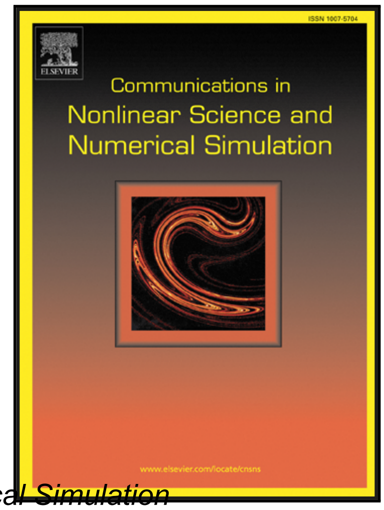
Accepted Manuscript



Stability analysis of Caputo–like discrete fractional systems

Dumitru Baleanu, Guo–Cheng Wu, Yun–Ru Bai, Fu–Lai Chen

PII: S1007-5704(17)30002-3
DOI: 10.1016/j.cnsns.2017.01.002
Reference: CNSNS 4073

To appear in: *Communications in Nonlinear Science and Numerical Simulation*

Received date: 11 June 2016
Revised date: 28 November 2016
Accepted date: 2 January 2017

Please cite this article as: Dumitru Baleanu, Guo–Cheng Wu, Yun–Ru Bai, Fu–Lai Chen, Stability analysis of Caputo–like discrete fractional systems, *Communications in Nonlinear Science and Numerical Simulation* (2017), doi: 10.1016/j.cnsns.2017.01.002

Highlights

- A theorem of asymptotical stability is proposed for discrete fractional systems.
- Implicit numerical formulae are derived and numerical solutions are obtained by use of Newton's method.
- Asymptotical behavior is illustrated according to the stability condition.

Stability analysis of Caputo–like discrete fractional systems

Dumitru Baleanu[1,2*], Guo–Cheng Wu[3†], Yun–Ru Bai[3], Fu–Lai Chen[4]

[1]Department of Mathematics, Cankaya University, 06530 Balgat, Ankara, Turkey;

[2]Institute of Space Sciences, Magurele–Bucharest, Romania;

[3]Data Recovery of Key Laboratory of Sichuan Province, College of Mathematics and Information Science, Neijiang Normal University, Neijiang 641100, PR China;

[4]Department of Mathematics, Xiangnan University, Chenzhou 423000, PR China.

January 6, 2017

Abstract

This study investigates stability of Caputo delta fractional difference equations. Solutions' monotonicity and asymptotic stability of a linear fractional difference equation are discussed. A stability theorem for a discrete fractional Lyapunov direct method is proved. Furthermore, an inequality is extended from the continuous case and a sufficient condition is given. Some linear, nonlinear and time varying examples are illustrated and the results show wide prospects of the stability theorems in fractional control systems of discrete time.

Keywords: Fractional difference equations; Monotonicity; Asymptotic stability; Caputo–like delta difference

1 Introduction

Fractional calculus is playing a crucial role in modern control areas. Stability theory of fractional differential equations is frequently used in fractional controllers. But due to the memory effects of fractional operators, it's not an easy task to directly extend the normal Lyapunov stability results to fractional cases since the Leibniz law becomes complicated and cannot hold generally.

Matignon [1] in his seminal work gave an explicit stability condition for a linear fractional equation. He derived a stability condition which depends on the order of a fractional derivative. Chen, Li et al. [2] proposed a fractional Lyapunov direct method. The authors defined and discussed Mittag–Leffler stability of fractional dynamical systems. Aguila–Camacho et al. [3] gave a useful inequality for sufficient conditions of Lyapunov stability without using the complicated Leibniz law. The methodology shows a broad prospect in fractional calculus and some recent applications are given in [4–10].

Fractional difference equations have been paid much attention recently, for example, initial value problem [11], boundary value problem [12–15], discrete fractional calculus of variation [16] and Laplace

*Email: dumitru@cankaya.edu.tr (D. Baleanu)

†Corresponding author, email: wuguocheng@gmail.com (G.C. Wu)

transform [17]. Some results of stability and other applications have been provided in [18–32]. In this paper, we use the idea [3] and give a direct method to analyze stability of the following discrete fractional systems

$$ {}_h^C\Delta_a^\nu x(t) = f(t+\nu h, x(t+\nu h)),\ 0<\nu\le 1, \tag{1}$$

where ${}_h^C\Delta_a^\nu x(t)$ denotes the Caputo–like delta fractional h–difference of $x(t)$ on sets of discrete time $(h\mathbb{N})_{a+(1-\nu)h} = \{a+(1-\nu)h, a+(2-\nu)h, ...\}$.

2 Preliminaries

The following definitions for the discrete fractional calculus are introduced.

