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Analytical Solutions of the Electrical RLC Circuit via Liouville–Caputo Operators with Local and Non-Local Kernels

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Abstract: In this work we obtain analytical solutions for the electrical RLC circuit model defined with Liouville–Caputo, Caputo–Fabrizio and the new fractional derivative based in the Mittag-Leffler function. Numerical simulations of alternative models are presented for evaluating the effectiveness of these representations. Different source terms are considered in the fractional differential equations. The classical behaviors are recovered when the fractional order α is equal to 1.

Keywords: fractional-order circuits; Liouville–Caputo fractional operator; Caputo–Fabrizio fractional operator; Atangana–Baleanu fractional operator

1. Introduction

In several works, fractional order operators are used to represent the behavior of electrical circuits; for example, fractional differential models serve to design analog and digital filters of fractional-order, and some works concern the fractional-order description of magnetically-coupled coils or the behavior of circuits and systems with memristors, meminductors or memcapacitors [1–16]. These research works address the study of the described electrical systems. These models have been extended to the scope of fractional derivatives using Riemann–Liouville and Liouville–Caputo derivatives with fractional order; however, these two derivatives have a kernel with singularity [17]. Caputo and Fabrizio proposed a novel definition without singular kernel. The resulting fractional operator is based on the exponential function [18–27]; however, the derivative proposed by Caputo and Fabrizio it is not a fractional derivative, its corresponding kernel is local. To solve the problem, Atangana and Baleanu suggested two news derivatives with Mittag-Leffler kernel, these operators in Liouville–Caputo and Riemann–Liouville have non-singular and non-local kernel and preserve the benefits of the Riemann–Liouville, Liouville–Caputo and Caputo–Fabrizio fractional operators [28–33].

This work aims to represent the fractional electrical RLC circuit with the Liouville–Caputo, Caputo–Fabrizio and the new representation with Mittag-Leffler kernel in the Liouville–Caputo sense, considering different sources terms in order to assess and compare their efficacy to describe a real world problem.

2. Fractional Derivatives

The Liouville–Caputo operator (C) with fractional order is defined for $(\gamma > 0)$ as [34]

$$ {}_{0}^{C}\mathcal{D}_{t}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{0}^{t} (t-s)^{n-\alpha-1} f^{(n)}(s) ds. \tag{1} $$

The Laplace transform of (1) has the form

$$ \mathscr{L}[{}_{0}^{C}D_{t}^{\alpha} f(t)] = s^{\alpha} F(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} f^{(k)}(0), \tag{2} $$

where $n = [\Re(\alpha)] + 1$. From this expression we have two particular cases

$$ \mathscr{L}[{}_{0}^{C}D_{t}^{\alpha} f(t)] = s^{\alpha} F(s) - s^{\alpha-1} f(0) \qquad 0 < \alpha \leq 1, \tag{3} $$

$$ \mathscr{L}[{}_{0}^{C}D_{t}^{\alpha} f(t)] = s^{\alpha} F(s) - s^{\alpha-1} f(0) - s^{\alpha-2} f'(0) \qquad 1 < \alpha \leq 2. \tag{4} $$

The Mittag-Leffler function is defined as

$$ E_{\alpha,\theta}(t) = \sum_{m=0}^{\infty} \frac{t^m}{\Gamma(\alpha m + \theta)}, \quad (\alpha > 0), (\theta > 0). \tag{5} $$

Some common Mittag-Leffler functions are described in [34]

$$ E_{1/2,1}(\pm\alpha) = \exp(\alpha^2)[1 \pm \operatorname{erfc}(\alpha)], \tag{6} $$

$$ E_{1,1}(\pm\alpha) = \exp(\pm\alpha), \tag{7} $$

$$ E_{2,1}(-\alpha^2) = \cos(\alpha), \tag{8} $$

$$ E_{3,1}(\alpha) = \frac{1}{2}\Big[\exp(\alpha^{1/3}) + 2\exp(-(1/2)\alpha^{1/3}) \cos\Big(\frac{\sqrt{3}}{2}\alpha^{1/3}\Big)\Big]. \tag{9} $$

The Caputo–Fabrizio fractional operator (CF) is defined as follows [18,19]

$$ {}_{0}^{CF}\mathcal{D}_{t}^{\alpha} f(t) = \frac{B(\alpha)}{1-\alpha} \int_{0}^{t} \dot{f}(\theta) \exp\Big[-\frac{\alpha(t-\theta)}{1-\alpha}\Big] d\theta, \tag{10} $$

where $B(\alpha)$ is a normalization function such that $M(0) = M(1) = 1$.

If $n \geq 1$ and $\alpha \in [0,1]$, CF operator of order $(n+\alpha)$ is defined by

$$ {}_{0}^{CF}\mathcal{D}_{t}^{(\alpha+n)} f(t) = {}_{0}^{CF}\mathcal{D}_{t}^{(\alpha)} ({}_{0}^{CF}\mathcal{D}_{t}^{(n)} f(t)). \tag{11} $$

The Laplace transform of (11) is defined as follows

$$ \mathscr{L}[{}_{0}^{CF}\mathcal{D}_{t}^{(\alpha+n)} f(t)] = \frac{s^{n+1}\mathscr{L}[f(t)] - s^n f(0) - s^{n-1} f'(0) \ldots - f^{(n)}(0)}{s + \alpha(1-s)}. \tag{12} $$

From this expression we have

$$ \mathscr{L}[{}_{0}^{CF}\mathcal{D}_{t}^{\alpha} f(t)] = \frac{s\mathscr{L}[f(t)] - f(0)}{s + \alpha(1-s)}, \qquad n = 0, \tag{13} $$

$$\mathscr{L}[{}_0^{CF}\mathcal{D}_t^{(\alpha+1)} f(t)] = \frac{s^2 \mathscr{L}[f(t)] - sf(0) - \dot{f}(0)}{s + \alpha(1-s)}, \qquad n = 1. \tag{14}$$

The Atangana–Baleanu fractional operator in Liouville–Caputo sense (ABC) is defined as follows [28–33]

$${}_a^{ABC}\mathcal{D}_t^{\alpha} f(t) = \frac{B(\alpha)}{1-\alpha} \int_a^t \dot{f}(\theta) E_\alpha \Big[-\alpha \frac{(t-\theta)^\alpha}{1-\alpha} \Big] d\theta, \tag{15}$$

where $B(\alpha)$ has the same properties as in the above case.

