A LIE GROUP APPROACH TO SOLVE THE FRACTIONAL POISSON EQUATION

M.S. HASHEMI[1, a, *], D. BALEANU[2,3], M. PARTO-HAGHIGHI[1]

[1]Department of Mathematics, Basic Science Faculty, University of Bonab, Bonab 55517, Iran
[2]Department of Mathematics, Çankaya University, Ogretmenler Caddesi 14, Balgat, 06530 Ankara, Turkey
[3]Institute of Space Sciences, Magurele-Bucharest, Romania
E-mail[a]: hashemi_math396@yahoo.com, hashemi@bonabu.ac.ir

Received January 12, 2015

In the present paper, approximate solutions of fractional Poisson equation (FPE) have been considered using an integrator in the class of Lie groups, namely, the fictitious time integration method (FTIM). Based on the FTIM, the unknown dependent variable $u(x,t)$ is transformed into a new variable with one more dimension. We use a fictitious time τ as the additional dimension (fictitious dimension), by transformation: $v(x,t,\tau) := (1+\tau)^{\kappa}u(x,t)$, where $0 < \kappa \leq 1$ is a parameter to control the rate of convergency in the FTIM. Then the group preserving scheme (GPS) is used to integrate the new fractional partial differential equations in the augmented space $\mathbb{R}^{2+1}$. The power and the validity of the method are demonstrated using two examples.

Key words: Fractional Poisson equation, Fictitious time integration method, Caputo derivative, Group-preserving scheme.

PACS: 02.70.Bf,02.20.Sv, 05.30.Pr.

1. INTRODUCTION

Partial differential equations of elliptic type have a significant role in the many branches of science and engineering such as in the theory of electromagnetic potentials or in the search of vibration modes of elastic structures. The Poisson equation is one of the well-known elliptic partial differential equations, which plays an important role in the modeling of a large variety of phenomena, often of stationary nature. The Poisson equation has the general form:

$$\Delta u = f, \;\; u \in \Omega \subset \mathbb{R}^n, \tag{1}$$

where f is the source function. The wave and diffusion equations model evolution phenomena, whereas the homogeneous Poisson equation (Laplace equation, *i.e.* $\Delta u = 0$, is a special case of Eq. (1), which is obtained when the source function f is zero.), describes the corresponding steady state, in which the solution does not depend on time anymore. Moreover, Eq. (1) plays an important role in the theory of

*Corresponding author

conservative fields (magnetic, electrical, gravitational, etc.) where the vector field is derived from the gradient of a potential.

On the other hand, the fractional calculus started recently to become very significant in several branches of science and engineering [1–5]. Many important phenomena, *e.g.* electro-chemistry, control processing, acoustics, electro-magnetics, anomalous diffusion, and visco-elasticity are well described by fractional differential equations. An important task in the fractional calculus area is to find the exact and approximate solutions for the equations involving the fractional calculus derivatives. Some of the exact approaches dealing with the fractional differential equations can be found in [6]-[20].

We recall that some of the properties of the fractional derivatives are very different those of the classical ones, therefore there exist a huge motivation to dig into area of finding the solutions of some generic equations like the fractional Poisson equation. It is well known that new fractional order models are more satisfactory than previously used integer order models and this is our motivation to consider the fractional Poisson equation. Eq. (1) with fractional derivatives in the domain $\Omega := \{(x,t): \ a < x < b, \ 0 < t < T\}$, is as following [21]:

$$\begin{cases} D_x^{\alpha} u(x,t) + D_t^{\beta} u(x,t) = f(x,t), \ \ (x,t) \in \Omega, \ \alpha, \beta \in (1,2], \\ u(x,0) = 0, \ \ a \leq x \leq b, \\ u(x,T) = 0, \ \ a \leq x \leq b, \\ u(a,t) = 0, \ \ 0 \leq t \leq T, \\ u(b,t) = 0, \ \ 0 \leq t \leq T, \end{cases} \tag{2}$$

where the time-fractional derivative D_t^{β} and space-fractional derivative D_x^{α} are defined in the Caputo sense.

