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On the solutions of electrohydrodynamic flow with fractional differential equations by reproducing kernel method

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Abstract: In this manuscript we investigate electrodynamic flow. For several values of the intimate parameters we proved that the approximate solution depends on a reproducing kernel model. Obtained results prove that the reproducing kernel method (RKM) is very effective. We obtain good results without any transformation or discretization. Numerical experiments on test examples show that our proposed schemes are of high accuracy and strongly support the theoretical results.

Keywords: kernel functions; electrohydrodynamic flow; approximate solutions

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1 Introduction

The electrohydrodynamic flow of a fluid is governed by a non-linear ordinary differential equation. The degree of non-linearity is stated by a nondimensional variable α and the equation can be approached by two different linear equations for very small or very large values of α respectively. The electrohydrodynamic flow of a fluid has been researched by McKee [21]. The governing equations were turned to the following problem [20]:

$$\frac{d^\gamma \varphi}{dr^\gamma} + \frac{1}{r}\frac{d^\beta \varphi}{dr^\beta} + H^2\left(1 - \frac{\varphi}{1-\alpha\varphi}\right) = 0, \quad 0 < r < 1, \quad (1)$$

with the boundary conditions

$$\varphi'(0) = \varphi(1) = 0, \quad (2)$$

where $\varphi(r)$ is the fluid speed, r is the radial range from the center of the cylindrical conduit, H is the Hartmann electric number, the parameter α is the size of the power of the nonlinearity and $\gamma = 2$, $\beta = 1$. Paullet [23] showed the existence and uniqueness of a solution to (1)–(2), and explored an error in the perturbative and numerical solutions given in [21] for large values of α.

Fractional calculus is a 300 years old and has been enhanced progressively up to now. The concept of differentiation to fractional order was described in 19th century by Rieman and Liouville. In several problems of physics, mechanics and engineering, fractional differential equations have been demonstrated to be a valuable tool in modeling many phenomena. However, most fractional order equations do not have analytic solutions. Therefore, there has been an important interest in developing numerical methods for the solutions of fractional-order differential equations [18]. Fractional differential equations, as an important research branch, have attracted much interest recently [9]. We recall that a general solution technique for fractional differential equations has not yet been constituted. Most of the solution methods in this area have been enhanced for significant sorts of problems. Consequently, a single standard method for problems related fractional calculus has not been found. Thus, determining credible and affirmative solution methods along with fast application techniques is beneficial and worthy of further examination [3]. For more details see [10–14, 24, 30].

The goal of this paper is to give approximate solutions to (1)–(2) for all values of the relevant variables using the RKM. Recently, much interest has been dedicated to the work of the RKM to research several scientific models [4]. The book [6] presents an overview for the

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RKM. Many problems such as population models and complex dynamics have been solved in the reproducing kernel spaces [7, 17, 26, 27]. For more details of this method see [1, 2, 5, 8, 15, 16, 19, 25, 28, 29].

This study is arranged as follows. Section 2 presents useful reproducing kernel functions. Solutions in $W_2^3[0,1]$ and a related linear operator are given in Section 3. This section demonstrates the fundamental results. The exact and approximate solutions of (1)–(2) are given in this section. Examples are shown in Section 4. Some conclusions are given in the final section.

Definition 1.1. A Hilbert space H which is defined on a nonempty set E is denominated a *reproducing kernel Hilbert space* if there exists a reproducing kernel function $K : E \times E \to \mathbb{C}$.

2 Construction of reproducing kernel space

Definition 2.1. $G_2^1[0,1]$ is defined by:

$$G_2^1[0,1] = \{\varphi \in AC[0,1] : \varphi' \in L^2[0,1]\}.$$

$$\langle \varphi, \psi \rangle_{G_2^1} = \varphi(0)\psi(0) + \int_0^1 \varphi'(r)\psi'(r)\mathrm{d}r, \quad \varphi, \psi \in G_2^1[0,1]$$

and

$$\|\varphi\|_{G_2^1} = \sqrt{\langle \varphi, \varphi \rangle_{G_2^1}}, \quad \varphi \in G_2^1[0,1],$$

are the inner product and the norm in $G_2^1[0,1]$.

