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Research Article

Some Fixed Point Results for *TAC*-Type Contractive Mappings

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We prove some fixed point results for new type of contractive mappings using the notion of cyclic admissible mappings in the framework of metric spaces. Our results extend, generalize, and improve some well-known results from literature. Some examples are given to support our main results.

1. Introduction

In the fixed point theory, a well-known theorem of Banach [1] states that if T is a self-mapping on a complete metric space (X, d) and satisfies $d(Tx, Ty) \leq kd(x, y)$, for some $k \in [0, 1)$ and all $x, y \in X$, then T has a unique fixed point. Thereafter, various researchers generalized this result for different type of nonlinear contractive mappings and prove some interesting fixed point results (see [2–27] and references cited therein).

Recently, Samet et al. [23] introduced the concept of α-ψ-contractive type mappings and established various fixed point theorems for such mappings in complete metric spaces. Thereafter, a lot of researchers worked on it and generalized the results under certain contractive conditions (see [5, 9, 14, 18, 22] and references cited therein).

Using the concept of Samet et al. [23], we prove some fixed point results for a new type of contractive mappings. Our results extend, generalize, and improve some well-known results from literature. Some examples are given to support our main results.

2. Preliminaries

Let X be a nonempty set and let $T : X \to X$ be an arbitrary mapping. We say that $x \in X$ is a fixed point for T, if $x = Tx$. We denote Fix (T) the set of all fixed points of T.

Definition 1 (see [18]). Let $T : X \to X$ be a mapping and let $\alpha, \beta : X \to \mathbb{R}^+$ be two functions. One can say that T is a cyclic (α, β)-admissible mapping if

(i) $\alpha(x) \geq 1$ for some $x \in X$ implies $\beta(Tx) \geq 1$,

(ii) $\beta(x) \geq 1$ for some $x \in X$ implies $\alpha(Tx) \geq 1$.

Example 2 (see [18]). Let $T : \mathbb{R} \to \mathbb{R}$ be defined by $T(-x) = -T(x)$. Suppose that $\alpha, \beta : \mathbb{R} \to \mathbb{R}^+$ are given by $\beta(x) = 5^x$, for all $x \in \mathbb{R}$, and $\alpha(y) = 5^{-y}$, for all $y \in \mathbb{R}$. Then, T is a cyclic (α, β)-admissible mapping.

Let Ψ denote the set of all monotone increasing continuous functions $\psi : [0, \infty) \to [0, \infty)$, with $\psi^{-1}(\{0\}) = 0$.

Let Φ denote the set of all continuous functions $\phi : [0, \infty) \to [0, \infty)$, with $\lim_{n\to\infty}\phi(t_n) = 0 \Rightarrow \lim_{n\to\infty} t_n = 0$.

Lemma 3 (see [19]). *Suppose that (X, d) is a metric space. Let $\{x_n\}$ be a sequence in X such that $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$. If $\{x_n\}$ is not a Cauchy sequence, then there exist an $\epsilon > 0$ and sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ with $m(k) > n(k) > k$ such that $d(x_{m(k)}, x_{n(k)}) \geq \epsilon$, $d(x_{m(k)-1}, x_{n(k)}) \leq \epsilon$, and*

(i) $\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)+1}) = \epsilon$,

(ii) $\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}) = \epsilon$,

(iii) $\lim_{k\to\infty} d(x_{m(k)-1}, x_{n(k)}) = \epsilon$.

Remark 4. In the same way as the proof of Lemma 3, we get $\lim_{k\to\infty} d(x_{n(k)+1}, x_{m(k)+1}) = \epsilon$.

3. Main Results

In 2014, the concept of C-class functions (see Definition 5) was introduced by Ansari in [6] and is important; for example, see numbers (1), (2) from Example 6. Also, see [7, 8, 12, 13].

Definition 5 (see [6]). One can say that $f : [0,\infty)^2 \to \mathbb{R}$ is called *C-class* function if it is continuous and satisfies the following axioms:

(1) $f(s,t) \le s$.

(2) $f(s,t) = s$ implies that either $s = 0$ or $t = 0$.

for all $s, t \in [0,\infty)$.

Note that $f(0,0) = 0$.

One can denote C-class functions as $\mathscr{C}$.

