Singular Dirac systems in the Sobolev space

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Abstract: In this paper we construct Weyl's theory for the singular left-definite Dirac systems. In particular, we prove that there exists at least one solution of the system of equations that lies in the Sobolev space. Moreover, we describe the behavior of the solution belonging to the Sobolev space around the singular point.

Key words: Dirac systems, left-definite operators, Weyl theory

1. Introduction

Consider the equation

$$y'' + y = 0.$$

One can immediately obtain that the solution is described by

$$y = c_1 \sin(x + c_2),$$

where c_1 and c_2 are arbitrary constants. However, a different approach to obtain the solution from the ordinary way is possible. In fact, multiplying the equation by y' one can find

$$(y')^2 + y^2 = c^2 \tag{1.1}$$

and

$$y' = \sqrt{c^2 - y^2}.$$

Considering $y = c \sin \gamma$ it is obtained that $\gamma = x + d$. Consequently, the solution is $y = c \sin(x + d)$ [5].

The main point of this construction of the solution is the equation (1.1), which can arise naturally in Sturm–Liouville equations, called *left-definite* Sturm–Liouville equations. The name left-definite comes from the left-side of the equation

$$-(py')' + qy = \lambda w y, \ x \in (a, b) \subseteq \mathbb{R}, \tag{1.2}$$

in contrast to the standard *right-definite* case. That is, as is well known, in the right-definite case we impose the condition to the *weight* function w to be positive, which gives rise to the standard Lebesgue space $L_w^2(a, b)$ with the inner product

$$(y, z) = \int_a^b y \bar{z} w \, dx.$$

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However, in general, in the left-definite case the function w is allowed to change the sign on the interval (a, b). In this case, the positiveness conditions are imposed on the functions p and q. Namely, in this case, the inner product is defined as

$$\langle y, z \rangle = \int_a^b [py'\overline{z}' + qy\overline{z}] \, dx,$$

which generates the Sobolev space $H^1(a, b; p, q)$.

If one of the functions in the equation (1.2) increases boundedness then it is hard to describe the behavior of the solutions around a neighborhood of the singular point. This problem was solved by Weyl in 1910 [16] for the right-definite second order Sturm–Liouville equation with his extraordinary way. In 1995, Krall and Race [6] studied the singular left-definite Sturm–Liouville equations in the Sobolev space and they showed that there always exists at least one solution belonging to the Sobolev space under some conditions. Moreover, one can find some papers that contain the left-definite case [7–10]. In particular, in [10], Krall investigated the self-adjointness of the regular left-definite Hamiltonian systems.

As is well known, Dirac systems are of the form [12], [13]

$$y_2' + p(x)y_1 + r(x)y_2 = \lambda e(x)y_1 + \lambda f(x)y_2,$$
$$-y_1' + r(x)y_1 + q(x)y_2 = \lambda f(x)y_1 + \lambda h(x)y_2,$$

(1.3)

where λ is a complex parameter and p, r, q, e, f, h are real-valued and locally integrable functions on $(a, b) \subseteq \mathbb{R}$. The right-definite case corresponds with the positiveness of the matrix

$$\begin{bmatrix} e(x) & f(x) \\ f(x) & h(x) \end{bmatrix} > 0, \ x \in (a, b),$$

and has been investigated in the literature (see [1–5,11–15]). We should note that the system (1.3) plays a central role in relativistic quantum theory. Namely, the system (1.3) coincides with Dirac's radial relativistic wave equation for a particle in a central field. In this paper we investigate the singular left-definite Dirac systems and we describe the solution of Dirac systems belonging to the Sobolev space. Moreover, we investigate the behavior of the solution of Dirac system belonging to the Sobolev space around a neighborhood of the singular point.

2. Singular Dirac systems

To investigate the left-definite case we consider the following special Dirac systems on $(a, b) \subseteq \mathbb{R}$

$$y_2' + p(x)y_1 + r(x)y_2 = \lambda e(x)y_1,$$
$$-y_1' + r(x)y_1 + q(x)y_2 = 0,$$

(2.1)

or

$$Ly = W^{-1}(x)\ell(y) = By' + Q(x)y = \lambda y, x \in [a, b) \subseteq \mathbb{R}, \tag{2.2}$$

where p, r, q, e are real-valued, Lebesgue measurable functions on $[a, b)$, a is the regular point and b is the singular point for (2.1) or (2.2), $p > 0$, $q < 0$, $e > 0$ on $[a, b)$, $\rho e \leq p$ for some $\rho > 0$, e and p are integrable

functions on $[a, b)$, $W^{-1}(x)W(x) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and

$$B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} p & r \\ r & q \end{bmatrix}, \quad W = \begin{bmatrix} e & 0 \\ 0 & 0 \end{bmatrix}, \quad y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}.$$

As is well known, the ordinary inner product is defined as

$$(y, z) = \int_a^b z^* W y dx,$$

in which the Lebesgue space $L_W^2(a, b)$ is equipped with this inner product.

