{-j\Delta r k_m}}{\gamma_k + 2a + (a + \mu_j) e^{j\Delta r k_m} + (a - \mu_j) e^{-j\Delta r k_m}}. \tag{25}$$

Remark: Note that for all $0 \le j \le M$, $0 \le k \le N$, $a > 0$, $\gamma_k > 0$, thus it follows without any doubt that

$$\left| \frac{\xi_j^{k+1}}{\xi_j^k} \right| = \left| \frac{\gamma_k - 2a + (a + \mu_j) e^{j\Delta r k_m} + (a - \mu_j) e^{-j\Delta r k_m}}{\gamma_k + 2a + (a + \mu_j) e^{j\Delta r k_m} + (a - \mu_j) e^{-j\Delta r k_m}} \right| < 1.$$

We can conclude that the Crank-Nicholson scheme is unconditionally stable for the generalized groundwater flow equation using the beta derivative.

3.2. ANALYTICAL SOLUTION OF THE GENERALIZED GROUNDWATER FLOW EQUATION

In the following we shall see if with this fractional derivative, the groundwater flow equation can be solved analytically. We shall use in this section the so-called *method of separation of variables*. To solve equation (8) with this method, we assume that the solution can be found in the following form

$$s(r,t)=f(r)g(r)\,. \tag{26}$$

We obtain the following equation

$${}_0^A D_t^{\beta} g(t)=-\lambda^2\frac{T}{S}g(t)\,. \tag{27}$$

The second equation is given by

$$\partial_{rr}^2\left(f(r)\right)+\frac{1}{r}\partial_r\left(s(r)\right)-\lambda^2 f(r)=0\,. \tag{28}$$

The exact solution of equation (27) is given as

$$g(t)=cExp\left[-\lambda^2\frac{T\left(t+\frac{1}{\Gamma(\beta)}\right)^{2-\beta}}{2-\beta}\right]. \tag{29}$$

The solution of equation (28) has been provided in [4] as

$$f(r)=J_0\left(\lambda r\right)=\sum_{k=0}^{\infty}\frac{\left(-1\right)^k}{k!\Gamma\left(k+1\right)}\left(\frac{\lambda r}{2}\right)^{2k}. \tag{30}$$

Thus the exact solution of equation (8) can be written as

$$s(r,t)=c\sum_{n=0}^{\infty}Exp\left[-\lambda_n^{\ 2}\frac{T\left(t+\frac{1}{\Gamma(\beta)}\right)^{2-\beta}}{2-\beta}\right]\sum_{k=0}^{\infty}\frac{\left(-1\right)^k}{k!\Gamma\left(k+1\right)}\left(\frac{r\lambda_{\ n}}{2}\right)^{2k}. \tag{31}$$

However by applying the initial condition on (31), we obtain that

$$c=\frac{Q}{4\pi T}.$$

Then the solution of generalized groundwater flow equation using the $\beta-$derivative is given by

$$s(r,t)=\frac{Q}{4\pi T}\sum_{n=0}^{\infty}Exp\left[-\lambda_n^{\ 2}\frac{T\left(t+\frac{1}{\Gamma(\beta)}\right)^{2-\beta}}{2-\beta}\right]\sum_{k=0}^{\infty}\frac{\left(-1\right)^k}{k!\Gamma\left(k+1\right)}\left(\frac{r\lambda_{\ n}}{2}\right)^{2k}. \tag{32}$$

Figure 3 shows the numerical simulation of the modified groundwater flow equation for different values of the order of the β-derivative.

Here we chose Q = 350, T = 700, S = 0.001, r = 50. It is clear from the figure that the solution of this equation does not depend only on the aquifer parameters, but also on the new parameter β. When the parameter β is equal to 1, we recover the exact solution of the classical groundwater flow equation; this solution is overestimating the change of level of water during the time of removal of water from the confined aquifer, because the classical equation does not take into account any presence of uncertainties in the physical problem.

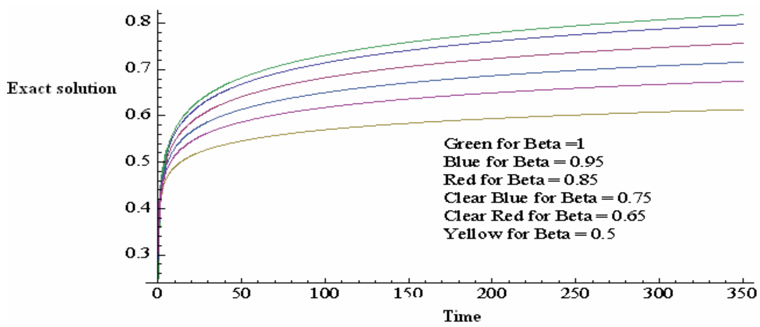

Fig. 3 – Solution of the groundwater flow equation for different values of β.