Example 6 (see [6]). The following functions $f : [0,\infty)^2 \to \mathbb{R}$ are elements of $\mathscr{C}$:

(1) $f(s,t) = s - t$, $f(s,t) = s \Rightarrow t = 0$.

(2) $f(s,t) = ks$, $k \in (0,1)$, $f(s,t) = s \Rightarrow s = 0$.

(3) $f(s,t) = s/(1+t)$, $f(s,t) = s \Rightarrow s = 0$ or $t = 0$.

(4) $f(s,t) = \log(t + a^s)/(1+t)$, $a > 1$, $f(s,t) = s \Rightarrow s = 0$ or $t = 0$.

(5) $f(s,t) = \ln(1 + a^s)/2$, $a > e$, $f(s,t) = s \Rightarrow s = 0$.

(6) $f(s,t) = (s+l)^{(1/(1+t))} - l$, $l > 1$, $f(s,t) = s \Rightarrow t = 0$.

Definition 7. Let (X,d) be a metric space and let $\alpha, \beta : X \to \mathbb{R}^+$ be two functions. One can say that $T : X \to X$ is a *TAC*-contractive mapping if

$$\begin{gathered}\alpha(x)\beta(y) \ge 1 \Longrightarrow \\ \psi(d(Tx,Ty)) \le f(\psi(d(x,y)), \phi(d(x,y)))\end{gathered} \tag{1}$$

for $x, y \in X$, where $f \in \mathscr{C}$, $\psi \in \Psi$, and $\phi \in \Phi$.

Now, we are ready to prove our first theorem.

Theorem 8. *Let (X,d) be a complete metric space and let $T : X \to X$ be a cyclic (α,β)-admissible mapping. Assume that T is a TAC-contractive mapping. Suppose that there exists $x_0 \in X$ such that $\alpha(x_0) \ge 1$ and $\beta(x_0) \ge 1$ and either of the following conditions hold:*

(a) *T is continuous.*

(b) *if $\{x_n\}$ is a sequence in X such that $x_n \to x$ and $\beta(x_n) \ge 1$, for all n, then $\beta(x) \ge 1$.*

Then, T has a fixed point.

Moreover, if $\alpha(x) \ge 1$ and $\beta(y) \ge 1$, for all $x, y \in \operatorname{Fix}(T)$, then T has a unique fixed point.

Proof. Define a sequence $\{x_n\}$ by $x_n = T^n x_0 = Tx_{n-1}$, for all $n \in \mathbb{N}$. Since T is a cyclic (α,β)-admissible mapping and $\alpha(x_0) \ge 1$, then $\beta(x_1) = \beta(Tx_0) \ge 1$ which implies $\alpha(x_2) = \alpha(Tx_1) \ge 1$. By continuing this process, we get $\alpha(x_{2n}) \ge 1$ and $\beta(x_{2n-1}) \ge 1$, for all $n \in \mathbb{N}$. Again, since T is a cyclic (α,β)-admissible mapping and $\beta(x_0) \ge 1$, by the similar method, we have $\beta(x_{2n}) \ge 1$ and $\alpha(x_{2n-1}) \ge 1$, for all $n \in \mathbb{N}$. That is, $\alpha(x_n) \ge 1$ and $\beta(x_n) \ge 1$, for all $n \in \mathbb{N} \cup \{0\}$. Equivalently, $\alpha(x_{n-1})\beta(x_n) \ge 1$, for all $n \in \mathbb{N}$. From (1), we have

$$\begin{aligned}\psi(d(x_n, x_{n+1})) &\le f(\psi(d(x_{n-1}, x_n)), \phi(d(x_{n-1}, x_n))) \\ &\le \psi(d(x_{n-1}, x_n)).\end{aligned} \tag{2}$$

Using monotonicity of ψ, we get

$$d(x_n, x_{n+1}) \le d(x_{n-1}, x_n), \tag{3}$$

for all $n \in \mathbb{N}$. Hence, the sequence $\{d(x_n, x_{n+1})\}$ is a decreasing sequence. So for the nonnegative decreasing sequence $\{d(x_n, x_{n+1})\}$, there exists some $r \ge 0$, such that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = r. \tag{4}$$