Definition 2.1. *(Fractional sum [11, 33]) Let $x:(h\mathbb{N})_a \to \mathbb{R}$ and $0<\nu$ be given. a is a starting point. The ν–th order h–sum is given by*

$$ {}_h\Delta_a^{-\nu} x(t) := \frac{h}{\Gamma(\nu)}\sum_{s=\frac{a}{h}}^{\frac{t}{h}-\nu}(t-\sigma(sh))_h^{(\nu-1)}x(sh),\quad \sigma(sh)=(s+1)h, a\in\mathbb{R},\ t\in(h\mathbb{N})_{a+\nu h} \tag{2}$$

where the h–falling factorial function is defined as

$$ t_h^{(\nu)} = h^\nu\frac{\Gamma(\frac{t}{h}+1)}{\Gamma(\frac{t}{h}+1-\nu)},\ t,\nu\in\mathbb{R}. \tag{3}$$

Definition 2.2. *(Caputo delta difference [34–36]) For $x(t)$ defined on $(h\mathbb{N})_a$ and $0<\alpha, \alpha\notin\mathbb{N}$, the Caputo–like difference is defined by*

$$ {}_h^C\Delta_a^\alpha x(t) := \Delta_a^{-(m-\alpha)}\Delta^m x(t),\ t\in(h\mathbb{N})_{a+(m-\alpha)h}, \tag{4}$$

where $\Delta x(t) = \frac{x(t+h)-x(t)}{h}$ and $m=[\alpha]+1$.

If the fractional order α is a positive integer m, then we have the following definition

$$ {}_h^C\Delta_a^\alpha x(t) := \Delta^m x(t),\ t\in(h\mathbb{N})_a.$$

Property 2.3. *The following properties are useful in this paper:*

Fractional Caputo difference of a constant x [35]

$$ {}_h^C\Delta_a^\nu x(t) = 0,\ 0<\nu\le 1; \tag{5}$$

Delta difference of the h–falling factorial function [36]

$$ \Delta_s(t-sh)_h^{(\nu)} = -\nu(t-\sigma(sh))_h^{(\nu-1)}; \tag{6}$$

Discrete Leibniz integral law [35, 36]

$$ {}_h\Delta_{a+(1-\nu)h}^{-\nu}\,{}_h^C\Delta_a^\nu x(t) = x(t)-x(a),\ 0<\nu\le 1,\ t\in(h\mathbb{N})_{a+h}; \tag{7}$$

Summation by parts

$$ \sum_{s=\frac{a}{h}}^{\frac{t}{h}+\nu-1} g(sh+h)\Delta f(sh) = -\sum_{s=\frac{a}{h}}^{\frac{t}{h}+\nu-1} f(sh)\Delta g(sh) + \frac{1}{h}(f(sh)g(sh))\,\big|_{s=\frac{a}{h}}^{\frac{t}{h}+\nu},\quad t\in(h\mathbb{N})_{a+(1-\nu)h}. \tag{8}$$

For the following fractional difference equation

$$ {}_h^C\Delta_a^\nu x(t) = f(t + (\nu - 1)h, x(t + (\nu - 1)h)),\ t \in (h\mathbb{N})_{a+(1-\nu)h}, \tag{9} $$

some results have been obtained, for example, the definition of the Lyapunov stability in [37], existence and asymptotic results in [21, 22], asymptotic stability of discrete time varying systems [23] and applications of stability results [26]. Eq. (1) is a new fractional difference equation which is distinct from (9). See the numerical examples in Section 3. It's evident that Eq. (1) is an implicit relationship with respect to $x(n+1)$. To the best of our knowledge, few work has been done. In this section, we give some basics and stability results of the discrete fractional equations.

Definition 2.4. *Let $\mathbf{x} = 0$ be an equilibrium point of Eq. (1) and $t_0 = a$ is the initial time. The equilibrium point is said to be stable if for all $\epsilon > 0$ there exits a $\delta = \delta(t_0, \epsilon) > 0$ such that if*

$$ \|\mathbf{x}(t_0)\| < \delta \ \text{ then } \|\mathbf{x}(t)\| < \epsilon,\ \forall t > t_0,\ t \in (h\mathbb{N})_{t_0}. $$

Remark 2.5. *In this paper, we state that all the results are considered when the equilibrium point is $\mathbf{x} = 0$. If the equilibrium point is $\mathbf{x}^* \neq 0$, we can use a change of variable $\mathbf{y} = \mathbf{x} - \mathbf{x}^*$ and derive the Caputo difference of $\mathbf{y}$ as*

$$ {}_h^C\Delta_a^\nu \mathbf{y} = {}_h^C\Delta_a^\nu(\mathbf{x} - \mathbf{x}^*) = f(t, \mathbf{x}) = f(t, \mathbf{y} + \mathbf{x}^*) = m(t, \mathbf{y}),\ 0 < \nu \leq 1, \tag{10} $$

and we can see $m(t, 0) = f(t, \mathbf{x}^*) = 0$. The newly obtained system (10) has the equilibrium point $\mathbf{y} = 0$. As a result, we only need to consider the case of the equilibrium point $\mathbf{x} = 0$ in the original system.

Definition 2.6. *An equilibrium point is asymptotically stable if there exits a $\delta = \delta(t_0) > 0$ such that*

$$ \|\mathbf{x}(t_0)\| < \delta \ \text{ implies } \lim_{t\to\infty} \mathbf{x}(t) = 0,\ t \in (h\mathbb{N})_{t_0}. $$

Existence results of Riemann–Liouville (R–L) fractional difference equations were given in [22] by use of fixed point theorems. Considering the relationship between the R–L difference and the Caputo difference, we do not give the proof here. Readers are suggested to consult [21, 22] and the references therein. We can say, the following linear fractional difference equation

$$ {}_h^C\Delta_a^\nu u(t) = \lambda u(t + \nu h),\ u(a) = u_a,\ 0 < \nu \leq 1,\ t \in (h\mathbb{N})_{a+(1-\nu)h} \tag{11} $$

has a unique solution.