The Laplace transform of (15) is defined as follows

$$\begin{aligned} \mathscr{L}[{}_a^{ABC}\mathcal{D}_t^{\alpha} f(t)](s) &= \tfrac{B(\alpha)}{1-\alpha} \mathscr{L}\Big[\int_a^t \dot{f}(\theta) E_\alpha \Big[-\alpha \tfrac{(t-\theta)^\alpha}{1-\alpha} \Big] d\theta \Big] \\ &= \tfrac{B(\alpha)}{1-\alpha} \tfrac{s^\alpha \mathscr{L}[f(t)](s) - s^{\alpha-1} f(0)}{s^\alpha + \frac{\alpha}{1-\alpha}}. \end{aligned} \tag{16}$$

Atangana and Baleanu also suggest another fractional derivative in Riemann–Liouville sense (ABR) [28–33]:

$${}_a^{ABR}\mathcal{D}_t^{\alpha} f(t) = \frac{B(\alpha)}{1-\alpha} \frac{d}{dt} \int_b^t f(\theta) E_\alpha \Big[-\alpha \frac{(t-\theta)^\alpha}{1-\alpha} \Big] d\theta, \tag{17}$$

where $B(\alpha)$ is a normalization function as in the previous definition.

The Laplace transform of (17) is defined as follows

$$\begin{aligned} \mathscr{L}[{}_a^{ABR}\mathcal{D}_t^{\alpha} f(t)](s) &= \tfrac{B(\alpha)}{1-\alpha} \mathscr{L}\Big[\tfrac{d}{dt} \int_a^t f(\theta) E_\alpha \Big[-\alpha \tfrac{(t-\theta)^\alpha}{1-\alpha} \Big] d\theta \Big] \\ &= \tfrac{B(\alpha)}{1-\alpha} \tfrac{s^\alpha \mathscr{L}[f(t)](s)}{s^\alpha + \frac{\alpha}{1-\alpha}}. \end{aligned} \tag{18}$$

3. RLC Electrical Circuit

In this work, an auxiliary parameter σ was introduced with the finality to preserve the dimensionality of the temporal operator [14]

$$\frac{d}{dt} \to \frac{1}{\sigma^{1-\alpha}} \cdot \mathcal{D}_t^{\alpha}, \quad \nu - 1 < \alpha \leq \nu, \quad \nu = 1, 2, 3, \ldots \tag{19}$$

and

$$\frac{d^2}{dt^2} \to \frac{1}{\sigma^{2(1-\alpha)}} \cdot \mathcal{D}_t^{2\alpha}, \quad \nu - 1 < \alpha \leq \nu, \quad \nu = 1, 2, 3, \ldots \tag{20}$$

where σ has the dimension of seconds. This parameter is associated with the temporal components of the system [14], when $\alpha = 1$ the expressions (19) and (20) are recovered in the traditional sense. Applying Kirchhoff's laws, the equation of the RLC circuit represented in Figure 1 is given by

$$D_t^2 I(t) + \frac{R}{L}\, D_t I(t) + \frac{1}{LC} I(t) = \frac{1}{L}\, E(t), \tag{21}$$

where L is the inductance, R is the resistance and the source voltage is $E(t)$.

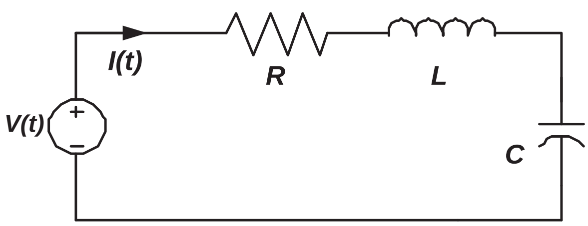

Figure 1. RLC circuit.

3.1. RLC Electrical Circuit via Liouville–Caputo Fractional Operator

Considering (19) and (20), the fractional equation corresponding to (21) in the Liouville–Caputo sense is given by:

$$ {}_0^C D_t^{2\alpha} I(t) + A\, {}_0^C D_t^{\alpha} I(t) = B\,C\,E(t) - BI(t), \quad 0 < \alpha \leq 1, \tag{22}$$

where $A = \frac{R}{L}\sigma^{1-\alpha}$ and $B = \frac{\sigma^{2(1-\alpha)}}{LC}$. Now we obtain the analytical solution of Equation (22) considering different source terms $E(t)$.

