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Dirac systems with regular and singular transmission effects

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Abstract: In this paper, we investigate the spectral properties of singular eigenparameter dependent dissipative problems in Weyl's limit-circle case with finite transmission conditions. In particular, these transmission conditions are assumed to be regular and singular. To analyze these problems we construct suitable Hilbert spaces with special inner products and linear operators associated with these problems. Using the equivalence of the Lax–Phillips scattering function and Sz-Nagy–Foiaş characteristic functions we prove that all root vectors of these dissipative operators are complete in Hilbert spaces.

Key words: Dissipative operator, first-order system, transmission condition, scattering function, characteristic function

1. Introduction

As is known, Dirac systems are of the form

$$\begin{aligned} y_2' + p(x)y_1 + r(x)y_2 &= \lambda y_1, \\ -y_1' + r(x)y_1 + q(x)y_2 &= \lambda y_2, \end{aligned} \tag{1}$$

where λ is a complex parameter, and p, q, and r are real-valued and locally integrable functions on some interval $(a, b) \subseteq \mathbb{R}$. The system (1) plays a central role in relativistic quantum theory. In fact, the system (1) corresponds to Dirac's radial relativistic wave equation for a particle in a central field [12,14]. One of the important problems of the system (1) is to describe the solutions belonging to squarely integrable space on some singular intervals, that is, intervals in which at least one of the potentials p, q, and r increase boundedlessly. In 1910, Weyl showed with his extraordinary method that at least one of the linearly independent solutions of a singular second-order differential equation must belong to a squarely integrable space [18]. Moreover, two linearly independent solutions and combinations of them may belong to a squarely integrable space. These cases are known as limit-point and limit-circle cases, respectively. Weyl's method was adopted by Levitan and Sargsjan to the first-order (Dirac) system (1) [12]. Therefore, the behavior of the coefficients p, q, and r at singular point(s) describes the solutions belonging to a squarely integrable space (or not).

In some boundary value problems, eigenparameters occur at both differential equation (system) and boundary conditions. In this situation, the corresponding operator associated with the problem is unusually defined but with operator-theoretic formulation. This formulation is from Friedman [5]. It is better to note that a lot of authors have used this formulation to investigate regular and singular eigenparameter dependent selfadjoint (symmetric) and nonselfadjoint problems [3,6,8].

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An important class of nonselfadjoint operators is the class of dissipative operators. A well-known result is that all eigenvalues of dissipative operators belong to the closed upper half-plane. In the literature there are some methods to complete the spectral analysis of dissipative operators. Functional model theory from Sz.-Nagy and Foiaş [17] is one of the basic methods to study the spectral properties of a dissipative operator. This method can be used once the characteristic function of the corresponding contractive operator is established. On the other hand, there is an equivalence between the characteristic function of a contractive operator and *abstract* scattering function. In fact, Lax and Phillips established the abstract scattering theory to analyze the scattering problems of acoustic waves off compact obstacles [11]. Originally this theory was constructed for hyperbolic partial differential equations. Adamyan and Arov showed that the Lax–Phillips scattering function and Sz.-Nagy–Foiaş characteristic function can be handled as equivalent [1]. This equivalence has been used in many papers (for example, see [3,4,13]).

Recently, a new type of operator has been studied intensively called operators with transmission conditions. These transmission conditions occur between end points of disjoint intervals. It is better to note that operators with transmission conditions appear as a natural description of observed evolution phenomena of several real-world problems. Many physical, chemical, and biological phenomena involving thresholds; bursting rhythm models in medicine, pharmacokinetics, and frequency modulated systems; and mathematical models in economics exhibit transmission effects [10]. Therefore, the theory of differential operators with transmission conditions is a new and important branch of operator theory that has extensive physical, chemical, and realistic mathematical models.

In this paper, we investigate the spectral properties of two main first-order differential systems with finite regular and singular transmission points. Finally, showing the absence of the singular factor in the factorization of the characteristic function, we prove the completeness theorem.

2. First-order system with finite regular transmission conditions

We consider the system (1) on the multi-interval $I := \bigcup_{k=1}^{n+1} I_k$ in the following form:

$$\tau(y) := By' + P(x)y = \lambda y, \tag{2}$$

where $I_k = (\zeta_{k-1}, \zeta_k)$ and

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad P(x) = \begin{pmatrix} p(x) & r(x) \\ r(x) & q(x) \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

Basic assumptions on (2) and the intervals I_k are as follows:

(i) $-\infty < \zeta_0 < \zeta_1 < \cdots < \zeta_{n+1} \leq \infty$,

(ii) p, q, and r are real-valued and Lebesgue measurable functions on I_k, $k = \overline{1, n+1} := 1, 2, ..., n+1$,

(iii)

$$\int\limits_{I_m} \{|p(x)| + |r(x)| + |q(x)|\}\, dx < \infty, \ m = \overline{1, n},$$

and

$$\int\limits_{I_{n+1}} \{|p(x)| + |r(x)| + |q(x)|\}\, dx = \infty.$$

Let $L^2(I,\mathbb{C}^2)$ denote the Hilbert space consisting of all vector-valued functions $y=\binom{y_1}{y_2}$ in $\mathbb{C}^2$ satisfying $\int_I \left(|y_1|^2+|y_2|^2\right)dx<\infty$ with the usual inner product

$$(y,\chi)=\int\limits_I y^T\overline{\chi}dx,$$

where y^T denotes the transpose of the vector $y=\binom{y_1}{y_2}$.

Consider the set $D(I,\mathbb{C}^2)$ consisting of all vector-valued functions $y=\binom{y_1}{y_2}\in L^2(I,\mathbb{C}^2)$ in which y_1 and y_2 are locally integrable functions on all $I_k,\ k=\overline{1,n+1}$, and $\tau(y)\in L^2(I,\mathbb{C}^2)$.

For arbitrary two vector-valued functions $y,\chi\in D(I,\mathbb{C}^2)$ we have the following Green's formula:

$$(\tau(y),\chi)-(y,\tau(\chi))=\sum_{k=1}^{n+1}[y,\chi]_{\zeta_{k-1}+}^{\zeta_k-},$$

where $[y,\chi]_{\zeta_{k-1}+}^{\zeta_k-}=[y,\chi](\zeta_k-)-[y,\chi](\zeta_{k-1}-)$ and $[y,\chi](x)=y_2(x)\overline{\chi_1}(x)-y_1(x)\overline{\chi_2}(x),\ x\in I_k,\ k=\overline{1,n+1}$. Green's formula implies that at singular point ζ_{n+1} the value $[y,\chi](\zeta_{n+1}-)$ for arbitrary $y,\chi\in D(I,\mathbb{C}^2)$ exists and is finite.

We assume that Weyl's limit-circle case holds at singular point ζ_{n+1} for (2) [12], [16].