Assume that $r > 0$. On letting $n \to \infty$ in (2), using the continuity of ψ and f and (4), we obtain

$$\psi(r) \le f(\psi(r), \phi(r)) \le \psi(r), \tag{5}$$

and thus $f(\psi(r), \phi(r)) = \psi(r)$. Now, by using Definition 5, we get that either $\psi(r) = 0$ or $\phi(r) = 0$; in both cases, it follows that $r = 0$, which implies

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{6}$$

Now, we shall prove that $\{x_n\}$ is a Cauchy sequence. If possible, let $\{x_n\}$ not be a Cauchy sequence. Then, by Lemma 3 and Remark 4, there exist a $\delta > 0$ and two sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ with $n(k) > m(k) > k$ such that

$$\lim_{n\to\infty} d(x_{m(k)}, x_{n(k)}) = \lim_{n\to\infty} d(x_{m(k)+1}, x_{n(k)+1}) = \delta. \tag{7}$$

Now, by setting $x = x_{m_k}$ and $y = x_{n_k}$ in (1), and using $\alpha(x_{n(k)})\beta(x_{m(k)}) \ge 1$, we obtain

$$\begin{aligned}&\psi(d(x_{m(k)+1}, x_{n(k)+1})) \\ &\quad \le f(\psi(d(x_{m(k)}, x_{n(k)})), \phi(d(x_{m(k)}, x_{n(k)}))).\end{aligned} \tag{8}$$

On letting $k \to \infty$, using (7), we obtain

$$\psi(\delta) \le f(\psi(\delta), \phi(\delta)) \le \psi(\delta), \tag{9}$$

$\psi(\delta) = 0$, or $\phi(\delta) = 0$; that is, $\delta = 0$, which is a contradiction. This shows that $\{x_n\}$ is a Cauchy sequence. Since (X,d) is a complete metric space, then there is $z \in X$ such that $x_n \to z$ as $n \to \infty$.

Now, first we suppose that T is continuous. Hence,

$$Tz = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = z. \tag{10}$$

So z is a fixed point of T.

In the second part, we suppose that condition (b) holds; that is, $\alpha(x_n)\beta(z) \geq 1$. So, we have

$$\begin{aligned}\psi(d(x_{n+1}, Tz)) &\leq f(\psi(d(x_n, z)), \phi(d(x_n, z))) \\ &\leq \psi(d(x_n, z)).\end{aligned} \tag{11}$$

By taking the limit $n \to \infty$ and using the properties of ψ, we obtain $d(z, Tz) = 0$. Hence, z is a fixed point of T.

To prove the uniqueness of fixed point, suppose that z_1 and z_2 are two fixed points of T. Since $\alpha(z_1)\beta(z_2) \geq 1$, from (1), we have

$$\begin{aligned}\psi(d(z_1, z_2)) &= \psi(d(Tz_1, Tz_2)) \\ &\leq f(\psi(d(z_1, z_2)), \phi(d(z_1, z_2))) \\ &\leq \psi(d(z_1, z_2)).\end{aligned} \tag{12}$$

Hence, by using the properties of f, we have $z_1 = z_2$. □

Example 9. Let $X = \mathbb{R}$ be endowed with the usual metric $d(x, y) = |x - y|$, for all $x, y \in X$, and let $T : X \to X$ be defined by

$$T(x) = \begin{cases} -\dfrac{x}{4}, & x \in [-2, 1] \\ 3x, & \mathbb{R} \setminus [-2, 1] \end{cases} \tag{13}$$

and let $\alpha, \beta : X \to \mathbb{R}^+$ be given by

$$\begin{aligned}\alpha(x) &= \begin{cases} 2 & x \in [-2, 0], \\ 0 & \mathbb{R} \setminus [-2, 0], \end{cases} \\ \beta(x) &= \begin{cases} 1, & x \in [0, 1], \\ 0, & \mathbb{R} \setminus [0, 1]. \end{cases}\end{aligned} \tag{14}$$

Also, define $\psi \in \Psi$ as $\psi(t) = t$, $\phi \in \Phi$ as $\phi(t) = 1/3$, and $F \in \mathscr{C}$ as $F(s, t) = s/(1 + t)$.