Case 1. *Unit step source, $E(t) = u(t)$, $I(0) = I_0$, $(I_0 > 0)$, $\dot{I}(0) = 0$, (22) is defined as follows*

$$ {}_0^C D_t^{2\alpha} I(t) + A\, {}_0^C D_t^{\alpha} I(t) = B\,C\,u(t) - BI(t). \tag{23}$$

Applying the Laplace transform (12) to (23), we have

$$\widetilde{I}(s) = \frac{s^{2\alpha-1} I_0 + A s^{\alpha-1} I_0 + BC(1/s)}{s^2 + A s^\alpha + B}. \tag{24}$$

Taking the inverse Laplace transform of (24), *we obtain:*

$$\begin{aligned} I(t) &= I_0 \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-B)^n(-A)^k \binom{n+k}{k}}{\Gamma(\alpha k+2\alpha(n+1)-2\alpha+1)} \cdot \tau^{\alpha(k+2n)} \\ &+ A I_0 t^\alpha \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-B)^n(-A)^k \binom{n+k}{k}}{\Gamma(\alpha k+2\alpha(n+1)-2\alpha+1)} \cdot \tau^{\alpha(k+2n)} \\ &+ \frac{BC}{\Gamma(\alpha+1)} \int_0^t (t-\tau)^\alpha \tau^{\alpha-1} \cdot \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-B)^n(-A)^k \binom{n+k}{k}}{\Gamma(\alpha k+2\alpha(n+1)-\alpha+1)} \cdot \tau^{\alpha(k+2n)}. \end{aligned} \tag{25}$$

Case 2. *Exponential source, $E(t) = e^{-at}$, $I(0) = I_0$, $(I_0 > 0)$, $\dot{I}(0) = 0$, (22) is defined as follows*

$$ {}_0^C D_t^{2\alpha} I(t) + A\, {}_0^C D_t^{\alpha} I(t) = B\,C\,e^{-at} - BI(t). \tag{26}$$

Applying the Laplace transform (12) to (26), the expression for the current is

$$\widetilde{I}(s) = \frac{s^{2\alpha-1} I_0 + A s^{\alpha-1} I_0 + BC(1/s+c)}{s^2 + A s^\alpha + B}. \tag{27}$$

Taking the inverse Laplace transform to (27), the analytical solution is:

$$\begin{aligned} I(t) &= BC \int_0^t \tau^{\alpha-1} \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-B)^n(-A)^k \binom{n+k}{k}}{\Gamma(\alpha k+2\alpha(n+1)-\alpha)} \cdot \tau^{\alpha(k+2n)} \cdot E_{\alpha,1}(-c(t-\tau))d\tau \\ &+ I_0 \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-B)^n(-A)^k \binom{n+k}{k}}{\Gamma(\alpha k+2\alpha(n+1)-2\alpha+1)} \cdot \tau^{\alpha(k+2n)} \\ &+ A I_0 t^\alpha \cdot \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-B)^n(-A)^k \binom{n+k}{k}}{\Gamma(\alpha k+2\alpha(n+1)-\alpha+1)} \cdot \tau^{\alpha(k+2n)}. \end{aligned} \tag{28}$$

Case 3. *Periodic source, $E(t) = \sin(\varphi t)$, $I(0) = I_0$, $(I_0 > 0)$, $\dot{I}(0) = 0$, (32) is defined as follows*

$$ {}_0^C D_t^{2\alpha} I(t) + A\, {}_0^C D_t^{\alpha} I(t) = B\,C\,\sin(\varphi t) - BI(t). \tag{29} $$

Applying the Laplace transform (12) to (29), the expression for the current is

$$ \widetilde{I}(s) = \frac{BC}{s^{2\alpha} + As^{\alpha} + B} \cdot \frac{\varphi}{s^2 + \varphi^2} + \frac{I_0 s^{2\alpha-1}}{s^{2\alpha} + As^{\alpha} + B} + \frac{AI_0 s^{\alpha-1}}{s^{2\alpha} + As^{\alpha} + B}. \tag{30} $$

Taking the inverse Laplace transform to (30), the analytical solution is:

$$ \begin{aligned} I(t) = BC \int_0^t \tau^{\alpha-1} \textstyle\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} & \frac{(-B)^n(-A)^k \begin{pmatrix} n+k \\ k \end{pmatrix}}{\Gamma(\alpha k + 2\alpha(n+1) - \alpha)} \cdot \tau^{\alpha(k+2n)} \cdot \sin(\varphi(t-\tau))d\tau \\ + I_0 \textstyle\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} & \frac{(-B)^n(-A)^k \begin{pmatrix} n+k \\ k \end{pmatrix}}{\Gamma(\alpha k + 2\alpha(n+1) - 2\alpha + 1)} \cdot \tau^{\alpha(k+2n)} \\ + AI_0 t^{\alpha} \cdot \textstyle\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} & \frac{(-B)^n(-A)^k \begin{pmatrix} n+k \\ k \end{pmatrix}}{\Gamma(\alpha k + 2\alpha(n+1) - \alpha + 1)} \cdot \tau^{\alpha(k+2n)}. \end{aligned} \tag{31} $$

3.2. RLC Electrical Circuit via Caputo–Fabrizio Fractional Operator

Considering (19) and (20), the fractional equation corresponding to (21) in the Caputo–Fabrizio sense is given by:

$$ {}_0^{CF} D_t^{2\alpha} I(t) + A\, {}_0^{CF} D_t^{\alpha} I(t) = B\,C\,E(t) - BI(t), \quad 0 < \alpha \leq 1, \tag{32} $$

we obtain the analytical solutions of Equation (32) considering different source terms.