Lemma 2.7. *If $u(t)$ is a solution of Eq. (11) and $\lambda < 0$, then $\Delta u(t) < 0$ for all $t \in (h\mathbb{N})_a$.*

Proof. Arguing by the contradiction, we assume that there exists a first point t_1 such that $\Delta u(t) \geq 0$ in the set $[t_1,\ t_2] \bigcap (h\mathbb{N})_{a+h}$, where $t_1, t_2 \in (h\mathbb{N})_a$ and $\Delta u(t) < 0$ in the set $[a,\ t_1] \bigcap (h\mathbb{N})_{a+h}$.

Let $t_2^* = t_2 - \nu h$ and $t_1^* = t_1 - \nu h$. We can calculate that

$$
\begin{aligned}
&{}_h^C\Delta_a^\nu u(t_2^*) - {}_h^C\Delta_a^\nu u(t_1^*) \\
&= \frac{h}{\Gamma(1-\nu)} \sum_{s=\frac{a}{h}}^{\frac{t_2^*}{h}+\nu-1} (t_2^* - \sigma(sh))_h^{(-\nu)} \Delta_s u(sh) - \frac{h}{\Gamma(1-\nu)} \sum_{s=\frac{a}{h}}^{\frac{t_1^*}{h}+\nu-1} (t_1^* - \sigma(sh))_h^{(-\nu)} \Delta_s u(sh) \\
&= \frac{h}{\Gamma(1-\nu)} \sum_{s=\frac{a}{h}}^{\frac{t_1^*}{h}+\nu-1} ((t_2^* - \sigma(sh))_h^{(-\nu)} - (t_1^* - \sigma(sh))_h^{(-\nu)}) \Delta_s u(sh) \\
&+ \frac{h}{\Gamma(1-\nu)} \sum_{s=\frac{t_1^*}{h}+\nu}^{\frac{t_2^*}{h}+\nu-1} (t_2^* - \sigma(sh))_h^{(-\nu)} \Delta_s u(sh).
\end{aligned}
\tag{12}
$$

Since $(t_2^* - \sigma(sh))_h^{(-\nu)} > 0$ and $(t_2^* - \sigma(sh))_h^{(-\nu)} - (t_1^* - \sigma(sh))_h^{(-\nu)} < 0$, we can derive that

$$
{}_h^C\Delta_a^\nu u(t_2^*) - {}_h^C\Delta_a^\nu u(t_1^*) \geq 0.
$$

On the other hand, due to $\lambda < 0$ and $u(t_1) < u(t_2)$, we have

$$
{}_h^C\Delta_a^\nu u(t_2^*) - {}_h^C\Delta_a^\nu u(t_1^*) = \lambda(u(t_2^* + \nu h) - u(t_1^* + \nu h)) = \lambda(u(t_2) - u(t_1)) < 0,
$$

which results a contradiction and we conclude that the lemma holds.

□

Lemma 2.8. *If $\lambda < 0$ and $u(a) > 0$, the solution $u(t)$ of Eq. (11) holds that $u(t) > 0$ for all $t \in (h\mathbb{N})_a$.*

Proof. From Lemma 2.7, we can see that $\Delta u(t) < 0$ leads to

$$
{}_h^C\Delta_a^\nu u(t) < 0,\ 0 < \nu \leq 1,\ \forall t \in (h\mathbb{N})_{a+(1-\nu)h}.
$$

As a result, $u(t)$ of the r.h.s of (11) should be positive when $\lambda < 0$.

□

Lemma 2.9. *If $\lambda < 0$ and $u(a) > 0$, the solution $u(t)$ of Eq. (11) has the limit as*

$$
\lim_{t\to\infty} u(t) = 0,\ t \in (h\mathbb{N})_a.
$$

Proof. From Lemma 2.7 and Lemma 2.8, we can see the limit exists. Arguing by the contradiction, we assume $\lim_{t\to\infty} u(t) = c > 0$ for $t \in (h\mathbb{N})_a$.

Taking the fractional sum to both sides of (11), we have

$$
u(t) - u(a) = \frac{\lambda h}{\Gamma(\nu)} \sum_{s=\frac{a}{h}+1-\nu}^{\frac{t}{h}-\nu} (t - \sigma(sh))_h^{(\nu-1)} u(sh + \nu h), t \in (h\mathbb{N})_{a+h} \tag{13}
$$

Considering each $u(sh + \nu h) \geq u(t) > 0, s = \frac{a}{h} + 1 - \nu, ..., \frac{t}{h} - \nu$ and $\lambda < 0$, we get

$$
u(t) - u(a) \leq u(t) \frac{\lambda h}{\Gamma(\nu)} \sum_{s=\frac{a}{h}+1-\nu}^{\frac{t}{h}-\nu} (t - \sigma(sh))_h^{(\nu-1)} = \lambda \frac{u(t)(t + (\nu-1)h - a)_h^{(\nu)}}{\Gamma(1+\nu)},\ t \in (h\mathbb{N})_{a+h}. \tag{14}
$$

