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An algorithm for Hopf bifurcation analysis of a delayed reaction–diffusion model

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Abstract We present an algorithm for determining the existence of a Hopf bifurcation of a system of delayed reaction–diffusion equations with the Neumann boundary conditions. The conditions on parameters of the system that a Hopf bifurcation occurs as the delay parameter passes through a critical value are determined. These conditions depend on the coefficients of the characteristic equation corresponding to linearization of the system. Furthermore, an algorithm to obtain the formulas for determining the direction of the Hopf bifurcation, the stability, and period of the periodic solution is given by using the Poincaré normal form and the center manifold theorem. Finally, we give several examples and some numerical simulations to show the effectiveness of the algorithm proposed.

Keywords Hopf bifurcation · Delay differential equations · Reaction–diffusion equation · Stability · Time delay · Periodic solutions

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1 Introduction

In order to produce a better understanding for the world around us and to find solutions to technical problems, representing the events by using mathematical terms is called mathematical modeling. The model is initially kept as simple as possible. After adding new terms and variables to the model at later stages, it becomes more realistic. The models obtained are often used to understand dynamics of the systems which change with respect to time. However, this is not mostly enough to understand the whole story. For example, some problems in real life usually depend on time, but they may also depend on various independent variables such as location or age. In an assemblage of particles (for example, cells, bacteria, chemicals, and animals), each particle usually moves around in a random way. The particles spread out as a result of this irregular individual particle motion. When this microscopic irregular movement results in some macroscopic or gross regular motion of the group, we can think of it as a diffusion process. Therefore, a model which contains diffusion process can be improved mathematically by adding a spatial variable to the model. A reaction diffusion system which is a mechanism proposed as a model for the chemical basis of morphogenesis by Turing [41] is an example of such models. Such systems have been studied widely since 1970 (see, for example, [23,28,30,33,39,40,42,44,46], and references therein).

On the other hand, in many applications, it is generally assumed that the system under consideration is governed by a principle of causality; that is, the future state of the system is independent of the past and is determined solely by the present. One should keep in mind that this is only a first approximation to the true situation. Time delays occur so often (in almost every situation) that to ignore them is to ignore reality. In the context of neural network, for example, time delay can come from close loop for autapse connected to neuron, and it is classified as propagation time delay and response time delay (see, for example, [24,25,34]). Therefore, to create a somewhat more realistic description and to form a more realistic model, a time delay should be included into the model, which results in a delay differential equation [21]. Delay differential equations have been used to model real life problems arising in various practical applications such as population dynamics (most often accounting for maturation/gestation periods), immunology (incubation/latent periods), and physiological and pharmaceutical kinetics (glucose–insulin regulation, blood pressure regulation). Studies show that exploring the dynamical behaviors of the models involving some of the past histories has attracted very much interest in chemistry, physics, mathematical biology, medicine, ecology, neuroscience, economics, and other fields (see, for example, [2,5,7,11,20,21,29,31,32,35,38,43], and references therein).

Mathematical models often contain parameters, such as delay term, which have impact on the dynamics of systems. Bifurcation is answer of the question: "How the dynamic of a system, for example (1) below, changes when parameter μ varies?"

$$X' = F(X, \mu) \quad X \in \mathbb{R}^m, \quad \mu \in \mathbb{R}. \tag{1}$$

Mathematically, the appearance of topologically nonequivalent phase portraits under variation of parameter is called bifurcation [22]. We can talk about bifurcation if qualitative or topological structure of system (1) varies, such as change in the number of or stability structure of equilibrium point and appearance or disappearance of periodic solutions when the value of the bifurcation parameter varies. A specific value of the parameter at which topological structure of the system changes is called bifurcation value [1]. There are several types of bifurcations. The bifurcation which corresponds to purely imaginary eigenvalues $\lambda_{1,2} = \pm i\omega_0$ such that $\omega_0 > 0$ of a linearized system at bifurcation value is called Hopf bifurcation [22]. The main feature of Hopf bifurcation is appearance of periodic solutions. In particular, the properties of periodic solutions appearing through the Hopf bifurcation in delayed reaction–diffusion systems are of great interest.

Determining Hopf bifurcation in a system is important to show the existence of periodic solutions of the system. However, it is not an easy work to determine whether Hopf bifurcation occurs in the system or not. Studies on the existence of periodic solutions are important for the research problems arising in different research areas ranging from mathematical biology to applied economics (see, for example, [19,22,27] and references therein).

The methods often used to analyze existence and direction of the Hopf bifurcation for a 2×2 differential equation system can be found in [19] and [22]; for a 2×2 delay differential equation system can be found in [18] and [19]; for a 2×2 reaction–diffusion system can be found in [19,22,42] and [45]. For many cases, reaction–diffusion models involving delay are more realistic than other models. However, the aforementioned methods for Hopf bifurcation analysis in the literature do not directly give a method for existence and direction analysis of the Hopf bifurcation of a 2×2 delayed reaction–diffusion system with the Neumann boundary conditions below:

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = d_1 \frac{\partial^2 u(x,t)}{\partial x^2} \\ +f(u(x,t), u(x,t-\tau), v(x,t), v(x,t-\tau)), \quad x \in \Omega, \quad t > 0, \\ \frac{\partial v(x,t)}{\partial t} = d_2 \frac{\partial^2 v(x,t)}{\partial x^2} \\ +g(u(x,t), u(x,t-\tau), v(x,t), v(x,t-\tau)), \quad x \in \Omega, \quad t > 0, \\ \frac{\partial u}{\partial \vec{n}} = \frac{\partial v}{\partial \vec{n}} = 0, x \in \partial\Omega, \quad t \geq 0, \\ u(x,t) = u^*(x,t), \quad v(x,t) = v^*(x,t), \quad x \in \overline{\Omega}, \quad t \in [-\tau, 0], \end{cases} \tag{2}$$

where $f, g : \mathbb{R}^4 \to \mathbb{R}$ are C^k ($k \geq 2$), Ω is an open bounded domain in $\mathbb{R}^m$, $m \geq 1$, with smooth boundaries $\partial\Omega$, $\vec{n}$ is the unit outer normal to $\partial\Omega$ and $u^*, v^* \in C^2((\Omega, [-\tau, 0])) \cap C\big((\overline{\Omega}, [-\tau, 0])\big)$. Here, d_1 and d_2, which are positive constants, are diffusion coefficients of u and v, respectively, and $\tau > 0$ is delay parameter. For simplicity, we chose the spatial domain as $\Omega = (0, \ell\pi) \subset \mathbb{R}$, but all calculations can be extended for the higher dimensions.

In this paper, we derive an algorithm to provide a simple and quick way to check the existence and direction of a Hopf bifurcation of a 2×2 delayed reaction–diffusion system (2). This general algorithm can be also used for a 2×2 system of delay differential equations by taking diffusion coefficients zero in (2). In literature, there are two common methods to calculate the normal form of a Hopf bifurcation, namely, Hassard's method (see, for example, [19,20,28,45]) and Faria's method ([13–15]). For example, in [39] and [40], the authors studied Turing–Hopf bifurcation in the reaction–diffusion equations without delay and a predator–prey model without delay that involves herd behavior and cross-diffusion, respectively, by utilizing Faria's method for the calculation of normal forms. Turing–Hopf bifurcation is degenerate case, and the characteristic equation has a pair of simple purely imaginary roots and also a simple zero root; however, in Hopf bifurcation the characteristic equation has only a pair of simple purely imaginary roots. In this paper, we use Hassard's method for the normal form calculation to derive the algorithm for determining the direction and also some other properties of the Hopf bifurcation arising from the reaction–diffusion equations with delay.

The paper is organized as follows. In Sect. 2, we show that the 2×2 system of delayed ordinary differential equations and the 2×2 delayed reaction–diffusion systems have the same form of characteristic equations with different coefficients. In Sect. 3, an algorithm for Hopf bifurcation analysis of the system which has characteristic equation of this form is investigated, and a table for existence of a Hopf bifurcation is obtained. In Sect. 4, following Hassard's method, an algorithm to obtain the formulas for determining the direction of the Hopf bifurcation and the stability and period of the periodic solution is derived by using the Poincaré normal form and the center manifold theorem [19]. In Sect. 5, we implement these algorithms to three different models and perform some numerical simulations to show the usefulness of them. Paper ends with some concluding remarks.

2 Form of characteristic equations

2.1 A general DDE model with one delay

Consider the following 2×2 system of delay differential equations:

$$\begin{cases} \frac{du(t)}{\mathrm{d}t} = f(u(t), u(t-\tau), v(t), v(t-\tau)), & t > 0, \\ \frac{dv(t)}{\mathrm{d}t} = g(u(t), u(t-\tau), v(t), v(t-\tau)), & t > 0, \\ u(t) = u^*(t), \quad v(t) = v^*(t), \quad t \in [-\tau, 0], \end{cases} \tag{3}$$

where $f, g : \mathbb{R}^4 \to \mathbb{R}$ are C^k $(k \geq 2)$ with $f(P_0) = g(P_0) = 0$, $P_0 = (u_0, v_0)$ and $u^*, v^* \in C([-\tau, 0])$.

Jacobian matrix of (3) at P_0 is

$$J(P_0) = \begin{pmatrix} k_1 + k_2 e^{-\lambda\tau} & k_3 + k_4 e^{-\lambda\tau} \\ l_1 + l_2 e^{-\lambda\tau} & l_3 + l_4 e^{-\lambda\tau} \end{pmatrix}, \tag{4}$$

where

$$\begin{aligned} &k_1 = f_u(P_0), \quad k_2 = f_{u_\tau}(P_0), \\ &k_3 = f_v(P_0), \quad k_4 = f_{v_\tau}(P_0), \\ &l_1 = g_u(P_0), \quad l_2 = g_{u_\tau}(P_0), \\ &l_3 = g_v(P_0), \quad l_4 = g_{v_\tau}(P_0), \end{aligned} \tag{5}$$

and

$$f_{u_\tau} = \frac{\partial f}{\partial u(t-\tau)}, \quad f_{v_\tau} = \frac{\partial f}{\partial v(t-\tau)},$$
$$g_{u_\tau} = \frac{\partial g}{\partial u(t-\tau)}, \quad g_{v_\tau} = \frac{\partial g}{\partial v(t-\tau)}.$$

Then the characteristic equation of $J(P_0)$ is

$$\lambda^2 + a\lambda + be^{-\lambda\tau} + c\lambda e^{-\lambda\tau} + d + he^{-2\lambda\tau} = 0, \tag{6}$$

where

$$\begin{aligned} &a = -(k_1 + l_3), \quad b = (k_1 l_4 + k_2 l_3 - k_3 l_2 - k_4 l_1), \\ &c = -(k_2 + l_4), \quad d = (k_1 l_3 - k_3 l_1), \\ &h = (k_2 l_4 - k_4 l_2). \end{aligned}$$

2.2 A general PDE model with one delay

Let us now consider the following 2×2 delayed reaction–diffusion system with the Neumann boundary condition:

$$
\begin{cases}
\frac{\partial u(x,t)}{\partial t} = d_1 \frac{\partial^2 u(x,t)}{\partial x^2} \\
\quad + f(u(x,t), u(x,t-\tau), v(x,t), v(x,t-\tau)), \\
x \in (0, \ell\pi), \quad t > 0, \\
\frac{\partial v(x,t)}{\partial t} = d_2 \frac{\partial^2 v(x,t)}{\partial x^2} \\
\quad + g(u(x,t), u(x,t-\tau), v(x,t), v(x,t-\tau)), \\
x \in (0, \ell\pi), \quad t > 0, \\
u_x(x,t) = v_x(x,t) = 0, \\
x \in \{0, \ell\pi\}, \quad t \geq 0, \\
u(x,t) = u^*(x,t), \quad v(x,t) = v^*(x,t), \\
x \in [0, \ell\pi], \quad t \in [-\tau, 0],
\end{cases}
\tag{7}
$$

where $\ell, d_1, d_2 \in \mathbb{R}^+$ and $f, g : \mathbb{R}^4 \to \mathbb{R}$ are C^k $(k \geq 2)$ with $f(P_0) = g(P_0) = 0$, $P_0 = (u_0, v_0)$.

We first shift the equilibrium point P_0 to the origin via the transformations $\widetilde{u} = u - u_0$ and $\widetilde{v} = v - v_0$, and then linearize the new system around zero. For convenience, we continue our calculations by taking u for $\widetilde{u}$ and v for $\widetilde{v}$, then we have the following system:

$$
\begin{cases}
\frac{\partial u(x,t)}{\partial t} = d_1 \frac{\partial^2 u(x,t)}{\partial x^2} + k_1 u(x,t) + k_2 u(x, t-\tau) \\
\quad + k_3 v(x,t) + k_4 v(x, t-\tau) + \text{h.o.t.}, \\
\frac{\partial v(x,t)}{\partial t} = d_2 \frac{\partial^2 v(x,t)}{\partial x^2} + l_1 u(x,t) + l_2 u(x, t-\tau) \\
\quad + l_3 v(x,t) + l_4 v(x, t-\tau) + \text{h.o.t.},
\end{cases}
\tag{8}
$$

where the term h.o.t. denotes the higher-order terms and k_i, l_i for $i = 1, 2, 3, 4$ are defined in (5).