Lemma 2.2 (See [6, page 17]). *Reproducing kernel function Q_θ of $G_2^1[0,1]$ is obtained as:*

$$Q_\theta(r) = \begin{cases} 1 + r, & 0 \le r \le \theta \le 1, \\ 1 + \theta, & 0 \le \theta < r \le 1. \end{cases}$$

Definition 2.3. We denote the space $W_2^3[0,1]$ by

$$W_2^3[0,1] = \Big\{\varphi \in AC[0,1] : \varphi', \varphi'' \in AC[0,1], \varphi^{(3)} \in L^2[0,1], \ \varphi'(0) = 0 = \varphi(1)\Big\}.$$

$$\langle \varphi, \psi \rangle_{W_2^3} = \sum_{i=0}^{2} \varphi^{(i)}(0)\psi^{(i)}(0) + \int_0^1 \varphi^{(3)}(r)\psi^{(3)}(r)\mathrm{d}r, \quad \varphi, \psi \in W_2^3[0,1]$$

and

$$\|\varphi\|_{W_2^3} = \sqrt{\langle \varphi, \varphi \rangle_{W_2^3}}, \quad \varphi \in W_2^3[0,1],$$

are the inner product and the norm in $W_2^3[0,1]$.

Theorem 2.4. *Reproducing kernel function B_θ of $W_2^3[0,1]$ is acquired as*

$$B_\theta(r) = \begin{cases} \frac{5}{624}r^2\theta^4 - \frac{1}{624}r^2\theta^3 - \frac{5}{312}r^2\theta^3 + \frac{21}{104}r^2\theta^2 \\ +\frac{5}{1872}r^3\theta^4 - \frac{1}{1872}r^3\theta^5 - \frac{5}{936}r^3\theta^3 + \frac{7}{104}r^3\theta^2 \\ -\frac{5}{3744}r^4\theta^4 + \frac{1}{3744}r^4\theta^5 + \frac{5}{1872}r^4\theta^3 + \frac{5}{624}r^4\theta^2 \\ -\frac{5}{24}r^4\theta + \frac{1}{3744}r^5\theta^4 - \frac{1}{18720}r^5\theta^5 - \frac{1}{1872}r^5\theta^3 \\ -\frac{1}{624}r^5\theta^2 - \frac{5}{26}r^2 - \frac{5}{78}r^3 + \frac{5}{156}r^4 - \frac{5}{26}\theta^2 - \frac{5}{78}\theta^3 \\ +\frac{5}{156}\theta^4 - \frac{1}{156}\theta^5 + \frac{3}{13} + \frac{1}{520}r^5, \quad 0 \le r \le \theta \le 1, \\ \\ \frac{5}{624}\theta^2 r^4 - \frac{1}{624}\theta^2 r^3 - \frac{5}{312}\theta^2 r^3 + \frac{21}{104}\theta^2 r^2 \\ +\frac{5}{1872}\theta^3 r^4 - \frac{1}{1872}\theta^3 r^5 - \frac{5}{936}\theta^3 r^3 + \frac{7}{104}\theta^3 r^2 \\ -\frac{5}{3744}\theta^4 r^4 + \frac{1}{3744}\theta^4 r^5 + \frac{5}{1872}\theta^4 r^3 + \frac{5}{624}\theta^4 r^2 \\ -\frac{5}{24}\theta^4 r + \frac{1}{3744}\theta^5 r^4 - \frac{1}{18720}\theta^5 r^5 - \frac{1}{1872}\theta^5 r^3 \\ -\frac{1}{624}\theta^5 r^2 - \frac{5}{26}\theta^2 - \frac{5}{78}\theta^3 + \frac{5}{156}\theta^4 - \frac{5}{26}r^2 - \frac{5}{78}r^3 \\ +\frac{5}{156}r^4 - \frac{1}{156}r^5 + \frac{3}{13} + \frac{1}{520}\theta^5, \quad 0 \le \theta < r \le 1. \end{cases} \tag{3}$$