Let t tend to infinity. The l.h.s. of Eq. (14) becomes

$$\lim_{t\to\infty}(u(t)-u(a)) = c - u(a) < 0, \tag{15}$$

while the r.h.s. results

$$\lim_{t\to\infty} u(t)\frac{\lambda(t+(\nu-1)h-a)_h^{(\nu)}}{\Gamma(1+\nu)} = -\infty. \tag{16}$$

This yields a contradiction and we conclude that $\lim_{t\to\infty} u(t) = 0$. $\square$

Lemma 2.10. *Let $u(t)$ and $g(t)$ satisfy*

$${}_h^C\Delta_a^\nu u(t) = \lambda u(t+\nu h),\ 0<\nu\le 1 \tag{17}$$

and the inequality

$${}_h^C\Delta_a^\nu g(t) \le \lambda g(t+\nu h), \tag{18}$$

respectively. If $\lambda < 0$ and $g(a) = u(a)$, then $g(t)\le u(t)$ for all $t\in(h\mathbb{N})_a$.

Proof. Using the discrete Leibniz integral law (7), we can equivalently derive that

$$u(t) = u(a) + \frac{\lambda h^\nu}{\Gamma(\nu)}\sum_{s=\frac{a}{h}+1-\nu}^{\frac{t}{h}-\nu}\frac{\Gamma(\frac{t}{h}-s)}{\Gamma(\frac{t}{h}-s+\nu+1)}u(sh), t\in(h\mathbb{N})_{a+h} \tag{19}$$

and

$$g(t) \le g(a) + \frac{\lambda h^\nu}{\Gamma(\nu)}\sum_{s=\frac{a}{h}+1-\nu}^{\frac{t}{h}-\nu}\frac{\Gamma(\frac{t}{h}-s)}{\Gamma(\frac{t}{h}-s+\nu+1)}g(sh), t\in(h\mathbb{N})_{a+h}. \tag{20}$$

Let $t = a+nh$, $1\le n$ and $u(a+n) = u(a+nh)$ for simplicity in the rest of the paper. We have numerical formulae of the equations as

$$u(a+n) = u(a) + \frac{\lambda h^\nu}{\Gamma(\nu)}\sum_{j=0}^{n-1}\frac{\Gamma(n-1-j+\nu)}{\Gamma(n-j)}u(a+j+1) \tag{21}$$

and

$$g(a+n) \le g(a) + \frac{\lambda h^\nu}{\Gamma(\nu)}\sum_{j=0}^{n-1}\frac{\Gamma(n-1-j+\nu)}{\Gamma(n-j)}g(a+j+1), \tag{22}$$

respectively.

Since both sides appear $u(a+n)$ and $g(a+n)$, we solve Eq. (21) and (22) to obtain

$$u(a+n) = \frac{u(a)}{1-\lambda h^\nu} + \frac{\lambda h^\nu}{1-\lambda h^\nu}\frac{1}{\Gamma(\nu)}\sum_{j=0}^{n-2}\frac{\Gamma(n-1-j+\nu)}{\Gamma(n-j)}u(a+j+1),$$

and (23)

$$g(a+n) \le \frac{g(a)}{1-\lambda h^\nu} + \frac{\lambda h^\nu}{1-\lambda h^\nu}\frac{1}{\Gamma(\nu)}\sum_{j=0}^{n-2}\frac{\Gamma(n-1-j+\nu)}{\Gamma(n-j)}g(a+j+1).$$

Let $n = 1$ in (23). We can obtain $u(a+1)$ and $g(a+1)$ as

$$u(a+1) = \frac{u(a)}{1-\lambda h^\nu}, \ g(a+1) \le \frac{g(a)}{1-\lambda h^\nu} \tag{24}$$

which lead to $g(a+1) \le u(a+1)$ due to the fact $g(a) = u(a)$.

We assume for $n = k$, the inequality holds $g(a+k) \le u(a+k), \ 1 < k$. For $n = k+1$, from Eq. (23), we have

$$u(a+k+1) = \frac{u(a)}{1-\lambda h^\nu} + \frac{\lambda h^\nu}{1-\lambda h^\nu}\frac{1}{\Gamma(\nu)}\sum_{j=0}^{k-1}\frac{\Gamma(k-j+\nu)}{\Gamma(k+1-j)}u(a+j+1),$$

and (25)

$$g(a+k+1) \le \frac{g(a)}{1-\lambda h^\nu} + \frac{\lambda h^\nu}{1-\lambda h^\nu}\frac{1}{\Gamma(\nu)}\sum_{j=0}^{k-1}\frac{\Gamma(k-j+\nu)}{\Gamma(k+1-j)}g(a+j+1).$$

We compare each term of the r.h.s. of (25). Since $g(a) = u(a), \ g(a+1) \le u(a+1), ..., \ g(a+k) \le u(a+k)$, we get $g(a+k+1) \le u(a+k+1)$. This completes the proof.

□

Discrete class $\mathcal{K}$ functions, positive definite and decrescent functions were defined as Definition 3.3 and Definition 3.4 in [37]. Lemma 2.7 – Lemma 2.10 still hold for the fractional difference equation

$${}_h^C\Delta_a^\nu U^*(t) = -\gamma(U^*(t+\nu h)), \ t \in (h\mathbb{N})_{a+(1-\nu)h}, \tag{26}$$

where $U^*(t)$ is a positive definite and decrescent scalar function and γ is a discrete class $\mathcal{K}$ function. Then we present the following theorem of a discrete fractional Lyapunov direct method for Eq. (1).

