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On the Kolmogorov forward equations within Caputo and Riemann-Liouville fractions derivatives

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Abstract

In this work, we focus on the fractional versions of the well-known Kolmogorov forward equations. We consider the problem in two cases. In case 1, we apply the left Caputo fractional derivatives for $\alpha \in (0\,,\,1]$ and in case 2, we use the right Riemann-Liouville fractional derivatives on R_+, for $\alpha \in (1\,,\,+\infty)$. The exact solutions are obtained for the both cases by Laplace transforms and stable subordinators.

1 Introduction and preliminaries

In the last decades, attention of scientists has been attracted to generalizations of classical processes and differential equations by the fractional order for derivatives. For instance, existence and uniqueness of the fractional differential equations [1], the fractional integro-differential equations [2], the fractional diffusions [3, 4, 5, 6], the fractional telegraph equation [7] and fractional Poisson processes [8, 9, 10, 11]. In this work we consider generalization of the well-known Kolmogorov forward equations [12] in two cases: in the first case, the left Caputo fractional derivative is applied and in the second case the right fractional Riemann-Liouville derivative is used. These models in special cases reduce to the well-known fractional relaxation equations [13] and discrete version of the fractional master equation [14].

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We consider here the fractional Kolmogorov forward equations (FKFE):

$$\frac{d^{\alpha}}{dt^{\alpha}}B_k^{\alpha}(t)=\begin{cases}-\lambda B_1^{\alpha}(t), & k=1,\\ -\lambda B_k^{\alpha}(t)+\sum_{i=1}^{k-1}\lambda b_i B_{k-i}^{\alpha}(t)\ , & k\in\{2,3,\ldots\},\end{cases} \tag{1}$$

for $t\geq 0,\ \alpha>0$, with initial conditions

$$B_k^{\alpha}(0)=\begin{cases}1 & k=1,\\ 0 & k\in\{2,3,\ldots\}\end{cases}\ , \tag{2}$$

where $\lambda\in\mathbb{R}$ and $b_i\ ,\ i\in\{1,2,\ldots\}$ are the distribution of some compounding random variable with values in $i\in\{1,2,\ldots\}$ and $\frac{d^{\alpha}}{dt^{\alpha}}$ is fractional derivativ. The exact solutions to (1)-(2) will be obtained for diferent values of $\alpha\in(0\,,+\infty)$.

The problem (1)-(2) is analyzed with two the different fractional derivative operators that be defined as follows (refer [15, 16, 17, 18] to see these definitions and properties of them):

1. The left Caputo fractional derivative of order α:

$${}_0^CD_t^{\alpha}\ f(t):=\frac{1}{\Gamma\left(1-\alpha\right)}\int_0^t\left(t-\tau\right)^{-\alpha}f\,'(\tau)d\tau,\qquad t>0,\ \alpha\in\left(0\,,\,1\right].$$

2. The right Riemann-Liouville fractional derivative on R_+ of order α:

$${}_t^{RL}D_{+\infty}^{\alpha}\ f(t):=\frac{1}{\Gamma\left(n-\alpha\right)}\left(-\frac{d}{dt}\right)^n\int_t^{+\infty}\frac{f(\tau)}{\left(\tau-t\right)^{\alpha-n+1}}d\tau,\ t>0,\alpha\in\left(1\,,\,+\infty\right)$$

where $n=\lceil\alpha\rceil$.

Note that, for $\alpha=1$, we have ${}_0^CD_t^{\alpha}=\frac{d}{dt}$ and ${}_t^{RL}D_{+\infty}^{\alpha}=-\frac{d}{dt}$.

Remark 1.1. For $\alpha=1$, equation (1) reduces to the well-known Kolmogorov forward equations [10]. For $k=1$, the equation (1) coincides with the well-known fractional relaxation equations [11]. For $k>1$, they can be seen as a discrete version of the fractional master equation [12].

2 Original results

In this section, we use some notations as follows:
i) $L\left[f(t);s\right]:=\int_0^{\infty}e^{-s\,t}f(t)\,dt.$
ii) $E_{\alpha,\beta}(t)=\sum_{j=0}^{\infty}\frac{t^j}{\Gamma(\alpha j+\beta)},\quad x\in\mathbb{R},\ \ \alpha,\beta\in C,\ \ \Re(\alpha),\,\Re(\beta)>0$ (The Mittag-Leffler function).

Theorem 2.1. The exact solution of the FKFE

$$ {}_0^C D_t^\alpha B_k^\alpha(t) = \begin{cases} -\lambda_1 B_1^\alpha(t), & k=1, \\ -\lambda B_k^\alpha(t) + \sum_{i=1}^{k-1} \lambda b_i B_{k-i}^\alpha(t) \ , & k \in \{2,3,\ldots\}, \end{cases} \tag{3} $$

for $t>0, \ 0<\alpha\leq 1$ and with initial conditions

$$ B_k^\alpha(0) = \begin{cases} 1 & k=1, \\ 0 & k\in\{2,3,\ldots\} \end{cases} , \tag{4} $$

for $\lambda \in \mathbb{R}$ and $b_i \, (i=1,2,\ldots)$that are the distribution of some compounding random variable with values in $i\in\{1,2,\ldots\}$, is given by