Now, first we prove that T is a cyclic (α, β)-admissible mapping.

If $\alpha(x) \geq 1$, then $x \in [-2, 0]$ and $Tx \in [0, 1]$. Therefore, $\beta(Tx) \geq 1$. Similarly, if $\beta(x) \geq 1$, then $\alpha(Tx) \geq 1$. Then, T is a cyclic (α, β)-admissible mapping.

Now, we check the hypotheses (b) of Theorem 8.

Let $\{x_n\} \subseteq X$ such that $\beta(x_n) \geq 1$ and $x_n \to x$. Therefore, $x_n \in [0, 1]$. Hence, $x \in [0, 1]$,

Let $\alpha(x)\beta(y) \geq 1$. Then, $x \in [-2, 0]$ and $y \in [0, 1]$ and so we have $\psi(d(Tx, Ty)) = |Tx - Ty| = (1/4)|x - y| \leq (3/4)|x - y| = |x - y|/(1 + 1/3) = \psi(d(x, y))/(1 + \phi(d(x, y)))$. Hence, inequality (1) is satisfied. Therefore, by Theorem 8, T has a fixed point.

Corollary 10. *Let (X, d) be a complete metric space and let $T : X \to X$ be a cyclic (α, β)-admissible mapping. Assume that T is an (α, β)-contractive mapping; that is, for all $x, y \in X$,*

$$\begin{aligned}&\alpha(x)\beta(y)\psi(d(Tx, Ty)) \\ &\quad \leq f(\psi(d(x, y)), \phi(d(x, y))).\end{aligned} \tag{15}$$

Suppose that there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$ and either of the following conditions hold:

(a) *T is continuous.*

(b) *if $\{x_n\}$ is a sequence in X such that $x_n \to x$ and $\beta(x_n) \geq 1$, for all n, then $\beta(x) \geq 1$.*

Then, T has a fixed point.

Moreover, if $\alpha(x) \geq 1$ and $\beta(y) \geq 1$, for all $x, y \in \mathrm{Fix}(T)$, then T has a unique fixed point.

Proof. Let $\alpha(x)\beta(y) \geq 1$, for $x, y \in X$. Hence, by using (15), we have the fact that T is a TAC-contractive mapping. Therefore, by applying Theorm 8, we have the result. □

Definition 11. Let (X, d) be a metric space and let $\alpha, \beta : X \to \mathbb{R}^+$ be two functions. A mapping $T : X \to X$ is called a weak TAC-rational contraction if $\alpha(x)\beta(y) \geq 1$, for some $x, y \in X$, implies

$$d(Tx, Ty) \leq f(M(x, y), \phi(M(x, y))), \tag{16}$$

where $f \in \mathscr{C}$, $\phi \in \Phi$, and

$$\begin{aligned}&M(x, y) \\ &\quad = \max\left\{d(x, y), \frac{[1 + d(x, Tx)]\, d(y, Ty)}{d(x, y) + 1}\right\}.\end{aligned} \tag{17}$$

Theorem 12. *Let (X, d) be a complete metric space and let $T : X \to X$ be a cyclic (α, β)-admissible mapping. Suppose that T is a weak TAC-rational contraction. Assume that there exists $x_0 \in X$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$ and one of the following assertions holds:*

(a) *T is continuous.*

(b) *if $\{x_n\}$ is a sequence in X such that $x_n \to x$ and $\beta(x_n) \geq 1$, for all n, then $\beta(x) \geq 1$.*

Then, T has a fixed point.

Moreover, if $\alpha(x) \geq 1$ and $\beta(y) \geq 1$, for all $x, y \in \mathrm{Fix}(T)$, then T has a unique fixed point.