Let the linear operator Δ be defined by $\Delta := \operatorname{diag}\left\{\frac{\partial^2}{\partial x^2}, \frac{\partial^2}{\partial x^2}\right\}$ and $U(t) := (u(t), v(t))^T = (u(\cdot, t), v(\cdot, t))^T$. With this notation, system (8) can be rewritten as an abstract ordinary differential equation in Banach Space $C = C([-\tau, 0], X)$ where $X = \left\{(u, v) : u, v \in W^{2,2}(0, \pi); \quad \frac{du}{dx} = \frac{dv}{dx} = 0, \quad x = 0, \pi\right\}$ as follows:

$$
\frac{d}{dt} U(t) = d\Delta U(t) + L(U_t) + \text{h.o.t.}, \tag{9}
$$

where $d = (d_1, d_2)^T$, $U_t(\theta) = U(t + \theta)$, $-\tau \leq \theta \leq 0$, $L : C \to X$. Here, L is defined by

$$
L(\varphi) = \begin{pmatrix} k_1\varphi_1(0) + k_2\varphi_1(-\tau) & k_3\varphi_2(0) + k_4\varphi_2(-\tau) \\ l_1\varphi_1(0) + l_2\varphi_1(-\tau) & l_3\varphi_2(0) + l_4\varphi_2(-\tau) \end{pmatrix} \tag{10}
$$

for $\varphi(\theta) = U_t(\theta)$, $\varphi = (\varphi_1, \varphi_2)^T \in C$. The characteristic equation of (9) is

$$
\lambda y - d\Delta y - L(e^{\lambda} y) = 0, \tag{11}
$$

where $y \in \operatorname{dom}(\Delta)$ and $y \neq 0$, $\operatorname{dom}(\Delta) \subset X$. From properties of the Laplacian operator defined on a bounded domain, the operator Δ has eigenvalues $-\frac{n^2}{\ell^2}$, where $n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$. The corresponding eigenfunctions for each n are given by

$$
\beta_n^1 = \begin{pmatrix} \gamma_n \\ 0 \end{pmatrix}, \; \beta_n^2 = \begin{pmatrix} 0 \\ \gamma_n \end{pmatrix}, \quad \gamma_n = \cos\left(\frac{n}{\ell} x\right),
$$

where $n \in \mathbb{N}_0$. It is easy to see that $\left\{\beta_n^1, \beta_n^2\right\}_{n=0}^{\infty}$ forms a basis for the phase space X. Therefore, any arbitrary y in X can be written as a Fourier Series in the following form:

$$
y = \sum_{n=0}^{\infty} Y_n^T \begin{pmatrix} \beta_n^1 \\ \beta_n^2 \end{pmatrix}, \quad Y_n^T = \begin{pmatrix} < y, \beta_n^1 > \\ < y, \beta_n^2 > \end{pmatrix}. \tag{12}
$$

One can easily show that

$$
L(\varphi^T \begin{pmatrix} \beta_n^1 \\ \beta_n^2 \end{pmatrix}) = L(\varphi)^T \begin{pmatrix} \beta_n^1 \\ \beta_n^2 \end{pmatrix}, \quad n \in \mathbb{N}_0. \tag{13}
$$

From (12) and (13) one can show that equation (11) is equivalent to

$$
\sum_{n=0}^{\infty} Y_n^T \left[\begin{array}{l} \left(\lambda + d\frac{n^2}{\ell^2}\right) I_2 \\ -\begin{pmatrix} k_1 + k_2 e^{-\lambda\tau} & k_3 + k_4 e^{-\lambda\tau} \\ l_1 + l_2 e^{-\lambda\tau} & l_3 + l_4 e^{-\lambda\tau} \end{pmatrix} \end{array} \right] \begin{pmatrix} \beta_n^1 \\ \beta_n^2 \end{pmatrix} = 0, \tag{14}
$$

where I_2 is the 2×2 identity matrix here. Notice that the sum in (14) is zero if and only if the determinant of the matrix in brackets is zero, i.e., $\det(\lambda I_2 - J(P_0)) = 0$, where

$$
J(P_0) = \begin{pmatrix} -d_1\frac{n^2}{\ell^2} + k_1 + k_2 e^{-\lambda\tau} & k_3 + k_4 e^{-\lambda\tau} \\ l_1 + l_2 e^{-\lambda\tau} & -d_2\frac{n^2}{\ell^2} + l_3 + l_4 e^{-\lambda\tau} \end{pmatrix}. \tag{15}
$$

This means that eigenvalues corresponding to the linearization of (7) is determined by the characteristic equation of (15) which is equal to

$$\lambda^2 + \widetilde{a}\lambda + \widetilde{b}e^{-\lambda\tau} + \widetilde{c}\lambda e^{-\lambda\tau} + \widetilde{d} + \widetilde{h}e^{-2\lambda\tau} = 0, \tag{16}$$

where

$$\widetilde{a} = (d_1 + d_2)\tfrac{n^2}{\ell^2} - (k_1 + l_3),$$
$$\widetilde{b} = \left(k_1 l_4 + k_2 l_3 - k_3 l_2 - k_4 l_1 - k_2 d_2 \tfrac{n^2}{\ell^2} - l_4 d_1 \tfrac{n^2}{\ell^2}\right),$$
$$\widetilde{c} = -(k_2 + l_4),$$
$$\widetilde{d} = d_1 d_2 \tfrac{n^4}{\ell^4} - (k_1 d_2 + l_3 d_1)\tfrac{n^2}{\ell^2} + k_1 l_3 - k_3 l_1,$$
$$\widetilde{h} = (k_2 l_4 - k_4 l_2)\,.$$

Thus, we conclude the following result.

Lemma 1 *Characteristic equations of systems* (3) *and* (7) *have the same form below:*

$$\lambda^2 + A\lambda + Be^{-\lambda\tau} + C\lambda e^{-\lambda\tau} + D + He^{-2\lambda\tau} = 0,$$

where the coefficients are determined by their linearizations.

3 Existence of Hopf bifurcation

We have shown that systems (3) and (7) have the following form of characteristic equation:

$$\lambda^2 + A\lambda + Be^{-\lambda\tau} + C\lambda e^{-\lambda\tau} + D + He^{-2\lambda\tau} = 0, \tag{17}$$

where distribution of its roots will determine stability properties of each system. The characteristic equations of many differential equation systems arising in mathematical biology, chemistry, and other fields have the form of (17), in particular, of the form of (18) below (see, for example, [7,16,28,43]). Studies on distribution of the roots of this equation have attracted very much attention. Some stability/instability properties of the zeros of (17) for $H = 0$ have been given by Bellman and Cooke [4], Cooke and Driessche [9], Mahaffy [26] and Nayfeh [32]; a similar analysis has been also done for $D = 0$, $H = 0$ by Cooke and Grossman [10] and Baptistini and Taboas [3]. Ghosh et al. [17] analyzed (17) for $C = 0$, $D = 0$, $H = 0$. Existence of Hopf bifurcation has been investigated by Bodnar and Fory's [6] for $C = 0$, $H = 0$, and by Sen et al. [37] for $C = 0$, $D = 0$, $H = 0$. Chen et. al [8] determined the conditions under which equilibrium point is stable or unstable, and also showed that Hopf bifurcation occurs under the assumption at least one of C and D is not zero beside H is not zero. However, the results obtained in [8] is valid only when $H \neq 0$. In other words, if you have a characteristic equation of the form (17) in which $H = 0$, the conclusions in [8] cannot be used. The distribution of the roots of the characteristic equation of the form (17) in which $H = 0$ was partially analyzed by Ruan [36]. He also applied the result to show the existence of the Hopf bifurcation of two specific ODE models. However, none of these works give a complete and a systematic way of analyzing Hopf bifurcation in a 2×2 general delayed reaction–diffusion system.

In this section, we drive an explicit algorithm which gives a methodical way for determining the existence of Hopf bifurcation of a 2×2 system of general reaction–diffusion equation including delay whose linearization yields the characteristic equation of the following form:

$$\lambda^2 + A\lambda + Be^{-\lambda\tau} + C\lambda e^{-\lambda\tau} + D = 0. \tag{18}$$

First, we remind that either type (supercritical or subcritical) of Hopf bifurcation has the property that equilibrium point at which Hopf bifurcation arises loses its stability when bifurcation parameter (which is delay term τ for the system we are interested in here) passes through a critical bifurcation value [22]. That is why, let us first examine (18) for $\tau = 0$. In other words, we look for the conditions at which equilibrium point is stable when $\tau = 0$. Then, if we take $\tau = 0$ in the characteristic equation (18), it turns into the following quadratic equation:

$$\lambda^2 + (A + C)\lambda + B + D = 0. \tag{19}$$

The roots of (19) are obtained by the following formula:

$$\lambda_{1,2} = \frac{-(A + C) \pm \sqrt{(A + C)^2 - 4(B + D)}}{2}. \tag{20}$$

Let us assume that the following hypotheses hold:

(H1) $(A + C) > 0$,
(H2) $(B + D) > 0$.

One can easily check that equilibrium point is asymptotically stable if (H1) and (H2) holds when $\tau = 0$. As τ increases from 0 to ∞, stability of the equilibrium point changes when (18) has either a simple zero root or a pair of purely imaginary roots which is not repeating. Note that if (18) had a zero root, then

$(B+D)$ had to be zero which contradicts to (H2). The other possibility is that (18) has a pair of purely imaginary roots for some $\tau > 0$. Therefore, if (H1) and (H2) hold, and there is no purely imaginary root of the characteristic equation (18) at a value of $\tau > 0$, then equilibrium point stays stable for all $\tau \geq 0$. This underlines that if the characteristic equation (18) does not have a pair of pure imaginary roots at a positive value of τ, which is called the critical bifurcation value, Hopf bifurcation for systems (3) and (7) does not occur at all. Because of that, we need to determine the condition on parameters at which (18) has a pair of pure imaginary roots.

Let us assume that (18) has a pair of pure imaginary roots, i.e., it has a root $\lambda = i\omega$, $\omega \in R$ and $\omega > 0$, when $\tau \neq 0$. To determine the conditions on parameters at which a pair of purely imaginary roots arises, we substitute $\lambda = i\omega$ into the characteristic equation (18) and then obtain the following transcendental equation:

$$-\omega^2 + iA\omega + (B + iC\omega)e^{-i\omega\tau} + D = 0. \tag{21}$$

Utilizing Euler's formula and separating its real and imaginary parts, we get the following two equations in ω:

$$\begin{cases} \omega^2 - D = B\cos(\omega\tau) + C\omega\sin(\omega\tau), \\ A\omega = B\sin(\omega\tau) - C\omega\cos(\omega\tau). \end{cases} \tag{22}$$

First taking square of both sides and then summing them up lead to

$$\omega^4 + (A^2 - C^2 - 2D)\omega^2 + D^2 - B^2 = 0 \tag{23}$$

so that we have

$$\omega^2 = \frac{-X \pm \sqrt{X^2 - 4Y}}{2}, \tag{24}$$

where $X = A^2 - C^2 - 2D$ and $Y = D^2 - B^2$.

Recall that we look for a ω satisfying the conditions: $\omega \in R$ and $\omega > 0$. Examining (24) yields the result that a positive real ω arises under the following conditions:

1. $X^2 - 4Y = 0$ and $X < 0$ that yields a single value: $\omega_1 = \sqrt{\frac{-X}{2}} > 0$,
2. $Y < 0$ that yields a single value: $\omega_2 = \sqrt{\frac{-X+\sqrt{X^2-4Y}}{2}} > 0$,
3. $Y = 0$ and $X < 0$ that yields a single value: $\omega_3 = \sqrt{-X} > 0$,
4. $X^2 - 4Y > 0$, $Y > 0$ and $X < 0$ that give two values: $\omega_4 = \sqrt{\frac{-X+\sqrt{X^2-4Y}}{2}} > 0$, $\omega_5 = \sqrt{\frac{-X-\sqrt{X^2-4Y}}{2}} > 0$.

Thus, we conclude the following results.

Lemma 2 *Assume that (H1) and (H2) hold. Then the equilibrium point* $P_0 = (u_0, v_0)$ *of systems* (3) *and* (7) *is asymptotically stable when* $\tau = 0$.

Lemma 3 *Assume that (H1) and (H2) hold. If one of the following conditions holds:*

(i) $X^2 - 4Y < 0$,
(ii) $X^2 - 4Y = 0$, $X > 0$,
(iii) $X = Y = 0$,
(iv) $X > 0$, $Y = 0$,
(v) $X^2 - 4Y > 0$, $X > 0$, $Y > 0$,

then all roots of (18) *have negative real parts for all* $\tau \geq 0$. *Furthermore, the equilibrium point* $P_0 = (u_0, v_0)$ *of systems* (3) *and* (7) *is asymptotically stable for all* $\tau \geq 0$.

Proof (24) implies that (23) has no positive root if one of the conditions (i)–(v) holds. Hence, the characteristic equation (18) has no purely imaginary root. Since the equilibrium point is stable when $\tau = 0$, all roots of (18) have negative real parts by the continuity. □

Lemma 4 *If one of the conditions (1)–(4) holds, then the characteristic equation* (18) *has a purely imaginary root of the form* $\lambda = i\omega_j$ $(j = 1, 2, 3, 4, 5)$ *at some* τ.