Proof. Let $\varphi \in W_2^3[0,1]$ and $0 \le \theta \le 1$. By using the definition 3 and integrating by parts, we acquire

$$\begin{aligned} \langle \varphi, B_\theta \rangle_{W_2^3} &= \sum_{i=0}^{2} \varphi^{(i)}(0)B_\theta^{(i)}(0) + \int_0^1 \varphi^{(3)}(r)B_\theta^{(3)}(r)\mathrm{d}r \\ &= \varphi(0)B_\theta(0) + \varphi'(0)B_\theta'(0) + \varphi''(0)B_\theta''(0) \\ &+ \varphi''(1)B_\theta^{(3)}(1) - \varphi''(0)B_\theta^{(3)}(0) - \varphi'(1)B_\theta^{(4)}(1) \\ &+ \varphi'(0)B_\theta^{(4)}(0) + \int_0^1 \varphi'(r)B_\theta^{(5)}(r)\mathrm{d}r. \end{aligned}$$

After substituting the values of $B_\theta(0)$, $B_\theta'(0)$, $B_\theta''(0)$, $B_\theta^{(3)}(0)$, $B_\theta^{(4)}(0)$, $B_\theta^{(3)}(1)$, $B_\theta^{(4)}(1)$ into the above equation we get

$$\langle \varphi, B_\theta \rangle_{W_2^3} = \varphi(\theta).$$

This completes the proof. □

3 Representation of the solutions

The solution of (1)–(2) is acquired in the $W_2^3[0,1]$. We describe the linear operator $T : W_2^3[0,1] \to G_2^1[0,1]$ by

$$T\varphi = \frac{d^\gamma \varphi}{dr^\gamma} + \frac{1}{r}\frac{d^\beta \varphi}{dr^\beta}, \quad \varphi \in W_2^3[0,1]. \tag{4}$$

The problem (1)–(2) alters to the problem

$$\begin{cases} T\varphi = z(r, \varphi), \\ \varphi(1) = 0, \quad \varphi'(0) = 0, \end{cases} \tag{5}$$

where $z(r, \varphi) = -H^2\left(1 - \frac{\varphi}{1-\alpha\varphi}\right)$.

Theorem 3.1. *T is a bounded linear operator.*

Proof. We will show $\|T\varphi\|^2_{G^1_2} \le K\|\varphi\|^2_{W^3_2}$. We get

$$\|T\varphi\|^2_{G^1_2} = \langle T\varphi, T\varphi\rangle_{G^1_2} = \left[T\varphi(0)\right]^2 + \int_0^1 \left[T\varphi'(r)\right]^2 \mathrm{d}r,$$

by definition 2.1. By the reproducing property, we obtain

$$\varphi(r) = \left\langle \varphi(\cdot), B_r(\cdot)\right\rangle_{W^3_2},$$

and

$$T\varphi(r) = \left\langle \varphi(\cdot), TB_r(\cdot)\right\rangle_{W^3_2}.$$

Thus,

$$\left|T\varphi(r)\right| \le \|\varphi\|_{W^3_2}\|TB_r\|_{W^3_2} = K_1\|\varphi\|_{W^3_2},$$

where $K_1 > 0$. Therefore,

$$\left[(T\varphi)(0)\right]^2 \mathrm{d}r \le K_1^2\|\varphi\|^2_{W^3_2}.$$

Considering that

$$(T\varphi)'(r) = \left\langle \varphi(\cdot), (TB_r)'(\cdot)\right\rangle_{W^3_2},$$

then

$$\left|(T\varphi)'(r)\right| \le \|\varphi\|_{W^3_2}\left\|(TB_r)'\right\|_{W^3_2} = K_2\|\varphi\|_{W^3_2},$$

where $K_2 > 0$. Thus, we acquire

$$\left[(T\varphi)'(r)\right]^2 \le K_2^2\|\varphi\|^2_{W^3_2},$$

and

$$\int_0^1 \left[(T\varphi)'(r)\right]^2 \mathrm{d}r \le K_2^2\|\varphi\|^2_{W^3_2}.$$