Proof. Define a sequence $\{x_n\}$ by $x_n = T^n x_0 = Tx_{n-1}$, for all $n \in N$. Since T is a cyclic (α, β)-admissible mapping and $\alpha(x_0) \geq 1$, then $\beta(x_1) = \beta(Tx_0) \geq 1$ which implies $\alpha(x_2) = \alpha(Tx_1) \geq 1$. By continuing this process, we get $\alpha(x_{2n}) \geq 1$ and $\beta(x_{2n-1}) \geq 1$, for all $n \in \mathbb{N}$. Again, since T is a cyclic (α, β)-admissible mapping and $\beta(x_0) \geq 1$, by the similar method, we have $\beta(x_{2n}) \geq 1$ and $\alpha(x_{2n-1}) \geq 1$, for all $n \in \mathbb{N}$. That is, $\alpha(x_n) \geq 1$ and $\beta(x_n) \geq 1$, for all $n \in \mathbb{N} \cup \{0\}$. Equivalently, $\alpha(x_{n-1})\beta(x_n) \geq 1$, for all $n \in \mathbb{N}$. Therefore, by (16), we have

$$d(x_n, x_{n+1}) \leq f(M(x_{n-1}, x_n), \phi(M(x_{n-1}, x_n))), \tag{18}$$

where $M(x_{n-1}, x_n) = \{d(x_{n-1}, x_n, d(x_n, x_{n+1}))\}$.

Now, suppose that there exists $n_0 \in \mathbb{N}$ such that $d(x_{n_0}, x_{n_0+1}) > d(x_{n_0-1}, x_{n_0})$. Therefore, $M(x_{n_0-1}, x_{n_0}) = d(x_{n_0}, x_{n_0+1})$ and so, from (18), we get

$$d\left(x_{n_0}, x_{n_0+1}\right) \leq f\left(d\left(x_{n_0}, x_{n_0+1}\right), \phi\left(d\left(x_{n_0}, x_{n_0+1}\right)\right)\right) \quad (19)$$

$$\leq d\left(x_{n_0}, x_{n_0+1}\right). \quad (20)$$

This implies that $d(x_{n_0}, x_{n_0+1}) = 0$, or $\phi(d(x_{n_0}, x_{n_0+1})) = 0$; that is, $d(x_{n_0}, x_{n_0+1}) = 0$, which is a contradiction. Hence, $d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n)$, for all $n \in \mathbb{N}$. Hence, the sequence $\{d(x_n, x_{n+1})\}$ is a decreasing sequence. So for the nonnegative decreasing sequence $\{d(x_n, x_{n+1})\}$, there exists some $r \geq 0$, such that

$$\lim_{n\to\infty} d\left(x_n, x_{n+1}\right) = r. \quad (21)$$

Assume that $r > 0$. On letting $n \to \infty$ in (19), using the continuity of f and (21), we obtain

$$r \leq f\left(r, \phi\left(r\right)\right) \leq r, \quad (22)$$

which implies that either $r = 0$ or $\phi(r) = 0$; that is, in both cases, it follows that $r = 0$, which implies

$$\lim_{n\to\infty} d\left(x_n, x_{n+1}\right) = 0. \quad (23)$$

Now, we shall prove that $\{x_n\}$ is a Cauchy sequence. If possible, let $\{x_n\}$ not be a Cauchy sequence. Then, by Lemma 3 and Remark 4, there exist $\delta > 0$ and two sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ with $n(k) > m(k) > k$ such that

$$\lim_{n\to\infty} d\left(x_{m(k)}, x_{n(k)}\right) = \lim_{n\to\infty} d\left(x_{m(k)+1}, x_{n(k)+1}\right) = \delta. \quad (24)$$

Now, by setting $x = x_{n_k+1}$ and $y = y_{m_k+1}$ in (16), and using $\alpha(x_{n(k)})\beta(x_{m(k)}) \geq 1$, we obtain

$$\begin{aligned} & d\left(x_{m_k+1}, x_{n_k+1}\right) \\ & \quad \leq f\left(M\left(x_{n_k}, x_{m_k}\right), \phi\left(M\left(x_{n_k}, x_{m_k}\right)\right)\right), \end{aligned} \quad (25)$$

where

$$\begin{aligned} M\left(x_{n_k}, x_{m_k}\right) = \max \Bigg\{ & d\left(x_{n_k}, x_{m_k}\right), \\ & \frac{\left[1 + d\left(x_{n_k}, x_{n_k+1}\right)\right] d\left(x_{m_k}, x_{m_k+1}\right)}{d\left(x_{n_k}, x_{m_k}\right) + 1} \Bigg\}. \end{aligned} \quad (26)$$

On letting $k \to \infty$, using (24) and (25), we obtain

$$\delta \leq f\left(\delta, \phi\left(\delta\right)\right), \quad (27)$$

So, $\phi(\delta) = 0$; that is, $\delta = 0$, which is a contradiction. This shows that $\{x_n\}$ is a Cauchy sequence. Since (X, d) is a complete metric space, then there exists $z \in X$ such that $x_n \to z$ as $n \to \infty$.