Next we determine $\tau = \tau_j$ associated with each ω_j $(j = 1, 2, 3, 4, 5)$. If we solve the equations in (22) for $\cos(\omega\tau)$ and $\sin(\omega\tau)$, we have the following identities:

$$\begin{cases} \cos(\omega\tau) = \dfrac{(B - AC)\omega^2 - BD}{B^2 + C^2\omega^2}, \\ \sin(\omega\tau) = \dfrac{C\omega^3 + (AB - CD)\omega}{B^2 + C^2\omega^2}, \end{cases} \tag{25}$$

that leads to

$$\tau_{j,k} = \frac{1}{\omega_j}\arctan\left(\frac{C\omega_j^3 + (AB - CD)\omega_j}{(B - AC)\omega_j^2 - BD}\right) + \frac{k\pi}{\omega_j}, \tag{26}$$

where $j = 1, 2, 3, 4, 5$; $k \in \mathbb{N}_0$. It is also possible to calculate $\tau_{j,k}$ via either

$$\tau_{j,k} = \frac{1}{\omega_j} \arcsin\left(\frac{C\omega_j^3 + (AB - CD)\omega_j}{B^2 + C^2\omega_j^2}\right) + \frac{2k\pi}{\omega_j}, \tag{27}$$

or

$$\tau_{j,k} = \frac{1}{\omega_j} \arccos\left(\frac{(B - AC)\omega_j^2 - BD}{B^2 + C^2\omega_j^2}\right) + \frac{2k\pi}{\omega_j}. \tag{28}$$

Note that $\mathrm{Re}(\lambda(\tau_{j,k})) = 0$ and $\omega(\tau_{j,k}) = \omega_j > 0$. Note also that for each $j^* \in \{1, 2, 3, 4, 5\}$ we uniquely determine $\tau_{j^*,k}$ such that $\lambda(\tau_{j^*,k}) = i\omega_{j^*}$. This underlines that all other roots of the characteristic equation (18) have nonzero real parts at $\tau = \tau_{j^*,k}$.

Now, we check transversality condition for each $\lambda = i\omega_j$ to show that it is a simple root of (18), i.e., by [9] we need to show $\mathrm{Re}\left(\frac{d\lambda}{d\tau}\Big|_{\tau=\tau_{j,0}}\right) \neq 0$, $j = 1, 2, 3, 4, 5$. Differentiating the characteristic equation (18) with respect to τ we get

$$\frac{d\lambda}{d\tau} = \frac{B\lambda + C\lambda^2}{(2\lambda + A)\, e^{\lambda\tau} + C - B\tau - C\lambda\tau}. \tag{29}$$

Substituting $\lambda = i\omega_j$, $(j = 1, 2, 3, 4, 5)$ into (29), we have

$$\frac{d\lambda}{d\tau}\Big|_{\tau=\tau_{j,0}} = \frac{iB\omega_j - C\omega_j^2}{(i2\omega_j + A)\, e^{i\omega_j\tau_{j,0}} + C - B\tau_{j,0} - iC\omega_j\tau_{j,0}}. \tag{30}$$

Since $\mathrm{Re}\left(\frac{d\lambda}{d\tau}\Big|_{\tau=\tau_{j,0}}\right) \neq 0$ if and only if $\mathrm{Re}\left(\left(\frac{d\lambda}{d\tau}\Big|_{\tau=\tau_{j,0}}\right)^{-1}\right) \neq 0$, it is enough to show latter one. From (30) one has

$$\left(\frac{d\lambda}{d\tau}\Big|_{\tau=\tau_{j,0}}\right)^{-1} = \frac{P_j}{Q_j},$$

where

$$\begin{aligned} P_j &= A\cos(\omega_j\tau_{j,0}) - 2\omega_j \sin(\omega_j\tau_{j,0}) + C - B\tau_{j,0} \\ &\quad + i\left(A\sin(\omega_j\tau_{j,0}) + 2\omega_j\cos(\omega_j\tau_{j,0}) - C\omega_j\tau_{j,0}\right) \\ Q_j &= -C\omega_j^2 + iB\omega_j. \end{aligned} \tag{31}$$

Let us define

$$\mathrm{Re}\left(\left(\frac{d\lambda}{d\tau}\Big|_{\tau=\tau_{j,0}}\right)^{-1}\right) := \mathrm{Re}\left(\frac{P_j}{Q_j}\right) = \frac{P_j^*}{Q_j^*}, \tag{32}$$

where from (31)

$$\begin{aligned} P_j^* &= \left(-AC\omega_j^2 + 2B\omega_j^2\right)\cos(\omega_j\tau_{j,0}) \\ &\quad + \left(2C\omega_j^3 + AB\omega_j\right)\sin(\omega_j\tau_{j,0}) - C^2\omega_j^2 \end{aligned} \tag{33}$$

and

$$Q_j^* = C^2\omega_j^4 + B^2\omega_j^2.$$

Substituting (25) into (33) one obtains

$$P_j^* = \omega_j^2\left(2\omega_j^2 + A^2 - C^2 - 2D\right).$$

Recalling that $X = A^2 - C^2 - 2D$, we have

$$\mathrm{Re}\left(\left(\frac{d\lambda}{d\tau}\Big|_{\tau=\tau_{j,0}}\right)^{-1}\right) = \frac{P_j^*}{Q_j^*} = \frac{2\omega_j^2 + X}{B^2 + C^2\omega_j^2}. \tag{34}$$

Let us now check whether $\mathrm{Re}\left(\left(\frac{d\lambda}{d\tau}\Big|_{\tau=\tau_{j,0}}\right)^{-1}\right)$ is nonzero. Since $Q_j^* > 0$ for all $j = 1, 2, 3, 4, 5$, it is enough to check sign of P_j^* for $j = 1, 2, 3, 4, 5$.

1. If $X^2 - 4Y = 0$ and $X < 0$, then $\omega_1 = \sqrt{\frac{-X}{2}} > 0$ so that
$$P_1^* = 2\omega_1^2 + X = 0.$$
2. If $Y < 0$, then $\omega_2 = \sqrt{\frac{-X + \sqrt{X^2 - 4Y}}{2}} > 0$ so that
$$P_2^* = 2\omega_2^2 + X = \sqrt{X^2 - 4Y} > 0.$$
3. If $Y = 0$ and $X < 0$, then $\omega_3 = \sqrt{-X} > 0$ so that
$$P_3^* = 2\omega_3^2 + X = -X > 0.$$

4. If $X^2 - 4Y > 0$, $Y > 0$ and $X < 0$, then $\omega_4 = \sqrt{\frac{-X+\sqrt{X^2-4Y}}{2}} > 0$ and $\omega_5 = \sqrt{\frac{-X-\sqrt{X^2-4Y}}{2}} > 0$ so that

$$P_4^* = 2\omega_4^2 + X = \sqrt{X^2 - 4Y} > 0,$$

$$P_5^* = 2\omega_5^2 + X = -\sqrt{X^2 - 4Y} < 0.$$

Thus, the transversality conditions hold for $j = 2, 3, 4, 5$. Moreover,

$$sign\,\mathrm{Re}\left(\left(\frac{d\lambda}{d\tau}\bigg|_{\tau=\tau_{j,k}}\right)^{-1}\right) = sign\,\mathrm{Re}\left(\left(\frac{d\lambda}{d\tau}\bigg|_{\tau=\tau_{j,0}}\right)^{-1}\right)$$

for $j = 1, 2, 3, 4, 5$; $k \in \mathbb{N}_0$. By the general Hopf bifurcation theorem [19], we have the following results:

Theorem 1 *Assume (H1) and (H2) hold. Equilibrium point $P_0 = (u_0, v_0)$ is asymptotically stable for $\tau \in [0, \tau_{j,0})$ for $j = 2, 3$. Moreover, systems (3) and (7) which have characteristic equation of the form*

$$\lambda^2 + A\lambda + Be^{-\lambda\tau} + C\lambda e^{-\lambda\tau} + D = 0$$

undergo a Hopf bifurcation at the equilibrium point $P_0 = (u_0, v_0)$ if one of the following conditions in the first column of Table 1 holds as the delay parameter τ passes through $\tau = \tau_{j,0}$ where from (26)

$$\tau_{j,0} = \frac{1}{\omega_j}\arctan\left(\frac{C\omega_j^3 + (AB - CD)\omega_j}{(B - AC)\omega_j^2 - BD}\right)$$

and ω_j is given in the second column of Table 1 for $j = 2, 3$.

Theorem 2 *Assume (H1) and (H2) hold. For systems (3) and (7) which have characteristic equation of the form*

$$\lambda^2 + A\lambda + Be^{-\lambda\tau} + C\lambda e^{-\lambda\tau} + D = 0$$

there is a positive integer r such that the equilibrium point $P_0 = (u_0, v_0)$ is stable when $\tau \in [0, \tau_{4,0}) \cup (\tau_{5,0}, \tau_{4,1}) \cup \cdots \cup (\tau_{5,r-1}, \tau_{4,r})$ and unstable when $\tau \in (\tau_{4,0}, \tau_{5,0}) \cup (\tau_{4,1}, \tau_{5,1}) \cup \cdots \cup (\tau_{4,r-1}, \tau_{5,r-1}) \cup (\tau_{4,r}, \infty)$ if the condition in the first column of Table 2 holds. This means that the stability switches of the equilibrium point $P_0 = (u_0, v_0)$ from stability to instability to stability occur as τ increases, and the equilibrium point $P_0 = (u_0, v_0)$ eventually becomes unstable for $\tau \in (\tau_{4,r}, \infty)$. Moreover, systems (3) and (7) undergo a Hopf bifurcation at the equilibrium point $P_0 = (u_0, v_0)$ when $\tau = \tau_{4,0}$ where from (26)

$$\tau_{4,0} = \frac{1}{\omega_4}\arctan\left(\frac{C\omega_4^3 + (AB - CD)\omega_4}{(B - AC)\omega_4^2 - BD}\right)$$

and ω_4 is given in Table 2.

Table 1 This table shows the conditions on parameters of systems (3) and (7) to have Hopf bifurcation. Here, $X = A^2 - C^2 - 2D$ and $Y = D^2 - B^2$

X, Y	ω	Transversality Condition
$Y < 0$	$\omega_2 = \sqrt{\frac{-X+\sqrt{X^2-4Y}}{2}}$	$\mathrm{Re}\left(\frac{d\lambda}{d\tau}\big\|_{\tau=\tau_{2,0}}\right) > 0,$
$Y = 0$, $X < 0$	$\omega_3 = \sqrt{-X}$	$\mathrm{Re}\left(\frac{d\lambda}{d\tau}\big\|_{\tau=\tau_{3,0}}\right) > 0.$

Table 2 This table shows the condition on parameters of systems (3) and (7) to have Hopf bifurcation and stability switches. Here, $X = A^2 - C^2 - 2D$ and $Y = D^2 - B^2$

X, Y	ω, Transversality Condition
$X^2 - 4Y > 0,\ Y > 0,\ X < 0$	$\omega_4 = \sqrt{\frac{-X+\sqrt{X^2-4Y}}{2}}$, $\mathrm{Re}\left(\frac{d\lambda}{d\tau}\big\|_{\tau=\tau_{4,0}}\right) > 0.$
	$\omega_5 = \sqrt{\frac{-X-\sqrt{X^2-4Y}}{2}}$, $\mathrm{Re}\left(\frac{d\lambda}{d\tau}\big\|_{\tau=\tau_{5,0}}\right) < 0.$

4 Direction analysis

In the former section, we have shown that systems (3) and (7) undergo a Hopf bifurcation at the equilibrium point $P_0 = (u_0, v_0)$ when $\tau = \tau_{j,0}$ ($j = 2, 3, 4$) under the conditions given in Theorem 5 and Theorem 6. In this section, we will give a complete and systematic way of analyzing the direction of the Hopf bifurcation, the stability and period of bifurcating periodic solutions by applying normal form theory and the center manifold theorem. To do this, we first translate systems (3) and (7) into the following systems (35) and (36), respectively, by the transitions $\widetilde{u} = u - u_0$, $\widetilde{v} = v - v_0$, and then linearize the systems around zero. For convenience, we still use u and v for $\widetilde{u}$ and $\widetilde{v}$, respectively, in the new systems, so we have

$$\begin{cases} \frac{du(t)}{\mathrm{d}t} = k_1 u(t) + k_2 u(t-\tau) + k_3 v(t) \\ \quad + k_4 v(t-\tau) + \text{h.o.t.}, \\ \frac{dv(t)}{\mathrm{d}t} = l_1 u(t) + l_2 u(t-\tau) + l_3 v(t) \\ \quad + l_4 v(t-\tau) + \text{h.o.t.}, \end{cases} \tag{35}$$

and

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = d_1 \frac{\partial^2 u(x,t)}{\partial x^2} + k_1 u(x,t) + k_2 u(x,t-\tau) \\ \quad + k_3 v(x,t) + k_4 v(x,t-\tau) + \text{h.o.t.}, \\ \frac{\partial v(x,t)}{\partial t} = d_2 \frac{\partial^2 v(x,t)}{\partial x^2} + l_1 u(x,t) + l_2 u(x,t-\tau) \\ \quad + l_3 v(x,t) + l_4 v(x,t-\tau) + \text{h.o.t.}, \end{cases} \tag{36}$$

where the term h.o.t. denotes the higher-order terms and k_i, l_i for $i = 1, 2, 3, 4$ are defined in (5).

The aim of this section is to derive an algorithm to analyze the direction of the Hopf bifurcation, the stability and period of bifurcating periodic solutions of a general 2×2 delayed reaction–diffusion system (36). This general algorithm can be also applied for a general 2×2 system of delay differential equations (35) by taking diffusion coefficients zero in (36).