$$ B_k^\alpha(t) = \begin{cases} \mathrm{E}_{\alpha\,,\,1}(-\lambda t^\alpha)\,, & k=1 \\ \sum_{r=1}^{k-1} \left(\sum_{(i_1\,,\,i_2\,,\,\ldots\,,\,i_r)\,\in W_r^{k-1}} \left(\prod_{j=1}^{r} b_{\,i_j} \right) \right) \frac{(\lambda t^\alpha)^r}{r!} \, \mathrm{E}_{\alpha\,,\,1}^{(r)}(-\lambda\, t^\alpha)\ , \\ \quad k=2\,,\,3\,,\,\ldots \end{cases} \tag{5} $$

where

$$ W_r^k = \left\{ (i_1,\, i_2,\, \ldots\,,\, i_r) \;\middle|\; i_j \in \{1,2,\ldots k-r+1\} \ \ and \ \ \sum_{j=1}^{r} i_j = k \right\}. \tag{6} $$

Proof. We use induction method to prove the result (5). We start by $k=1$, so we have

$$ {}_0^C D_t^\alpha B_1^\alpha(t) = -\lambda B_1^\alpha(t)\,, \quad B_1^\alpha(0) = 1. $$

Then, from the Laplace transform

$$ s^\alpha L[B_1^\alpha(t);s] - s^{\alpha-1}\, B_1^\alpha(0) = -\lambda\, L[B_1^\alpha(t);s] \quad \Rightarrow \ L[B_1^\alpha(t);s] = \frac{s^{\alpha-1}}{s^\alpha+\lambda} \ . $$

Since

$$ L\left[t^{n\alpha} \mathrm{E}_{\alpha\,,\,1}^{(n)}(-\lambda t^\alpha); s \right] = \frac{n! s^{\alpha-1}}{(s^\alpha+\lambda)^{n+1}}\,, \quad n=0,1,\ldots, \tag{7} $$

we have

$$B_1^\alpha(t) = \mathrm{E}_{\alpha\,,\,1}(-\lambda\, t^\alpha)\,.$$

For $k = 2$, equation (3) becomes

$${}_0^C D_t^\alpha B_2^\alpha(t) = -\lambda B_2^\alpha(t) + \lambda b_1 B_1^\alpha(t), \qquad B_2^\alpha(0) = 0.$$

The Laplace transform gives

$$s^\alpha L[B_2^\alpha(t); s] - s^{\alpha-1}\, B_2^\alpha(0) = -\lambda\, L[B_2^\alpha(t); s] + \lambda b_1 \frac{s^{\alpha-1}}{s^\alpha + \lambda}$$

$$\Rightarrow \quad L[B_2^\alpha(t); s] = \frac{\lambda b_1\, s^{\alpha-1}}{\left(s^\alpha + \lambda\right)^2}.$$

So, the inverse Laplace's Transform applied in (7) give

$$B_2^\alpha(t) = b_1 \lambda\, t^\alpha E_{\alpha\,,\,1}^{(1)}(-\lambda\, t^\alpha)\,.$$

Therefore, the equation (3) is satisfied for $k = 2$.
For $k = 3$, equation (3) becomes

$${}_0^C D_t^\alpha B_3^\alpha(t) = -\lambda B_3^\alpha(t) + \lambda b_1 B_2^\alpha(t) + \lambda b_2 B_1^\alpha(t), \qquad B_3^\alpha(0) = 0.$$

By the Laplace transform, we have

$$s^\alpha L\left[B_3^\alpha(t); s\right] - s^{\alpha-1} B_3^\alpha(0) = -\lambda L[B_3^\alpha(t); s] + \lambda b_1 \frac{\lambda b_1\, s^{\alpha-1}}{\left(s^\alpha + \lambda\right)^2} + \lambda b_2 \frac{s^{\alpha-1}}{(s^\alpha + \lambda)}.$$

So, we get

$$L\left[B_3^\alpha(t); s\right] = \frac{\lambda^2 b_1^2\, s^{\alpha-1}}{\left(s^\alpha + \lambda\right)^3} + \frac{\lambda b_2 s^{\alpha-1}}{\left(s^\alpha + \lambda\right)^2}.$$

Now, we apply the inverse Laplace's Transform with (7), thus

$$B_3^\alpha(t) = \; b_2 \lambda t^\alpha \mathrm{E}_{\alpha\,,\,1}^{(1)}(-\lambda\, t^\alpha) + b_1^2 \frac{\left(\lambda t^\alpha\right)^2}{2} \mathrm{E}_{\alpha\,,\,1}^{(2)}(-\lambda\, t^\alpha)\,.$$

By the same way for $k = 4$, we can write

$$B_4^\alpha(t) = \; b_3 \lambda t^\alpha \mathrm{E}_{\alpha\,,\,1}^{(1)}(-\lambda\, t^\alpha) + 2 b_1 b_2 \frac{\left(\lambda t^\alpha\right)^2}{2} \mathrm{E}_{\alpha\,,\,1}^{(2)}(-\lambda\, t^\alpha) + b_1^3 \frac{\left(\lambda t^\alpha\right)^3}{6} \mathrm{E}_{\alpha\,,\,1}^{(3)}(-\lambda\, t^\alpha)\,.$$