In this paper, we construct a simple and accurate numerical method of FTIM to solve the FPE. This method was firstly proposed by Liu [22] to solve an inverse Sturm-Liouville problem. In the sense of stability and accuracy of FTIM, it would be very interesting that it is much better than other conventional numerical approaches. The above idea has been proposed and used by Liu and his coworkers in Refs. [23–27], and by Hashemi *et al.* in [28], whose numerical results are very good.

2. THE FICTITIOUS TIME INTEGRATION METHOD

There are several mathematical definitions about the fractional derivative [1–5]. Here we adopt the usually used definition, namely the Caputo time and space fractional derivatives given by:

$$D_x^{\alpha}u(x,t) = \frac{1}{\Gamma(2-\alpha)}\int_a^x \frac{u_{ss}(s,t)}{(x-s)^{\alpha-1}}ds,$$
$$D_t^{\beta}u(x,t) = \frac{1}{\Gamma(2-\beta)}\int_0^t \frac{u_{ss}(x,s)}{(t-s)^{\beta-1}}ds,$$

where $1<\alpha,\beta\leq 2$. By using these fractional derivatives we can write Eq. (2) as:

$$\frac{1}{\Gamma(2-\alpha)}\int_a^x \frac{u_{ss}(s,t)}{(x-s)^{\alpha-1}}\mathrm{d}s + \frac{1}{\Gamma(2-\beta)}\int_0^t \frac{u_{ss}(x,s)}{(t-s)^{\beta-1}}\mathrm{d}s = f(x,t). \tag{3}$$

Now, we introduce a fictitious damping coefficient $\nu_0>0$ into Eq. (3) to increase the stability of numerical integration:

$$\frac{\nu_0}{\Gamma(2-\alpha)}\int_a^x \frac{u_{ss}(s,t)}{(x-s)^{\alpha-1}}\mathrm{d}s + \frac{\nu_0}{\Gamma(2-\beta)}\int_0^t \frac{u_{ss}(x,s)}{(t-s)^{\beta-1}}\mathrm{d}s - \nu_0 f(x,t) = 0. \tag{4}$$

Then, we are ready to propose the following pivotal transformation:

$$v(x,t,\tau) = (1+\tau)^{\kappa}u(x,t), \quad 0<\kappa\leq 1. \tag{5}$$

So, Eq. (3) will transform to the following form:

$$\frac{\nu_0}{(1+\tau)^{\kappa}}\left[\frac{1}{\Gamma(2-\alpha)}\int_a^x \frac{v_{ss}(s,t,\tau)}{(x-s)^{\alpha-1}}\mathrm{d}s + \frac{1}{\Gamma(2-\beta)}\int_0^t \frac{v_{ss}(x,s,\tau)}{(t-s)^{\beta-1}}\mathrm{d}s - f(x,t)\right] = 0. \tag{6}$$

One time differentiation of Eq. (6) with respect to the fictitious coordinate τ, yields:

$$\frac{\partial v}{\partial \tau} = \kappa(1+\tau)^{\kappa-1}u(x,t), \tag{7}$$

which by using Eq. (4) and adding it on both sides of Eq. (6), we have

$$\begin{aligned}\frac{\partial v}{\partial \tau} = {} & \frac{\nu_0}{(1+\tau)^{\kappa}}\left[\frac{1}{\Gamma(2-\alpha)}\int_a^x \frac{v_{ss}(s,t,\tau)}{(x-s)^{\alpha-1}}\mathrm{d}s\right.\\ & \left. + \frac{1}{\Gamma(2-\beta)}\int_0^t \frac{v_{ss}(x,s,\tau)}{(t-s)^{\beta-1}}\mathrm{d}s - f(x,t)\right] + \kappa(1+\tau)^{\kappa-1}u.\end{aligned} \tag{8}$$