In order to determine the direction and the stability of the Hopf bifurcation, we consider the following system which is equivalent to system (36) [see (15)]

$$\begin{pmatrix} \frac{du}{\mathrm{d}t} \\ \frac{dv}{\mathrm{d}t} \end{pmatrix} = B \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} + C \begin{pmatrix} u(t-\tau) \\ v(t-\tau) \end{pmatrix} + F, \tag{37}$$

where $u(t) = u(.,t)$, $v(t) = v(.,t)$ and

$$B = \begin{pmatrix} -d_1 \frac{n^2}{\ell^2} + k_1 & k_3 \\ l_1 & -d_2 \frac{n^2}{\ell^2} + l_3 \end{pmatrix}, \quad C = \begin{pmatrix} k_2 & k_4 \\ l_2 & l_4 \end{pmatrix}, \quad F = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}. \tag{38}$$

We normalize the delay with the scaling $t \to (t/\tau)$. Using $\tau = \tau_{j,0} + \mu$, (37) can be rewritten as follows:

$$\begin{pmatrix} \frac{du}{\mathrm{d}t} \\ \frac{dv}{\mathrm{d}t} \end{pmatrix} = \left(\tau_{j,0} + \mu\right) \left(B \begin{pmatrix} u(t) \\ v(t) \end{pmatrix} + C \begin{pmatrix} u(t-1) \\ v(t-1) \end{pmatrix} + F \right). \tag{39}$$

Let $U(t) = (u(t), v(t))^T$. Then (39) can be rewritten as

$$U'(t) = \left(\tau_{j,0} + \mu\right) \left(BU(t) + CU(t-1) + F\right).$$

For $\phi = (\phi_1, \phi_2)^T \in \mathcal{C} = C\left([-1,0], \mathbb{R}^2\right)$ we can define $L_\mu : \mathcal{C} \to \mathbb{R}^2$ as follows:

$$L_\mu(\phi) = \left(\tau_{j,0} + \mu\right) \left(B\phi(0) + C\phi(-1)\right). \tag{40}$$

Now, system (39) can be rewritten as a functional differential equation in $C\left([-1,0], \mathbb{R}^2\right)$ as

$$U'(t) = L_\mu(U_t) + h(\mu, U_t), \tag{41}$$

where $U_t(\theta) = U(t+\theta) = (u(t+\theta), v(t+\theta))^T$, $\theta \in [-1, 0]$ and $h : \mathbb{R} \times \mathcal{C} \to \mathbb{R}^2$ where

$$h(\phi, \mu) = \left(\tau_{j,0} + \mu\right) F(\phi) \tag{42}$$

in which

$$F = \begin{pmatrix} F_1(\phi(\theta)) \\ F_2(\phi(\theta)) \end{pmatrix}, \tag{43}$$

where

$$\begin{aligned} F_1(\phi(\theta)) = {} & m_{11}\phi_1^2(0) + m_{12}\phi_1(0)\phi_1(-1) \\ & + m_{13}\phi_1(0)\phi_2(0) + m_{14}\phi_1(0)\phi_2(-1) \\ & + m_{22}\phi_1^2(-1) + m_{23}\phi_1(-1)\phi_2(0) \\ & + m_{24}\phi_1(-1)\phi_2(-1) + m_{33}\phi_2^2(0) \\ & + m_{34}\phi_2(0)\phi_2(-1) + m_{44}\phi_2^2(-1) \\ & + \text{h.o.t.}, \\ F_2(\phi(\theta)) = {} & r_{11}\phi_1^2(0) + r_{12}\phi_1(0)\phi_1(-1) \\ & + r_{13}\phi_1(0)\phi_2(0) + r_{14}\phi_1(0)\phi_2(-1) \\ & + r_{22}\phi_1^2(-1) + r_{23}\phi_1(-1)\phi_2(0) \\ & + r_{24}\phi_1(-1)\phi_2(-1) + r_{33}\phi_2^2(0) \\ & + r_{34}\phi_2(0)\phi_2(-1) + r_{44}\phi_2^2(-1) + \text{h.o.t.}, \end{aligned}$$

and

$$\begin{aligned}
&m_{11}=\tfrac{1}{2}f_{uu}(P_0),\quad m_{12}=f_{uu_\tau}(P_0),\\
&m_{13}=f_{uv}(P_0),\quad m_{14}=f_{uv_\tau}(P_0),\\
&m_{22}=\tfrac{1}{2}f_{u_\tau u_\tau}(P_0),\quad m_{23}=f_{u_\tau v}(P_0),\\
&m_{24}=f_{u_\tau v_\tau}(P_0),\quad m_{33}=\tfrac{1}{2}f_{vv}(P_0),\\
&m_{34}=f_{vv_\tau}(P_0),\quad m_{44}=\tfrac{1}{2}f_{v_\tau v_\tau}(P_0),\\
&r_{11}=\tfrac{1}{2}g_{uu}(P_0),\quad r_{12}=g_{uu_\tau}(P_0),\\
&r_{13}=g_{uv}(P_0),\quad r_{14}=g_{uv_\tau}(P_0),\\
&r_{22}=\tfrac{1}{2}g_{u_\tau u_\tau}(P_0),\quad r_{23}=g_{u_\tau v}(P_0),\\
&r_{24}=g_{u_\tau v_\tau}(P_0),\quad r_{33}=\tfrac{1}{2}g_{vv}(P_0),\\
&r_{34}=g_{vv_\tau}(P_0),\quad r_{44}=\tfrac{1}{2}g_{v_\tau v_\tau}(P_0).
\end{aligned}\tag{44}$$

Notice that system (41) has two different unknown functions, namely, $U(t)$ and $U_t=U(t+\theta)$. By the Riesz Representation theorem, there exists 2×2 matrix $\eta(\theta,\mu)$, $\theta\in[-1,0]$ whose elements are of bounded variation functions such that

$$L_\mu(\phi)=\int_{-1}^{0}\mathrm{d}\eta(\mu,\theta)\phi(\theta)\ \text{for}\ \phi\in\mathcal{C}.\tag{45}$$

In fact, we can choose

$$\mathrm{d}\eta(\mu,\theta)=\left(\tau_{j,0}+\mu\right)\left(B\delta(\theta)+C\delta(\theta+1)\right)d\theta,\tag{46}$$

where $\delta(\theta)$ is the Dirac delta function here. Then, for $\phi\in\mathcal{C}$, we define

$$A(\mu)\phi=\begin{cases}\dfrac{d\phi(\theta)}{d\theta}, & \theta\in[-1,0)\\ \int_{-1}^{0}\mathrm{d}\eta(\mu,\theta)\phi(\theta)=L_\mu(\phi), & \theta=0\end{cases}\tag{47}$$

and

$$R(\mu)\phi=\begin{cases}0, & \theta\in[-1,0)\\ h(\phi,\mu), & \theta=0.\end{cases}\tag{48}$$

Then system (41) is equivalent to the following abstract differential equation which involves only one unknown function

$$\dot{U}_t=A(\mu)U_t+R(\mu)U_t,\tag{49}$$

where $U_t(\theta)=U(t+\theta)$ for $\theta\in[-1,0]$. In order to construct center manifold coordinates, we need an inner product. For $\psi\in\mathbf{C}[0,1]$ and $\phi\in\mathbf{C}[-1,0]$, it is defined by

$$<\psi,\phi>=\overline{\psi}(0)\cdot\phi(0)-\int_{\theta=-1}^{0}\int_{\xi=0}^{\theta}\overline{\psi}^T(\xi-\theta)\times\mathrm{d}\eta(\mu,\theta)\phi(\xi)d\xi.\tag{50}$$

Let $q(\theta)$ be an eigenvector of $A(0)$ corresponding to $\lambda(0)=i\omega_j\tau_{j,0}$ and $q^*(s)$ be an eigenvector of $A^*(0)$ corresponding to $\overline{\lambda(0)}=-i\omega_j\tau_{j,0}$ satisfying

$$\begin{cases}\langle q^*(s),q(\theta)\rangle=1,\\ \langle q^*(s),\overline{q}(\theta)\rangle=0,\end{cases}\tag{51}$$

$$\begin{cases}A(0)q(\theta)=i\omega_j\tau_{j,0}q(\theta),\\ A^*(0)q^*(s)=-i\omega_j\tau_{j,0}q^*(s),\end{cases}\tag{52}$$

where $A^*(\mu)$ is adjoint operator of $A(\mu)$ defined as

$$A^*(\mu)\phi=\begin{cases}-\dfrac{d\phi(s)}{ds}, & s\in(0,1]\\ \int_{-1}^{0}\mathrm{d}\eta^T(\mu,s)\phi(-s), & s=0.\end{cases}\tag{53}$$

First, we determine $q(\theta)$ from $A(0)q(\theta)=i\omega_j\tau_{j,0}q(\theta)$ in (52). It will be done in two cases as follows:

Case A1: If $\theta\in[-1,0)$, then, by (47),

$$A(0)q(\theta)=\frac{dq(\theta)}{d\theta}=i\omega_j\tau_{j,0}q(\theta)\tag{54}$$

so that we obtain that $q(\theta)=\begin{pmatrix}1\\c\end{pmatrix}e^{i\omega_j\tau_{j,0}\theta}$ from (54) where c will be determined in Case A2.

Case A2: When $\theta=0$, utilizing (47) we have

$$\begin{aligned}
&A(0)q(0)\\
&=\int_{-1}^{0}\mathrm{d}\eta(0,\theta)q(\theta)\\
&=\tau_{j,0}\left(B\int_{-1}^{0}\delta(\theta)q(\theta)d\theta+C\int_{-1}^{0}\delta(\theta+1)q(\theta)d\theta\right)\\
&=\tau_{j,0}\begin{pmatrix}k_1+c\left(k_3+k_4e^{-i\omega_j\tau_{j,0}}\right)+k_2e^{-i\omega_j\tau_{j,0}}-d_1\frac{n^2}{\ell^2}\\ l_1+l_2e^{-i\omega_j\tau_{j,0}}+c\left(l_3+l_4e^{-i\omega_j\tau_{j,0}}-d_2\frac{n^2}{\ell^2}\right)\end{pmatrix},
\end{aligned}$$

and we also have $A(0)q\,(0)=i\omega_j\tau_{j,0}q\,(0)$. Hence

$$\begin{pmatrix} k_1+c\left(k_3+k_4e^{-i\omega_j\tau_{j,0}}\right)+k_2e^{-i\omega_j\tau_{j,0}}-d_1\frac{n^2}{\ell^2} \\ l_1+l_2e^{-i\omega_j\tau_{j,0}}+c\left(l_3+l_4e^{-i\omega_j\tau_{j,0}}-d_2\frac{n^2}{\ell^2}\right) \end{pmatrix} = \begin{pmatrix} i\omega_j \\ i\omega_j c \end{pmatrix}.$$

From the calculations above one obtains c as follows:

$$c=\frac{i\omega_j-\left(k_1-d_1\frac{n^2}{\ell^2}+k_2e^{-i\omega_j\tau_{j,0}}\right)}{\left(k_3+k_4e^{-i\omega_j\tau_{j,0}}\right)}. \tag{55}$$

Second, we determine $q^*(s)$ from $A^*(0)q^*(s)=-i\omega_j\tau_{j,0}q^*(s)$ in (52). Once again, it will be done in two cases as follows:

Case B1: If $s\in(0,1]$, then, by (53), one has

$$A^*(0)q^*(s)=-\frac{dq^*(s)}{ds}=-i\omega_j\tau_{j,0}q^*(s)$$

so that one obtains that $q^*(s)=E\begin{pmatrix} c^* \\ 1\end{pmatrix}e^{i\omega_j\tau_{j,0}s}$. The constant c^* will be calculated below.

Case B2: When $s=0$, we have [see (53)]

$$\begin{aligned} &A^*(0)q^*(0)\\ &=\int_{-1}^{0}\mathrm{d}\eta^T(0,s)\phi(-s)\\ &=\tau_{j,0}\left(B^T\int_{-1}^{0}\delta(s)\,q^*(-s)ds+C^T\int_{-1}^{0}\delta(s+1)\,q^*(-s)ds\right)\\ &=E\tau_{j,0}\begin{pmatrix} l_1+l_2e^{i\omega_j\tau_{j,0}}+c^*\left(k_1+k_2e^{i\omega_j\tau_{j,0}}-d_1\frac{n^2}{\ell^2}\right)\\ l_3+l_4e^{i\omega_j\tau_{j,0}}-d_2\frac{n^2}{\ell^2}+c^*\left(k_3+k_4e^{i\omega_j\tau_{j,0}}\right)\end{pmatrix}, \end{aligned}$$

and we also have $A^*(0)q^*(0)=-i\omega_j\tau_{j,0}q^*(0)$. Hence

$$\begin{pmatrix} l_1+l_2e^{i\omega_j\tau_{j,0}}+c^*\left(k_1+k_2e^{i\omega_j\tau_{j,0}}-d_1\frac{n^2}{\ell^2}\right)\\ l_3+l_4e^{i\omega_j\tau_{j,0}}-d_2\frac{n^2}{\ell^2}+c^*\left(k_3+k_4e^{i\omega_j\tau_{j,0}}\right)\end{pmatrix} = \begin{pmatrix} -i\omega_j c^* \\ -i\omega_j\end{pmatrix}.$$

These calculations yield that c^* has the following form:

$$c^*=-\left(\frac{i\omega_j+l_3+l_4e^{i\omega_j\tau_{j,0}}-d_2\frac{n^2}{\ell^2}}{k_3+k_4e^{i\omega_j\tau_{j,0}}}\right). \tag{56}$$

These two eigenvectors must satisfy the properties given in (51). Since $\lambda(0)=i\omega_j\tau_{j,0}$ is a simple eigenvalue of $A(0)$, one can show that $\langle q^*(s),\overline{q}(\theta)\rangle=0$ (see [19] and [42]). Let us now choose E such that $\langle q^*(s),q(\theta)\rangle=1$. By the definition of inner product [see (50)], one has

$$\begin{aligned} &\langle q^*(s),q(\theta)\rangle\\ &=\overline{q^*}(0)\cdot q(0)-\int_{\theta=-1}^{0}\int_{\xi=0}^{\theta}\overline{q^*}^T(\xi-\theta)\mathrm{d}\eta(0,\theta)q(\xi)d\xi\\ &=\overline{E}\left(\overline{c^*}+c\right)-\overline{E}\left(\overline{c^*}\quad 1\right)\left(\int_{-1}^{0}\mathrm{d}\eta(0,\theta)e^{i\omega_j\tau_{j,0}\theta}\theta\right)\begin{pmatrix}1\\ c\end{pmatrix}. \end{aligned}$$

First, we calculate the integral on the right-hand side of the latter equation as follows:

$$\begin{aligned} &\left(\int_{-1}^{0}\mathrm{d}\eta(0,\theta)e^{i\omega_j\tau_{j,0}\theta}\theta\right)\\ &=\tau_{j,0}\left(\int_{-1}^{0}B\delta(\theta)\,e^{i\omega_j\tau_{j,0}\theta}\theta d\theta\right.\\ &\quad\left.+\int_{-1}^{0}C\delta(\theta+1)e^{i\omega_j\tau_{j,0}\theta}\theta d\theta\right)\\ &=-e^{-i\omega_j\tau_{j,0}}\tau_{j,0}\begin{pmatrix}k_2 & k_4\\ l_2 & l_4\end{pmatrix}. \end{aligned}$$

Second, we substitute the result into the equation above to determine $\overline{E}$, and we obtain $\overline{E}$ as follows:

$$\overline{E}=\left(\overline{c^*}+c+e^{-i\omega_j\tau_{j,0}}\tau_{j,0}\left(\overline{c^*}k_2+\overline{c^*}ck_4+l_2+cl_4\right)\right)^{-1} \tag{57}$$

since $\langle q^*(s),q(\theta)\rangle=1$.