Consider the solutions

$$u(x)=\begin{cases}u_1(x),\ x\in I_1\\ u_2(x),\ x\in I_2\\ \vdots\\ u_{n+1}(x),\ x\in I_{n+1}\end{cases},\quad z(x)=\begin{cases}z_1(x),\ x\in I_1\\ z_2(x),\ x\in I_2\\ \vdots\\ z_{n+1}(x),\ x\in I_{n+1}\end{cases}$$

of the equation

$$\tau(y)=0,\ x\in I,$$

satisfying the conditions

$$\begin{cases}u_{k1}(\zeta_{k-1}+)=0, & u_{k2}(\zeta_{k-1}+)=1,\\ z_{k1}(\zeta_{k-1}+)=1, & z_{k2}(\zeta_{k-1}+)=0,\end{cases}$$

where

$$u_k(x)=\begin{pmatrix}u_{k1}(x)\\ u_{k2}(x)\end{pmatrix},\quad z_k(x)=\begin{pmatrix}z_{k1}(x)\\ z_{k2}(x)\end{pmatrix}$$

and $k=\overline{1,n+1}$.

Clearly one can infer from Green's formula that for two solutions $y(x,\lambda)$ and $\chi(x,\lambda)$ of (2) for the same value of λ the Wronskian of y and χ defined as $W[y,\chi]:=-[y,\overline{\chi}]=y_1\chi_2-y_2\chi_1$ does not depend on x

and depends only on λ on each $I_k,\ k=\overline{1,n+1}$. Moreover, they are linearly independent if and only if their Wronskian is nonzero.

Since $W[z_k,u_k]\equiv 1$ on each $I_k,\ k=\overline{1,n+1},\ \ z$ and u are linearly independent solutions of (2). Further they belong to $D(I,\mathbb{C}^2)$. This implies that for arbitrary $y\in D(I,\mathbb{C}^2)$ the values $[y,z](\zeta_{n+1}-)$ and $[y,u](\zeta_{n+1}-)$ exist and are finite.

Note that for $y,\chi\in D(I,\mathbb{C}^2)$, a direct calculation shows that

$$[y_k,\chi_k](x)=[y_k,u_k](x)[\overline{\chi}_k,z_k](x)-[y_k,z_k](x)[\overline{\chi}_k,u_k](x),\ x\in I_k. \tag{3}$$

In sections 3 and 4 we investigate the spectral properties of the following boundary value transmission problem (BVTP):

$$\tau(y)=\lambda y,\ y\in D(I,\mathbb{C}^2),\ x\in I, \tag{4}$$

$$(a_1y_{11}(\zeta_0+)-a_2y_{12}(\zeta_0+))-\lambda\left(a_1'y_{11}(\zeta_0+)-a_2'y_{12}(\zeta_0+)\right)=0, \tag{5}$$

$$y_{m1}(\zeta_m-)=b_my_{(m+1)1}(\zeta_m+), \tag{6}$$

$$y_{m2}(\zeta_m-)=b_m'y_{(m+1)2}(\zeta_m+), \tag{7}$$

$$[y_{n+1},u_{n+1}](\zeta_{n+1}-)-c[y_{n+1},z_{n+1}](\zeta_{n+1}-)=0, \tag{8}$$

where $m=\overline{1,n},\ \lambda$ and c are complex numbers with $\Im c>0,\ a_1,a_2,a_1',a_2',b_m,b_m'$ are real numbers with $b_mb_m'>0$ and

$$a:=\begin{vmatrix} a_1 & a_2\\ a_1' & a_2'\end{vmatrix}>0.$$

It should be noted that without transmission conditions the 1d-Hamiltonian system has been studied in [3].

3. Dissipative operator

Let $H=K\oplus\mathbb{C}$, where $K=\bigoplus_{k=1}^{n+1}K_k,\ K_k=L^2(I_k,\mathbb{C}^2)$, be the Hilbert space equipped with the following inner product

$$\langle\mathbf{v},\mathbf{w}\rangle_{\mathbf{H}}=\int\limits_I v^T(x)\overline{w(x)}d\mu(x)+\frac{1}{a}v_0^T\overline{w_0}$$

for

$$\mathbf{v}=\begin{bmatrix} v(x)\\ v_0\end{bmatrix}\in H,\quad \mathbf{w}=\begin{bmatrix} w(x)\\ w_0\end{bmatrix}\in H,$$

where

$$v(x) = \begin{cases} v_1(x), x \in I_1 \\ v_2(x), x \in I_2 \\ \vdots \\ v_{n+1}(x), x \in I_{n+1} \end{cases}, w(x) = \begin{cases} w_1(x), x \in I_1 \\ w_2(x), x \in I_2 \\ \vdots \\ w_{n+1}(x), x \in I_{n+1} \end{cases} \in H,$$

$$\mu(x) = \begin{cases} x, x \in I_1 \\ b_{(1)}x, x \in I_2 \\ \vdots \\ \prod_{m=1}^{n} b_{(m)}x, x \in I_{n+1} \end{cases},$$

$v_0, w_0 \in \mathbb{C},\ v_k = \binom{v_{k1}}{v_{k2}},\ w_k = \binom{w_{k1}}{w_{k2}},\ b_{(m)} := b_m b'_m > 0$. Clearly $v_0^T = v_0$, since v_0 is a complex number. However, this formulation will allow us to obtain the resolvent operator explicitly. It should be noted that such a representation has been given in [9].

Consider the set $Dom(T)$ in H consisting of all functions $\mathbf{v} = [\begin{smallmatrix} v(x) \\ v_0 \end{smallmatrix}]$ such that $v_{k1}, v_{k2},\ k = \overline{1, n+1}$, are locally absolutely continuous functions on all I_k satisfying $\tau(v) \in H,\ B_0[v] = 0,\ B'_0[v] = v_0,\ B_m[v] = 0, B'_m[v] = 0,\ m = \overline{1, n}$, and $B_{n+1}[v] = 0$, where $B_0[v] := a_1 v_{11}(\zeta_0+) - a_2 v_{12}(\zeta_0+),\ B'_0[v] := a'_1 v_{11}(\zeta_0+) - a'_2 v_{12}(\zeta_0+),\ B_m[v] := v_{m1}(\zeta_m-) - b_m v_{(m+1)1}(\zeta_m+),\ B'_m[v] := v_{m2}(\zeta_m-) - b'_m v_{(m+1)2}(\zeta_m+),\ B_{m+1}[v] := [y_{n+1}, z_{n+1}](\zeta_{n+1}-) - c[y_{n+1}, u_{n+1}](\zeta_{n+1}-)$. Then we define the operator T on $Dom(T)$ as