Then, Eq. (8) can be converted into a new type of functional partial differential equation (PDE) for v, by using $u=\frac{v}{(1+\tau)^{\kappa}}$:

$$\begin{aligned}\frac{\partial v}{\partial \tau} = {} & \frac{\nu_0}{(1+\tau)^{\kappa}}\left[\frac{1}{\Gamma(2-\alpha)}\int_a^x \frac{v_{ss}(s,t,\tau)}{(x-s)^{\alpha-1}}\mathrm{d}s\right.\\ & \left. + \frac{1}{\Gamma(2-\beta)}\int_0^t \frac{v_{ss}(x,s,\tau)}{(t-s)^{\beta-1}}\mathrm{d}s - f(x,t)\right] + \frac{\kappa v}{1+\tau}.\end{aligned} \tag{9}$$

Upon using

$$\frac{\partial}{\partial\tau}\left(\frac{v}{(1+\tau)^{\kappa}}\right)=\frac{v_{\tau}}{(1+\tau)^{\kappa}}-\frac{\kappa v}{(1+\tau)^{1+\kappa}}, \tag{10}$$

and after multiplying the integrating factor $\frac{1}{(1+\tau)^{\kappa}}$ on both sides of Eq. (9), one can obtain:

$$\begin{aligned}\frac{\partial}{\partial\tau}\left(\frac{v}{(1+\tau)^{\kappa}}\right)= & \frac{\nu_0}{(1+\tau)^{2\kappa}}\left[\frac{1}{\Gamma(2-\alpha)}\int_a^x\frac{v_{ss}(s,t,\tau)}{(x-s)^{\alpha-1}}\mathrm{d}s\right.\\ & \left.+\frac{1}{\Gamma(2-\beta)}\int_0^t\frac{v_{ss}(x,s,\tau)}{(t-s)^{\beta-1}}\mathrm{d}s-f(x,t)\right],\end{aligned} \tag{11}$$

Now, by using $u=\frac{v}{(1+\tau)^{\kappa}}$ again, we obtain a new type of functional PDE for u as follows:

$$\begin{aligned}u_{\tau}= & \frac{\nu_0}{(1+\tau)^{\kappa}}\left[\frac{1}{\Gamma(2-\alpha)}\int_a^x\frac{u_{ss}(s,t,\tau)}{(x-s)^{\alpha-1}}\mathrm{d}s\right.\\ & \left.+\frac{1}{\Gamma(2-\beta)}\int_0^t\frac{u_{ss}(x,s,\tau)}{(t-s)^{\beta-1}}\mathrm{d}s-f(x,t)\right].\end{aligned} \tag{12}$$

Here, we must stress that above τ is a fictitious time, and is used to embed Eq. (3) into a new functional PDE in the space of $\mathbb{R}^{2+1}$. As well as, u is an unknown function with $u=u(x,t,\tau)$ subjecting to the constraints in Eq. (2), for all $\tau\geq 0$, and $u(x,t,\tau)|_{\tau=0}$ is given initially by a guess.

Let $u_i^j(\tau):=u(x_i,t_j,\tau)$ be a numerical value of u at a fictitious time τ and a grid point (x_i,t_j). Applying a semi-discretization to the Eq. (12) yields:

$$\begin{aligned}\frac{d}{d\tau}u_i^j(\tau)= & \frac{\nu_0}{(1+\tau)^{\kappa}}\left[\frac{1}{\Gamma(2-\alpha)}\int_a^{x_i}\frac{u_{ss}(s,t_j,\tau)}{(x_i-s)^{\alpha-1}}\mathrm{d}s\right.\\ & \left.+\frac{1}{\Gamma(2-\beta)}\int_0^{t_j}\frac{u_{ss}(x_i,s,\tau)}{(t_j-s)^{\beta-1}}\mathrm{d}s-f(x_i,t_j)\right],\end{aligned} \tag{13}$$

where the two appeared integral terms can be calculated using a numerical integration as follows:

$$\begin{aligned}\int_a^{x_i}\frac{u_{ss}(s,t_j,\tau)}{(x_i-s)^{\alpha-1}}\mathrm{d}s= & \frac{u(x_3,t_j,\tau)-2u(x_2,t_j,\tau)+u(x_1,t_j,\tau)}{\Delta x(x_i-x_1)^{\alpha-1}}\\ & +\sum_{l=2}^{i-1}\frac{u(x_{l+1},t_j,\tau)-2u(x_l,t_j,\tau)+u(x_{l-1},t_j,\tau)}{\Delta x(x_i-x_l)^{\alpha-1}},\end{aligned} \tag{14}$$