So, we have shown that (5) is satisfied for $k = 1, 2, 3, 4$.
Now, from supposition of induction, we assume the (5) is true for $B_i^{\alpha}(t),\ i \in \{1, \ldots, k-1\}$. Then we will prove that (5) holds for $B_k^{\alpha}(t)$.
We apply the Laplace transform on (3) with $k \geq 2$, so we have

$$s^{\alpha} L[B_k^{\alpha}(t); s] - s^{\alpha-1} B_k^{\alpha}(0) = -\lambda L[B_k^{\alpha}(t); s] + \lambda \sum_{i=1}^{k-1} b_i \, L[B_{k-i}^{\alpha}(t); s] \,.$$

Therefore by (5) and (7) we have

$$\begin{aligned}&(s^{\alpha} + \lambda) \, L\left[B_k^{\alpha}(t); s\right] \\ &= \lambda \textstyle\sum_{i=1}^{k-2} b_i \left(\sum_{r=1}^{k-i-1} \left(\sum_{(i_1, \ldots, i_r) \in W_r^{k-i-1}} \left(\prod\limits_{j=1}^{r} b_{i_j} \right) \right) \frac{\lambda^r s^{\alpha-1}}{(s^{\alpha}+\lambda)^{r+1}} \right) \\ &+ \lambda b_{k-1} \tfrac{s^{\alpha-1}}{s^{\alpha}+\lambda}.\end{aligned}$$

and

$$\begin{aligned}&L\left[B_k^{\alpha}(t); s\right] \\ &= \textstyle\sum_{i=1}^{k-2} b_i \left(\sum_{r=1}^{k-i-1} \left(\sum_{(i_1, \ldots, i_r) \in W_r^{k-i-1}} \left(\prod\limits_{j=1}^{r} b_{i_j} \right) \right) \frac{\lambda^{r+1} s^{\alpha-1}}{(s^{\alpha}+\lambda)^{r+2}} \right) \\ &+ \lambda b_{k-1} \tfrac{s^{\alpha-1}}{(s^{\alpha}+\lambda)^2}.\end{aligned}$$

Now, by reversing of transform (7), the above equation is reduced to:

$$\begin{aligned}&B_k^{\alpha}(t) \\ &= \textstyle\sum_{i=1}^{k-2} b_i \left(\sum_{r=1}^{k-i-1} \left(\sum_{(i_1, \ldots, i_r) \in W_r^{k-i-1}} \left(\prod\limits_{j=1}^{r} b_{i_j} \right) \right) \frac{(\lambda t^{\alpha})^{r+1}}{(r+1)!} \mathrm{E}_{\alpha\,,\,1}^{(r+1)}(-\lambda\, t^{\alpha}) \right) \\ &+ \lambda t^{\alpha} b_{k-1} \mathrm{E}_{\alpha\,,\,1}^{(1)}(-\lambda\, t^{\alpha}).\end{aligned}$$

By replacing the first sigma to the second sigma, we have

$$\begin{aligned}&B_k^{\alpha}(t) \\ &= \textstyle\sum_{r=1}^{k-2} \left(\left(\sum_{i=1}^{k-r-1} b_i \left(\sum_{(i_1, \ldots, i_r) \in W_r^{k-i-1}} \left(\prod\limits_{j=1}^{r} b_{i_j} \right) \right) \right) \frac{(\lambda t^{\alpha})^{r+1}}{(r+1)!} \mathrm{E}_{\alpha\,,\,1}^{(r+1)}(-\lambda\, t^{\alpha}) \right) \\ &+ \lambda t^{\alpha} b_{k-1} \mathrm{E}_{\alpha\,,\,1}^{(1)}(-\lambda\, t^{\alpha}).\end{aligned}$$

Note that for $r = 1, 2, \ldots, k-2$

$$\sum_{i=1}^{k-r-1} b_i \left(\sum_{(i_1, \ldots, i_r) \in W_r^{k-i-1}} \left(\prod_{j=1}^{r} b_{i_j} \right) \right) = \sum_{(i_1, \ldots, i_{r+1}) \in W_{r+1}^{k-1}} \left(\prod_{j=1}^{r+1} b_{i_j} \right).$$

So,

$$B_k^\alpha(t) = \sum_{r=1}^{k-2}\left(\left(\sum_{(i_1,\,\dots\,,\,i_{r+1})\,\in\,W_{r+1}^{k-1}}\left(\prod_{j=1}^{r+1} b_{i_j}\right)\right)\frac{(\lambda t^\alpha)^{r+1}}{(r+1)!}\mathrm{E}_{\alpha\,,\,1}^{(r+1)}(-\lambda\, t^\alpha)\right)$$
$$+\lambda t^\alpha b_{k-1}\mathrm{E}_{\alpha\,,\,1}^{(1)}(-\lambda\, t^\alpha).$$

Now, by replacing $r+1$ to r we have

$$B_k^\alpha(t) = \sum_{r=2}^{k-1}\left(\left(\sum_{(i_1,\,\dots\,,\,i_r)\,\in\,W_r^{k-1}}\left(\prod_{j=1}^{r} b_{i_j}\right)\right)\frac{(\lambda t^\alpha)^{r}}{r!}\mathrm{E}_{\alpha\,,\,1}^{(r)}(-\lambda\, t^\alpha)\right)$$
$$+\lambda t^\alpha b_{k-1}\mathrm{E}_{\alpha\,,\,1}^{(1)}(-\lambda\, t^\alpha).$$