First, we consider that T is continuous. Hence,

$$Tz = \lim_{n\to\infty} Tx_n = \lim_{n\to\infty} x_{n+1} = z. \quad (28)$$

Therefore, z is a fixed point of T.

In the second part, we suppose that condition (b) holds; that is, $\alpha(x_n)\beta(z) \geq 1$. So, we have

$$\begin{aligned} d\left(x_{n+1}, Tz\right) & \leq f\left(M\left(x_n, z\right), \phi\left(M\left(x_n, z\right)\right)\right) \\ & \leq M\left(x_n, z\right), \end{aligned} \quad (29)$$

where

$$\begin{aligned} & M\left(x_n, z\right) \\ & \quad = \max\left\{d\left(x_n, z\right), \frac{\left[1 + d\left(x_n, x_{n+1}\right)\right] d\left(z, Tz\right)}{d\left(x_n, z\right) + 1}\right\}. \end{aligned} \quad (30)$$

By taking the limit $n \to \infty$ and using the properties of f, we obtain $d(z, Tz) = 0$. Hence, z is a fixed point of T.

To prove the uniqueness of fixed point, suppose that z_1 and z_2 are two fixed points of T. Since $\alpha(z_1)\beta(z_2) \geq 1$, from (16), we have

$$\begin{aligned} d\left(z_1, z_2\right) & = d\left(Tz_1, Tz_2\right) \\ & \leq f\left(M\left(z_1, z_2\right), \phi\left(M\left(z_1, z_2\right)\right)\right) \\ & \leq M\left(z_1, z_2\right), \end{aligned} \quad (31)$$

where

$$\begin{aligned} & M\left(z_1, z_2\right) \\ & \quad = \max\left\{d\left(z_1, z_2\right), \frac{\left[1 + d\left(z_1, Tz_1\right)\right] d\left(z_2, Tz_2\right)}{d\left(z_1, z_2\right) + 1}\right\}. \end{aligned} \quad (32)$$

This implies that $d(z_1, z_2) = 0$ or $\phi(d(z_1, z_2)) = 0$ and hence $z_1 = z_2$. □

Example 13. Let $X = [0, +\infty)$ be endowed with the usual metric $d(x, y) = |x - y|$, for all $x, y \in X$ and let $T : X \to X$ be defined by

$$T\left(x\right) = \begin{cases} \dfrac{x}{8}, & x \in [0, 1] \\ \dfrac{1}{2}, & x \in (1, +\infty) \end{cases} \quad (33)$$

and let $\alpha, \beta : X \to \mathbb{R}^+$ be given by

$$\alpha\left(x\right) = \beta\left(x\right) = \begin{cases} 1 & x \in [0, 1] \\ 0 & \text{otherwise.} \end{cases} \quad (34)$$

Also, define $\phi \in \Phi$ as $\phi(t) = t/2$ and $F \in \mathscr{C}$ as $F(s, t) = s - t$.

It is easy to verify that T is a cyclic (α, β)-admissible mapping.

Now, we check the hypotheses (b) of Theorem 12.

Let $\{x_n\} \subseteq X$ such that $\beta(x_n) \geq 1$ and $x_n \to x$. Therefore, $x_n \in [0,1]$. Hence, $x \in [0,1]$,

Let $\alpha(x)\beta(y) \geq 1$. Then, $x \in [0,1]$ and $y \in [0,1]$ and so we have

$$\begin{aligned} d(Tx, Ty) &= |Tx - Ty| = \frac{1}{8}|x - y| \\ &\leq 2^{-1} \max\left\{d(x, y), \frac{[1 + d(x, Tx)]\, d(y, Ty)}{d(x, y) + 1}\right\}. \end{aligned} \tag{35}$$

Hence, inequality (16) is satisfied. Therefore, by Theorem 12, T has a fixed point; that is, 0 is a fixed point of T.