$$T\mathbf{v} = \tau_1(\mathbf{v}),$$

where

$$\tau_1(\mathbf{v}) = \begin{bmatrix} \tau(v) \\ B_0[v] \end{bmatrix}.$$

Thus the BVTP (4)–(8) can be handled in H as

$$T\mathbf{v} = \lambda \mathbf{v},\ \mathbf{v} \in Dom(T),\ x \in I.$$

Let

$$\varphi(x,\lambda) = \begin{cases} \varphi_1(x,\lambda),\ x \in I_1 \\ \varphi_2(x,\lambda),\ x \in I_2 \\ \vdots \\ \varphi_{n+1}(x,\lambda),\ x \in I_{n+1} \end{cases}, \quad \theta(x,\lambda) = \begin{cases} \theta_1(x,\lambda),\ x \in I_1 \\ \theta_2(x,\lambda),\ x \in I_2 \\ \vdots \\ \theta_{n+1}(x,\lambda),\ x \in I_{n+1} \end{cases}$$

be the solutions of (4) satisfying the conditions

$$\varphi_{11}(\zeta_0+,\lambda) = \alpha_2 - \lambda\alpha'_2, \qquad \varphi_{12}(\zeta_0+,\lambda) = \alpha_1 - \lambda\alpha'_1,$$
$$\varphi_{(m+1)1}(\zeta_m+,\lambda) = b_m^{-1}\varphi_{m1}(\zeta_m-,\lambda), \quad \varphi_{(m+1)2}(\zeta_m+,\lambda) = b_m'^{-1}\varphi_{m2}(\zeta_m-,\lambda),$$

$m=\overline{1,n},$ and

$$[\theta_{n+1},u_{n+1}](\zeta_{n+1}-)=c, \qquad [\theta_{n+1},z_{n+1}](\zeta_{n+1}-)=1,$$
$$\theta_{s1}(\zeta_s-,\lambda)=b_s\theta_{(s+1)1}(\zeta_s+,\lambda), \quad \theta_{s2}(\zeta_s-,\lambda)=b_s'\theta_{(s+1)2}(\zeta_s+,\lambda),$$

$s=\overline{n,1}.$

Define the function $\Delta_k(\lambda)=W[\theta_k,\varphi_k](x),\ x\in I_k,\ k=\overline{1,n+1}.$ Constant of Wronskians on each I_k and transmission conditions give the following equalities:

$$\Delta(\lambda):=\Delta_1(\lambda)=b_{(1)}\Delta_2(\lambda)=\ldots=\prod_{m=1}^{n}b_{(m)}\Delta_{n+1}(\lambda). \tag{9}$$

It is clear that the zeros of Δ coincide with the eigenvalues of T (see [15]) and Δ is an entire function.

We shall recall that a linear operator L with dense domain $D(L)$ acting on some Hilbert space H is called dissipative if for all $y\in D(L)$ the inequality

$$\Im\langle Ly,y\rangle_H\geq 0$$

holds and is called maximal dissipative if it does not have any proper dissipative extension [7].

Theorem 3.1 *T is dissipative in H.*

Proof For

$$\mathbf{v}=\begin{bmatrix} v(x)\\ v_0\end{bmatrix}=\begin{bmatrix} v(x)\\ B_0'[v]\end{bmatrix}\in Dom(T)$$

a direct calculation gives

$$\begin{aligned}\langle T\mathbf{v},\mathbf{v}\rangle_H-\langle\mathbf{v},T\mathbf{v}\rangle_H &= [v,v]_{\zeta_0+}^{\zeta_1-}+b_{(1)}[v,v]_{\zeta_1+}^{\zeta_2-}+\cdots+\prod_{m=1}^{n}b_{(m)}[v,v]_{\zeta_n+}^{\zeta_{n+1}-}\\ &\quad +a^{-1}\left(B_0[v]\overline{B_0'[v]}-B_0'[v]\overline{B_0[v]}\right).\end{aligned} \tag{10}$$

One can obtain the equation

$$B_0[v]\overline{B_0'[v]}-B_0'[v]\overline{B_0[v]}=a[v,v](\zeta_0+). \tag{11}$$

The conditions $B_m[v]=0,\ B_m'[v]=0,\ m=\overline{1,n},$ give

$$[v,v](\zeta_1-)=b_{(1)}[v,v](\zeta_1+),\ldots,[v,v](\zeta_n-)=b_{(n)}[v,v](\zeta_n+). \tag{12}$$

Using (3) and the condition $B_{n+1}[v]=0,$ the following equality is obtained:

$$[v,v](\zeta_{n+1}-)=2i\Im c\left|[v,z](\zeta_{n+1}-)\right|^2. \tag{13}$$

Substituting (11)–(13) in (10) we have

$$\Im\langle T\mathbf{v},\mathbf{v}\rangle_H=\prod_{m=1}^{n}b_{(m)}\Im c\left|[v,z](\zeta_{n+1}-)\right|^2$$

and this completes the proof. □

Corollary 3.2 *All eigenvalues of T lie in the closed upper half-plane.*

Theorem 3.3 *T is maximal dissipative in H.*

Proof To prove that T is maximal dissipative in H, it is sufficient to show that the equality

$$(T-\lambda I)Dom(T)=H,\ \Im\lambda<0, \tag{14}$$

is true (see [7]).

Let

$$\mathbf{v}=\begin{bmatrix} v(x) \\ v_0 \end{bmatrix}\in Dom(T),\ \mathbf{g}=\begin{bmatrix} g(x) \\ g_0 \end{bmatrix}\in H.$$

Then the equation

$$(T-\lambda I)\mathbf{v}=\mathbf{g},\ x\in I,\ \Im\lambda<0,$$

is equivalent to the nonhomogeneous differential equation

$$\tau[v]-\lambda v=g(x),\ x\in I, \tag{15}$$

subject to the conditions

$$\begin{aligned}
&B_0[v]-\lambda B_0'[v]=g_0,\\
&B_m[v]=0,\\
&B_m'[v]=0,\ m=\overline{1,n},\\
&B_{n+1}[v]=0.
\end{aligned}$$

We may represent the general solution of the homogeneous differential equation as

$$v(x,\lambda)=\begin{cases} s_1\varphi_1(x,\lambda)+l_1\theta_1(x,\lambda),\ x\in I_1\\ s_2\varphi_2(x,\lambda)+l_2\theta_2(x,\lambda),\ x\in I_2\\ \vdots\\ s_{n+1}\varphi_{n+1}(x,\lambda)+l_{n+1}\theta_{n+1}(x,\lambda),\ x\in I_{n+1}\end{cases}$$

in which all s_k and l_k, $k=\overline{1,n+1}$, are arbitrary constants. Using the method of variation of constants and the conditions $B_m[v]=0,\ B'_m[v]=0,\ m=\overline{1,n},$ (see [2]) $v(x,\lambda)$ is found as