Theorem 2.11. *Let $x = 0$ be an equilibrium point of Eq. (1). If there exists a positive definite and decrescent scalar function $V(t, x(t))$, discrete class–$\mathcal{K}$ functions γ_1, γ_2 and γ_3 such that*

$$\gamma_1(\| x(t) \|) \le V(t, x(t)) \le \gamma_2(\| x(t) \|), \ t \in (h\mathbb{N})_a \tag{27}$$

and

$${}_h^C\Delta_a^\nu V(t, x(t)) \le -\gamma_3(\| x(t+\nu h) \|), \ t \in (h\mathbb{N})_{a+(1-\nu)h}, \tag{28}$$

then the equilibrium point is asymptotically stable.

Proof. From Eqs. (27) and (28), we have

$${}_h^C\Delta_a^\nu V(t, x(t)) \le -\gamma_3(\gamma_2^{-1}(V(t+\nu h, x(t+\nu h))), \ t \in (h\mathbb{N})_{a+(1-\nu)h}, \tag{29}$$

where γ_2^{-1} denotes the inverse of γ_2. It is evident that $\gamma_3 \circ \gamma_2^{-1}$ is a discrete class $\mathcal{K}$ function.

Considering a fractional difference equation

$${}_h^C\Delta_a^\nu U(t, x(t)) = -\gamma_3(\gamma_2^{-1}(U(t+\nu h, x(t+\nu h))), \ t \in (h\mathbb{N})_{a+(1-\nu)h} \tag{30}$$

$V(t, x(t))$ is bounded by the solution $U(t, x(t))$ from Lemma 2.10. $V(t, x(t))$ is a positive scalar function and we can derive $\lim_{t\to\infty} V(t, x(t)) = 0$ according to Lemma 2.9. Since γ_1 is a discrete class $\mathcal{K}$ function, it follows that $\lim_{t\to\infty} x(t) = 0$ due to the fact $\gamma_1(\| x(t) \|) \le V(t, x(t)), t \in (h\mathbb{N})_a$. This completes the proof. □

3 Stability Analysis of Fractional Difference Equations

Theorem 2.11 can be used to analyze stability of Eq. (1). But it's not easy to construct γ_i functions directly. In this section, we present another Lemma and sufficient condition. We first introduce the following Lemma. In the rest of the paper, we assume the initial point $a = 0$ for simplicity.

Lemma 3.1. *(Discrete comparison principle) For* $0 < \nu \leq 1$, *if* ${}_h^C\Delta_a^\nu x(t) \geq {}_h^C\Delta_a^\nu y(t)$, $\forall t \in (h\mathbb{N})_{a+(1-\nu)h}$ *and* $x(a) = y(a)$, *then*

$$x(t + \nu h) \geq y(t + \nu h). \tag{31}$$

Proof. Let $F(t) = x(t) - y(t)$. Due to ${}_h^C\Delta_a^\nu x \geq {}_h^C\Delta_a^\nu y$, we have

$${}_h^C\Delta_a^\nu F(t) \geq 0, \tag{32}$$

which can be written explicitly

$$\frac{h}{\Gamma(1-\nu)} \sum_{s=\frac{a}{h}}^{\frac{t}{h}+\nu-1} (t - \sigma(sh))_h^{(-\nu)} \Delta_s F(sh) \geq 0, \ t \in (h\mathbb{N})_{a+(1-\nu)h}. \tag{33}$$

Using the summation by parts (8), let $g(sh) = \frac{(t-sh)_h^{(-\nu)}}{\Gamma(1-\nu)}$ and $f(sh) = F(sh)$. We derive

$$\frac{(t - sh)_h^{(-\nu)}}{\Gamma(1-\nu)} F(sh) \mid_{s=\frac{a}{h}}^{\frac{t}{h}+\nu} - \frac{h}{\Gamma(1-\nu)} \sum_{s=\frac{a}{h}}^{\frac{t}{h}+\nu-1} (t - \sigma(sh))_h^{(-\nu-1)} F(sh) \geq 0. \tag{34}$$

Considering $F(sh) \mid_{s=\frac{a}{h}} = x(a) - y(a) = 0$ and $\frac{(t-sh)_h^{(-\nu)}}{\Gamma(1-\nu)} \mid_{s=\frac{t}{h}+\nu} = h^{-\nu}$, we can obtain

$$F(t + \nu h) \geq \frac{1}{\Gamma(1-\nu)} \sum_{s=\frac{a}{h}}^{\frac{t}{h}+\nu-1} \frac{\Gamma(\frac{t}{h} - s)}{\Gamma(\frac{t}{h} - s + \nu + 1)} F(sh). \tag{35}$$

For $t = a + (1-\nu)h$, we have $F(t + \nu h) = F(a + h) \geq \frac{F(a)}{\Gamma(2)} = 0$. Since the discrete kernel holds

$$\frac{\Gamma(\frac{t}{h} - s)}{\Gamma(1-\nu)\Gamma(\frac{t}{h} - s + \nu + 1)} > 0, \ t \in (h\mathbb{N})_{a+(1-\nu)h},$$

it successively follows that

$$x(t + \nu h) - y(t + \nu h) = F(t + \nu h) \geq \frac{1}{\Gamma(1-\nu)} \sum_{s=\frac{a}{h}}^{\frac{t}{h}+\nu-1} \frac{\Gamma(\frac{t}{h} - s)}{\Gamma(\frac{t}{h} - s + \nu + 1)} F(sh) \geq 0$$

and this completes the proof.