Case 4. *Unit step source, $E(t) = u(t)$, $I(0) = I_0$, $(I_0 > 0)$, $\dot{I}(0) = 0$, (32) is defined as follows*

$$ {}_0^{CF} D_t^{2\alpha} I(t) + A\, {}_0^{CF} D_t^{\alpha} I(t) = B\,C\,u(t) - BI(t). \tag{33} $$

Applying the Laplace transform (12) to (33), the expression for the current is:

$$ \begin{aligned} \widetilde{I}(s) = & \tfrac{sI_0}{s^2K+sL+M} + \tfrac{AI_0(1-\alpha)s}{s^2K+sL+M} + \tfrac{AI_0\alpha}{s^2K+sL+M} + \tfrac{s(1-\alpha)^2BC}{s^2K+sL+M} \\ & + \tfrac{2\alpha(1-\alpha)BC}{s^2K+sL+M} + \tfrac{BC\alpha^2}{s^2K+sL+M} \cdot \tfrac{1}{s}. \end{aligned} \tag{34} $$

Taking the inverse Laplace transform of (34), we obtain the following solution:

$$ \begin{aligned} I(t) = [I_0 + AI_0(1-\alpha)] \cdot \textstyle\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} & \frac{(-M)^n(-L)^k \begin{pmatrix} n+k \\ k \end{pmatrix}}{\Gamma(\alpha K + 2\alpha(n+1) - 1)} \cdot \tau^{(k+2n)} \\ + [AI_0\alpha + 2\alpha(1-\alpha)BC]t \cdot \textstyle\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} & \frac{(-M)^n(-L)^k \begin{pmatrix} n+k \\ k \end{pmatrix}}{\Gamma(\alpha K + 2\alpha(n+1))} \cdot \tau^{(k+2n)} \\ + (BC(1-\alpha)^2) \cdot \textstyle\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} & \frac{(-M)^n(-L)^k \begin{pmatrix} n+k \\ k \end{pmatrix}}{\Gamma(\alpha K + 2\alpha(n+1) - 1)} \cdot \tau^{(k+2n)} \\ + BC\alpha^2 \int_0^t (t-\tau) \cdot \textstyle\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} & \frac{(-M)^n(-L)^k \begin{pmatrix} n+k \\ k \end{pmatrix}}{\Gamma(\alpha K + 2\alpha(n+1) - 1)} \cdot (t-\tau)^{(k+2n)} d\tau, \end{aligned} \tag{35} $$

where

$$\begin{aligned} &M = \alpha^2 B, \\ &K = 1 + A(1-\alpha) + B - 2\alpha B + \alpha^2 B, \\ &L = A\alpha + 2\alpha B - 2\alpha^2 B. \end{aligned} \tag{36}$$

Case 5. *Exponential source, $E(t) = e^{-at}$, $I(0) = I_0$, $(I_0 > 0)$, $\dot{I}(0) = 0$, (32) is defined as follows*

$$ {}_0^{CF}D_t^{2\alpha} I(t) + A\, {}_0^{CF}D_t^{\alpha} I(t) = B\ C\ e^{-at} - BI(t). \tag{37}$$

Applying the Laplace transform (12) to (37), the expression for the current is:

$$\begin{aligned} \widetilde{I}(s) = & \frac{sI_0}{s^2K+sL+M} + \frac{AI_0(1-\alpha)s}{s^2K+sL+M} + \frac{AI_0\alpha}{s^2K+sL+M} + \frac{s(1-\alpha)^2BC}{s^2K+sL+M} \\ & + \frac{BC(1-\alpha)^2s^2}{s^2K+sL+M} \cdot \frac{1}{s+a} + \frac{2\alpha(1-\alpha)s}{s^2K+sL+M} \cdot \frac{BC}{s+a} + \frac{BC(\alpha)^2}{s^2K+sL+M} \cdot \frac{1}{s+a}. \end{aligned} \tag{38}$$

Taking the inverse Laplace transform to (38), the analytical solution is:

$$\begin{aligned} I(t) = & [I_0 + AI_0(1-\alpha)] \cdot \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-M)^n(-L)^k \begin{pmatrix} n+k \\ k \end{pmatrix}}{\Gamma(K+2\alpha(n+1)-1)} \cdot \tau^{(k+2n)} \\ & + (AI_0\alpha)t \cdot \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-M)^n(-L)^k \begin{pmatrix} n+k \\ k \end{pmatrix}}{\Gamma(K+2\alpha(n+1))} \cdot \tau^{k+2n} \\ & + BC(1-\alpha)^2 \cdot \int_0^t (t-\tau)^{-1} \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-M)^n(-L)^k \begin{pmatrix} n+k \\ k \end{pmatrix}}{\Gamma(K+2\alpha(n+1)-2)} \cdot (t-\tau)^{(k+2n)} \cdot E_{\alpha,1}(-a\tau)d\tau \\ & + 2BC\alpha(1-\alpha) \int_0^t (t-\tau) \cdot \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-M)^n(-L)^k \begin{pmatrix} n+k \\ k \end{pmatrix}}{\Gamma(K+2\alpha(n+1))} \cdot (t-\tau)^{(k+2n)} \cdot E_{\alpha,1}(-a\tau)d\tau, \end{aligned} \tag{39}$$

where M, K and L are given by (36).