$$v(x,\lambda)=\begin{cases} -\varphi_1(x,\lambda)\left(\frac{1}{\Delta_1(\lambda)}\int\limits_x^{\zeta_1}\theta_1^T g_1 dt+\frac{1}{\Delta_2(\lambda)}\int\limits_{I_2}\theta_2^T g_2 dt+\cdots+\frac{1}{\Delta_{n+1}(\lambda)}\int\limits_{I_{n+1}}\theta_{n+1}^T g_{n+1} dt\right) \\ -\theta_1(x,\lambda)\left(\frac{1}{\Delta_1(\lambda)}\int\limits_{\zeta_0}^{x}\theta_2^T g_1 dt-\frac{g_0}{\Delta_1(\lambda)}\right),\ x\in I_1 \\ -\varphi_2(x,\lambda)\left(\frac{1}{\Delta_2(\lambda)}\int\limits_x^{\zeta_2}\theta_2^T g_2 dt+\frac{1}{\Delta_3(\lambda)}\int\limits_{I_3}\theta_3^T g_3 dt+\cdots+\frac{1}{\Delta_{n+1}(\lambda)}\int\limits_{I_{n+1}}\theta_{n+1}^T g_{n+1} dt\right) \\ -\theta_2(x,\lambda)\left(\frac{1}{\Delta_1(\lambda)}\int\limits_{I_1}\varphi_1^T g_1 dt+\frac{1}{\Delta_2(\lambda)}\int\limits_{\zeta_1}^{x}\varphi_2^T g_2 dt-\frac{g_0}{\Delta_1(\lambda)}\right),\ x\in I_2 \\ \vdots \\ -\frac{\varphi_{n+1}(x,\lambda)}{\Delta_{n+1}(\lambda)}\int\limits_x^{\zeta_{n+1}}\theta_{n+1}^T g_{n+1} dt-\theta_{n+1}(x,\lambda)\left(\frac{1}{\Delta_1(\lambda)}\int\limits_{I_1}\varphi_1^T g_1 dt+\cdots+\right. \\ \left.\frac{1}{\Delta_n(\lambda)}\int\limits_{I_n}\varphi_n^T g_n dt+\frac{1}{\Delta_{n+1}(\lambda)}\int\limits_{\zeta_n}^{x}\varphi_{n+1}^T g_{n+1} dt-\frac{g_0}{\Delta_1(\lambda)}\right),\ x\in I_{n+1} \end{cases}, \tag{16}$$

Using the equalities given in (9) and setting the kernel

$$G(x,t,\lambda)=\begin{cases} -\frac{1}{\Delta(\lambda)}\theta(x,\lambda)\varphi^T(x,\lambda);\zeta_0\le t\le x\le\zeta_{n+1};x,t\ne\zeta_m,m=\overline{1,n} \\ -\frac{1}{\Delta(\lambda)}\varphi(x,\lambda)\theta^T(x,\lambda);\zeta_0\le x\le t\le\zeta_{n+1};x,t\ne\zeta_m,m=\overline{1,n} \end{cases}$$

we reduce (16) to

$$v(x,\lambda)=\int\limits_I G(x,t,\lambda)\overline{g}(t)d\mu(t)+\frac{1}{\Delta(\lambda)}\theta(x,\lambda)g_0. \tag{17}$$

On the other hand, the equality

$$B'_0[G^T(x,t,\lambda)]=\frac{a}{\Delta(\lambda)}\theta^T(x,\lambda) \tag{18}$$

holds.

Let

$$\mathcal{G}_{x,t,\lambda}=\begin{bmatrix} G^T(x,t,\lambda) \\ B'_0[G^T] \end{bmatrix}.$$

Then from (17) and (18) one gets that

$$\mathbf{v}=\left\langle\mathcal{G}_{x,t,\lambda},\overline{g}(t)\right\rangle_H.$$

Therefore the equality

$$K\mathbf{g}:=\left\langle\mathcal{G}_{x,t,\lambda},\overline{g}(t)\right\rangle_H=\mathbf{v}$$

holds for arbitrary $\mathbf{g}\in H$. Therefore (14) is satisfied and the theorem is proved. □

4. Scattering function

We shall add the incoming and outgoing channels to the Hilbert space H and form the main Hilbert space as follows:

$$\mathcal{H} = L^2(\mathbb{R}_-) \oplus H \oplus L^2(\mathbb{R}_+),$$

where $\mathbb{R}_- := (-\infty, 0]$ and $\mathbb{R}_+ := [0, \infty)$.

Let $Dom(S)$ be the set in $\mathcal{H}$ consisting of all vectors

$$V = (\chi_-, \mathbf{v}, \chi_+),$$

where $\chi_\mp \in W_2^1(\mathbb{R}_\mp)$ (W_2^1 is the Sobolev space), $\mathbf{v} = \left[\begin{smallmatrix} v(x) \\ v_0 \end{smallmatrix}\right] \in H$, $v_0 = B_0'[v]$, satisfying

$$B_l[v] = 0, \tag{19}$$

$$B_l'[v] = 0, \tag{20}$$

$$[v, u](\zeta_{n+1}-) - c[v, z](\zeta_{n+1}-) = \left(\prod_{m=1}^{n} b_{(m)}\right)^{-1/2} \sigma\chi_-(0), \tag{21}$$

$$[v, u](\zeta_{n+1}-) - \overline{c}[v, z](\zeta_{n+1}-) = \left(\prod_{m=1}^{n} b_{(m)}\right)^{-1/2} \sigma\chi_+(0), \tag{22}$$

where $l = \overline{1, n}$, $\sigma^2 := 2\Im c$, $\sigma > 0$. We define the operator S on $Dom(S)$ as

$$SV = \widetilde{S}V,$$

where

$$\widetilde{S}V = \widetilde{S}(\chi_-, \mathbf{v}, \chi_+) = \left(i\frac{d\chi_-}{dr}, \tau_1[\mathbf{v}], i\frac{d\chi_+}{ds}\right).$$

Theorem 4.1 *S is selfadjoint in $\mathcal{H}$.*

Proof Let $V = (\chi_-, \mathbf{v}, \chi_+), W = (\psi_-, \mathbf{w}, \psi_+) \in Dom(S)$. Then with the help of the conditions (19)–(22) we get that

$$\begin{aligned}
&(SV, W)_{\mathcal{H}} - (V, SW)_{\mathcal{H}} = [v, w]_{\zeta_0+}^{\zeta_1-} + b_{(1)}[v, w]_{\zeta_1+}^{\zeta_2-} + \ldots + \prod_{m=1}^{n} b_{(m)}[v, w]_{\zeta_n+}^{\zeta_{n+1}-} \\
&+a^{-1}\left(B_0[v]\overline{B_0'[w]} - B_0'[v]\overline{B_0[w]}\right) + i\chi_-(0)\overline{\psi}_-(0) - i\chi_+(0)\overline{\psi}_+(0) \\
&= \prod_{m=1}^{n} b_{(m)}[v, w](\zeta_{n+1}-) + i\chi_-(0)\overline{\psi}_-(0) - i\chi_+(0)\overline{\psi}_+(0) = 0.
\end{aligned}$$

This implies that $Dom(S) \subseteq Dom(S^*)$, where $Dom(S^*)$ is the domain of the adjoint operator S^* of S.