4. Some Cyclic Contraction via Cyclic (α, β)-Admissible Mapping

In this section, in a natural way, we apply Theorem 8 for proving a fixed point theorem involving a cyclic mapping.

Theorem 14. *Let A and B be two closed subsets of complete metric space (X, d) such that $A \cap B \neq \emptyset$ and let $T : A \cup B \to A \cup B$ be a mapping such that $TA \subset B$ and $TB \subset A$. Assume that*

$$\psi(d(Tx, Ty)) \leq f(\psi(d(x, y)), \phi(d(x, y))), \tag{36}$$

for all $x \in A$ and $y \in B$, where $f \in \mathscr{C}$, $\psi \in \Psi$, and $\phi \in \Phi$. Then, T has a unique fixed point in $A \cap B$.

Proof. Define $\alpha, \beta : X \to \mathbb{R}^+$ by

$$\begin{aligned} \alpha(x) &= \begin{cases} 1, & x \in A \\ 0, & \text{otherwise}, \end{cases} \\ \beta(x) &= \begin{cases} 1, & x \in B \\ 0, & \text{otherwise}. \end{cases} \end{aligned} \tag{37}$$

Let $\alpha(x)\beta(y) \geq 1$. Then, $x \in A$ and $y \in B$. Hence, by (36), we have

$$\psi(d(Tx, Ty)) \leq f(\psi(d(x, y)), \phi(d(x, y))), \tag{38}$$

for all $x, y \in A \cup B$.

Let $\alpha(x) \geq 1$ for some $x \in X$; then, $x \in A$. Hence, $Tx \in B$ and so $\beta(Tx) \geq 1$. Now, let $\beta(x) \geq 1$ for some $x \in X$, so $x \in B$. Hence, $Tx \in A$ and then $\alpha(Tx) \geq 1$. Therefore, T is a cyclic (α, β)-admissible mapping. Since $A \cap B$ is nonempty, then there exists $x_0 \in A \cap B$ such that $\alpha(x_0) \geq 1$ and $\beta(x_0) \geq 1$.

Now, let $\{x_n\}$ be a sequence in X such that $\beta(x_n) \geq 1$, for all $n \in \mathbb{N}$ and $x_n \to x$; then, $x_n \in B$, for all $n \in \mathbb{N}$. Therefore, $x \in B$. This implies that $\beta(x) \geq 1$. So the condition (b) of Theorem 8 holds. Therefore, T has a fixed point in $A \cup B$, for example, z. Since $z \in A$, then $z = Tz \in B$, and since $z \in B$, then $z = Tz \in A$. Therefore, $z \in A \cap B$. The uniqueness of the fixed point follows easily from (36). □

Example 15. Let $X = \mathbb{R}$ be endowed with the usual metric $d(x, y) = |x - y|$, for all $x, y \in X$, and let $T : A \cup B \to A \cup B$ be defined by $Tx = -x/3$, where $A = [-1, 0]$ and $B = [0, 1]$. Also, define ψ, $\phi : [0, \infty) \to [0, \infty)$ by $\psi(t) = t$ and $\phi(t) = (2/3)t$. Indeed, for all $x \in A$ and all $y \in B$, we have $\psi(d(Tx, Ty)) = |Tx - Ty| = (1/3)|x - y| = \psi(d(x, y)) - \phi(d(x, y)) = f(\psi(d(x, y)), \phi(d(x, y)))$. Therefore, the conditions of Theorem 14 hold and T has a unique fixed point; that is, 0 is a fixed point of T.

Corollary 16. *Let A and B be two closed subsets of complete metric space (X, d) such that $A \cap B \neq \emptyset$, and let $T : A \cup B \to A \cup B$ be a mapping such that $TA \subset B$ and $TB \subset A$. Assume that*

$$d(Tx, Ty) \leq f(d(x, y), \phi(d(x, y))), \tag{39}$$

for all $x \in A$ and $y \in B$, where $f \in \mathscr{C}$, and $\phi \in \Phi$. Then, T has a unique fixed point in $A \cap B$.

Conflict of Interests

The authors declare that they have no competing interests.

Authors' Contribution

All authors contributed equally and significantly to writing of this paper. All authors read and approved the final paper.

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