Case 6. *Periodic source, $E(t) = \sin(\varphi t)$, $I(0) = I_0$, $(I_0 > 0)$, $\dot{I}(0) = 0$, (32) is defined as follows*

$$ {}_0^{CF}D_t^{2\alpha} I(t) + A\, {}_0^{CF}D_t^{\alpha} I(t) = B\ C\ \sin(\varphi t) - BI(t). \tag{40}$$

Applying the Laplace transform (12) to (40), the expression for the current is:

$$\begin{aligned} \widetilde{I}(s) = & \frac{sI_0}{s^2K+sL+M} + \frac{AI_0(1-\alpha)s}{s^2K+sL+M} + \frac{AI_0\alpha}{s^2K+sL+M} + \frac{s(1-\alpha)^2BC}{s^2K+sL+M} \\ & + \frac{BC(1-\alpha)^2s^2}{s^2K+sL+M} \cdot \frac{\varphi}{s^2+\varphi} + \frac{2\alpha(1-\alpha)s}{s^2K+sL+M} \cdot BC\frac{\varphi}{s^2+\varphi} + \frac{BC(\alpha)^2}{s^2K+sL+M} \cdot \frac{\varphi}{s^2+\varphi}. \end{aligned} \tag{41}$$

Taking the inverse Laplace transform to (41), the analytical solution is:

$$\begin{aligned} I(t) = & [I_0 + AI_0(1-\alpha)] \cdot \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-M)^n(-L)^k \begin{pmatrix} n+k \\ k \end{pmatrix}}{\Gamma(K+2\alpha(n+1)-1)} \cdot \tau^{(k+2n)} \\ & + (AI_0\alpha)t \cdot \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-M)^n(-L)^k \begin{pmatrix} n+k \\ k \end{pmatrix}}{\Gamma(K+2\alpha(n+1))} \cdot \tau^{k+2n} \\ & + BC(1-\alpha)^2 \cdot \int_0^t (t-\tau)^{-1} \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-M)^n(-L)^k \begin{pmatrix} n+k \\ k \end{pmatrix}}{\Gamma(K+2\alpha(n+1)-2)} \cdot (t-\tau)^{(k+2n)} \cdot \sin(\varphi(t-\tau))d\tau \\ & + 2BC\alpha(1-\alpha) \int_0^t (t-\tau) \cdot \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-M)^n(-L)^k \begin{pmatrix} n+k \\ k \end{pmatrix}}{\Gamma(K+2\alpha(n+1))} \cdot (t-\tau)^{(k+2n)} \cdot \sin(\varphi(t-\tau))d\tau, \end{aligned} \tag{42}$$

where M, K and L are given by (36).

3.3. RLC Electrical Circuit Involving the Fractional Operator with Mittag-Leffler Kernel

Considering (19) and (20), the fractional equation corresponding to (21) via the fractional operator with Mittag-Leffler kernel is given by

$$ {}_{0}^{ABC}D_t^{2\alpha} I(t) + A\, {}_{0}^{ABC}D_t^{\alpha} I(t) = B\, C\, E(t) - BI(t), \quad 0 < \alpha \leq 1, \tag{43}$$

we obtain the analytical solutions of (43) considering different source terms.

Case 7. *Unit step source, $E(t) = u(t)$, $I(0) = I_0$, $(I_0 > 0)$, $\dot{I}(0) = 0$, (43) is defined as follows:*

$$ {}_{0}^{ABC}D_t^{2\alpha} I(t) + A\, {}_{0}^{ABC}D_t^{\alpha} I(t) = B\, C\, u(t) - BI(t). \tag{44}$$

Applying the Laplace transform (16) to (44), the expression for the current is:

$$\begin{aligned} \widetilde{I}(s) = B \cdot &\left[\frac{(1-\alpha)s^{2\alpha-1}}{s^{2\alpha}K+s^{\alpha}L+M} + \frac{2\alpha(1-\alpha)s^{\alpha-1}}{s^{2\alpha}K+s^{\alpha}L+M} + \frac{\alpha^2}{s^{2\alpha}K+s^{\alpha}L+M} \cdot \frac{1}{s} \right] \\ &+ B(\alpha)^2 \cdot \frac{s^{2\alpha-1} I_0}{s^{2\alpha}K+s^{\alpha}L+M} + AB(\alpha) I_0 \cdot \frac{s^{\alpha-1}(s^{\alpha}(1-\alpha)+\alpha)}{s^{2\alpha}K+s^{\alpha}L+M}. \end{aligned} \tag{45}$$

Taking the inverse Laplace transform of (45), *the solution is:*

$$\begin{aligned} I(t) = &\left[B(1-\alpha) + \frac{B(\alpha)^2 I_0}{K} + \frac{AB(\alpha)^2 I_0(1-\alpha)}{K} \right] \\ &\cdot \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-C)^n(-H)^k \begin{pmatrix} n+k \\ k \end{pmatrix}}{\Gamma[k\alpha+(n+1)2\alpha-2\alpha+1]} t^{\alpha(k+2n)} + t^{\alpha} \cdot \left[2\alpha(1-\alpha) + \frac{AB(\alpha) I_0 \alpha}{K} \right] \\ &\cdot \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-C)^n(-H)^k \begin{pmatrix} n+k \\ k \end{pmatrix}}{\Gamma[k\alpha+(n+1)2\alpha-2\alpha+1]} t^{\alpha(k+2n)} \\ &+ \frac{\alpha^2}{K} \int_0^t \tau^{2\alpha(n+1)-1} \cdot \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{(-C)^n(-H)^k \begin{pmatrix} n+k \\ k \end{pmatrix}}{\Gamma[k\alpha+(n+1)2\alpha]} \tau^{k\alpha} d\tau, \end{aligned} \tag{46}$$

where

$$\begin{aligned} &K = B(\alpha)^2 + AB(\alpha)(1-\alpha) + D(1-\alpha)^2, \\ &L = AB(\alpha) + 2D(\alpha)(1-\alpha), \\ &M = D(\alpha)^2, \\ &C = \frac{M}{K}, \\ &H = \frac{L}{K}. \end{aligned} \tag{47}$$

Case 8. *Exponential source, $E(t) = e^{-at}$, $I(0) = I_0$, $(I_0 > 0)$, $\dot{I}(0) = 0$, (43) is defined as follows:*

$$ {}_{0}^{ABC}D_t^{2\alpha} I(t) + A\, {}_{0}^{ABC}D_t^{\alpha} I(t) = B\, C\, e^{-at} - BI(t). \tag{48}$$