In the fractional calculus, Aguila–Camacho et al. [3] proposed a useful inequality for Lyapunov functions. we extend this result to the following discrete case.

Lemma 3.2. *For any discrete time* $t \in (h\mathbb{N})_{a+(1-\nu)h}$, *the following inequality holds*

$${}_h^C\Delta_a^\nu x^2(t) \leq 2x(t + \nu h){}_h^C\Delta_a^\nu x(t), \ 0 < \nu \leq 1. \tag{36}$$

Proof. We need to equivalently prove

$$ {}_h^C\Delta_a^\nu x^2(t) - 2x(t+\nu h) {}_h^C\Delta_a^\nu x(t) \leq 0. \tag{37}$$

The l.h.s. can be rewritten explicitly as

$$\frac{h}{\Gamma(1-\nu)} \sum_{s=\frac{a}{h}}^{\frac{t}{h}+\nu-1} (t-\sigma(sh))_h^{(-\nu)} \Delta_s (x^2(s) - 2x(t+\nu h)x(sh)). \tag{38}$$

Considering $\Delta_s x^2(t+\nu h) = 0$, we add it into (38) such that

$$ {}_h^C\Delta_a^\nu x^2(t) - 2x(t+\nu h) {}_h^C\Delta_a^\nu x(t) = \frac{h}{\Gamma(1-\nu)} \sum_{s=\frac{a}{h}}^{\frac{t}{h}+\nu-1} (t-\sigma(sh))_h^{(-\nu)} \Delta_s (x(sh) - x(t+\nu h))^2. \tag{39}$$

Using the summation by parts (8) again, let $g(sh) = (t-sh)_h^{(-\nu)}$ and $f(sh) = (x(sh) - x(t+\nu h))^2$. We obtain

$$\begin{aligned}
&{}_h^C\Delta_a^\nu x^2(t) - 2x(t+\nu h) {}_h^C\Delta_a^\nu x(t) \\
&= \frac{-h}{\Gamma(1-\nu)} \sum_{s=\frac{a}{h}}^{\frac{t}{h}+\nu-1} (x(sh) - x(t+\nu h))^2 \Delta_s (t-sh)_h^{(-\nu)} + \frac{(t-sh)_h^{(-\nu)}}{\Gamma(1-\nu)} (x(sh) - x(t+\nu h))^2 \Big|_{s=\frac{a}{h}}^{\frac{t}{h}+\nu} \\
&= \frac{-\nu h}{\Gamma(1-\nu)} \sum_{s=\frac{a}{h}}^{\frac{t}{h}+\nu-1} (x(sh) - x(t+\nu h))^2 (t-\sigma(sh))_h^{(-\nu-1)} - \frac{(t-a)_h^{(-\nu)}}{\Gamma(1-\nu)} (x(a) - x(t+\nu h))^2 \leq 0.
\end{aligned} \tag{40}$$

This completes the proof.

Remark 3.3. *For $\mathbf{x} = (x_1(t), ..., x_m(t))^T$, $t \in (h\mathbb{N})_{a+(1-\nu)h}$, Lemma 3.2 still hold. For example, we can have Lemma 3.2 as*

$$ {}_h^C\Delta_a^\nu (\mathbf{x}^T(t)\mathbf{x}(t)) \leq 2\mathbf{x}^T(t+\nu h) {}_h^C\Delta_a^\nu \mathbf{x}(t), \ t \in (h\mathbb{N})_{a+(1-\nu)h}. \tag{41}$$

Example 3.4. *Consider a discrete version of Example 2 in [3]*

$$\begin{cases} {}_h^C\Delta_a^\nu x_1(t) = -x_1(t+\nu h) + x_2^3(t+\nu h), 0 < \nu \leq 1, \\ {}_h^C\Delta_a^\nu x_2(t) = -x_1(t+\nu h) - x_2(t+\nu h), \ t \in (h\mathbb{N})_{a+(1-\nu)h} \end{cases} \tag{42}$$

with the initial conditions $x_1(0) = 0.4$ and $x_2(0) = 0.8$.

We use the Lyapunov function $V = \frac{1}{2}x_1^2(t) + \frac{1}{4}x_2^4(t)$. According to Lemma 3.2, we have

$$\begin{aligned}
{}_h^C\Delta_a^\nu V &\leq x_1(t+\nu h) {}_h^C\Delta_a^\nu x_1(t) + \frac{1}{2} x_2^2(t+\nu) {}_h^C\Delta_a^\nu x_2^2(t) \\
&\leq x_1(t+\nu h) {}_h^C\Delta_a^\nu x_1(t) + x_2^3(t+\nu) {}_h^C\Delta_a^\nu x_2(t) \\
&= -x_1^2(t+\nu h) - x_2^4(t+\nu h) < 0.
\end{aligned} \tag{43}$$

As a result, the system is asymptotically stable from Theorem 2.11.