Next, using a similar notation as in Hassard et al. [19], we compute the coordinates to describe the center manifold C_0 at $\mu=0$. Let U_t be the solution of equation (49) with $\mu=0$. Define

$$\begin{cases} z(t)=\langle q^*,U_t\rangle,\\ w(t,\theta)=U_t(\theta)-z(t)q(\theta)-\overline{z}(t)\overline{q}(\theta). \end{cases} \tag{58}$$

On the center manifold, $w(t,\theta)=w(z(t),\overline{z}(t),\theta)$ where

$$\begin{aligned} w(z,\overline{z},\theta)&=\sum\frac{1}{i!j!}w_{ij}(z)^i(\overline{z})^j\\ &=w_{20}(\theta)\frac{z^2}{2}+w_{11}(\theta)z\overline{z}+w_{02}(\theta)\frac{\overline{z}^2}{2}+\cdots, \end{aligned} \tag{59}$$

Indeed, z and $\overline{z}$ are local coordinates for the center manifold C_0 in $\mathcal{C} = C\left([-1,0], \mathbb{R}^2\right)$ in the direction of q^* and $\overline{q^*}$. For $U_t \in C_0$,

$$\dot{z}(t) = \left\langle q^*, \dot{U}_t \right\rangle = \left\langle q^*, AU_t + RU_t \right\rangle,$$

where '$\cdot$' means the derivative with respect to time here. Since $\mu = 0$,

$$\begin{aligned}\dot{z}(t) &= i\omega_j \tau_{j,0} \left\langle q^*, U_t \right\rangle + < q^*, h_0 > \\ &= i\omega_j \tau_{j,0} z(t) + g(z, \overline{z}),\end{aligned}$$

where

$$h_0 := h(z, \overline{z}, 0), \quad < q^*, h_0 >= \overline{q^*}(0) \cdot h_0.$$

We can represent $g(z, \overline{z})$ as

$$\begin{aligned}g(z, \overline{z}) &= \overline{q^*}(0) \cdot h_0 \\ &= g_{20}\frac{z^2}{2} + g_{11} z\overline{z} + g_{02}\frac{\overline{z}^2}{2} + g_{21}\frac{z^2\overline{z}}{2} + \cdots.\end{aligned} \tag{60}$$

Thus, at $\mu = 0$, system (49) is reduced to the following system in (z, w)-coordinates:

$$\begin{cases}\dot{z}(t) = i\omega_j \tau_{j,0} z + g(z, \overline{z}), \\ \dot{w}(t) = A(0)w + H(z, \overline{z}, \theta),\end{cases} \tag{61}$$

where

$$H(z, \overline{z}, \theta) = \begin{cases} - < q^*, h_0 > q(\theta) - \overline{< q^*, h_0 >}\overline{q}(\theta) & , \theta \in [-1, 0), \\ h_0 - < q^*, h_0 > q(0) - \overline{< q^*, h_0 >}\overline{q}(0) & , \theta = 0. \end{cases} \tag{62}$$

From (42) we have

$$h_0 = h(U_t, 0) = \begin{pmatrix} h_{0_1}(U_t) \\ h_{0_2}(U_t) \end{pmatrix}, \tag{63}$$

where

$$h_{0_1}(U_t) = \tau_{j,0} \begin{pmatrix} m_{11}u_t^2(0) + m_{12}u_t(0)u_t(-1) \\ + m_{13}u_t(0)v_t(0) + m_{14}u_t(0)v_t(-1) \\ + m_{22}u_t^2(-1) + m_{23}u_t(-1)v_t(0) \\ + m_{24}u_t(-1)v_t(-1) + m_{33}v_t^2(0) \\ + m_{34}v_t(0)v_t(-1) + m_{44}v_t^2(-1) + \text{h.o.t.}, \end{pmatrix}$$

$$h_{0_2}(U_t) = \tau_{j,0} \begin{pmatrix} r_{11}u_t^2(0) + r_{12}u_t(0)u_t(-1) \\ + r_{13}u_t(0)v_t(0) + r_{14}u_t(0)v_t(-1) \\ + r_{22}u_t^2(-1) + r_{23}u_t(-1)v_t(0) \\ + r_{24}u_t(-1)v_t(-1) + r_{33}v_t^2(0) \\ + r_{34}v_t(0)v_t(-1) + r_{44}v_t^2(-1) + \text{h.o.t.}, \end{pmatrix}$$

and m_{ij} and r_{ij} are defined in (44). From (58) and (59), we have

$$\begin{aligned}U_t(\theta) &= \begin{pmatrix} u_t(\theta) \\ v_t(\theta) \end{pmatrix} \\ &= zq(\theta) + \overline{zq}(\theta) + w_{20}(\theta)\frac{z^2}{2} + w_{11}(\theta)z\overline{z} \\ &\quad + w_{02}(\theta)\frac{(\overline{z})^2}{2} + \text{h.o.t.}.\end{aligned} \tag{64}$$

On the purpose of finding $h_0 = h(U_t, 0)$, we put $\theta = 0$ in (64) that leads to the following equations

$$\begin{aligned}u_t(0) &= z + \overline{z} + w_{20_1}(0)\frac{z^2}{2} + w_{11_1}(0)z\overline{z} \\ &\quad + w_{02_1}(0)\frac{(\overline{z})^2}{2} + \text{h.o.t.} \\ v_t(0) &= cz + \overline{cz} + w_{20_2}(0)\frac{z^2}{2} + w_{11_2}(0)z\overline{z} \\ &\quad + w_{02_2}(0)\frac{(\overline{z})^2}{2} + \text{h.o.t.}.\end{aligned}$$

Similarly, $u_t(-1)$ and $v_t(-1)$ can be obtained by plugging in $\theta = -1$ into (64), so we have the followings:

$$\begin{aligned}u_t(-1) &= ze^{-i\omega_j\tau_{j,0}} + \overline{z}e^{i\omega_j\tau_{j,0}} + w_{20_1}(-1)\frac{z^2}{2} \\ &\quad + w_{11_1}(-1)z\overline{z} + w_{02_1}(-1)\frac{(\overline{z})^2}{2} + \text{h.o.t.} \\ v_t(-1) &= cze^{-i\omega_j\tau_{j,0}} + \overline{cz}e^{i\omega_j\tau_{j,0}} + w_{20_2}(-1)\frac{z^2}{2} \\ &\quad + w_{11_2}(-1)z\overline{z} + w_{02_2}(-1)\frac{(\overline{z})^2}{2} + \text{h.o.t.}.\end{aligned}$$

Substituting now $u_t(0)$, $v_t(0)$, $u_t(-1)$ and $v_t(-1)$ into (63), one obtains $h_0(z, \overline{z})$ as follows:

$$\begin{aligned}h_0(z, \overline{z}) &= \begin{pmatrix} h_{0_1}(z, \overline{z}) \\ h_{0_2}(z, \overline{z}) \end{pmatrix} \\ &= \begin{pmatrix} K_{20}z^2 + K_{11}z\overline{z} + K_{02}(\overline{z})^2 + K_{21}z^2\overline{z} + \text{h.o.t.} \\ L_{20}z^2 + L_{11}z\overline{z} + L_{02}(\overline{z})^2 + L_{21}z^2\overline{z} + \text{h.o.t.} \end{pmatrix},\end{aligned} \tag{65}$$

where

$$K_{21}=\tau_{j,0}\begin{pmatrix}\left(2w_{11_1}(0)+w_{20_1}(0)\right)m_{11}\\+\begin{pmatrix}w_{11_1}(-1)+\frac{1}{2}w_{20_1}(-1)\\+e^{-i\omega_j\tau_{j,0}}w_{11_1}(0)\\+\frac{1}{2}e^{i\omega_j\tau_{j,0}}w_{20_1}(0)\end{pmatrix}m_{12}\\+\begin{pmatrix}cw_{11_1}(0)+\frac{\overline{c}}{2}w_{20_1}(0)\\+w_{11_2}(0)+\frac{1}{2}w_{20_2}(0)\end{pmatrix}m_{13}\\+\begin{pmatrix}e^{-i\omega_j\tau_{j,0}}cw_{11_1}(0)+w_{11_2}(-1)\\+\frac{e^{i\omega_j\tau_{j,0}}\overline{c}}{2}w_{20_1}(0)+\frac{1}{2}w_{20_2}(-1)\end{pmatrix}m_{14}\\+\begin{pmatrix}2e^{-i\omega_j\tau_{j,0}}w_{11_1}(-1)\\+e^{i\omega_j\tau_{j,0}}w_{20_1}(-1)\end{pmatrix}m_{22}\\+\begin{pmatrix}e^{-i\omega_j\tau_{j,0}}w_{11_2}(0)+cw_{11_1}(-1)\\+\frac{1}{2}e^{i\omega_j\tau_{j,0}}w_{20_2}(0)+\frac{\overline{c}}{2}w_{20_1}(-1)\end{pmatrix}m_{23}\\+\begin{pmatrix}e^{-i\omega_j\tau_{j,0}}cw_{11_1}(-1)\\+\frac{e^{i\omega_j\tau_{j,0}}\overline{c}}{2}w_{20_1}(-1)\\+e^{-i\omega_j\tau_{j,0}}w_{11_2}(-1)\\+\frac{e^{i\omega_j\tau_{j,0}}}{2}w_{20_2}(-1)\end{pmatrix}m_{24}\\+\left(2cw_{11_2}(0)+\overline{c}w_{20_2}(0)\right)m_{33}\\+\begin{pmatrix}e^{-i\omega_j\tau_{j,0}}cw_{11_2}(0)+cw_{11_2}(-1)\\+\frac{e^{i\omega_j\tau_{j,0}}\overline{c}}{2}w_{20_2}(0)+\frac{\overline{c}}{2}w_{20_2}(-1)\end{pmatrix}m_{34}\\+\begin{pmatrix}2e^{-i\omega_j\tau_{j,0}}cw_{11_2}(-1)\\+e^{i\omega_j\tau_{j,0}}\overline{c}w_{20_2}(-1)\end{pmatrix}m_{44}\end{pmatrix},$$

$$K_{02}=\tau_{j,0}\begin{pmatrix}m_{11}+e^{i\omega_j\tau_{j,0}}m_{12}\\+\overline{c}m_{13}+e^{i\omega_j\tau_{j,0}}\overline{c}m_{14}\\+e^{2i\omega_j\tau_{j,0}}m_{22}+e^{i\omega_j\tau_{j,0}}\overline{c}m_{23}\\+e^{2i\omega_j\tau_{j,0}}\overline{c}m_{24}+\overline{c}^2m_{33}\\+e^{i\omega_j\tau_{j,0}}\overline{c}^2m_{34}+e^{2i\omega_j\tau_{j,0}}\overline{c}^2m_{44}\end{pmatrix}=\overline{K_{20}},$$

$$K_{11}=\tau_{j,0}\begin{pmatrix}2m_{11}+\left(e^{-i\omega_j\tau_{j,0}}+e^{i\omega_j\tau_{j,0}}\right)m_{12}\\+(c+\overline{c})m_{13}+\left(e^{-i\omega_j\tau_{j,0}}c+e^{i\omega_j\tau_{j,0}}\overline{c}\right)m_{14}\\+2m_{22}+\left(e^{i\omega_j\tau_{j,0}}c+e^{-i\omega_j\tau_{j,0}}\overline{c}\right)m_{23}\\+(c+\overline{c})m_{24}+2c\overline{c}m_{33}\\+\left(e^{-i\omega_j\tau_{j,0}}+e^{i\omega_j\tau_{j,0}}\right)c\overline{c}m_{34}+2c\overline{c}m_{44}\end{pmatrix},$$

$$K_{20}=\tau_{j,0}\begin{pmatrix}m_{11}+e^{-i\omega_j\tau_{j,0}}m_{12}+cm_{13}\\+e^{-i\omega_j\tau_{j,0}}cm_{14}+e^{-2i\omega_j\tau_{j,0}}m_{22}\\+e^{-i\omega_j\tau_{j,0}}cm_{23}+e^{-2i\omega_j\tau_{j,0}}cm_{24}\\+c^2m_{33}+e^{-i\omega_j\tau_{j,0}}c^2m_{34}+e^{-2i\omega_j\tau_{j,0}}c^2m_{44}\end{pmatrix},$$