and

$$\int_0^{t_j}\frac{u_{ss}(x_i,s,\tau)}{(x_i-s)^{\beta-1}}\mathrm{d}s=\sum_{l=1}^{j-1}\frac{u(x_i,t_{l+2},\tau)-2u(x_i,t_{l+1},\tau)+u(x_i,t_1,\tau)}{\Delta t(t_j-t_l)^{\beta-1}}, \tag{15}$$

where $\Delta x = \frac{b-a}{n_1-1}$, $\Delta t = \frac{T}{n_2-1}$, $x_i = a+(i-1)\Delta x$ and $t_j = (j-1)\Delta t$.

3. THE GPS FOR DIFFERENTIAL EQUATIONS SYSTEM

Now by supposing $\mathbf{u} = (u_1^1, u_1^2, ..., u_{n_1}^{n_2})^T$, Eq. (13) converts to a system of ordinary differential equations (ODEs):

$$\frac{d\mathbf{u}}{d\tau} = \mathbf{f}(\mathbf{u},\tau),\ \mathbf{u} \in \mathbb{R}^M,\ \tau \in \mathbb{R}, \tag{16}$$

where $\mathbf{f}$ denotes a vector with ij-component being the right-hand side of Eq. (13), and $M = n_1 \times n_2$ is the number of total grid point inside the domain $\Omega = (a,b) \times (0,T]$.

Now, we can develop a GPS, introduced by Liu [29], to solve Eq. (16) as following: *

$$\mathbf{X}_{l+1} = \mathcal{G}(l)\mathbf{X}_l, \tag{17}$$

where $\mathbf{X}_l$ denotes the numerical value of $\mathbf{X}$ at τ_l and $\mathcal{G}(l) \in SO_0(M,1)$ is the group value of $\mathcal{G}$ at τ_l.

An exponential mapping of $\mathcal{A}(l)$ admits the closed-form representation:

$$\mathcal{G}_l = \exp[\Delta\tau\mathcal{A}(l)] = \begin{bmatrix} I_M + \frac{(\alpha_l-1)}{\|\mathbf{f}_l\|^2}\mathbf{f}_l\mathbf{f}_l^T & \frac{\beta_l \mathbf{f}_l}{\|\mathbf{f}_l\|} \\ \frac{\beta_l \mathbf{f}_l^T}{\|\mathbf{f}_l\|} & \alpha_l \end{bmatrix}, \tag{18}$$

where

$$\alpha_l = \cosh\left(\frac{\Delta\tau\|\mathbf{f}_l\|}{\|\mathbf{u}_l\|}\right),\ \ \beta_l = \sinh\left(\frac{\Delta\tau\|\mathbf{f}_l\|}{\|\mathbf{u}_l\|}\right). \tag{19}$$

Substituting Eq. (18) for $\mathcal{G}_l$ into Eq. (17), concludes:

$$\mathbf{u}_{l+1} = \mathbf{u}_l + \frac{(\alpha_l - 1)\mathbf{f}_l.\mathbf{u}_l + \beta_l\|\mathbf{u}_l\|\|\mathbf{f}_l\|}{\|\mathbf{f}_l\|^2}\mathbf{f}_l = \mathbf{u}_l + \eta_l\mathbf{f}_l. \tag{20}$$

This scheme (the GPS one) preserves the group properties for all $\Delta\tau > 0$. More details of this method can be found in [30–32].

Now we can utilize the GPS, by supposing the initial value of $u_i^j(0)$, to integrate Eq. (16) from the initial fictitious time $\tau = 0$ to a selected final fictitious time τ_f.