However, with the new derivative, the modified equation is clearly depending on the order of the derivative, which can be seen as contribution of uncertainties in the geological formation.

4. CONCLUSION

In this paper, we have presented a geometrical interpretation of the β-derivative with an example. The numerical approximation of this operator was presented and the stability analysis was investigated. The generalized groundwater flow equation within this β-derivative was investigated and the exact solution was also reported. As it can be seen from Fig. 3, the parameter β plays a crucial role in the numerical solution and it can be used to find new insights of the generalized groundwater flow equation.

REFERENCES

1. C. V. Theis, *The relation between the lowering of the piezometric surface and the rate and duration of discharge of a well using ground-water storage,* Transactions American Geophysical Union **16**, 519–524 (1935).

2. M. S. Hantush, C. E. Jacob, *Non-steady radial flow in an infinite leaky aquifer*, Transactions American Geophysical Union **36**, 95–100 (1955).
3. A. Cloot, J. F. Botha, *A generalised groundwater flow equation using the concept of non-integer order derivatives*, Water SA **32**, 1–7 (2006).
4. Y. Luchko, R. Gorenflo, *The initial value problem for some fractional differential equations with the Caputo derivative*, Preprint Series A08–98, Freie Universitat Berlin, Fachbreich Mathematik and Informatik, 1998.
5. A. Atangana, P. D. Vermeulen, *Analytical Solutions of a Space-Time Fractional Derivative of Groundwater Flow Equation*, Abstract and Applied Analysis **2014**, Article ID 381753 (2014).
6. A. Atangana, N. Bildik, *The Use of Fractional Order Derivative to Predict the Groundwater Flow*, Mathematical Problems in Engineering **2013**, Article ID 543026 (2013).
7. A. Atangana, *Drawdown in prolate spheroidal-spherical coordinates obtained via Green's function and perturbation methods,* Communications in Nonlinear Science and Numerical Simulation **19**, 1259–1269 (2014).
8. D. Baleanu, K. Diethelm, E. Scalas, J. J. Trujillo, *Fractional Calculus Models and Numerical Methods, Series on Complexity, Nonlinearity and Chaos*, World Scientific, Singapore, 2012.
9. J. F. Gomez Aguilar, D. Baleanu, *Solutions of the telegraph equations using a fractional calculus approach,* Proc. Romanian Acad. A **15**, 27–34 (2014).
10. X. J. Yang *et al*., *Transport equations in fractal porous media within fractional complex transform method*, Proc. Romanian Acad. A **14**, 287–292 (2013).
11. X. J. Yang *et al*., *Approximate solutions for diffusion equations on Cantor space-time,* Proc. Romanian Acad. A **14**, 127–133 (2013).
12. X. J. Yang, D. Baleanu, Y. Khan, S.T. Mohyud-Din, *Local Fractional Variational Iteration Method for Diffusion and Wave Equations On Cantor Sets*, Rom. J. Phys. **59**, 36–48 (2014).
13. A. M. O. Anwar *et al*., *Fractional Caputo Heat Equation within the Double Laplace Transform*, Rom. J. Phys. **58**, 15–22 (2013).
14. H. Jafari *et al*., *Exact Solutions of Boussinesq and KdV-mKdV Equations by Fractional Sub-Equation Method*, Rom. Rep. Phys. **65**, 1119–1124 (2013).
15. D. Rostamy *et al.*, *Solving multi-term orders fractional differential equations by operational matrices of BPs with convergence analysis*, Rom. Rep. Phys. **65**, 334–349 (2013).
16. R. Khalil, M. Al Horani, A. Yousef, M. Sababheh, *A new definition of fractional derivative*, Journal of Computational and Applied Mathematics **264**, 65–70 (2014).
17. M. Abu Hammad, R. Khalil, *Conformable fractional heat differential equation*, International Journal of Pure and Applied Mathematics **94**, 215-221 (2014).
18. A. Atangana, E. F. D. Goufo, *Extension of Match Asymptotic Method to Fractional Boundary Layer Problems*, Submitted in Mathematical Problems in Engineering, 2014.
19. J. Crank, P. Nicolson, *A practical method for numerical evaluation of solutions of partial differential equations of the heat-conduction type*, Proc. Cambridge Philos. Soc. **43**, 50–67 (1947).

J. G. Charney, R. Fjortoft, J. von Neumann, *Numerical Integration of the Barotropic Vorticity Equation*, Tellus **2,** 237–254 (1950).