and

$$L_{21}=\tau_{j,0}\begin{pmatrix}\left(2w_{11_1}(0)+w_{20_1}(0)\right)m_{11}\\+\begin{pmatrix}w_{11_1}(-1)+\frac{1}{2}w_{20_1}(-1)\\+e^{-i\omega_j\tau_{j,0}}w_{11_1}(0)\\+\frac{1}{2}e^{i\omega_j\tau_{j,0}}w_{20_1}(0)\end{pmatrix}m_{12}\\+\begin{pmatrix}cw_{11_1}(0)+\frac{\overline{c}}{2}w_{20_1}(0)\\+w_{11_2}(0)+\frac{1}{2}w_{20_2}(0)\end{pmatrix}m_{13}\\+\begin{pmatrix}e^{-i\omega_j\tau_{j,0}}cw_{11_1}(0)+w_{11_2}(-1)\\+\frac{e^{i\omega_j\tau_{j,0}}\overline{c}}{2}w_{20_1}(0)+\frac{1}{2}w_{20_2}(-1)\end{pmatrix}m_{14}\\+\begin{pmatrix}2e^{-i\omega_j\tau_{j,0}}w_{11_1}(-1)\\+e^{i\omega_j\tau_{j,0}}w_{20_1}(-1)\end{pmatrix}m_{22}\\+\begin{pmatrix}e^{-i\omega_j\tau_{j,0}}w_{11_2}(0)+cw_{11_1}(-1)\\+\frac{1}{2}e^{i\omega_j\tau_{j,0}}w_{20_2}(0)+\frac{\overline{c}}{2}w_{20_1}(-1)\end{pmatrix}m_{23}\\+\begin{pmatrix}e^{-i\omega_j\tau_{j,0}}cw_{11_1}(-1)\\+\frac{e^{i\omega_j\tau_{j,0}}\overline{c}}{2}w_{20_1}(-1)\\+e^{-i\omega_j\tau_{j,0}}w_{11_2}(-1)\\+\frac{e^{i\omega_j\tau_{j,0}}}{2}w_{20_2}(-1)\end{pmatrix}m_{24}\\+\left(2cw_{11_2}(0)+\overline{c}w_{20_2}(0)\right)m_{33}\\+\begin{pmatrix}e^{-i\omega_j\tau_{j,0}}cw_{11_2}(0)+cw_{11_2}(-1)\\+\frac{e^{i\omega_j\tau_{j,0}}\overline{c}}{2}w_{20_2}(0)+\frac{\overline{c}}{2}w_{20_2}(-1)\end{pmatrix}m_{34}\\+\begin{pmatrix}2e^{-i\omega_j\tau_{j,0}}cw_{11_2}(-1)\\+e^{i\omega_j\tau_{j,0}}\overline{c}w_{20_2}(-1)\end{pmatrix}m_{44}\end{pmatrix},$$

$$L_{02}=\tau_{j,0}\begin{pmatrix}r_{11}+e^{i\omega_j\tau_{j,0}}r_{12}\\+\overline{c}r_{13}+e^{i\omega_j\tau_{j,0}}\overline{c}r_{14}\\+e^{2i\omega_j\tau_{j,0}}r_{22}+e^{i\omega_j\tau_{j,0}}\overline{c}r_{23}\\+e^{2i\omega_j\tau_{j,0}}\overline{c}r_{24}+\overline{c}^2r_{33}\\+e^{i\omega_j\tau_{j,0}}\overline{c}^2r_{34}+e^{2i\omega_j\tau_{j,0}}\overline{c}^2r_{44}\end{pmatrix}=\overline{L_{20}},$$

$$L_{11}=\tau_{j,0}\begin{pmatrix}2r_{11}+\left(e^{-i\omega_j\tau_{j,0}}+e^{i\omega_j\tau_{j,0}}\right)r_{12}\\+(c+\overline{c})r_{13}+\left(e^{-i\omega_j\tau_{j,0}}c+e^{i\omega_j\tau_{j,0}}\overline{c}\right)r_{14}\\+2r_{22}+\left(e^{i\omega_j\tau_{j,0}}c+e^{-i\omega_j\tau_{j,0}}\overline{c}\right)r_{23}\\+(c+\overline{c})r_{24}+2c\overline{c}r_{33}\\+\left(e^{-i\omega_j\tau_{j,0}}+e^{i\omega_j\tau_{j,0}}\right)c\overline{c}r_{34}+2c\overline{c}r_{44}\end{pmatrix},$$

$$L_{20}=\tau_{j,0}\begin{pmatrix}r_{11}+e^{-i\omega_j\tau_{j,0}}r_{12}+cr_{13}\\+e^{-i\omega_j\tau_{j,0}}cr_{14}+e^{-2i\omega_j\tau_{j,0}}r_{22}\\+e^{-i\omega_j\tau_{j,0}}cr_{23}+e^{-2i\omega_j\tau_{j,0}}cr_{24}\\+c^2r_{33}+e^{-i\omega_j\tau_{j,0}}c^2r_{34}+e^{-2i\omega_j\tau_{j,0}}c^2r_{44}\end{pmatrix}.$$

By (60), we have

$$\sum \frac{1}{i!j!} g_{ij} (z)^i (\overline{z})^j = \overline{E} \left(\overline{c^*} h_{0_1} + h_{0_2}\right). \tag{66}$$

In order to determine the stability and the direction of the Hopf bifurcation, we need to evaluate the coefficient $c_1(0)$ of the Poincaré normal form that is given by the following formula

$$c_1(0) = \frac{i}{2\omega_j \tau_{j,0}} \left(g_{20} g_{11} - 2 |g_{11}|^2 - \frac{1}{3} |g_{02}|^2 \right) + \frac{g_{21}}{2}, \tag{67}$$

where from (66)

$$\begin{cases} g_{20} = 2\overline{E}(\overline{c^*} K_{20} + L_{20}), \\ g_{11} = \overline{E}(\overline{c^*} K_{11} + L_{11}), \\ g_{02} = 2\overline{E}(\overline{c^*} K_{02} + L_{02}), \\ g_{21} = 2\overline{E}(\overline{c^*} K_{21} + L_{21}). \end{cases} \tag{68}$$

To calculate g_{21}, we first need to find w_{20} and w_{11}. We can express $H(z, \overline{z}, \theta)$ as $H(z, \overline{z}, \theta) = \sum \frac{1}{i!j!} H_{ij}(\theta)(z)^i (\overline{z})^j$. Using (59) and substituting this expressions into

$$\dot{w}(t) = A(0)w + H(z, \overline{z}, \theta)$$

yields the following equalities:

$$H_{20}(\theta) = (2i\omega_j \tau_{j,0} - A(0)) w_{20}(\theta), \tag{69}$$

$$H_{11}(\theta) = -A(0) w_{11}(\theta), \tag{70}$$

$$w_{02}(\theta) = \overline{w_{20}(\theta)}. \tag{71}$$

First, we find w_{20}. From (62) H_{20} equals to

$$H_{20}(\theta) = \begin{cases} -g_{20} q(\theta) - \overline{g}_{02} \overline{q}(\theta), & \theta \in [-1, 0) \\ 2 \begin{pmatrix} K_{20} \\ L_{20} \end{pmatrix} - g_{20} q(0) - \overline{g}_{02} \overline{q}(0), & \theta = 0. \end{cases} \tag{72}$$

We analyze the right-hand side of the latter equation with respect to the position of θ as follows.

Case C1: If $\theta \in [-1, 0)$, then using (47) we can rewrite (69) as follows:

$$H_{20}(\theta) = 2i\omega_j \tau_{j,0} w_{20}(\theta) - \frac{dw_{20}(\theta)}{d\theta}. \tag{73}$$

Combining (72) and (73) one obtains the following differential equation:

$$\frac{dw_{20}(\theta)}{d\theta} - 2i\omega_j \tau_{j,0} w_{20}(\theta) = g_{20} q(\theta) + \overline{g}_{02} \overline{q}(\theta).$$

Its solution is

$$w_{20}(\theta) = S e^{2i\omega_j \tau_{j,0} \theta} - \frac{1}{i\omega_j \tau_{j,0}} q(0) e^{i\omega_j \tau_{j,0} \theta} g_{20} - \frac{1}{3i\omega_j \tau_{j,0}} \overline{q}(0) e^{-i\omega_j \tau_{j,0} \theta} \overline{g}_{02}. \tag{74}$$

Case C2: If $\theta = 0$, then from (72) we get

$$H_{20}(0) = 2 \begin{pmatrix} K_{20} \\ L_{20} \end{pmatrix} - g_{20} q(0) - \overline{g}_{02} \overline{q}(0). \tag{75}$$

Both (69) and (75) give us

$$A(0) w_{20}(0) = 2i\omega_j \tau_{j,0} w_{20}(0) + g_{20} q(0) + \overline{g}_{02} \overline{q}(0) - 2 \begin{pmatrix} K_{20} \\ L_{20} \end{pmatrix}. \tag{76}$$

From Case C1 we have a formula for $w_{20}(\theta)$, namely (74). By substituting $w_{20}(0)$ into (76) we obtain

$$A(0) w_{20}(0) = -g_{20} q(0) + \frac{1}{3} \overline{g}_{02} \overline{q}(0) + 2i\omega_j \tau_{j,0} S - 2 \begin{pmatrix} K_{20} \\ L_{20} \end{pmatrix}. \tag{77}$$

On the other hand, from definitions of the operator $d\eta(0, \theta)$, $A(0)$ [see (46), (47)] and the fact that $A(0)q\ (0) = i\omega_j \tau_{j,0} q\ (0)$, we get

$$A(0) w_{20}(0) = -g_{20} q(0) + \frac{1}{3} \overline{g}_{02} \overline{q}(0) + \tau_{j,0} \left(B + C e^{-2i\omega_j \tau_{j,0}} \right) S, \tag{78}$$

so that (77) and (78) yield the following equality that will give us

$$S = \left(2i\omega_j \tau_{j,0} I - \tau_{j,0} \left(B + C e^{-2i\omega_j \tau_{j,0}} \right) \right)^{-1} \tag{79}$$

$$\begin{pmatrix} 2K_{20} \\ 2L_{20} \end{pmatrix}. \tag{80}$$

Similarly, we will find w_{11} so that, from (62), H_{11} will be equal to

$$H_{11}(\theta) = \begin{cases} -g_{11} q(\theta) - \overline{g}_{11} \overline{q}(\theta), & \theta \in [-1, 0) \\ \begin{pmatrix} K_{11} \\ L_{11} \end{pmatrix} - g_{11} q(0) - \overline{g}_{11} \overline{q}(0), & \theta = 0 \end{cases} \tag{81}$$

To do this we consider two cases as follows.

Case D1: If $\theta \in [-1, 0)$, then because of definition of the operator $A(\theta)$ (see(47)) the equality (70) becomes

$$H_{11}(\theta) = -\frac{dw_{11}(\theta)}{d\theta}. \tag{82}$$

Both (81) and (82) give us

$$\frac{dw_{11}(\theta)}{d\theta} = g_{11}q(\theta) + \overline{g}_{11}\overline{q}(\theta)$$

so that we have

$$w_{11}(\theta) = G + \frac{1}{i\omega_j\tau_{j,0}}q(0)e^{i\omega_j\tau_{j,0}\theta}g_{11} - \frac{1}{i\omega_j\tau_{j,0}}\overline{q}(0)e^{-i\omega_j\tau_{j,0}\theta}\overline{g}_{11}, \tag{83}$$

where G will be determined in Case D2.

Case D2: If $\theta = 0$, then from (81) we have

$$H_{11}(0) = \begin{pmatrix} K_{11} \\ L_{11} \end{pmatrix} - g_{11}q(0) - \overline{g}_{11}\overline{q}(0). \tag{84}$$

Both (70) and (84) give us

$$A(0)w_{11}(0) = g_{11}q(0) + \overline{g}_{11}\overline{q}(0) - \begin{pmatrix} K_{11} \\ L_{11} \end{pmatrix}. \tag{85}$$

On the other hand, from definitions of the operator $d\eta(0,\theta)$, $A(0)$ [see (46), (47)] and the fact that $A(0)q\,(0) = i\omega_j\tau_{j,0}q\,(0)$ we get

$$A(0)w_{11}(0) = g_{11}q(0) + \overline{g}_{11}\overline{q}(0) + \tau_{j,0}\,(B + C)\,G. \tag{86}$$

Equating the right-hand sides of the equations (85) and (86), one obtains the following identity

$$G = -\frac{1}{\tau_{j,0}}\,(B + C)^{-1}\begin{pmatrix} K_{11} \\ L_{11} \end{pmatrix}. \tag{87}$$

Now we can compute all the unknowns in the equation of $c_1(0)$ which is given (67). Moreover, using these coefficients we can evaluate the following values:

$$\begin{aligned} \mathrm{Re}\,\{c_1(0)\} &= \mathrm{Re}\left(\frac{g_{21}}{2}\right) - \frac{1}{2\omega_j\tau_{j,0}}\,\mathrm{Re}(g_{20})\,\mathrm{Im}(g_{11}) \\ &\quad - \frac{1}{2\omega_j\tau_{j,0}}\,\mathrm{Im}(g_{20})\,\mathrm{Re}(g_{11}), \\ \mathrm{Im}\,\{c_1(0)\} &= \mathrm{Im}\left(\frac{g_{21}}{2}\right) + \frac{1}{2\omega_j\tau_{j,0}}\,(\mathrm{Re}(g_{20})\,\mathrm{Re}(g_{11})) \\ &\quad - \frac{1}{2\omega_j\tau_{j,0}} \\ &\quad \times \left(\mathrm{Im}(g_{20})\,\mathrm{Im}(g_{11}) + 2\,|g_{11}|^2 + \frac{1}{3}\,|g_{02}|^2\right), \end{aligned}$$

$$\begin{aligned} \mu_2 &= -\frac{\mathrm{Re}\,\{c_1(0)\}}{\mathrm{Re}\,\{\lambda'(\tau_{j,0})\}}, \\ \beta_2 &= 2\,\mathrm{Re}\,\{c_1(0)\}, \\ T_2 &= -\frac{\mathrm{Im}\,\{c_1(0)\} + \mu_2\,\mathrm{Im}\,\{\lambda'(\tau_{j,0})\}}{\omega_j\tau_{j,0}}. \end{aligned} \tag{88}$$

Finally, using the quantities above, some properties of Hopf bifurcation can be determined, which are given by the following theorem.