Let us consider the vector $V=(\chi_-,0,\chi_+)\in Dom(S)$ such that $\chi_\mp(0)=0$ and arbitrary vector $W=(\psi_-,\mathbf{w},\psi_+)\in Dom(S^*)$. Then we obtain that

$$\begin{aligned}(SV,W)_{\mathcal{H}} &= \left\langle \left(i\tfrac{d\chi_-}{dr},0,i\tfrac{d\chi_+}{ds}\right),(\psi_-,\mathbf{w},\psi_+)\right\rangle_{\mathcal{H}} \\ &= \left\langle (\chi_-,0,\chi_+),\left(i\tfrac{d\psi_-}{dr},\mathbf{w}^*,i\tfrac{d\psi_+}{ds}\right)\right\rangle_{\mathcal{H}},\end{aligned}$$

where $\psi_\mp\in W_2^1(\mathbb{R}_\mp)$, $\mathbf{w}^*=\left[\begin{smallmatrix} w^*(x)\\ w_0^*\end{smallmatrix}\right]\in Dom(T)$. Considering

$$B_l[w]=0,\ B_l'[w]=0,\ l=\overline{1,n}, \tag{23}$$

we have for arbitrary $V\in Dom(S)$ that $(SV,W)_{\mathcal{H}}=(V,SW)_{\mathcal{H}}$. Therefore using (21) and (22) we have

$$\begin{aligned}&\left(\prod_{m=1}^{n}b_{(m)}\right)^{1/2}\left\{\chi_-(0)\left[\left(\sigma+\tfrac{ic}{\sigma}\right)[\overline{w},z](\zeta_{n+1}-)-\tfrac{i}{\sigma}[\overline{w},u](\zeta_{n+1}-)\right]-\right.\\ &\left.-\chi_+(0)\left[\tfrac{ic}{\sigma}[\overline{w},z](\zeta_{n+1}-)-\tfrac{i}{\sigma}[\overline{w},u](\zeta_{n+1}-)\right]\right\}=i\chi_+(0)\overline{\psi}_+(0)-i\chi_-(0)\overline{\psi}_-(0).\end{aligned} \tag{24}$$

The coefficients of $\chi_-(0)$ and $\chi_+(0)$ in (24) give

$$[w,u](\zeta_{n+1}-)-c[w,z](\zeta_{n+1}-)=\left(\prod_{m=1}^{n}b_{(m)}\right)^{-1/2}\sigma\psi_-(0), \tag{25}$$

and

$$[w,u](\zeta_{n+1}-)-\overline{c}[w,z](\zeta_{n+1}-)=\left(\prod_{m=1}^{n}b_{(m)}\right)^{-1/2}\sigma\psi_+(0). \tag{26}$$

(23), (25), and (26) show that $Dom(S^*)\subseteq Dom(S)$ and this completes the proof. □

Consider the mappings

$$\begin{aligned} P^H:\ &\mathcal{H} \to H, & P^{\mathcal{H}}:\ & H \to \mathcal{H},\\ &(\chi_-,\mathbf{v},\chi_+)\to \mathbf{v}, & &\mathbf{v}\to (0,\mathbf{v},0).\end{aligned}$$

It is known that $U(t)=\exp(iSt)$, $t\in(-\infty,\infty)$, is an unitary group. Using this unitary group and mappings P^H and $P^{\mathcal{H}}$ we can construct a strongly continuous semigroup of completely nonunitary contractions on H as [13]

$$Z(t)=P^HU(t)P^{\mathcal{H}},\ t\in[0,\infty).$$

S is called the selfadjoint dilation of the generator A of $Z(t)$ [17], which is defined by

$$A=\lim_{t\to 0^+}\frac{Z(t)-I}{it}.$$

Note that A is maximal dissipative in H [13,17].

Theorem 4.2 *S is selfadjoint dilation of T.*

Proof Let us consider the equality

$$(S-\lambda I)^{-1} P^{\mathcal{H}}\mathbf{v} = W := (\psi_-, \mathbf{w}, \psi_+),$$

where $\mathbf{v} \in H,\ W \in Dom(S)$, and $\Im\lambda < 0$. Then we have

$$\begin{aligned} &\tau_1[w] - \lambda w = v, \\ &\psi_-(r) = \psi_-(0)\exp(-i\lambda r), \\ &\psi_+(s) = \psi_+(0)\exp(-i\lambda s). \end{aligned}$$

Since $\psi_- \in L^2(\mathbb{R}_-)$ and T is dissipative, one can write

$$(S-\lambda I)^{-1} P^{\mathcal{H}}\mathbf{v} = \left(0, (T-\lambda I)^{-1}\mathbf{v}, \left(\prod_{m=1}^{n} b_{(m)}\right)^{1/2} \sigma^{-1}\left([w,u](\zeta_{n+1}-) - c[w,z](\zeta_{n+1}-)\right)\exp(-i\lambda s)\right).$$

Therefore we have

$$P^{H}(S-\lambda I)^{-1} P^{\mathcal{H}}\mathbf{v} = (T-\lambda I)^{-1}\mathbf{v}. \tag{27}$$

On the other side we get for $\Im\lambda < 0$ that

$$\begin{aligned} P^{H}(S-\lambda I)^{-1} P^{\mathcal{H}} &= -iP^{H}\int_0^\infty U(t)\exp(-i\lambda t)dtP^{\mathcal{H}} = -i\int_0^\infty Z(t)\exp(-i\lambda t)dt \\ &= (A-\lambda I)^{-1} \end{aligned} \tag{28}$$

Hence (27) and (28) complete the proof. □

Let us consider the subspaces $D_- = (L^2(\mathbb{R}_-), 0, 0)$ and $D_+ = (0, 0, L^2(\mathbb{R}_+))$ of $\mathcal{H}$.

Lemma 4.3 *The subspaces D_- and D_+ with the unitary group $U(t)$, $t \in (-\infty, \infty)$, have the following properties:*

(i) $U(t)D_- \subset D_-,\ t \le 0;\ U(t)D_+ \subset D_+,\ t \ge 0,$

(ii) $\bigcap_{t\le 0} U(t)D_- = \bigcap_{t\ge 0} U(t)D_+ = \{0\},$

(iii) $\overline{\bigcup_{t\ge 0} U(t)D_-} = \overline{\bigcup_{t\le 0} U(t)D_+} = \mathcal{H},$

(iv) $D_- \perp D_+.$

Proof Let $V = (0, 0, \chi_+) \in D_+$. Then for $\Im\lambda < 0$ we get that

$$(S-\lambda I)^{-1} V = \left(0, 0, -i\exp(-i\lambda x)\int_0^x \exp(i\lambda t)\chi_+(t)dt\right) \in D_+.$$

Hence for $W \perp D_+$ and $\Im\lambda < 0$ we have

$$0 = \left\langle (S-\lambda I)^{-1} V, W\right\rangle_{\mathcal{H}} = -i\int_0^\infty \exp(-i\lambda t)\langle U(t)V, W\rangle_{\mathcal{H}}\, dt$$

and therefore $\langle U(t)V,W\rangle_{\mathcal{H}}=0,\ t\geq 0$. This implies for $t\geq 0$ that $U(t)D_+\subset D_+$. This proves the property (i) for D_+. A similar proof can be done for D_-.