Therefore, we get

$$\begin{aligned}\|T\varphi\|^2_{G^1_2} &\le \left[(T\varphi)(0)\right]^2 + \int_0^1 \left[(T\varphi)'(r)\right]^2 \mathrm{d}r \\ &\le \left(K_1^2 + K_2^2\right)\|\varphi\|^2_{W^3_2} = K\|\varphi\|^2_{W^3_2},\end{aligned}$$

where $K = K_1^2 + K_2^2 > 0$. This completes the proof. □

We denote $\varrho_i(r) = Q_{r_i}(r)$ and $\eta_i(r) = T^*\varrho_i(r)$. The orthonormal system $\left\{\widehat{\eta}_i(r)\right\}_{i=1}^{\infty}$ of $W^3_2[0,1]$ is obtained from Gram-Schmidt orthogonalization process of $\{\eta_i(r)\}_{i=1}^{\infty}$ and

$$\widehat{\eta}_i(r) = \sum_{k=1}^{i}\sigma_{ik}\eta_k(r), \quad (\sigma_{ii} > 0, \quad i = 1, 2, \ldots). \tag{6}$$

Theorem 3.2. *Let $\{r_i\}_{i=1}^{\infty}$ be dense in $[0,1]$ and $\eta_i(r) = T_\theta B_r(\theta)\big|_{\theta=r_i}$. Then, the sequence $\left\{\eta_i(r)\right\}_{i=1}^{\infty}$ is a complete system in $W^3_2[0,1]$.*

Proof. We obtain

$$\begin{aligned}\eta_i(r) &= (T^*\varrho_i)(r) = \left\langle (T^*\varrho_i)(\theta), B_r(\theta)\right\rangle = \left\langle(\varrho_i)(\theta), T_\theta B_r(\theta)\right\rangle \\ &= T_\theta B_r(\theta)\big|_{\theta=r_i}.\end{aligned}$$

Therefore, $\eta_i(r) \in W^3_2[0,1]$. For each fixed $\varphi(r) \in W^3_2[0,1]$, let $\left\langle \varphi(r), \eta_i(r)\right\rangle = 0$, $(i = 1, 2, \ldots)$, i.e.,

$$\left\langle \varphi(r), (T^*\varrho_i)(r)\right\rangle = \left\langle T\varphi(\cdot), \varrho_i(\cdot)\right\rangle = (T\varphi)(r_i) = 0.$$

Thus, $(T\varphi)(x) = 0$ and $\varphi \equiv 0$. This completes the proof. □

Theorem 3.3. *If $\varphi(r)$ is the exact solution of* (5)*, then*

$$\varphi(r) = T^{-1}z(r,\varphi) = \sum_{i=1}^{\infty}\sum_{k=1}^{i}\sigma_{ik}z(r_k, \varphi(r_k))\widehat{\eta}_i(r), \tag{7}$$

where $\{(r_i)\}_{i=1}^{\infty}$ is dense in $[0,1]$.

Proof. We acquire

$$\begin{aligned}\varphi(r) &= \sum_{i=1}^{\infty}\left\langle \varphi(r), \widehat{\eta}_i(r)\right\rangle_{W^3_2}\widehat{\eta}_i(r) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\sigma_{ik}\left\langle \varphi(r), \eta_k(r)\right\rangle_{W^3_2}\widehat{\eta}_i(r) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\sigma_{ik}\left\langle \varphi(r), T^*\varrho_k(r)\right\rangle_{W^3_2}\widehat{\eta}_i(r) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\sigma_{ik}\left\langle T\varphi(r), \varrho_k(r)\right\rangle_{G^1_2}\widehat{\eta}_i(r),\end{aligned}$$

from (6). By uniqueness of the solution of (5), we acquire

$$\begin{aligned}\varphi(r) &= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\sigma_{ik}\left\langle z(r,\varphi), Q_{r_k}\right\rangle_{G^1_2}\widehat{\eta}_i(r) \\ &= \sum_{i=1}^{\infty}\sum_{k=1}^{i}\sigma_{ik}z(r_k, \varphi(r_k))\widehat{\eta}_i(r).\end{aligned}$$