Finally, we get the result (5) as follows

$$B_k^\alpha(t) = \sum_{r=1}^{k-1}\left(\left(\sum_{(i_1,\,\dots\,,\,i_r)\,\in\,W_r^{k-1}}\left(\prod_{j=1}^{r} b_{i_j}\right)\right)\frac{(\lambda t^\alpha)^{r}}{r!}\mathrm{E}_{\alpha\,,\,1}^{(r)}(-\lambda\, t^\alpha)\right).$$

Therefore, the proof is completed.□

Now, we consider $b_i = (1-\rho)\rho^{i-1}$ as distribution of compounding random variable with values in $i \in \{1, 2, \dots\}$. So, we can see the behaviors of exact solutions $B_k^\alpha(t)$ for the problem (3) and (4) for $\rho = 0.5$, $\lambda = 0.5$, $k = 1, 2, 3$ and $\alpha = 0.7, 0.8, 0.9, 1$ in figures 1-3.

Remark 2.1. For $\alpha = 1$ and by writing $B_k^1 = B_k$ in Theorem 2.1, the solution of the following equations

$$\left\{\begin{array}{l}\frac{d}{dt}B_1(t) = -\lambda B_1(t)\\ \frac{d}{dt}B_k(t) = -\lambda B_k(t) + \lambda\sum_{i=1}^{k-1} b_i B_{k-i}(t)\ ,\quad k\in\{2,3,\dots\},\end{array}\right. \qquad t>0\,, \tag{8}$$

with

$$B_k(0) = \left\{\begin{array}{ll}1 & k=1,\\ 0 & k\in\{2,3,\dots\}\end{array}\right. , \tag{9}$$

is given by

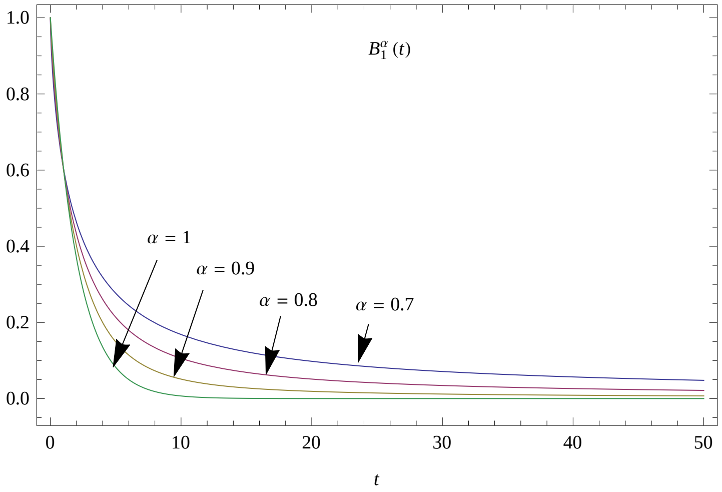

Figure 1: Plot of $B_1^\alpha(t)$ for $\rho = 0.5$, $\lambda = 0.5$ and $\alpha = 0.7$, 0.8, 0.9, 1.

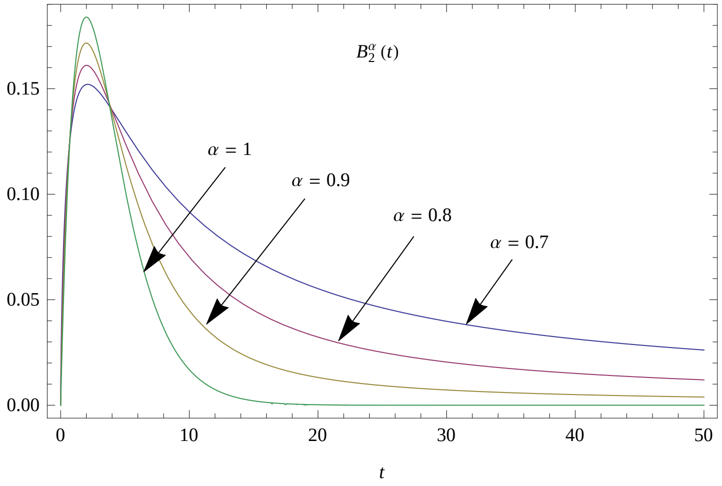

Figure 2: Plot of $B_2^\alpha(t)$ for $\rho = 0.5$, $\lambda = 0.5$ and $\alpha = 0.7$, 0.8, 0.9, 1.