Now consider the semigroup of isometries $U_+(t)=P_{L^2_+}U(t)P_{\mathcal{H}},\ t\geq 0,$ where

$$\begin{array}{llllllll} P_{L^2_+}: & \mathcal{H} & \rightarrow & L^2(\mathbb{R}_+), & P_{\mathcal{H}}: & L^2(\mathbb{R}_+) & \rightarrow & \mathcal{H}, \\ & (\chi_-,\mathbf{v},\chi_+) & \rightarrow & \chi_+, & & \chi_+ & \rightarrow & (0,0,\chi_+)\,. \end{array}$$

The generator A_+ of $U_+(t)$ is

$$A_+\chi=P_{L^2_+}SP_{\mathcal{H}}\chi=P_{L^2_+}S\left(0,0,i\frac{d\chi}{ds}\right)=i\frac{d\chi}{ds},$$

where $\chi\in W^1_2(\mathbb{R}_+)$ and $\chi(0)=0$. It is known that the generator of the one-sided shift, say $\widetilde{U}_+(t)$, in $L^2(\mathbb{R}_+)$ is the differential operator id/ds with the boundary condition $\chi(0)=0$. Since a semigroup is uniquely determined by its generator, we have $U_+(t)=\widetilde{U}_+(t)$. Therefore

$$\bigcap_{t\geq 0}U_+D_+=\left(0,0,\bigcap_{t\geq 0}\widetilde{U}_+(t)L^2(\mathbb{R}_+)\right)=\{0\}\,.$$

This proves (ii) for D_+. For D_-, a similar proof can be given.

Let

$$\mathcal{H}_-=\overline{\bigcup_{t\geq 0}U(t)D_-},\quad \mathcal{H}_+=\overline{\bigcup_{t\leq 0}U(t)D_+}.$$

It is better to recall that a linear operator L with domain $Dom(L)$ acting in a Hilbert space H is called completely nonselfadjoint if there is no invariant subspace $M\subseteq Dom(L)$, $M\neq\{0\}$, on which the restriction of L on M is selfadjoint. Our assertion is that the nonselfadjoint (dissipative) operator T is completely nonselfadjoint in H. In fact, if T_1 the restriction of T on a subspace H_1 of H is a selfadjoint part, then for $\mathbf{v}\in Dom(T_1)\cap H_1$, one obtains that

$$0=\langle T_1\mathbf{v},\mathbf{v}\rangle_H-\langle\mathbf{v},T_1\mathbf{v}\rangle_H=2i\prod_{m=1}^{n}b_{(m)}\Im c\left|[v,z](\zeta_{n+1}-)\right|^2,$$

and $[v,z](\zeta_{n+1}-)=0,\ x\in I_{n+1}$. This implies that $[v,u](\zeta_{n+1}-)=0,\ x\in I_{n+1}$, and $v_{n+1}\equiv 0,\ x\in I_{n+1}$. Transmission conditions $B_l[v]=0,\ B'_l[v]=0,\ l=\overline{1,n}$, give that all $v_l\equiv 0$ and consequently

$$\mathbf{v}=\left[\begin{array}{c} v(x) \\ v_0 \end{array}\right]\equiv 0,\ x\in I.$$

Using the expansion theorem in eigenvectors of the selfadjoint operator T_1 we have $H_1=\{0\}$. This proves the assertion. A consequence of this assertion is that

$$\mathcal{H}_-+\mathcal{H}_+=\mathcal{H}. \tag{29}$$

Otherwise, there would be a nontrivial subspace $\mathcal{H} \ominus (\mathcal{H}_- + \mathcal{H}_+)$ that would be invariant relative to the group $U(t)$ and the restriction of $U(t)$ to $\mathcal{H} \ominus (\mathcal{H}_- + \mathcal{H}_+)$ would be unitary and therefore the restriction of T on $\mathcal{H} \ominus (\mathcal{H}_- + \mathcal{H}_+)$ would be selfadjoint.

Consider the solution

$$\eta(x,\lambda) = \begin{cases} \eta_1(x,\lambda), \ x \in I_1 \\ \eta_2(x,\lambda), \ x \in I_2 \\ \vdots \\ \eta_{n+1}(x,\lambda), \ x \in I_{n+1} \end{cases}$$

of equation (4) satisfying

$$\eta_{11}(\zeta_0+,\lambda) = \tfrac{a_2'}{a}, \qquad \eta_{12}(\zeta_0+,\lambda) = \tfrac{a_1'}{a},$$
$$\eta_{(m+1)1}(\zeta_0+,\lambda) = b^{-1}\eta_{m1}(\zeta_m-,\lambda), \quad \eta_{(m+1)2}(\zeta_0+,\lambda) = b'^{-1}\eta_{m2}(\zeta_m-,\lambda),$$

where $m = \overline{1,n}$, and

$$\eta_k(x,\lambda) = \begin{pmatrix} \eta_{k1}(x,\lambda) \\ \eta_{k2}(x,\lambda) \end{pmatrix}, \ k = \overline{1,n+1}.$$

Let

$$\Psi_- = \left(\exp(-i\lambda r), \left(\prod_{m=1}^{n} b_{(m)} \right)^{-1/2} \sigma \frac{\tau(\lambda)}{(\alpha(\lambda)+c)[\eta,z](\zeta_{n+1}-))} U, \overline{\Theta}(\lambda)\exp(-i\lambda s) \right) \tag{30}$$

and

$$\Psi_+ = \left(\Theta(\lambda)\exp(-i\lambda r), \left(\prod_{m=1}^{n} b_{(m)} \right)^{-1/2} \sigma \frac{\tau(\lambda)}{(\alpha(\lambda)+\overline{c})[\eta,z](\zeta_{n+1}-))} U, \exp(-i\lambda s) \right), \tag{31}$$

where

$$U = \begin{bmatrix} \varphi(x,\lambda) \\ a \end{bmatrix}, \quad \tau(\lambda) = -\tfrac{[\eta,z](\zeta_{n+1}-)}{[\varphi,z](\zeta_{n+1}-)}, \quad \alpha(\lambda) = -\tfrac{[\varphi,u](\zeta_{n+1}-)}{[\varphi,z](\zeta_{n+1}-)}, \tag{32}$$

and

$$\Theta(\lambda) = \frac{\alpha(\lambda)+c}{\alpha(\lambda)+\overline{c}}. \tag{33}$$

Note that the vectors Ψ_- and Ψ_+ do not belong to $\mathcal{H}$ for real λ but they satisfy the equation $S\Psi = \lambda\Psi$ and corresponding boundary-transmission conditions for S. For $V = (\chi_-, \mathbf{v}, \chi_+)$ we define the Fourier transformations as follows:

$$\mathfrak{F}_- : V \to \mathfrak{F}_- V = \frac{1}{\sqrt{2\pi}} \langle V, \Psi_- \rangle_{\mathcal{H}} := \widetilde{V}_-(\lambda)$$

and

$$\mathfrak{F}_+ : V \to \mathfrak{F}_+ V = \frac{1}{\sqrt{2\pi}} \langle V, \Psi_+ \rangle_{\mathcal{H}} := \widetilde{V}_+(\lambda),$$

where χ_-, v, and χ_+ are smooth, compactly supported functions.