We assume that

$$By' + Qy = Wf$$

exists a.e. and $f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$ is in $L_W^2(a, b)$.

The ordinary inner product in $L_W^2(a, b)$ gives rise to the following:

$$(y, z) = \int_a^b z^* W y dx = \int_a^b \begin{pmatrix} \bar{z}_1 & \bar{z}_2 \end{pmatrix} \begin{pmatrix} e(x) & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} dx = \int_a^b e y_1 \bar{z}_1 dx.$$

If $By' + Qy = Wf$ is decomposed into components we find

$$y_2' + p(x)y_1 + r(x)y_2 = e(x)f_1,$$
$$-y_1' + r(x)y_1 + q(x)y_2 = 0.$$

On the other hand, $By' + Qy = Wf$ implies

$$(Ly, z) = \int_a^b z^* W L y dx = \int_a^b z^* W f dx = \int_a^b e f_1 \bar{z}_1 dx.$$

Therefore

$$\int_a^b e f_1 \bar{z}_1 dx = \int_a^b \bar{z}_1 (y_2' + p y_1 + r y_2) dx = y_2 \bar{z}_1 \big|_a^b - \int_a^b y_2 (\bar{z}_1' - r \bar{z}_1) dx + \int_a^b p y_1 \bar{z}_1 dx$$

$$= y_2 \bar{z}_1 \big|_a^b - \int_a^b q y_2 \bar{z}_2 dx + \int_a^b p y_1 \bar{z}_1 dx$$

provided that

$$-z_1' + r y_1 + q y_2 = 0.$$

Consequently, we can introduce a new inner product

$$\langle y, z \rangle = \int\limits_a^b p y_1 \overline{z}_1 dx - \int\limits_a^b q y_2 \overline{z}_2 dx = \int\limits_a^b z^* \begin{pmatrix} p & 0 \\ 0 & -q \end{pmatrix} y dx,$$

which gives rise to the Sobolev space $H^1(a, b; p, q)$.

We can therefore give the relation

$$(Ly, z) = \langle y, z \rangle + y_2 \overline{z}_1 \mid_a^b. \tag{2.3}$$

3. Sobolev space solutions

Consider the solutions

$$\varphi(x, \lambda) = \begin{bmatrix} \varphi_1(x, \lambda) \\ \varphi_2(x, \lambda) \end{bmatrix}, \quad \psi(x, \lambda) = \begin{bmatrix} \psi_1(x, \lambda) \\ \psi_2(x, \lambda) \end{bmatrix}$$

of the equation

$$By' + Qy = \lambda W y \tag{3.1}$$

satisfying the conditions

$$\varphi_1(a, \lambda) = \cos\alpha, \quad \varphi_2(a, \lambda) = \sin\alpha,$$
$$\psi_1(a, \lambda) = \sin\alpha, \quad \psi_2(a, \lambda) = -\cos\alpha,$$

where α is a fixed real number. We have $W[\varphi, \psi] = \varphi_2\psi_1 - \varphi_1\psi_2 = 1$. Therefore the general solution of (3.1) must be of the form

$$\chi(x, \lambda) = \varphi(x, \lambda) + m(\lambda)\psi(x, \lambda).$$

Now consider the following boundary condition at d, $a < d$, as follows:

$$\cos\theta y_1(d) + \sin\theta y_2(d) = 0, \tag{3.2}$$

where θ is a real number. Then

$$m(\lambda) = -\frac{\cos\theta\varphi_1(d, \lambda) + \sin\theta\varphi_2(d, \lambda)}{\cos\theta\psi_1(d, \lambda) + \sin\theta\psi_2(d, \lambda)}. \tag{3.3}$$

It is well known that for $0 \le \theta < \pi$, $m(\lambda)$ describes a circle in the complex plane. Equation (2.3) shows that

$$\lambda \int\limits_a^d e \left| y_1 \right|^2 dx = \int\limits_a^d p \left| y_1 \right|^2 dx - \int\limits_a^d q \left| y_2 \right|^2 dx + y_2 \overline{y}_1 \mid_a^d. \tag{3.4}$$