Theorem 3 *If* $\mathrm{Re}\,\{c_1(0)\} \neq 0$, *then*

a. *the quantity* μ_2 *determines the direction of Hopf bifurcation: If* $\mu_2 > 0$, *then Hopf bifurcation is supercritical and the bifurcating periodic solutions exist for* $\tau > \tau_{j,0}$; *and if* $\mu_2 < 0$, *then Hopf bifurcation is subcritical and the bifurcating periodic solutions exist for* $\tau < \tau_{j,0}$.
b. β_2 *determines the stability of the bifurcating periodic solutions: If (H1) and (H2) hold, then the bifurcating periodic solutions are stable if* $\beta_2 < 0$; *unstable if* $\beta_2 < 0$.
c. *the period of the bifurcated periodic solution is* $\frac{2\pi}{\omega_j\tau_{j,0}}$ *as* $\tau = \tau_{j,0}$ *and* T_2 *determines the period of the bifurcating solution: The period increases if* $T_2 > 0$ *and decreases if* $T_2 < 0$.

5 Applications and examples

In this section, we give several examples to show the effectiveness of the algorithms derived in the preceding sections.

5.1 Example 1

Bi and Ruan [5] studied dynamics of a delayed ODE model of tumor–immune system interaction of the following form:

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = x\left(t\right)\left[\nu\left(x\left(t-\tau\right)\right) - \phi\left(x\left(t\right), y\left(t\right)\right)\right], \\ \frac{\mathrm{d}y}{\mathrm{d}t} = \beta\left(x\left(t-\tau\right)\right) y\left(t\right) - \mu\left(x\left(t\right)\right) y\left(t\right) \\ +\sigma q\left(x\left(t\right)\right) + \theta\left(t\right), \end{cases} \tag{89}$$

where $x(t)$ and $y(t)$ are the densities of tumor cells and immune effector cells at time $t > 0$, respectively, and τ is positive constant. The authors implement their results to the following example:

$$\begin{cases} \frac{\mathrm{d}x}{\mathrm{d}t} = x\left(t\right)\left[1.636\left(1 - 0.002x\left(t-\tau\right)\right) - y\left(t\right)\right], \\ \frac{\mathrm{d}y}{\mathrm{d}t} = \frac{1.131x(t-\tau)}{20.19+x(t-\tau)} y\left(t\right) \\ -\left(0.00311x + 0.3743\right) y\left(t\right) + 0.1181. \end{cases} \tag{90}$$

This system has two positive equilibrium points, namely, microequilibrium point (8, 18971, 1.6092) and macroequilibrium point (447.134, 0.17298). They showed that system (90) has a supercritical Hopf bifurcation at the microequilibrium point (8, 18971, 1.6092) when $\tau = \tau_0 = 1.27248$ and the bifurcating periodic solution is stable.

Since (89) is a special case of system (3) with $d_1 = d_2 = 0$ and it has a characteristic equation of the form in (18) (i.e., $H = 0$), one can apply the algorithms given in Sects. 3 and 4 for the existence of Hopf bifurcation and its properties, respectively. First, applying Lemma 2 one can determine the local stability of the equilibria. To show the existence of Hopf bifurcation of (90), one needs to check the conditions in Sect. 3 which are

(H1) $0.1002 > 0$, (H2) $0.0706 > 0$ and
$Y = -0.0062 < 0$.

Hence, by Theorem 1, system (90) undergoes a Hopf bifurcation at the positive equilibrium (8, 18971, 1.6092) when τ passes through $\tau_{2,0} = 1.2725$ associated with $\omega_2 = 0.2618$. Using the algorithm given in Sect. 4 one can also determine the properties of the Hopf bifurcation via calculating the values below:

$$\mu_2 = 5.3436 > 0, \quad \beta_2 = -0.3996 < 0, \\ T_2 = 0.7035 > 0, \tag{91}$$

which yield the same result (by Theorem 3) for the direction of the Hopf bifurcation in [5]. Summary, the results agree with those in [5].

5.2 Example 2

Zou and Wei [47] considered a diffusive predator–prey model with delay of the form:

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} = d_1 \frac{\partial^2 u(x,t)}{\partial x^2} \\ \quad +u\left(x,t\right)\left[r_1 - a_{11}u\left(x,t-\tau\right) - a_{12}v\left(x,t\right)\right], \\ x \in (0,\pi), \quad t > 0, \\ \frac{\partial v(x,t)}{\partial t} = d_2 \frac{\partial^2 v(x,t)}{\partial x^2} \\ \quad +v\left(x,t\right)\left[-r_2 + a_{21}u\left(x,t\right) - a_{22}v\left(x,t\right)\right], \\ x \in (0,\pi), \quad t > 0, \\ u_x\left(x,t\right) = v_x\left(x,t\right) = 0, x \in \{0,\pi\}, \quad t \geq 0, \\ u(x,t) = \phi(x,t) \geq 0, \quad v(x,t) = \psi(x,t) \geq 0, \\ x \in [0,\pi], \quad t \in [-\tau, 0], \end{cases} \tag{92}$$

where $u(x,t)$ and $v(x,t)$ stand for the prey and predator densities, respectively. r_i and a_{ij} $(i, j = 1, 2)$ are positive constants; $\tau > 0$ denotes the generation time of the prey species. d_1 and d_2 denote the diffusion coefficients of prey and predator species, respectively.

In [47], stability of the positive equilibrium point

$$P_0 = \left(\frac{r_1 a_{22} + r_2 a_{12}}{a_{11}a_{22} + a_{12}a_{21}}, \frac{r_1 a_{21} - r_2 a_{11}}{a_{11}a_{22} + a_{12}a_{21}}\right),$$

where $(r_1 a_{21} - r_2 a_{11}) > 0$, and the existence and direction analysis of Hopf bifurcation have been investigated for system (92) when τ varies. For the following coefficients:

$$d_1 = 0.2, \quad d_2 = 3, \quad n = 1, \quad r_1 = r_2 = 1, \\ a_{11} = a_{12} = 1, \quad a_{21} = 8, \quad a_{22} = 7, \tag{93}$$

their analytical results yield that the positive equilibrium $P_0 = (0.5333, 0.4667)$ is asymptotically stable when $\tau < 21.3827$; however, the positive equilibrium point lose its stability and a Hopf bifurcation occurs when τ passes through the critical value 21.3827, i.e., a family of inhomogeneous $(n \neq 0)$ periodic solutions are bifurcating from P_0. Also, they determine that since $\mu_2 < 0$ and $\beta_2 > 0$, the direction of the Hopf bifurcation is subcritical and the bifurcating periodic solutions are unstable. Once again, (92) is in the form of system (7) and its characteristic equation is similar to the equation in (18) (i.e., $H = 0$). Hence, we can use the algorithm given in Sect. 3 to show the existence of

Hopf bifurcation. We first analyze system (92) under the conditions (93) with zero diffusion, i.e.,

$$\begin{aligned} &d_1 = 0, \quad d_2 = 0, \quad r_1 = r_2 = 1, \\ &a_{11} = a_{12} = 1, \quad a_{21} = 8, \quad a_{22} = 7. \end{aligned} \tag{94}$$

When there is no diffusion, since

$$\begin{aligned} &\text{(H1) } 3.8000 > 0 \text{ and (H2) } 3.7333 > 0, \\ &X = 6.4044 > 0 \; Y = 0.9292 > 0, \\ &X^2 - 4Y = 37.3002 > 0, \end{aligned}$$

by Lemma 3-(v) the equilibrium point $P_0 = (0.5333, 0.4667)$ is asymptotically stable for all $\tau \geq 0$, hence there is no Hopf bifurcation. Figures 1 and 2 support the analytical results.

However, if the model involves diffusion, then with the coefficients given in (93), one obtains that

$$\text{(H1) } 7 > 0, \text{ (H2) } 6.5867 > 0 \text{ and } Y = -0.6440 < 0.$$

Hence, by Theorem 1 system (92) undergoes a Hopf bifurcation at the positive equilibrium $P_0 = (0.5333, 0.4667)$ when τ passes across $\tau_{2,0} = 21.3827$ which is associated with $\omega_2 = 0.1355$. Figures 3 and 4 support to our findings. In addition, by the algorithm given in Sect. 4, we also get

$$\begin{aligned} &\mu_2 = -5921.5 < 0, \quad \beta_2 = 2640.7 > 0, \\ &T_2 = 17{,}778 > 0. \end{aligned} \tag{95}$$

Using Theorem 3, we achieve the same result for the direction of Hopf bifurcation in [47] which is subcritical. This example and its numerical simulations (see Fig. 1, 2, 3, 4) underline one of the effects of diffusion on the dynamics. Figures 2 and 4 represent that diffusion changes the stability of the equilibrium point from stable to unstable when $\tau > \tau_{2,0} = 21.3827$ so that a Hopf bifurcation arises as τ passes across $\tau_{2,0} = 21.3827$. In other words, if there is no diffusion, the ODE model does not have a Hopf bifurcation; however, the PDE model involving diffusion has a Hopf bifurcation (Fig. 5).

5.3 Example 3

In this example, we consider the following reaction–diffusion model with time delay under the Neumann

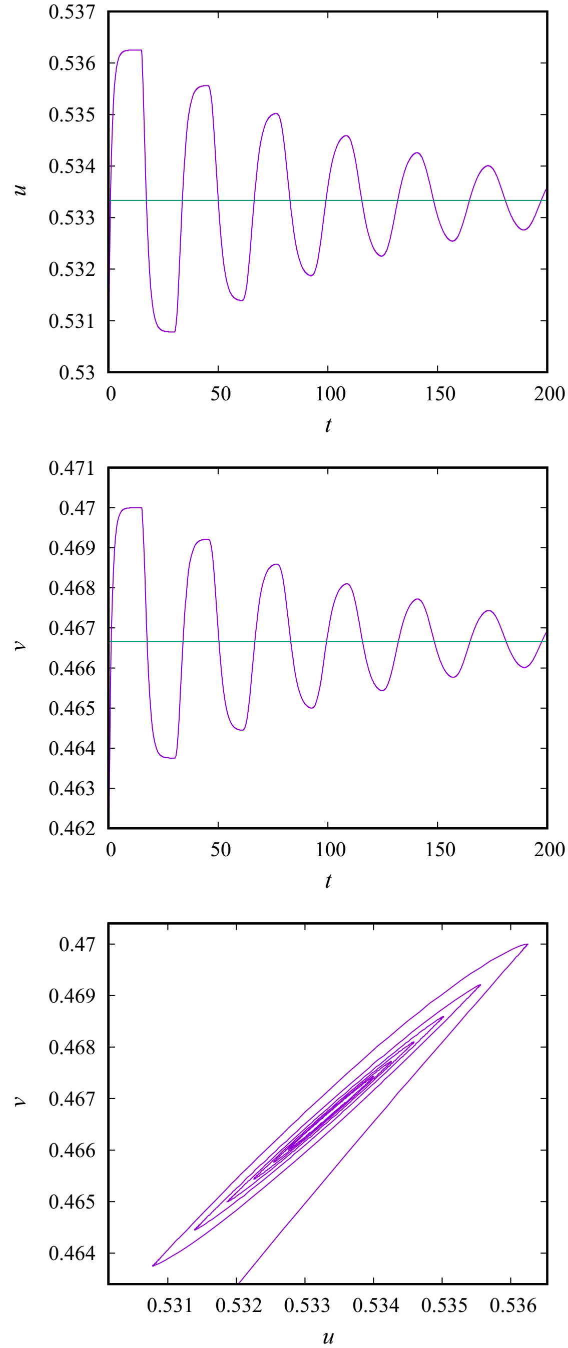

Fig. 1 Graphs of the solutions $u(t)$ and $v(t)$ and the phase portrait of system (92) with zero diffusion and the parameters in (94). For these simulations, we take initial condition as $u(0) = 0.5320$, $v(0) = 0.4630$, and $\tau = 15 < \tau_{2,0} = 21.3827$. Simulations show that the equilibrium point is asymptotically stable for $\tau < \tau_{2,0} = 21.3827$