Let $V=(\chi_-,0,0)\in D_-$. Then we get that

$$\widetilde{V}_-(\lambda)=\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{0}\chi_-(t)\exp(i\lambda t)dt\in H_-^2,$$

where H_-^2 (H_+^2) is the Hardy class in $L^2(\mathbb{R})$ consisting of all functions analytically extendible to the lower (upper) half-plane. Let $\widetilde{\mathcal{H}}_-$ be a dense set in $\mathcal{H}_-$ consisting of all vectors V such that χ_- is compactly supported in D_- and $V\in\widetilde{\mathcal{H}}_-$ if $V=U(\mathcal{T})V_0$, $V_0=(\chi_-,0,0)$, $\chi_-\in C_0^\infty(\mathbb{R})$, where $\mathcal{T}=\mathcal{T}_V$ is a nonnegative number. Then for $V,W\in\mathcal{H}_-$ we get that $U(-\mathcal{T})V,U(-\mathcal{T})W\in D_-$ and their first components are in $C_0^\infty(\mathbb{R}_-)$, where $\mathcal{T}>\mathcal{T}_V$, $\mathcal{T}>\mathcal{T}_W$. Therefore

$$\begin{aligned}\langle V,W\rangle_{\mathcal{H}}&=\langle U(-\mathcal{T})V,U(-\mathcal{T})W\rangle_{\mathcal{H}}=\langle\mathfrak{F}_-U(-\mathcal{T})V,\mathfrak{F}_-U(-\mathcal{T})W\rangle_{\mathcal{H}}\\&=\langle\exp(-i\lambda\mathcal{T})U(-\mathcal{T})V,\exp(-i\lambda\mathcal{T})U(-\mathcal{T})W\rangle_{\mathcal{H}}=\langle\mathfrak{F}_-V,\mathfrak{F}_-W\rangle_{\mathcal{H}}.\end{aligned}\tag{34}$$

Therefore from (34) we have Parseval equality for the whole $\mathcal{H}_-$. Moreover, the inversion formula

$$V=\frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{\infty}\widetilde{V}_-(\lambda)\Psi_-d\lambda$$

follows from the Parseval equality if all integrals are taken as limits in the mean of the intervals. Consequently we have

$$\mathfrak{F}_-H_-=\overline{\bigcup_{t\geq 0}\mathfrak{F}_-U(t)D_-}=\overline{\bigcup_{t\geq 0}\exp(-i\lambda t)H_-^2}=L^2(\mathbb{R}).$$

A similar argument can be given for $\mathcal{H}_+$. Hence we get that $\mathcal{H}_-$ and $\mathcal{H}_+$ are isometrically identical with $L^2(\mathbb{R})$. This result with (29) implies that $\mathcal{H}_-=\mathcal{H}_+=\mathcal{H}$. Therefore (iii) is proved.

Finally the inner product in $\mathcal{H}$ implies that D_- is orthogonal to D_+. □

Remark 4.4 *(i) $\alpha(\lambda)$ defined in (32) is a meromorphic function in $\mathbb{C}$ with a countable number of poles on $\mathbb{R}$. For all $\lambda\in\mathbb{C}$ except the real poles of $\alpha(\lambda)$, $\overline{\alpha(\lambda)}=\alpha(\overline{\lambda})$ and for all $\Im\lambda\neq 0$, $\Im\lambda\Im\alpha(\lambda)<0$,*

(ii) the transformations $\mathfrak{F}_-$ and $\mathfrak{F}_+$ are the incoming and outgoing spectral representations for $U(t)$, respectively. Moreover, $U(t)$ is transformed into $\exp(i\lambda t)$,

(iii) it is clear from (33) that for $\lambda\in\mathbb{R}$, $|\Theta(\lambda)|=1$. Hence (30), (31), and (33) imply for $\lambda\in\mathbb{R}$ that

$$\Psi_-=\overline{\Theta}(\lambda)\Psi_+$$

and

$$\mathfrak{F}_-\varPsi_-=\varTheta(\lambda)\mathfrak{F}_-\varPsi_+.$$

According to the Lax–Phillips scattering theory, the scattering function is the coefficient by which the $\mathfrak{F}_+$ representation must be multiplied for getting the $\mathfrak{F}_-$ representation. Therefore using Remark 4.4 we have the following theorem.

Theorem 4.5 $\overline{\Theta}(\lambda)$ *is the scattering function of* $U(t)$*.*

Unitary transformation $\mathfrak{F}_-$ allows us to obtain that

$$\mathcal{H} = D_- \oplus H \oplus D_- \to L^2(\mathbb{R}) = H_-^2 \oplus H \oplus \Theta(\lambda)H_+^2.$$

Therefore we have

$$H = H_+^2 \ominus \Theta(\lambda)H_+^2.$$

Since the operator $U(t)V$ is unitary equivalent under the transformation $\mathfrak{F}_-$ to $\exp(i\lambda t)\widetilde{V}(\lambda)$, it can be concluded that $\widetilde{Z}(t)z = P[\exp(i\lambda t)z(\lambda)]$, $t \geq 0$, where P is the orthogonal projection from H_+^2 onto H, is a semigroup of operators. Therefore the generator of $\widetilde{Z}(t)$

$$\widetilde{A} = \lim_{t\to 0^+} \frac{\widetilde{Z}(t) - I}{it}$$

is a maximal dissipative operator on H. $\widetilde{A}$ is called the model operator [17] and therefore $\Theta(\lambda)$ is the characteristic function. Since the characteristic functions of unitary equivalent dissipative operators coincide with each other, we have the following.