Applying the Laplace transform (16) to (48), the expression for the current is:

$$\begin{aligned} \widetilde{I}(s) = B \cdot &\left[\frac{1}{s+a} \cdot \left(\frac{(1-\alpha)s^{2\alpha-1}}{s^{2\alpha}K+s^{\alpha}L+M} + \frac{2\alpha(1-\alpha)s^{\alpha}}{s^{2\alpha}K+s^{\alpha}L+M} + \frac{\alpha^2}{s^{2\alpha}K+s^{\alpha}L+M} \right) \right] \\ &+ B(\alpha)^2 \frac{s^{2\alpha-1} I_0}{s^{2\alpha}K+s^{\alpha}L+M} + AB(\alpha) I_0 \frac{s^{\alpha-1}(s^{\alpha}(1-\alpha)+\alpha)}{s^{2\alpha}K+s^{\alpha}L+M}. \end{aligned} \tag{49}$$

Taking the inverse Laplace transform to (49), the solution is:

$$
\begin{aligned}
I(t) = {} & \left[\frac{B(\alpha)^2 I_0}{K} + \frac{AB(\alpha)I_0(1-\alpha)}{K}\right] \cdot \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-C)^n(-H)^k \binom{n+k}{k}}{\Gamma[k\alpha+(n+1)2\alpha-2\alpha+1]} t^{\alpha(k+2n)} \\
& + \frac{AB(\alpha)I_0(\alpha)}{K} t^{\alpha} \cdot \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-C)^n(-H)^k \binom{n+k}{k}}{\Gamma[k\alpha+(n+1)2\alpha-2\alpha+1]} t^{\alpha(k+2n)} \\
& + \frac{2B\alpha(1-\alpha)}{K} \int_0^t E_{\alpha,\alpha}(-a(t-\tau))\tau^{\alpha-1} \cdot \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-C)^n(-H)^k \binom{n+k}{k}}{\Gamma[k\alpha+(n+1)2\alpha]} \tau^{\alpha(k+2n)} d\tau \\
& + \frac{B(1-\alpha)}{K} \int_0^t E_{\alpha,\alpha}(-a(t-\tau))\tau^{-1} \cdot \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-C)^n(-H)^k \binom{n+k}{k}}{\Gamma[k\alpha+(n+1)2\alpha]} \tau^{\alpha(k+2n)} d\tau \\
& + \frac{B(\alpha)^2}{K} \int_0^t E_{\alpha,\alpha}(-a(t-\tau))\tau^{2\alpha(n+1)-1} \cdot \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-C)^n(-H)^k \binom{n+k}{k}}{\Gamma[k\alpha+(n+1)2\alpha]} \tau^{k\alpha} d\tau,
\end{aligned}
\tag{50}
$$

where K, L, M, C and H are given by (47).

Case 9. *Periodic source, $E(t) = \sin(\varphi t)$, $I(0) = I_0$, $(I_0 > 0)$, $\dot{I}(0) = 0$, (43) is defined as follows:*

$$
{}_0^{ABC}D_t^{2\alpha} I(t) + A\, {}_0^{ABC}D_t^{\alpha} I(t) = B\, C\, \sin(\varphi t) - BI(t). \tag{51}
$$

Applying the Laplace transform (16) to (51), the expression for the current is:

$$
\begin{aligned}
\widetilde{I}(s) = {} & \left[\frac{s^2\alpha(1-\alpha)^2 + 2\alpha(1-\alpha)s^{\alpha} + \alpha^2}{s^2\alpha K + s^{\alpha} L + M}\right] \cdot \frac{\varphi}{s^2+\varphi^2} \\
& + B(\alpha)^2 \cdot \frac{s^{2\alpha-1} I_0}{s^2\alpha K + s^{\alpha} L + M} + AB(\alpha) \cdot \frac{s^{\alpha-1} I_0 (s^{\alpha}(1-\alpha)+\alpha)}{s^2\alpha K + s^{\alpha} L + M}.
\end{aligned}
\tag{52}
$$

Taking the inverse Laplace transform to (52), the solution is:

$$
\begin{aligned}
I(t) = {} & \frac{(1-\alpha)^2}{K} \cdot \int_0^t \sin(\varphi(t-\tau))\tau^{\alpha-1} \cdot \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-C)^n(-H)^k \binom{n+k}{k}}{\Gamma[k\alpha+(n+1)2\alpha-2\alpha+1]} t^{\alpha(k+2n)} d\tau \\
& + \frac{2\alpha(1-\alpha)}{K} \cdot \int_0^t \sin(\varphi(t-\tau))\tau^{-1} \cdot \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-C)^n(-H)^k \binom{n+k}{k}}{\Gamma[k\alpha+(n+1)2\alpha-2\alpha+1]} t^{\alpha(k+2n)} d\tau \\
& + \frac{\alpha^2}{K} \cdot \int_0^t \sin(\varphi(t-\tau))\tau^{2\alpha(n+1)-1} \cdot \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-C)^n(-H)^k \binom{n+k}{k}}{\Gamma[k\alpha+(n+1)2\alpha-2\alpha+1]} t^{k\alpha} d\tau \\
& + \left[\frac{B(\alpha)^2 I_0}{K} + \frac{AB(\alpha)I_0(1-\alpha)}{K} + \frac{AB(\alpha)I_0\alpha}{K} t^{\alpha}\right] \\
& \cdot \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-C)^n(-H)^k \binom{n+k}{k}}{\Gamma[k\alpha+(n+1)2\alpha-2\alpha+1]} t^{\alpha(k+2n)},
\end{aligned}
\tag{53}
$$

where K, L, M, C and H are given by (47).