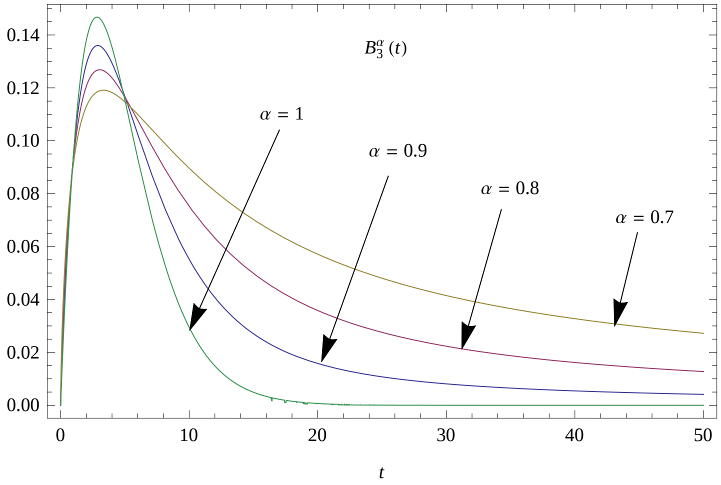

Figure 3: Plot of $B_3^\alpha(t)$ for $\rho = 0.5$, $\lambda = 0.5$ and $\alpha = 0.7,\ 0.8,\ 0.9,\ 1$.

$$B_k(t) = \begin{cases} e^{-\lambda\, t} & k = 1 \\ \sum_{r=1}^{k-1} \left(\sum_{(i_1\,, i_2\,, \ldots\,, i_r) \,\in W_r^{k-1}} \left(\prod_{j=1}^{r} b_{\,i_j} \right) \right) \frac{(\lambda t)^r}{r!} e^{-\lambda\, t} \\ k \in \{2, 3, \ldots\} \end{cases} \tag{10}$$

Using the notations of [19], we show the stable subordinator of index α by $A^\alpha(t)$. Furthermore its inverse (or hitting time) process is defined as $L^\alpha(t) = \inf\{z\,|z > 0,\ A^\alpha(z) > t\}$ for all $t \geq 0$.

Theorem 2.2. Let $N_\alpha(t)$ be the process defined as $N_1\left(H_\alpha(t)\right),\quad t \geq 0$, where $H_\alpha(t) = \begin{cases} L^\alpha(t), & \alpha \in (0\,,\,1)\,, \\ A^{\frac{1}{\alpha}}(t), & \alpha \in (1\,,\,+\infty)\,, \end{cases}$ and $H_\alpha(t) = t$ for $\alpha = 1$, under the assumption that N_1 and H_α are independent. Then the problem (1)-(2) is satisfied by the distribution $B_k^\alpha(t) = \Pr\left\{N_\alpha(t) = k\right\},\ k \geq 1$, for any $\alpha > 0$.

Proof. By noting that the definitions $l_\alpha(z,t)$ and $h_{\frac{1}{\alpha}}(z,t)$ ar the densities of $L^\alpha(t)$ and $A^{\frac{1}{\alpha}}(t)$, respectively, we have

$$B_k^{\alpha}(t) = \begin{cases} \int_0^{+\infty} B_k(z)\, l_{\alpha}(z,t)dz, & 0 < \alpha \leq 1, \\ \\ \int_0^{+\infty} B_k(z)\, h_{\frac{1}{\alpha}}(z,t)dz, & \alpha > 1\,, \end{cases}.$$

Now, we do the proof in two cases $\alpha \in (0\,,\, 1]$ and $\alpha \in (1\,, +\infty\,]$. In the first case, for $\alpha \in (0\,,\, 1]$ we define

$$a_r^k = \begin{cases} 1 & k=1,\ j=1, \\ \left(\sum_{(i_1\,,\, i_2\,,\, \dots\,,\, i_r)\,\in W_r^{k-1}} \left(\Pi_{j=1}^r b_{\,i_j}\right)\right) \frac{\lambda^r}{r!} & k \geq 2, \\ j = 1, \dots, k, \end{cases}$$

so the (5) can be reduced $B_k^{\alpha}(t) = \sum_{r=1}^k a_r^k\, t^{\alpha r}\, E_{\alpha,1}^{(r)}\,(-\lambda t^{\alpha})$. Let $F_{\alpha}(\nu,t) = \sum_{k=1}^{\infty} \nu^k B_k^{\alpha}(t)$ be the probability generating function, then we can write

$$\begin{aligned} &\int_0^{+\infty} e^{-st} F_{\alpha}(\nu,t)dt = \int_0^{+\infty} e^{-st} \sum_{k=1}^{\infty} \nu^k B_k^{\alpha}(t)dt \\ &= \int_0^{+\infty} e^{-st} \sum_{k=1}^{\infty} \nu^k \sum_{r=1}^k a_r^k t^{\alpha r} \underset{\alpha,1}{\overset{(r)}{E}}(-\lambda t^{\alpha})dt \\ &= \sum_{k=1}^{\infty} \nu^k \sum_{r=1}^k a_r^k \frac{r! s^{\alpha-1}}{(s^{\alpha}+\lambda)^{r+1}} = \sum_{k=1}^{\infty} \nu^k s^{\alpha-1} \sum_{r=1}^k a_r^k \int_0^{+\infty} e^{-\mu(s^{\alpha}+\lambda)} \mu^r\, d\mu \\ &= \int_0^{+\infty} \left(s^{\alpha-1} e^{-\mu s^{\alpha}} \sum_{k=1}^{\infty} \nu^k \sum_{r=1}^k a_r^k e^{-\mu\lambda} \mu^r\, ds \right) \\ &= \int_0^{+\infty} s^{\alpha-1} e^{-\mu s^{\alpha}} \sum_{k=1}^{\infty} \nu^k B_k^1(\mu)d\mu = \int_0^{+\infty} s^{\alpha-1} e^{-\mu s^{\alpha}} F_1(\nu,\mu)d\mu \\ &= \int_0^{+\infty} e^{-st} \int_0^{+\infty} F_1(\nu,\mu) l_{\alpha}(\mu,t)d\mu\, dt, \end{aligned}$$