Theorem 4.6 $\Theta(\lambda)$ *is the characteristic function of* T*.*

Lemma 4.7 *The characteristic function* $\Theta(\lambda)$ *is a Blaschke product except for a single point in the upper half-plane.*

Proof Since $\Theta(\lambda)$ is an inner function in the upper half-plane, it has the following form:

$$\Theta(\lambda) = B(\lambda)\exp(i\lambda t),$$

where $B(\lambda)$ is a Blaschke product and $t \geq 0$. Therefore we have

$$|\Theta(\lambda)| \leq \exp(-\Im\lambda.t), \ \Im\lambda \geq 0. \tag{35}$$

Moreover, from (33) one obtains that

$$\alpha(\lambda) = \frac{-\bar{c}\Theta(\lambda) + c}{\Theta(\lambda) - 1}.$$

Using (35) we get for $\lambda = is$ that

$$\lim_{s\to\infty} \alpha(is) = c_0.$$

Therefore t is zero except for a single point c_0. □

Using Lemma 4.7 and all the obtained results in sections 2–4 we introduce the following theorem.

Theorem 4.8 *Let* ζ_m, $m = \overline{1,n}$, *be the regular points and limit-circle case holds at singular point* ζ_{n+1} *for* τ*. Then* T *has purely discrete eigenvalues in the open upper half-plane. The possible limit points of these eigenvalues occur at infinity. All eigen- and associated functions of* T *are complete in* H *except possibly for a single point* c_0*.*

5. First-order system with finite singular transmission conditions

In this section we consider the system (1) on the multi-interval $J := \bigcup_{k=1}^{n+1} J_k$ as

$$\kappa(y) := By' + P(x)y = \lambda y, \tag{36}$$

where $J_k = (\zeta_{k-1}, \zeta_k)$ and

$$B = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad P(x) = \begin{pmatrix} p(x) & r(x) \\ r(x) & q(x) \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}.$$

We shall introduce the basic assumptions on (36) and the intervals J_k as follows:

(i) $-\infty \le \zeta_0 < \zeta_1 < \cdots < \zeta_{n+1} \le \infty$,

(ii) p, q, and r are real-valued and Lebesgue measurable functions on J_k, $k = \overline{1, n+1}$,

(iii)

$$\int\limits_{J_k} \{|p(x)| + |r(x)| + |q(x)|\}\, dx = \infty, \ k = \overline{1, n+1}.$$

Let $L^2(J, \mathbb{C}^2)$ denote the Hilbert space consisting of all vector-valued functions $y = \binom{y_1}{y_2}$ in $\mathbb{C}^2$ satisfying $\int_J \left(|y_1|^2 + |y_2|^2\right) dx < \infty$ with the usual inner product.

Let $D(J, \mathbb{C}^2)$ be a set in $L^2(J, \mathbb{C}^2)$ consisting of all vector-valued functions $y = \binom{y_1}{y_2}$ in which y_1 and y_2 are locally integrable functions on all J_k, $k = \overline{1, n+1}$, and $\kappa(y) \in L^2(J, \mathbb{C}^2)$. Then for arbitrary two vector-valued functions $y, \chi \in D(J, \mathbb{C}^2)$ we have the following Green's formula:

$$(\kappa(y), \chi) - (y, \kappa(\chi)) = \sum_{k=1}^{n+1} [y, \chi]_{\zeta_{k-1}+}^{\zeta_k -}.$$

Therefore we get that at all singular points ζ_l, $l = \overline{0, n+1}$, the values $[y, \chi](\zeta_k -)$, $k = \overline{1, n+1}$, and $[y, \chi](\zeta_s +)$, $= \overline{0, n}$, exist and are finite.

We assume that at all singular points ζ_l, $l = \overline{0, n+1}$, Weyl's limit-circle case holds for (36).

Let

$$u(x) = \begin{cases} u_1(x), \ x \in J_1 \\ u_2(x), \ x \in J_2 \\ \vdots \\ u_{n+1}(x), \ x \in J_{n+1} \end{cases}, \quad z(x) = \begin{cases} z_1(x), \ x \in J_1 \\ z_2(x), \ x \in J_2 \\ \vdots \\ z_{n+1}(x), \ x \in J_{n+1} \end{cases}$$

be the solutions of the equation

$$\kappa(y) = 0, \ x \in J,$$

satisfying the conditions

$$\begin{cases} u_{k1}(c_k) = 0, & u_{k2}(c_k) = 1, \\ z_{k1}(c_k) = 1, & z_{k2}(c_k) = 0, \end{cases}$$

where $c_k \in J_k$,

$$z_k(x) = \begin{pmatrix} z_{k1}(x) \\ z_{k2}(x) \end{pmatrix}, \quad u_k(x) = \begin{pmatrix} u_{k1}(x) \\ u_{k2}(x) \end{pmatrix}$$

and $k = \overline{1, n+1}$.

Green's formula implies that for two solutions $y(x,\lambda)$ and $\chi(x,\lambda)$ of (36) for the same value of λ the Wronskian of y and χ does not depend on x and depends only on λ on each J_k, $k = \overline{1, n+1}$. Moreover, they are linearly independent if and only if their Wronskian is nonzero.

Since $W[z_k, u_k] \equiv 1$ on each J_k, $k = \overline{1, n+1}$, z and u are linearly independent solutions of (36). Moreover, they belong to $D(J, \mathbb{C}^2)$. Therefore for arbitrary $y \in D(J, \mathbb{C}^2)$ all the values $[y, z](\zeta_k-)$, $[y, u](\zeta_k-)$, $k = \overline{1, n+1}$, and $[y, z](\zeta_s+)$, $[y, u](\zeta_s+)$, $s = \overline{0, n}$, exist and are finite.

Let us consider the following BVTP:

$$\kappa(y) = \lambda y, \ y \in D(J, \mathbb{C}^2), \ x \in J, \tag{37}$$

$$(a[y_1, u_1](\zeta_0+) - a_2[y_1, z_1](\zeta_0+)) - \lambda\,(a_1'[y_1, u_1](\zeta_0+) - a_2'[y_1, z_1](\zeta_0+)) = 0, \tag{38}$$

$$[y_m, u_m](\zeta_m-) = b_m[y_{(m+1)}, u_{m+1}](\zeta_m+), \tag{39}$$

$$[y_m, z_m](\zeta_m-) = b_m'[y_{(m+1)}, z_{m+1}](\zeta_m+), \tag{40}$$

$$[y_{n+1}, u_{n+1}](\zeta_{n+1}-) - c[y_{n+1}, z_{n+1}](\zeta_{n+1}-) = 0, \tag{41}$$

where $m = \overline{1, n}$, λ and c are complex numbers with $\Im c > 0$, $a_1, a_2, a_1', a_2', b_m, b_m'$ are real numbers with $b_m b_m' > 0$ and

$$\begin{vmatrix} a_1 & a_2 \\ a_1' & a_2' \end{vmatrix} > 0.$$

Following the same method given as in sections 2–4 we arrive at the following results.

Theorem 5.1 *Let ζ_l, $m = \overline{0, n+1}$, be singular points and limit-circle case holds at all singular points ζ_l for κ. Then the BVTP (37)–(41) has purely discrete eigenvalues in the open upper half-plane. The possible limit points of these eigenvalues occur at infinity. All eigen- and associated functions of the BVTP (37)–(41) are complete in H except possibly for a single point c_0.*

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