Example 1. *Consider the electrical circuit RLC with R = 100 Ω, $L = 10$ H, $C = 0.1$ F and $V(0) = 10$ V. Figures 2–4 show numerical simulations for the current in the inductor, for different particular cases of α using the Liouville–Caputo, Caputo–Fabrizio and the Atangana–Baleanu–Caputo fractional operator, respectively.*

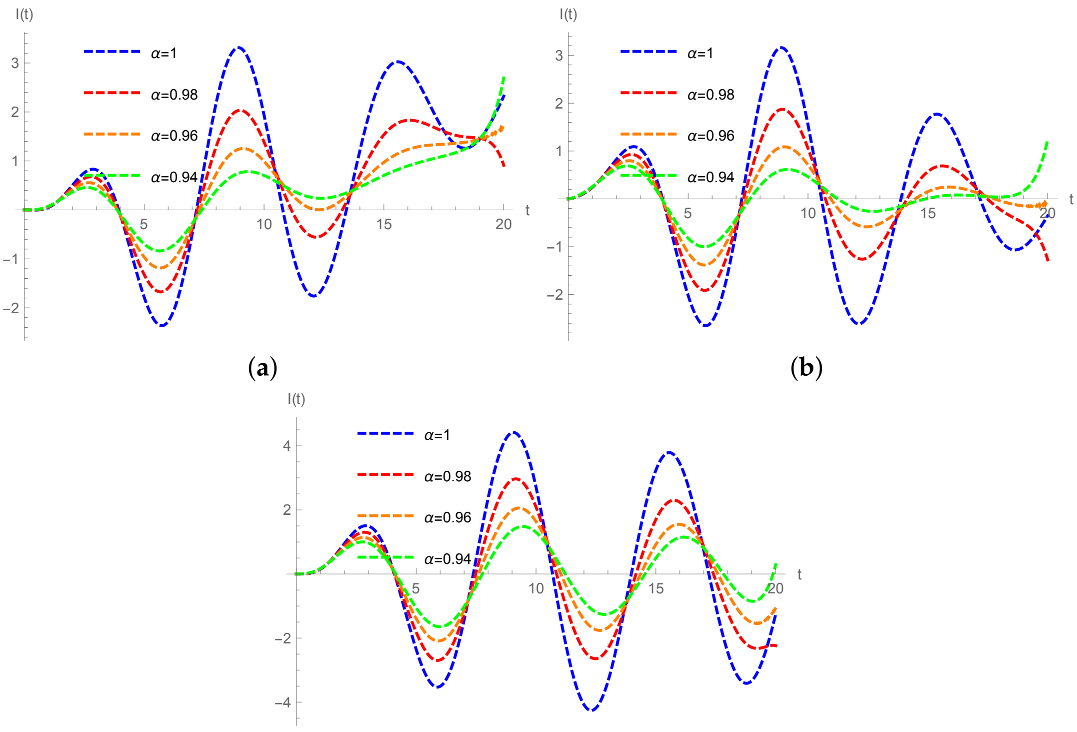

Figure 2. Numerical simulation for an RLC electrical circuit via Liouville–Caputo fractional operator, in (**a**) Equation (25), corresponding to a unit step source; in (**b**) Equation (28), corresponding to an exponential source; in (**c**) Equation (31), corresponding to periodic source; for all figures $I(t)$ is measured at Amperes and t is measured at seconds.

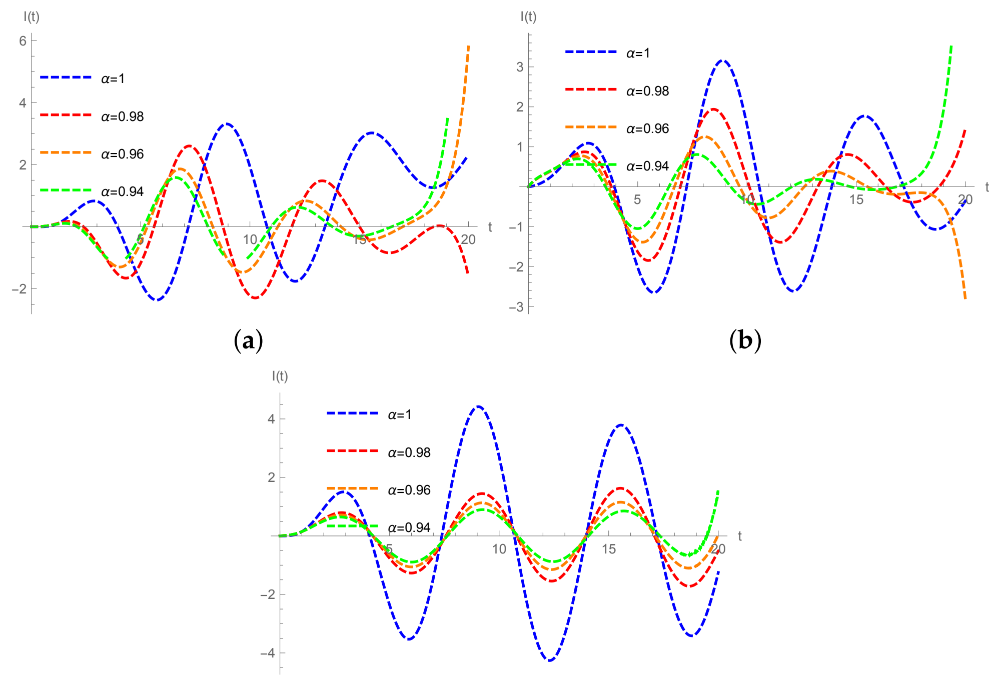

Figure 3. Numerical simulation for RLC electrical circuit via Caputo–Fabrizio fractional operator, in (**a**) Equation (35), corresponding to unit step source; in (**b**) Equation (39), corresponding to exponential source; in (**c**) Equation (42), corresponding to periodic source; for all figures $I(t)$ is measured at Amperes and t is measured at seconds.