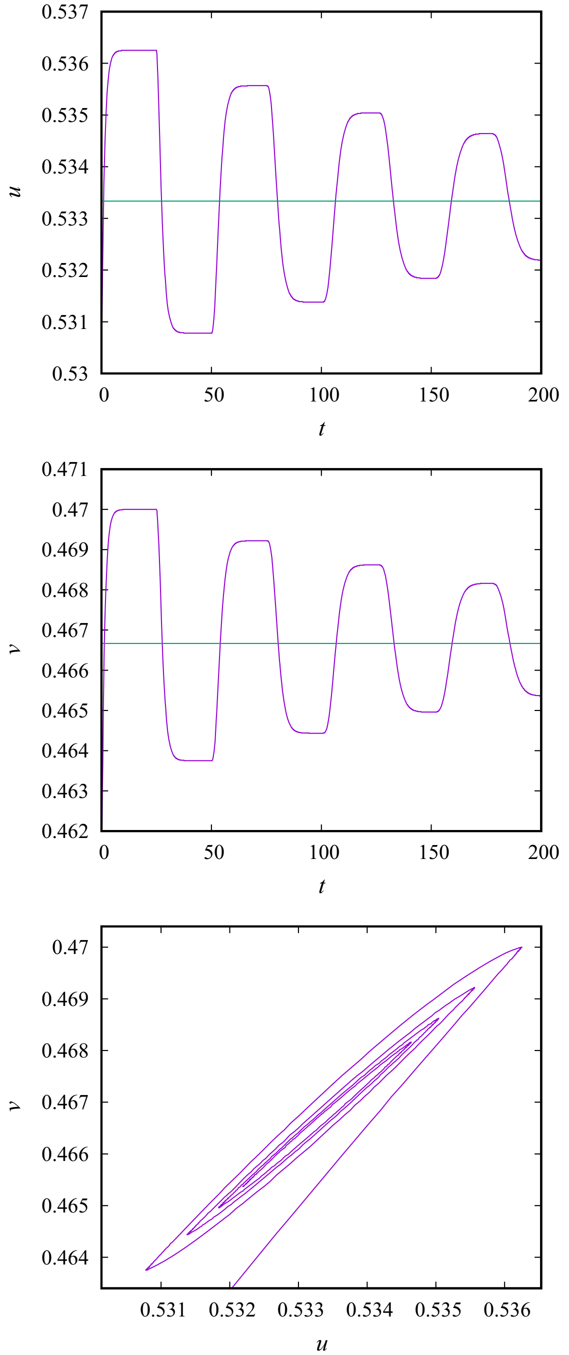

Fig. 2 Graphs of the solutions $u(t)$ and $v(t)$ and the phase portrait of system (92) with zero diffusion and the parameters in (94). For these simulations, we take initial condition as $u(0) = 0.5320$, $v(0) = 0.4630$, and $\tau = 25 > \tau_{2,0} = 21.3827$. Simulations show that the equilibrium point is still asymptotically stable when $\tau > \tau_{2,0} = 21.3827$

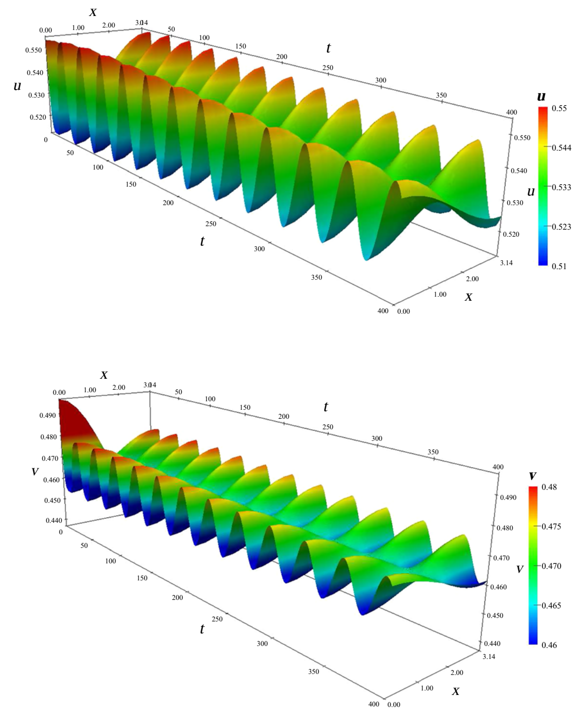

Fig. 3 Graphs of the solutions $u(x,t)$ (on the *top*) and $v(x,t)$ (in the *bottom*) of system (92) with nonzero diffusion and the parameters in (93). For these simulations, we take initial conditions as $u(x,0) = 0.5333 + 0.02cos(x)$, $v(x,0) = 0.4667 + 0.03cos(x)$, and $\tau = 15 < \tau_{2,0} = 21.3827$. Simulations show that the equilibrium point is asymptotically stable for $\tau < \tau_{2,0} = 21.3827$

boundary conditions studied by Merdan and Kayan [28]:

$$\begin{cases} \dfrac{\partial u(x,t)}{\partial t} = d_1 \dfrac{\partial^2 u(x,t)}{\partial x^2} \\ + a - u(x,t) - 4\dfrac{u(x,t)v(x,t-\tau)}{1+u^2(x,t)}, \\ x \in (0,\pi), \quad t > 0, \\ \dfrac{\partial v(x,t)}{\partial t} = d_2 \dfrac{\partial^2 v(x,t)}{\partial x^2} \\ + \sigma b \left(u(x,t) - \dfrac{u(x,t)v(x,t-\tau)}{1+u^2(x,t)} \right), \\ x \in (0,\pi), \quad t > 0, \\ u_x(x,t) = v_x(x,t) = 0, \quad x \in \{0,\pi\}, \quad t \geq 0, \\ u(x,t) = u^*(x,t), \quad v(x,t) = v^*(x,t), \\ x \in [0,\pi], \quad t \in [-\tau, 0], \end{cases} \tag{96}$$

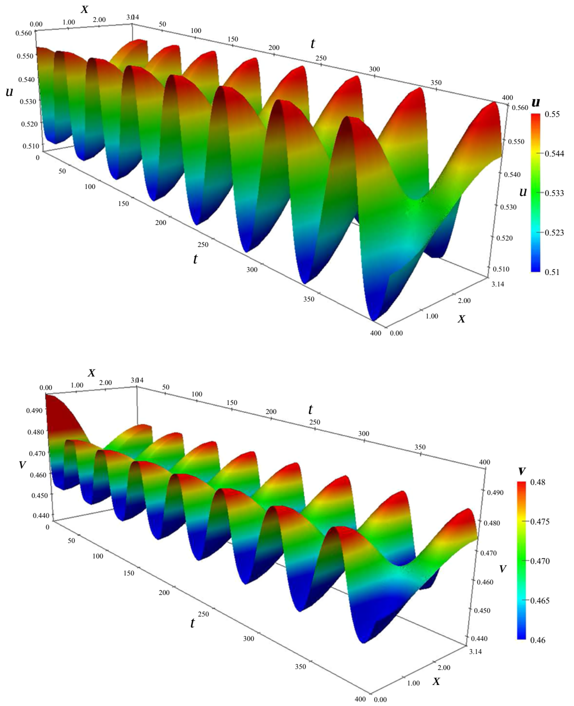

Fig. 4 Graphs of the solutions $u(x, t)$ (on the *top*) and $v(x, t)$ (in the *bottom*) of system (92) with nonzero diffusion and the parameters in (93). For these simulations, we choose initial conditions as $u(x, 0) = 0.5333 + 0.02cos(x)$, $v(x, 0) = 0.4667 + 0.03cos(x)$, and $\tau = 25 > \tau_{2,0} = 21.3827$. Simulations show that the equilibrium point is unstable when $\tau > \tau_{2,0} = 21.3827$

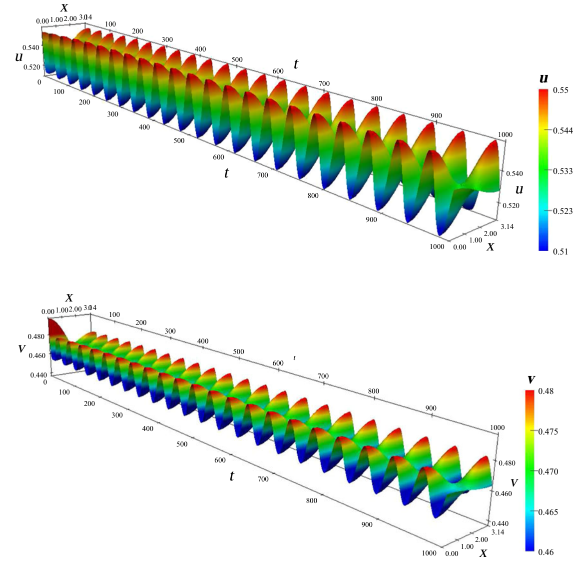

Fig. 5 Graphs of the solutions $u(x, t)$ (on the *top*) and $v(x, t)$ (in the *bottom*) of system (92) with nonzero diffusion and the parameters in (93). For these simulations, we take initial conditions as $u(x, 0) = 0.5333+0.02cos(x)$, $v(x, 0) = 0.4667+0.03cos(x)$, and $\tau = \tau_{2,0} = 21.3827$. These simulations show that there are periodic solutions for $\tau \in (\tau_{2,0} - \epsilon, \tau_{2,0}]$, where $\epsilon > 0$

where $u^*, v^* \in C^2(((0, \pi), [-\tau, 0])) \cap C(([0, \pi], [-\tau, 0]))$. When there is no time delay, system (96) reduces to the well-known Lengyel–Epstein reaction–diffusion model based on the chlorite-iodide-malonic acid chemical (CIMA) reaction (see [12,28] and the references therein for more details). In the model, $u(x, t)$ and $v(x, t)$ denote chemical concentrations of the activator iodide and the inhibitor chlorite, respectively. $a > 0$ and $b > 0$ are parameters related to the feed concentrations, $\sigma > 0$ is a rescaling parameter depending on the concentration of the starch. Here, the positive constants d_1 and d_2 are diffusion coefficients of the activator and the inhibitor, respectively, and τ is delay parameter.

In [28], the authors give a detail analysis of existence of Hopf bifurcation and its properties for system (96). In laboratory conditions, a sample of parameters is taken in the range $0 < a < 35$, $0 < b < 8$, $\sigma = 8$. When there is no diffusion, under the following values of parameters:

$$d_1 = 0, \quad d_2 = 0, \quad a = 15, \quad \sigma = 8 \text{ and } b = 1.2, \tag{97}$$

the characteristic equation of the equation is in the form of (18) since $H = 0$. The conditions are calculated as:

(H1) $0.6800 > 0$ and (H2) $14.4000 > 0$,

hence its equilibrium point $N_0 = (3, 10)$ is stable when $\tau = 0$. On the other hand, since $Y = -207.3600 < 0$, when $\tau = \tau_{2,0}$ there is a purely imaginary eigenvalue $\lambda\left(\tau_{2,0}\right) = i\omega_2$ where

$$\omega_2 = 4.0287 \text{ and } \tau_{2,0} = 0.0443.$$

Transversality condition is satisfied automatically as shown in Sect. 3. Now applying Theorem 1 one can show that system (96) with coefficients (97) undergoes a Hopf bifurcation at the equilibrium point N_0 as the delay parameter τ passes through $\tau_{2,0} = 0.0443$. Moreover, using algorithm given in Sect. 4, one can easily obtain the values to determine the properties of the Hopf bifurcation as follows:

$$\mu_2 = 0.000013488, \quad \beta_2 = -0.00024811, \\ T_2 = 0.0099, \ \text{Period} = 35.2195. \tag{98}$$

Using Theorem 3, we conclude that since $\mu_2 > 0$ and $\beta_2 < 0$, bifurcating spatially non-homogeneous periodic solutions (since $n = 1 \neq 0$) exist for $\tau > \tau_{2,0}$ and they are stable (i.e., supercritical). Also, because $T_2 > 0$, the period of the bifurcating solution increases.

To see the effects of diffusion, we choose second set of parameters as follows:

$$d_1 = 1, \quad d_2 = 13, \quad a = 15, \quad \sigma = 8 \text{ and } b = 1.2. \tag{99}$$

The characteristic equation of the PDE model is again in the form of equation (18) (i.e., $H = 0$). We first checked the conditions (H1) and (H2). The calculations yield that

(H1) $14.6800 > 0$ and (H2) $1.6800 > 0$,

so that $N_0 = (3, 10)$ is stable when $\tau = 0$. On the other hand, since $Y = -55.2384 < 0$, there is a purely imaginary eigenvalue $\lambda\left(\tau_{2,0}\right) = i\omega_2$ when $\tau = \tau_{2,0}$ where

$\omega_2 = 0.5831$ and $\tau_{2,0} = 0.8650$.

Thus, by Theorem 1, a Hopf bifurcation occurs at the equilibrium point N_0 as the delay parameter τ passes through $\tau_{2,0} = 0.8650$ for system (96) with coefficients (98). On the other hand, using Theorem 3, we conclude that periodic solutions exist for $\tau < \tau_{2,0}$. Moreover, they are unstable (i.e., subcritical) and the period of the bifurcating solution increases since $\mu_2 < 0$, $\beta_2 > 0$ and $T_2 > 0$ where

$$\mu_2 = -0.4117, \quad \beta_2 = 1.3782, \\ T_2 = 3.2607, \ \text{Period} = 12.4582. \tag{100}$$

By comparing (98) and (100), we observe that diffusion has effects on the dynamics of the model as follows: (i) diffusion changes the type of Hopf bifurcation from supercritical to subcritical, (ii) it changes the critical bifurcation value at which the Hopf bifurcation occurs, and (iii) it also changes the periods of the periodic solutions.

6 Conclusion

In this paper, we drive an algorithm which gives a methodical way for determining the existence and direction of Hopf bifurcation of a 2×2 system of reaction–diffusion equation with the Neumann boundary conditions incorporating delay. We determine the conditions on parameters of the system that Hopf bifurcation occurs as the delay parameter passes through a critical value. Once one determines the conditions at which equilibrium point of the system is stable when there is no delay, then one can decide whether Hopf bifurcation exists by checking the conditions (given in Tables 1, 2) on X and Y that depend on the coefficients of the characteristic equation associated with the linear part of the system including delay. These tables represent a quick and simple way for determining Hopf bifurcation of a system having the characteristic equation in a special form, and also give formulae for purely imaginary eigenvalues and the bifurcation value. Furthermore, one can determine the stability switches at an equilibrium point of the system by utilizing Table 2. Moreover, an algorithm to get the formulae that determine direction, period and stability of the periodic solution is generated using the normal form theory and the center manifold theorem.

We also observe that diffusion has effects on the dynamics of a model. Diffusion may change (i) the type of Hopf bifurcation from supercritical to subcritical or vice versa (ii) the critical bifurcation value at which the Hopf bifurcation occurs, and (iii) the periods of the periodic solutions (see Examples 5.2 and 5.3).

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