On the other hand, the numerical formulae can be given accordingly

$$\begin{cases} x_1(n+1) = x_1(0) + \frac{h^\nu}{\Gamma(\nu)} \sum_{j=0}^{n} \frac{\Gamma(n-j+\nu)}{\Gamma(n-j+1)} (-x_1(j+1) + x_2^3(j+1)), \ x_1(0) = 0.1, \\ x_2(n+1) = x_2(0) + \frac{h^\nu}{\Gamma(\nu)} \sum_{j=0}^{n} \frac{\Gamma(n-j+\nu)}{\Gamma(n-j+1)} (-x_1(j+1) - x_2(j+1)), \ x_2(0) = 0.2. \end{cases} \tag{44}$$

This is a nonlinear system and we cannot establish an explicit recurrence relationship for $x_1(n+1)$ and $x_2(n+1)$ as Examples 3.6–3.8. Besides, considering the predictor–corrector method [38, 39] for fractional differential equations, we need to design the predictors and correctors. Since it is impossible to give a predictor formula, we also cannot apply the predictor–corrector method. We adopt the famous Newton–iteration method for the solutions and all the calculation are performed with Maple. The asymptotic stability of the system is shown in Figs. 1 and 2 where $\nu = 0.9$, $N = 500$ and $h = 0.1$. We can see the Lemma 3.2's usefulness.

Theorem 3.5. *If $x = 0$ is an equilibrium point of (1) and $x(t+\nu h) f(t+\nu h, x(t+\nu h)) < 0$ for $\forall t \in (h\mathbb{N})_{a+(1-\nu)h}$, then Eq. (1) is asymptotically stable.*

Proof. Use a discrete Lyapunov candidate function $V(t, x(t)) = \frac{1}{2}x^2(t)$.
If $x(t+\nu h) f(t+\nu h, x(t+\nu h)) < 0$, then we have

$${}_h^C\Delta_a^\nu V \le x(t+\nu h) {}_h^C\Delta_a^\nu x(t) = x(t+\nu h) f(t+\nu h, x(t+\nu h)) < 0$$

which means the fractional difference of $V(t, x(t))$ is negative definite.

Considering the property (5) and the comparison principle (31), we can obtain

$$V(t, x(t)) < V(0, x(0)), \ t \in (h\mathbb{N})_a \tag{45}$$

or

$$\frac{x^2(t)}{2} < \frac{x^2(0)}{2}. \tag{46}$$

According to the definition of the stability 2.4, we can determine the origin is stable.

On the other hand, the fractional difference of the V function results negative definite. The used Lyapunov function is positive definite. As a result, the equilibrium point is asymptotically stable from Theorem 2.11. This completes the proof.

The following examples are used to show the convenience of Theorem 3.5.

Example 3.6. *Consider the linear discrete fractional equation*

$${}_h^C\Delta_a^\nu x(t) = -x(t+\nu h), \ x(0) = 0.1, \ 0 < \nu \le 1, \ t \in (h\mathbb{N})_{a+(1-\nu)h}. \tag{47}$$

We check that

$$x(t+\nu h) f(t+\nu h, x(t+\nu h)) = -x^2(t+\nu h) < 0, \quad t \in (h\mathbb{N})_{a+(1-\nu)h}. \tag{48}$$

Hence, the equilibrium point $x(t) = 0$ is asymptotically stable according to Theorem 3.5.

Example 3.7. *The linear fractional difference equations*

$$\begin{cases} {}_h^C\Delta_a^\nu x_1(t) = -x_1(t+\nu h) - x_2(t+\nu h), \ x_1(0) = 0.1, \ 0 < \nu \le 1, \\ {}_h^C\Delta_a^\nu x_2(t) = x_1(t+\nu h) - 2x_2(t+\nu h), \ x_2(0) = 0.2, \ t \in (h\mathbb{N})_{a+(1-\nu)h} \end{cases} \tag{49}$$

have a trivial solution $x(t) = 0$.

We calculate that

$$\mathbf{x}^T(t+\nu h)f(t+\nu h,\mathbf{x}(t+\nu h)) = -x_1^2(t+\nu h) - 2x_2^2(t+\nu h) < 0. \tag{50}$$

We can conclude that the system is asymptotically stable.

We now consider numerical simulations to show Theorem 3.5's validness. In fact, Eq. (49) can be rewritten in an implicit numerical formula

$$\begin{cases} x_1(n+1) = x_1(0) - \frac{h^\nu}{\Gamma(\nu)}\sum_{j=0}^{n}\frac{\Gamma(n-j+\nu)}{\Gamma(n-j+1)}(x_1(j+1)+x_2(j+1)), \\ x_2(n+1) = x_2(0) + \frac{h^\nu}{\Gamma(\nu)}\sum_{j=0}^{n}\frac{\Gamma(n-j+\nu)}{\Gamma(n-j+1)}(x_1(j+1)-2x_2(j+1)). \end{cases} \tag{51}$$