Equation (3.4) implies that

$$\begin{aligned}
(\mathrm{Re}\lambda + i\mathrm{Im}\lambda) \int\limits_a^d e \left| \psi_1 \right|^2 dx &= \int\limits_a^d p \left| \psi_1 \right|^2 dx - \int\limits_a^d q \left| \psi_2 \right|^2 dx - |K|^2 \cos\theta\sin\theta \\
&\quad + \left[|m|^2 - 1 \right] \sin\alpha\cos\alpha + m\cos^2\alpha - \overline{m}\sin^2\alpha,
\end{aligned} \tag{3.5}$$

where

$$\psi_1(a,\lambda) = \cos\alpha + m(\lambda)\sin\alpha,$$
$$\psi_2(a,\lambda) = \sin\alpha - m(\lambda)\cos\alpha,$$

and

$$\psi_1(d,\lambda) = K\sin\theta,$$
$$\psi_2(d,\lambda) = -K\cos\theta.$$

The imaginary part of (3.5) is

$$\mathrm{Im}\,m = \mathrm{Im}\lambda \int_a^d e\,|\psi_1|^2\,dx$$

and the real part of (3.5) is

$$\int_a^d p\,|\psi_1|^2\,dx - \int_a^d q\,|\psi_2|^2\,dx - |K|^2\cos\theta\sin\theta \quad = \quad \left[1 - |m|^2\right]\sin\alpha\cos\alpha - \mathrm{Re}\,m\cos^2\alpha$$
$$+\,\mathrm{Re}\,m\sin^2\alpha + \mathrm{Re}\lambda\frac{\mathrm{Im}\,m}{\mathrm{Im}\lambda}.$$

Now let $\theta = \frac{\pi}{2}$. In this case (3.2) and (3.3) become

$$\chi_2(d,\lambda) = 0$$

and

$$m(\lambda) = -\frac{\varphi_2(d,\lambda)}{\psi_2(d,\lambda)}.$$

In this case we obtain

$$\int_a^d p\,|\psi_1|^2\,dx - \int_a^d q\,|\psi_2|^2\,dx \quad = \quad \left[1 - |m|^2\right]\sin\alpha\cos\alpha - \mathrm{Re}\,m\cos^2\alpha$$

$$+\,\mathrm{Re}\,m\sin^2\alpha + \mathrm{Re}\lambda\frac{\mathrm{Im}\,m}{\mathrm{Im}\lambda},$$

where all m's are on the limit point or limit circle. Therefore we arrive at the following result.

Theorem 3.1 *There exists a solution*

$$\chi(x,\lambda) = \varphi(x,\lambda) + m_b(\lambda)\psi(x,\lambda),\ \ Im\lambda \neq 0,$$

of the equation (3.1), which lies in $H^1(a,b;p,q)$.

To describe the behavior of the solution of Dirac systems we need to restrict the system (2.1) to the following:

$$y_2' + p(x)y_1 = \lambda e(x)y_1,$$
$$-y_1' + q(x)y_2 = 0,$$
(3.6)

Then we obtain the following theorem.

Theorem 3.2 *Let* $\chi(x, \lambda) = \varphi(x, \lambda) + m_b(\lambda)\psi(x, \lambda)$ *be the solution of* (3.6) *in* $H^1(a, b; p, q)$ *generated by the approximation solution* $\widehat{\chi}(x, \lambda) = \widehat{\chi}_1(x, \lambda)\widehat{\chi}_2(x, \lambda)$ *satisfying* $\widehat{\chi}_2(x, \lambda) = 0$. *Then* $\lim\limits_{x \to b} \chi_2(x, \lambda) = 0$.

Proof   We have the following equation

$$\psi_2(d, \lambda) = \psi_2(a, \lambda) + \int\limits_a^d \psi'(x, \lambda)dx = \psi_2(a, \lambda) + \int\limits_a^d \left[\lambda e\psi_1 - p\psi_1\right] dx.$$

Therefore we get

$$|\psi_2(d, \lambda)| \;\leq\; |\psi_2(a, \lambda)| + K \int\limits_a^d e\,|\psi_1|^2\,dx$$

$$\leq\; |\psi_2(a, \lambda)| + K \left[\int\limits_a^d e\,dx\right]^{1/2} \left[\int\limits_a^d e\,|\psi_1|^2\,dx\right]^{1/2}.$$

On the other side one can write

$$\chi_2(d, \lambda) = \chi_2(d, \lambda) - \widehat{\chi}_2(d, \lambda) = (m_b - m_d)\psi_2(d, \lambda).$$

In the limit-point case

$$|m_b - m_d| < 2r_b = \frac{2}{|\text{Im}\lambda| \int\limits_a^d e\,|\psi_1|^2\,dx}$$

and hence

$$|m_b - m_d|\,|\psi_2(d, \lambda)| \leq \frac{A + B \left[\int\limits_a^d e\,|\psi_1|^2\,dx\right]^{1/2}}{\int\limits_a^d e\,|\psi_1|^2\,dx}$$

which approaches zero as d approaches b.

In the limit-circle case

$$|\psi_2(d, \lambda)| < K.$$

Since $m_d \to m_b$,

$$\lim\limits_{d \to b}(m_d - m_b)\psi_2(d, \lambda) = 0.$$

Therefore the proof is completed. □

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