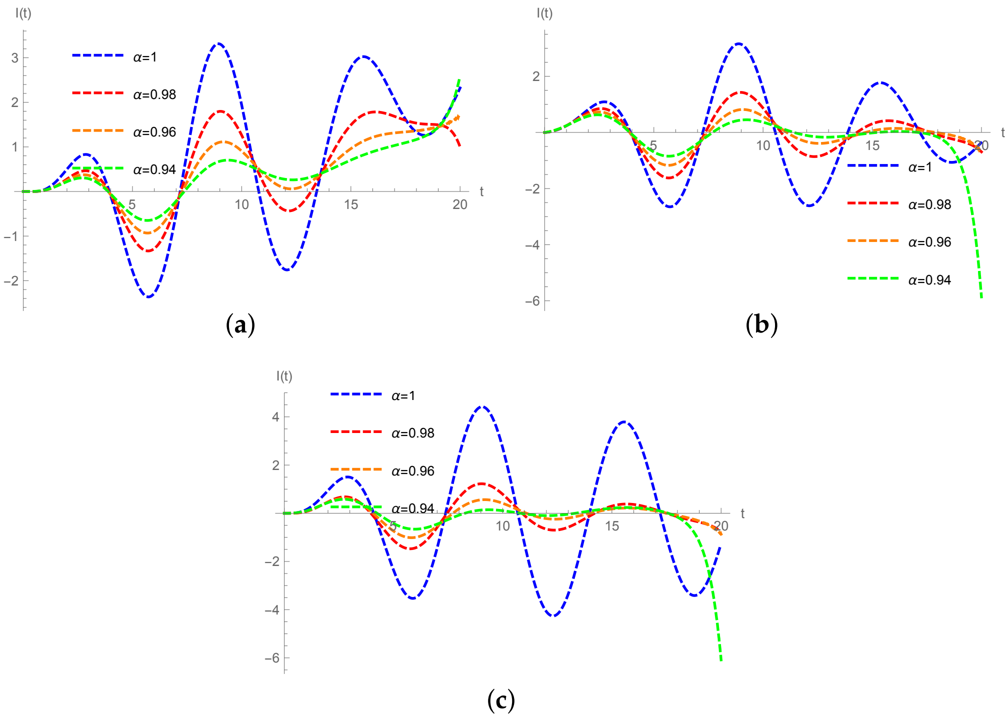

Figure 4. Numerical simulation for RLC electrical circuit via Atangana-Baleanu–Caputo fractional operator, in (**a**) Equation (46), corresponding to unit step source; in (**b**) Equation (50), corresponding to exponential source; in (**c**) Equation (53), corresponding to periodic source; for all figures $I(t)$ is measured at Amperes and t is measured at seconds.

4. Conclusions

In the present paper, analytical solutions of the electrical RLC circuit using the Liouville–Caputo, Caputo–Fabrizio and the Atangana–Baleanu–Caputo fractional operators were presented. The solutions obtained preserve the dimensionality of the studied system for any value of the exponent of the fractional derivative.

We can conclude that the decreasing value of α provides an attenuation of the amplitudes of the oscillations, the system increases its "damping capacity" and the current changes due to the order derivative (causing irreversible dissipative effects such as ohmic friction), the response of the system evolves from an under-damped behavior into an over-damped behavior. The fractional differentiation with respect to the time represents a non-local effect of dissipation of energy (internal friction) represented by the fractional order α. The electrical circuit RLC exhibits fractality in time to different scales and shows the existence of heterogeneities in the electrical components (resistance, capacitance and inductance). Due to the physical process involved (i.e., magnetic hysteresis), these components can present signs of nonlinear phenomena and non-locality in time, it is clear that the approximate solutions continuously depend on the time-fractional derivative α. In the classical case, where $\alpha = 1$, due to the absence of damping, the amplitude is maintained and the system displays the Markovian nature.

For the Liouville–Caputo fractional operator the solutions incorporate and describe long term memory effects (attenuation or dissipation), these effects are related to an algebraic decay related to the Mittag-Leffler function. However, this fractional operator involves a kernel with singularity. The Caputo–Fabrizio fractional operator is based on the exponential function; thus, the used kernel is local and may not be able to portray more accurately some systems. Nevertheless, due to their properties, some researchers have concluded that this operator can be viewed as a filter regulator [28]. Atangana and Baleanu presented a fractional derivative with Mittag-Leffler kernel. This derivative is the average of the given function and its Riemann–Liouville fractional integral. The Figures show that the system presents dissipative effects that correspond to the nonlinear situation of the physical process (realistic behavior that is non-local in time). Furthermore, the Figures show that the Liouville–Caputo fractional derivative is more affected by the past compared with the new fractional operator based on the Mittag-Leffler function which shows a rapid stabilization. Finally, the Caputo–Fabrizio approach is a particular case of the representation obtained using the fractional operator with thw Mittag-Leffler kernel in the Liouville–Caputo sense.

This methodology can be applied in the analysis of electromagnetic transients problems in electrical systems, machine windings, modeling of surface discharge in electrical equipment, transmission lines, power electronics, underground cables or partial discharge in insulation systems and control theory.

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