since $\mu^{\alpha-1} e^{-\mu s^{\alpha}} = \int_0^{+\infty} e^{-s\,t} l_{\alpha}(\mu,t)dt$(see [20]). Thus, we get

$$F_{\alpha}(\nu,t) = \int_0^{+\infty} F_1(\nu,\mu)\, l_{\alpha}(\mu,t)d\mu\ \Rightarrow B_k^{\alpha}(t) = \int_0^{+\infty} B_k^1(\mu)\, l_{\alpha}(\mu,t)d\mu.$$

In the second case for $\alpha \in (1\,,\, +\infty)$, we have
$B_k^{\alpha}(t) = \int_0^{+\infty} B_k(\tau)\, h_{\frac{1}{\alpha}}(\tau,t)d\tau$. So, we can write

$$\begin{aligned} {}_t^{RL}D_{+\infty}^{\alpha} B_k^{\alpha}(t) &= \int_0^{+\infty} B_k(\tau)\, {}_t^{RL}D_{+\infty}^{\alpha} h_{\frac{1}{\alpha}}(\tau,t)d\tau = \int_0^{+\infty} B_k(\tau)\, \tfrac{\partial}{\partial\tau} h_{\frac{1}{\alpha}}(\tau,t)d\tau \\ &= \Big[B_k(\tau)\, h_{\frac{1}{\alpha}}(\tau,t) \Big]_{z=0}^{+\infty} - \int_0^{+\infty} \tfrac{d}{d\tau} B_k(\tau)\ h_{\frac{1}{\alpha}}(\tau,t)d\tau \\ &= -\int_0^{+\infty} \left(-\lambda B_k(\tau) + \lambda \sum_{i=1}^{k-1} b_i B_{k-i}(\tau) \right)\ h_{\frac{1}{\alpha}}(\tau,t)d\tau \\ &= \lambda \int_0^{+\infty} B_k(\tau)\ h_{\frac{1}{\alpha}}(\tau,t)d\tau - \lambda \sum_{i=1}^{k-1} \int_0^{+\infty} b_i B_{k-i}(\tau) h_{\frac{1}{\alpha}}(\tau,t)d\tau \\ &= \lambda B_k^{\alpha}(t) - \lambda \sum_{i=1}^{k-1} b_i B_{k-i}(t). \end{aligned}$$

From [21], we have $\lim\limits_{\tau\to\infty} h_{\frac{1}{\alpha}}(\tau,t)=0$. Also the following equation govern on the law of $A^{\frac{1}{\nu}}(t)$:

$$ {}_t^{RL}D^{\alpha}_{+\infty} h_{\frac{1}{\alpha}}(\tau,t)=\frac{\partial}{\partial\tau}h_{\frac{1}{\alpha}}(\tau,t), \qquad \tau,t>0\ ,\ \alpha\in(1\,,\,+\infty)\,, $$

with

$$ \begin{cases} h_{\frac{1}{\alpha}}(0,t)=0, \\ \\ h_{\frac{1}{\alpha}}(\tau,0)=\delta(\tau). \end{cases} $$

In [22] and [23], this result is proved for $\alpha=n\in N$ and $\alpha>1$, respectively. So, since $\frac{d^{\alpha}}{dt^{\alpha}}:=-{}_t^{RL}D^{\alpha}_{+\infty}$ for $\alpha\in(1\,,+\infty)$, the proof is complete .□

Now, we focus on the exact solution to (1)-(2) for $\alpha\in(1\,,\,+\infty)$.

Theorem 2.3. The exact solution of FKFE (1)-(2) for $\alpha\in(1\,,\,+\infty)$ is

$$ B^{\alpha}_{k}(t)=\begin{cases} e^{-t\lambda^{\frac{1}{\alpha}}} & k=1, \\ \sum_{r=1}^{k-1}\left(\sum_{(i_1\,,\,i_2\,,\,\ldots\,,\,i_r)\,\in W_r^{k-1}}\left(\prod\limits_{j=1}^{r} b_{i_j}\right)\right)\frac{\lambda^r}{r!}(-1)^r\frac{d^r}{d\lambda^r}\left(e^{-t\lambda^{\frac{1}{\alpha}}}\right) \\ k\geq 2. \end{cases} $$