□

The approximate solution $\varphi_n(r)$ is achieved as

$$\varphi_n(r) = \sum_{i=1}^{n}\sum_{k=1}^{i}\sigma_{ik}z(r_k, \varphi(r_k))\widehat{\eta}_i(r). \tag{8}$$

Theorem 3.4. *Let φ be any solution of* (1) *in $W^3_2[0,1]$. Then*

$$\|\varphi_n - \varphi\|_{W^3_2} \to 0, \quad n \to \infty.$$

Moreover the sequence $\|\varphi_n - \varphi\|_{W_2^3}$ *is monotonically decreasing in* n.

Proof. We obtain

$$\|\varphi_n - \varphi\|_{W_2^3} = \left\| \sum_{i=n+1}^{\infty} \sum_{k=1}^{i} \sigma_{ik} z(r_k, \varphi(r_k)) \widehat{\eta}_i(r) \right\|_{W_2^3},$$

by (7) and (8). Therefore

$$\|\varphi_n - \varphi\|_{W_2^3} \to 0, \quad n \to \infty.$$

Furthermore

$$\|\varphi_n - \varphi\|_{W_2^3}^2 = \left\| \sum_{i=n+1}^{\infty} \sum_{k=1}^{i} \sigma_{ik} z(r_k, \varphi(r_k)) \widehat{\eta}_i(r) \right\|_{W_2^3}^2$$

$$= \sum_{i=n+1}^{\infty} \left(\sum_{k=1}^{i} \beta_{ik} \widehat{\Psi}_i \right)^2 .$$

Obviously, $\|\varphi_n - \varphi\|_{W_2^3}$ is monotonically decreasing in n. □

4 Numerical experiments

We solve (1)–(2) numerically in this section. Tables 1–2 present the approximate solutions of the problem (1)–(2) for different values of γ, β and α. Figures 1–2 show the approximate solutions for several values of the intimate variables. The results depend on both H and α. We use MAPLE to solve the BVP. In figures 1–2 we give numerical solutions of the BVP for values of $\alpha = 0.5, 1.0$ and $H^2 = 0.5, 1.0, 2.0$.

REMARK 4.1. A Spectral Method [22] and Homotopy analysis method [20] have been applied to the electrohydrodynamic flow. Our results are in good agreement with the results obtained by these methods. Therefore the RKM is a reliable method for electrohydrodynamic flow.

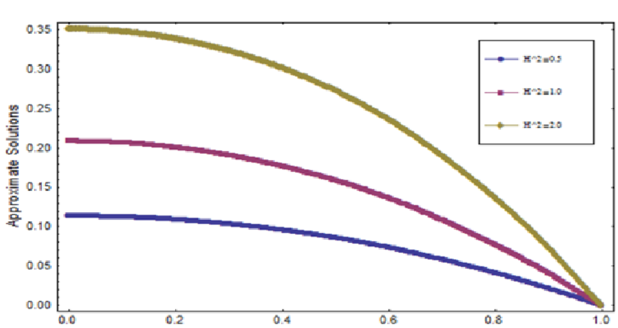

Figure 1: Graph of numerical results for $\alpha = 0.5, \gamma = 2, \beta = 1$ and several values of H.