$^*\mathbf{X} := (\mathbf{u}^T, \|\mathbf{u}\|)^T$ is a vector in the Minkowskian space which transforms Eq. (16) into $\frac{d\mathbf{X}}{d\tau} = \mathcal{A}\mathbf{X}$ where

$$\mathcal{A} = \begin{pmatrix} \mathbf{0}_{M\times M} & \frac{\mathbf{f}(\mathbf{u},\tau)}{\|\mathbf{u}\|} \\ \frac{\mathbf{f}^T(\mathbf{u},\tau)}{\|\mathbf{u}\|} & 0 \end{pmatrix} \in so(M,1),$$

is a Lie algebra of the proper orthochronous Lorentz group $SO_0(M,1)$.

Stopping criterion for this numerical integration is:

$$\sqrt{\sum_{i=1}^{n_1}\sum_{j=1}^{n_2}[u_i^j(l+1)-u_i^j(l)]^2} \leq \varepsilon, \tag{21}$$

where ε is a selected convergence criterion. The solution of u is obtainable from

$$u_i^j = \frac{v_i^j(\tau_0)}{(1+\tau_0)^\kappa} \tag{22}$$

where $\tau_0(\leq \tau_f)$ satisfies the above criterion. The parameters ν_0 and κ can strengthen the stability of numerical scheme and enhance the convergence speed of numerical integration, respectively.

4. NUMERICAL EXAMPLES

In this section, we examine the performance of our FTIM to solve the FPE.

Example 1: Consider the FPE (2) with fractional orders $\alpha = 1.9$, $\beta = 1.5$ and

$$f(x,t) = \frac{3x^{\frac{2}{3}}t(t-1)}{\Gamma(\frac{2}{3})} + \frac{4\sqrt{t}x(x-1)}{\sqrt{\pi}}.$$

In the semi-discretization of Eq. (12), we use $n_1 = 19$ and $n_2 = 19$ as the number of knots in each direction of x and t. The domain of FPE in this example is specified by $a = 0$, $b = 1$, $T = 1$ and homogeneous boundary conditions are imposed to the problem. During the application of GPS, we use $u_j^i(0) = 1e-3$ and $\Delta\tau = 1e-5$ as the initial guess and stepsize with respect to the fictitious dimension τ. In this example, we choose the values $\nu_0 = 0.401$ and $\kappa = 1e-4$ to control the stability and convergency of method. The exact solution of this problem is given by $u(x,t) = xt(x-1)(t-1)$.

Plots of exact and approximate solutions obtained from FTIM show that the results obtained by using this method are in good agreement. Also, the accuracy and the usefulness of FTIM for this equation is demonstrated by the contour plots in Fig. 1.

Example 2: In this example, we consider the Eq. (2) with fractional derivative orders $\alpha = 2$, $\beta = 1.5$ and

$$f(x,t) = -4\sin(2\pi x)\pi^2 t(t-\frac{1}{2}) + \frac{4\sqrt{t}\sin(\pi x)\cos(\pi x)(-3+4t)}{\sqrt{\pi}}.$$

The number of meshes are assumed to be $n_1 = n_2 = 29$. The initial guess and stepsize of GPS is considered the same as in the previous example. Homogeneous boundary conditions and $a = 0$, $b = 1$, and $T = 1$ specify this FPE. The parameters $\nu_0 = 124.7$