Proof. by Theorem 2.2 and (10) for $\alpha\in(1\,,\,+\infty)$, we obtain

$$ \begin{aligned} B^{\alpha}_{k}(t) &= \int_0^{+\infty} B^{1}_{k}(\tau)\,h_{\frac{1}{\alpha}}(\tau,t)d\tau \\ &= \begin{cases} \int_0^{+\infty} e^{-\lambda\,z} h_{\frac{1}{\alpha}}(\tau,t)d\tau & k=1, \\ \sum_{r=1}^{k-1}\left(\sum_{(i_1\,,\,i_2\,,\,\ldots\,,\,i_r)\,\in W_r^{k-1}}\left(\prod\limits_{j=1}^{r} b_{i_j}\right)\right)\frac{\lambda^r}{r!}\int_0^{+\infty} e^{-\lambda\,\tau}\tau^r h_{\frac{1}{\alpha}}(\tau,t)d\tau \\ k\geq 2, \end{cases} \\ &= \begin{cases} e^{-t\lambda^{\frac{1}{\alpha}}} & k=1, \\ \sum_{r=1}^{k-1}\left(\sum_{(i_1\,,\,i_2\,,\,\ldots\,,\,i_r)\,\in W_r^{k-1}}\left(\prod\limits_{j=1}^{r} b_{i_j}\right)\right)\frac{\lambda^r}{r!}(-1)^r\frac{d^r}{d\lambda^r}\left(e^{-t\lambda^{\frac{1}{\alpha}}}\right) \\ k\geq 2, \end{cases} \end{aligned} $$

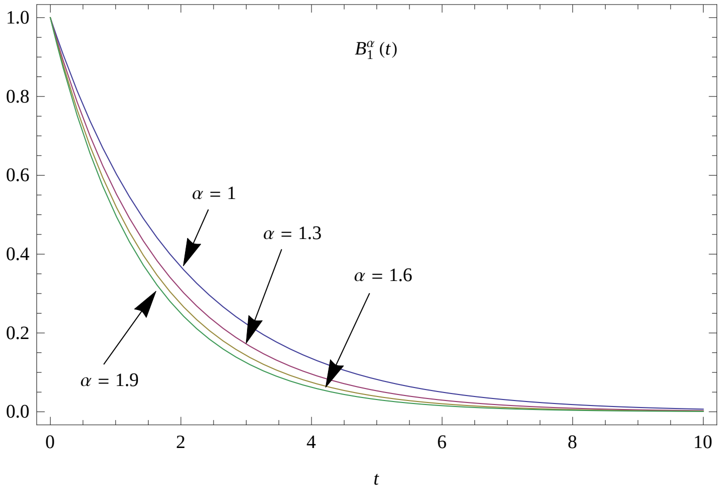

Figure 4: Plot of $B_1^\alpha(t)$ for $\rho = 0.5$, $\lambda = 0.5$ and $\alpha = 1$, 1.3, 1.6, 1.9.

because from [8], we know $\int_0^{+\infty} e^{-\lambda\tau} h_{\frac{1}{\alpha}}(\tau, t)\, d\tau = e^{-t\lambda^{\frac{1}{\alpha}}}$.
Also since $h_{\frac{1}{\alpha}}(z, 0) = \delta(z)$, for the initial conditions in (2) we can write:

$$\begin{aligned} B_k^\alpha(0) &= \int_0^{+\infty} B_k^1(\tau) h_{\frac{1}{\alpha}}(\tau, 0)\, d\tau \\ &= B_1^k(0) = \begin{cases} 1, & k = 1 \\ 0, & k \in \{2, 3, \dots\} \end{cases}. \end{aligned}$$

□

Similarly, let $b_i = (1-\rho)\rho^{i-1}$. Then, we show plots of exact solutions $B_k^\alpha(t)$ for the problem (1) and (2) for $\rho = 0.5$, $\lambda = 0.5$, $k = 1$, 2, 3 and $\alpha = 1$, 1.3, 1.6, 1.9 in figures 4-6.

3 Conclusion

In this work, by Laplace transforms and stable subordinators, we have obtained the analytical solutions of fractional Kolmogorov forward equations

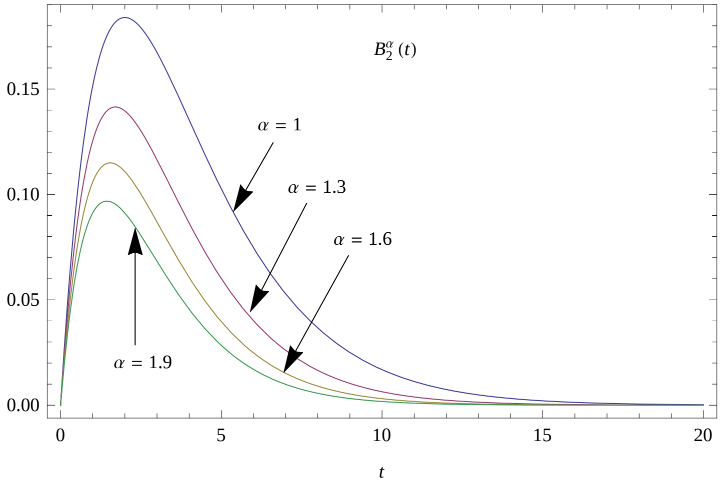

Figure 5: Plot of $B_2^\alpha(t)$ for $\rho = 0.5$, $\lambda = 0.5$ and $\alpha = 1$, 1.3, 1.6, 1.9.