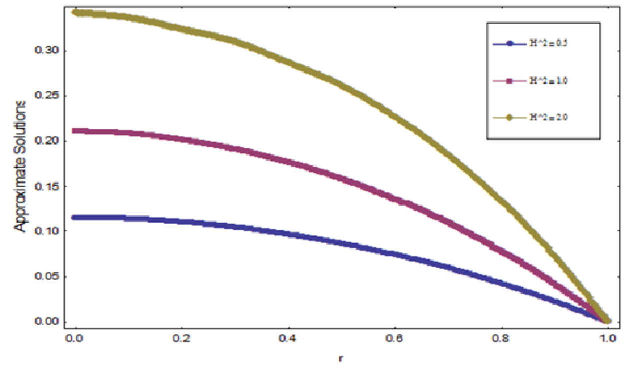

Figure 2: Graph of numerical results for $\gamma = 2, \beta = 1, \alpha = 1.0$ and several values of H.

Table 1: Approximate solutions of (1)–(2) when $\alpha = 0.5$.

r	$\gamma = 1.9, \beta = 0.9$	$\gamma = 1.9, \beta = 1.0$
0.0	0.381236310	0.3771173839
0.1	0.374950080	0.3713730420
0.2	0.359968470	0.3575225829
0.3	0.342105510	0.3401839144
0.4	0.315723980	0.3146921558
0.5	0.284128540	0.2835844577
0.6	0.244546771	0.2445250520
0.7	0.197105564	0.1974472101
0.8	0.141053509	0.1415539908
0.9	0.075613972	0.0760178037
1.0	2.9691×10^{-11}	-5.856×10^{-10}

Table 2: Approximate solutions of (1)–(2) when $\alpha = 1.0$.

r	$\gamma = 1.9, \beta = 0.9$	$\gamma = 1.9, \beta = 1.0$
0.0	0.317659270	0.3348212962
0.1	0.311127694	0.3288793293
0.2	0.297816605	0.3162240251
0.3	0.285820932	0.3028615829
0.4	0.265485319	0.2812364427
0.5	0.245783017	0.2579063018
0.6	0.215406643	0.2247358955
0.7	0.176723589	0.1834843558
0.8	0.128333833	0.1327839543
0.9	0.069612052	0.0718612844
1.0	2.858×10^{-11}	-7.983×10^{-9}

5 Conclusion

In this work, the reproducing kernel method (RKM) has been performed to acquire solutions for a nonlinear boundary value problems. We came across an important challenge in regard to attaining solutions however, we have shown that the solutions obtained are convergent. We obtained good results for different values of α, β and γ in (1)–(2). Reproducing kernel functions were found to be very useful to get these results and they prove that the RKM is very effective.

Competing interests: The authors declare that they have no competing interests.

References

[1] Abbasbandy S., B. Azarnavid, M. S. Alhuthali, A shooting reproducing kernel Hilbert space method for multiple solutions of nonlinear boundary value problems. J. Comput. Appl. Math., 2015, 279, 293–305

[2] Akgül A., New reproducing kernel functions, Math. Probl. Eng., pages Art. ID 158134, 2015, 10

[3] Akgül A., Inc M., Karatas E., Baleanu D., Numerical solutions of fractional differential equations of Lane-Emden type by an accurate technique, Adv. Difference Equ., 2015, 220

[4] Aronszajn N., Trans. Amer. Math. Soc., 1950, 68, 337–404

[5] Bushnaq S., Maayah B., Momani S., Alsaedi A., A reproducing kernel Hilbert space method for solving systems of fractional integro differential Equations, Abstr. Appl. Anal., pages Art. ID 103016, 2014, 6

[6] Cui M., Lin Y., Nova Science Publishers Inc., New York, 2009

[7] Du J., Cui M., Solving the forced Duffing equation with integral boundary conditions in the reproducing kernel space. Int. J. Comput. Math., 2010, 87(9), 2088–2100

[8] Geng F. Z., Qian S. P., Solving singularly perturbed multi pantograph delay equations based on the reproducing kernel method. Abstr. Appl. Anal., pages Art. ID 794716, 2014, 6

[9] Geng F., Cui M., A reproducing kernel method for solving nonlocal fractional boundary value problems. Appl. Math. Lett., 2012, 25(5), 818–823

[10] Hashemi M. S., Constructing a new geometric numerical integration method to the nonlinear heat transfer equations. Commun. Nonlinear Sci. Numer. Simul., 2015, 22(1-3), 990-1001