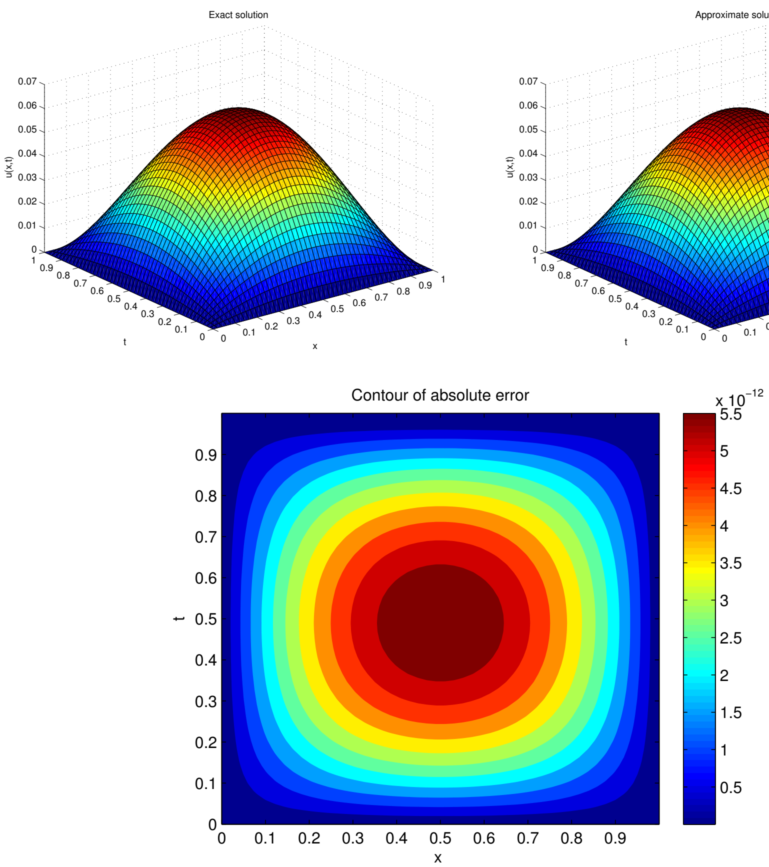

Fig. 1 – Plots of the exact solution and numerical solution and contour plots of absolute error for example 1.

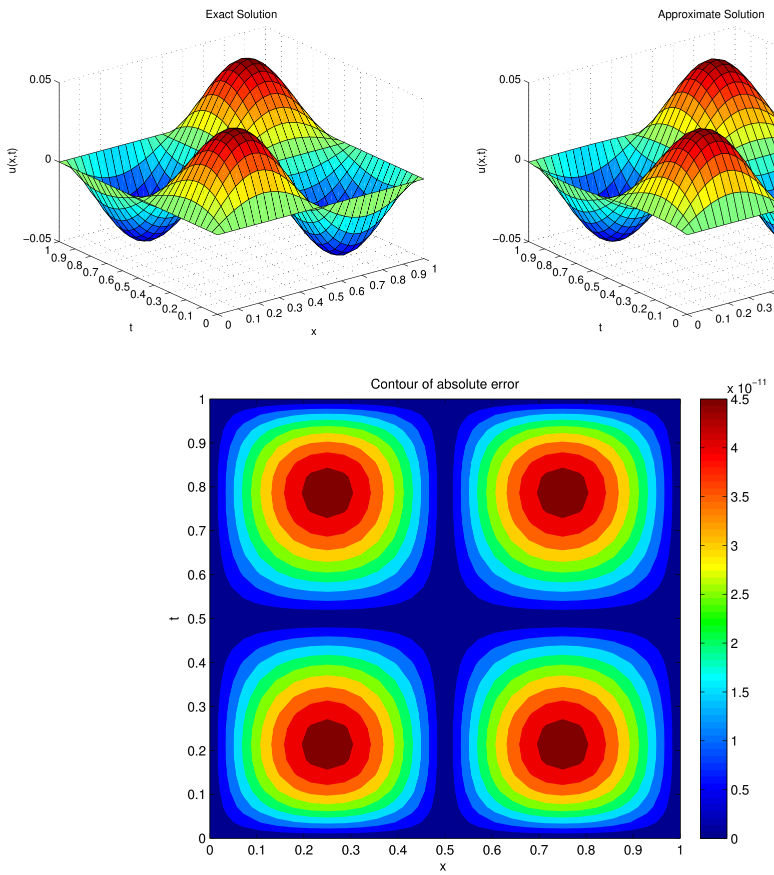

Fig. 2 – Plots of the exact solution and numerical solution and contour plots of absolute error for example 2.

and $\kappa = 0.1e-4$ are chosen for this example, and the exact solution is given by $u(x,t) = \sin(2\pi x)t(x-\frac{1}{2})(t-1)$. Fig. 2 demonstrates the validity of FTIM for Eq. (2), with a high accuracy.

5. CONCLUSION

Using a novel FTIM and introducing a fictitious variable, we have transformed the FPE into another type of functional PDE in a higher dimensional space. By employing the GPS we numerically integrated the discretized equations. Two numerical examples are given, which clearly demonstrate that FTIM is applicable to get the accurate numerical solutions of FPE.