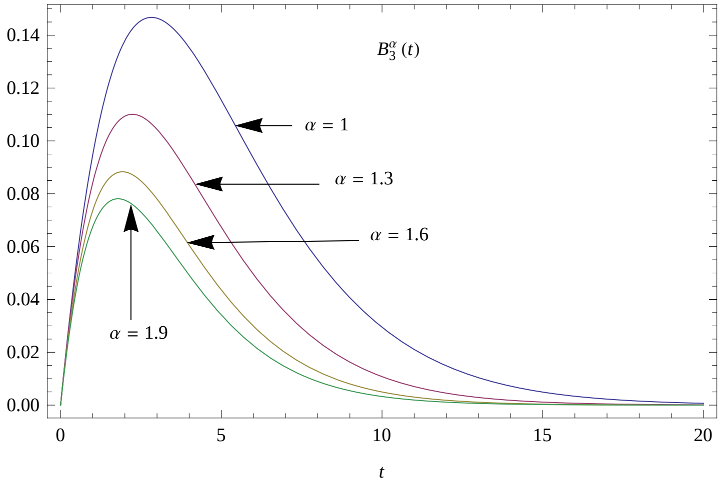

Figure 6: Plot of $B_3^\alpha(t)$ for $\rho = 0.5$, $\lambda = 0.5$ and $\alpha = 1$, 1.3, 1.6, 1.9.

with the left Caputo fractional derivative for $\alpha \in (0\,,\,1]$ and the right Riemann-Liouville fractional derivative on R_+ for $\alpha \in (1\,,\,+\infty)$. These problems cover the well-known Kolmogorov forward equations, fractional relaxation equations and discrete version of the fractional master equation.

References

[1] Anber A., Belarbi S., Dahmani Z. (2013), New existence and uniqueness results for fractional Differential equations, *An. St. Univ. Ovidius Constanta*, 21, n.3, 33-41.

[2] Cernea A. (2015), Some remarks on a fractional integro-differential inclusion with boundary conditions, *An. St. Univ. Ovidius Constanta*, 23, n.1, 73-82.

[3] Mainardi F. (1996), The fundamental solutions for the fractional diffusion-wave equation, *Appl. Math. Lett.*, 9, n.6, 23-28.

[4] Mainardi F. (1996), Fractional relaxation-oscillation and fractional diffusion-wave phenomena, *Chaos, Solitons and Fractals*, 7, n.9, 1461-1477.

[5] Angulo J.M., Ruiz-Medina, M.D., Anh V.V., Greckosch W. (2000), Fractional diffusion and fractional heat equation, *Adv. in Appl. Probab.*, 32, 1077-1099.

[6] Orsingher E., Beghin L. (2009), Fractional diffusion equations and processes with randomly-varying time, *Ann. Probab.*, 37, n.1, 206-249.

[7] Orsingher, E., Beghin, L. (2004), Time-fractional equations and telegraph processes with Brownian time, *Probability Theory and Related Fields*, 128, 141-160.

[8] Laskin, N. (2003), Fractional Poisson process, *Communications in Nonlinear Science and Numerical Simulation,* , 8, 201-213.

[9] Mainardi F., Gorenflo R., Scalas E. (2004), A fractional generalization of the Poisson processes, *Vietnam J. Math.*, 32, 53-64.

[10] Beghin L., Orsingher E. (2009), Fractional Poisson processes and related planar random motions, *Electron. J. Probab.*, 14, n.61, 1790-1826.

[11] Meerschaert M. M. Nane, E., Veillaisamy P. (2011), The fractional Poisson process and the inverse stable subordinator. *Electron. J. Probab.*, 59, 1600-1620.

[12] Karlin, S., Taylor, H.M. (1981), A Second Course in Stochastic Processes. Academic Press, London.

[13] Beghin. L. (2012),Fractional relaxation equations and Brownian crossing probabilities of a random boundary. *Adv. in Appl. Probab.*, 44, 479-505.

[14] Hilfer, R., Anton, L. (1995), Fractional master equations and fractal random walks. Phys. Rev. E 51 848–851.

[15] Kilbas A.A., Srivastava H.M., Trujillo J.J. (2006), *Theory and Applications of Fractional Differential Equations*, vol. 204 of North-Holland Mathematics Studies, Elsevier Science B.V., Amsterdam.

[16] Podlubny I. (1999), *Fractional Differential Equations*, San Diego:Academic Press.

[17] Baleanu D, Diethelm K, Scalas E, Trujillo J. (2012), *Fractional calculus models and numerical methods*. Singapore: World Scientific.

[18] G. Samko, A. Kilbas, O. Marichev, (1993), *Fractional integrals and derivatives: Theory and applications*, Gordon and Breach.

[19] Samorodnitsky G., Taqqu M.S. (1994), *Stable Non-Gaussian Random Processes*. Chapman Hall, New York.

[20] Hahn M.G., Kobayashi K., Umarov S. (2001), Fokker-Planck-Kolmogorov equations associated with time-changed fractional Brownian motion, *Proc. Amer. Math. Soc.*, 139, n.2, 691-705.

[21] Uchaikin V.V., Zolotarev V.M. (1999), *Chance and Stability: Stable Distributions and their Applications*, VSP, Utrecht.

[22] D'Ovidio M. (2011), On the fractional counterpart of the higher-order equations, *Statist. Probab. Lett.*, 81, 1929-1939.

[23] Beghin L., Macci C. (2012), Alternative forms of compound fractional Poisson processes, *Abstr. Appl. Anal.*, 2012, 1-30.

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