[11] Hashemi M. S., Abbasbandy S., A geometric approach for solving troesch's problem, Bulletin of the Malaysian Mathematical Sciences Society, 2015, 1-20

[12] Hashemi M. S., Baleanu D., Numerical approximation of higher-order time fractional telegraph equation by using a combination of a geometric approach and method of line, J. Comput. Phys., 2016, 316, 10–20

[13] Hashemi M. S., Baleanu D., Haghighi M. P., A lie group approach to solve the fractional poisson equation, Romanian Journal of Physics, 2015, 60, 1289–1297

[14] Hashemi M. S., Baleanu D., Haghighi M. P., Darvishi E., Solving the timefractional diffusion equation using a lie group integrator, Therm. Sci., 2015, 19, 77–83

[15] Inc M., Akgül A., Kilicman A., Numerical solutions of the second order One-dimensional telegraph equation based on reproducing kernel Hilbert space method. Abstr. Appl. Anal., pages Art. ID 768963, 2013, 13

[16] Javadi S., Babolian E., Moradi E., New implementation of reproducing kernel Hilbert space method for solving a class of functional integral equations, Commun. Numer. Anal., pages Art. ID 00205, 2014, 7

[17] Jiang W., Cui M., Solving nonlinear singular pseudoparabolic equations with nonlocal mixed conditions in the reproducing kernel space, Int. J. Comput. Math., 2010, 87(15), 3430-3442

[18] Jiang W., Tian T., Numerical solution of nonlinear Volterra integro differential equations of fractional order by the reproducing kernel method. Appl. Math. Model., 2015, 39(16), 4871-4876

[19] Komashynska I., Smadi M. A. , Iterative reproducing kernel method for solving second-order integrodifferential equations of Fredholm type, J. Appl. Math., pages Art. ID 459509, 2014, 11

[20] Mastroberardino A., Homotopy analysis method applied to electrohydrodynamic flow. Commun. Nonlinear Sci. Numer. Simul., 2011, 16(7), 2730-2736

[21] McKee S., Watson R., Cuminato J. A., Caldwell J., Chen M. S., Calculation of electrohydrodynamic flow in a circular cylindrical conduit, Z. Angew. Math. Mech., 1997, 77(6), 457-465

[22] Moghtadaei M., Nik H. S., Abbasbandy S., A spectral method for the electrohydrodynamic flow in a circular cylindrical conduit, Chin. Ann. Math. Ser. B, 2015, 36(2), 307- 322

[23] Paullet J. E., On the solutions of electrohydrodynamic flow in a circular cylindrical Conduit, ZAMM Z. Angew. Math. Mech., 1999, 79(5), 357-360

[24] Sakar M. G., Iterative reproducing kernel Hilbert spaces method for Riccati differential equations, J. Comput. Appl. Math., 2017, 309, 163–174

[25] Shawagfeh N., Arqub O. A., Momani S., Analytical solution of nonlinear second-order periodic boundary value problem using reproducing kernel method, J. Comput. Anal. Appl., 2014, 16(4), 750-762

[26] Kumar B., Vangeepuram S., Reproducing Kernel Space Embeddings and Metrics on Probability Measures, ProQuest LLC, Ann Arbor, MI, Thesis (Ph.D.)–University of California, San Diego, 2010

[27] Wu B., Li X., Application of reproducing kernel method to third order three-point boundary value problems. Appl. Math. Comput., 2010, 217(7), 3425-3428

[28] Xu L., Luo B., Tang Y., Ma X., An efficient multiple kernel learning in reproducing kernel Hilbert spaces (RKHS), Int. J. Wavelets Multiresolut. Inf. Process., 13(2):1550008, 2015, 13

[29] Zayed A. I., Solution of the energy concentration problem in reproducingkernel Hilbert space, SIAM J. Appl. Math., 2015, 75(1), 21–37

[30] Zhang R., Lin Y. , A novel method for nonlinear boundary value problems. J. Comput. Appl. Math., 2015, 282, 77-82