REFERENCES

1. I. Podlubny, *Fractional Differential Equations* (Academic Press, New York, 1999).
2. A. Kilbas, H. Srivastava, J. Trujillo, *Theory and Applications of Fractional Differential Equations* (Elsevier, 2006).
3. R. Hilfer, *Applications of Fractional Calculus in Physics* (Academic Press, Orlando, 1999).
4. K. Diethelm, *The Analysis of Fractional Differential Equations, An Application-Oriented Exposition Using Differential Operators of Caputo Type* (Springer-Verlag Berlin Heidelberg 2010).
5. K.S. Miller, B. Ross, *An introduction to the fractional calculus and fractional differential equations* (John Wiley & Sons, Inc. 1993).
6. J. Juan Rosales Garcia *et al.*, Proc. Romanian Acad. A **14**, 42–47 (2013).
7. G.W. Wang, T.Z. Xu, Nonlinear Dyn. **76**, 571–580 (2014).
8. X.J. Yang, D. Baleanu, J.H. He, Proc. Romanian Acad. A **14**, 287–292 (2013).
9. Q. Huang, R. Zhdanov, Physica A, **409**, 110–118 (2014).
10. H. Jafari *et al.*, Rom. Rep. Phys. **65**, 1119–1124 (2013).
11. J. Hu, Y. Ye, S. Shen, J. Zhang, Appl. Math. Comput. **233**, 439-444 (2014).
12. H.Z. Liu, Stud. Appl. Math. **131**, 317–330 (2013).
13. A. Biswas *et al.*, Rom. J. Phys. **59**, 433–442 (2014).
14. R. Sahadevan, T. Bakkyaraj, J. Math. Anal. Appl. **393**, 341–347 (2012).
15. G. W. Wang, T. Z. Xu, Rom. J. Phys. **59**, 636–645 (2014).
16. G. W. Wang, T. Z. Xu, Rom. Rep. Phys. **66**, 595–602 (2014).
17. D. Rostamy *et al.*, Rom. Rep. Phys. **65**, 334–349 (2013).
18. A.H. Bhrawy, A. A. Al-Zahrani, Y.A. Alhamed, D. Baleanu, Rom. J. Phys. **59**, 646-657 (2014).
19. X.J. Yang, D. Baleanu, Y. Khan, S.T. Mohyud-Din, Rom. J. Phys. **59**, 36-48 (2014).
20. J. F. Gomez Aguilar, D. Baleanu, Proc. Romanian Acad. A **15**, 27-34 (2014).
21. M.H. Heydari, M.R. Hooshmandasl, F.M. Maalek Ghaini, F. Fereidouni, Engin. Anal. Bound. Elem. **37**, 1331-1338 (2013).
22. C.S. Liu, Bound. Value Problems, **2008**, Article ID 749865 (2008).
23. C.W. Chang, C.S. Liu, Int. J. Heat Mass Trans. **53**, 5552-5569 (2010).
24. Y.W. Chen, C.M. Chang, C.S. Liu, J.R. Chang, IMA J. Numer. Anal. **34**, 362-389 (2014).
25. C.S. Liu, S.N. Atluri, CMES. **34**(2), 155–178 (2008).
26. C.S. Liu, S.N. Atluri, CMES. **41**(3), 243–261 (2009).
27. C.C. Tsai, C.S. Liu, W.C. Yeih, CMES. **56**(2), 131–151 (2010).
28. M.S. Hashemi, M. Sariri, J. Math. Comput. Sci. **14**, 87–96 (2015).
29. C.S. Liu, Int. J. Non-Linear Mech. **36**, 1047–1068 (2001).
30. S. Abbasbandy, M. Hashemi, Eng. Anal. Bound. Elem. **35**, 1003–1009 (2011).
31. S. Abbasbandy, M. Hashemi, C.S. Liu, Commun. Nonlin. Sci. **16**, 4238–4249 (2011).
32. M. Hashemi, Commun. Nonlin. Sci. **22**, 990–1001 (2015).