Since both sides contain $x_1(n+1)$ and $x_2(n+1)$ and the system is linear, we can solve them and obtain

$$\begin{aligned} &\begin{pmatrix} 1 & \frac{h^\nu}{1+h^\nu} \\ -\frac{h^\nu}{1+2h^\nu} & 1 \end{pmatrix}\begin{pmatrix} x_1(n+1) \\ x_2(n+1) \end{pmatrix} \\ &= \begin{pmatrix} \frac{x_1(0)}{1+h^\nu} - \frac{h^\nu}{(1+h^\nu)\Gamma(\nu)}\sum_{j=0}^{n-1}\frac{\Gamma(n-j+\nu)}{\Gamma(n-j+1)}(x_1(j+1)+x_2(j+1)) \\ \frac{x_2(0)}{1+2h^\nu} + \frac{h^\nu}{(1+2h^\nu)\Gamma(\nu)}\sum_{j=0}^{n-1}\frac{\Gamma(n-j+\nu)}{\Gamma(n-j+1)}(x_1(j+1)-2x_2(j+1)) \end{pmatrix} \end{aligned} \tag{52}$$

or

$$\begin{aligned} &\begin{pmatrix} x_1(n+1) \\ x_2(n+1) \end{pmatrix} \\ &= \begin{pmatrix} 1 & \frac{h^\nu}{1+h^\nu} \\ -\frac{h^\nu}{1+2h^\nu} & 1 \end{pmatrix}^{-1}\begin{pmatrix} \frac{x_1(0)}{1+h^\nu} - \frac{h^\nu}{(1+h^\nu)\Gamma(\nu)}\sum_{j=0}^{n-1}\frac{\Gamma(n-j+\nu)}{\Gamma(n-j+1)}(x_1(j+1)+x_2(j+1)) \\ \frac{x_2(0)}{1+2h^\nu} + \frac{h^\nu}{(1+2h^\nu)\Gamma(\nu)}\sum_{j=0}^{n-1}\frac{\Gamma(n-j+\nu)}{\Gamma(n-j+1)}(x_1(j+1)-2x_2(j+1)) \end{pmatrix}, \end{aligned} \tag{53}$$

from which we can successively derive the numerical values of $x_1(n)$ and $x_2(n)$. Set $\nu = 0.6$, $h = 1$ and $n = 200$. Plot the solutions $x_1(n)$ and $x_2(n)$ in Figs. 3 and 4, respectively. We can see $x_1(n)$ and $x_2(n)$ tend to zero as $n \to \infty$, respectively.

Example 3.8. *Consider a discrete time varying system of fractional order*

$$\begin{cases} {}_h^C\Delta_a^\nu x_1(t) = -2x_1(t+\nu h) + tx_2(t+\nu h),\ x_1(0) = 0.4,\ 0 < \nu \le 1, \\ {}_h^C\Delta_a^\nu x_2(t) = -tx_1(t+\nu h) - x_2(t+\nu h),\ x_2(0) = 0.8,\ t \in (h\mathbb{N})_{a+(1-\nu)h}. \end{cases} \tag{54}$$

We can calculate

$$\mathbf{x}^T(t+\nu h)f(t+\nu h,\mathbf{x}(t+\nu h)) = -2x_1^2 - x_2^2 < 0.$$

It is evident that the system is asymptotically stable from Theorem 3.5.

Conclusions

Fractional difference equations are a new class of discrete equations. They hold discrete memory effects and have been used in fractional modeling. However, on the other hand, one cannot directly use properties of normal difference equations because of the memory. This paper defines the Lyapunov stability for a class of implicit fractional difference equations and presents a sufficient condition to distinguish the stable area of fractional systems. Furthermore, the fractional difference equations' numerical formulae are given and the numerical simulation supports the theories. The result can be further used in the fractional controlling or design of fractional controllers of discrete time, for example, discrete fractional network, chaos synchronization of discrete time and so on. We will consider these possible applications in our future work.

Acknowledgements

The study was supported by Sichuan Provincial Education Department (14CZ0026), and the Construct Program of the Key Discipline in Hunan Province.

References

[1] D. Matignon, Stability properties for generalized fractional differential systems, ESAIM: Proc. 5 (1998) 145–158.

[2] Y. Li, Y.Q. Chen, I. Podlubny, Stability of fractional–order nonlinear dynamic systems: Lyapunov direct method and generalized Mittag–Leffler stability, Comput. Math. Appl. 59 (2010) 1810–1821.

[3] N. Aguila–Camacho, M.A. Duarte–Mermoud, J.A. Gallegos, Lyapunov functions for fractional order systems, Commun. Nonlinear Sci. Numer. Simulat. 19 (2014) 2951–2957.

[4] J. A. Gallegos, M.A. Duarte–Mermoud, On the Lyapunov theory for fractional order systems, Appl. Math. Comput. 287–288 (2016) 161–170.

[5] M. A. Duarte–Mermoud, N. Aguila–Camacho, J.A. Gallegos, R. Castro–Linares, Using general quadratic Lyapunov functions to prove Lyapunov uniform stability for fractional order systems, Commun. Nonlinear Sci. Numer. Simulat. 22 (2015) 650–659.

[6] S. Shao, M. Chen, X. Yan, Adaptive sliding mode synchronization for a class of fractional–order chaotic systems with disturbance, Nonlinear Dyn. 83 (2016) 1855–1866.

[7] S. Liang, R.C. Wu, L.P. Chen, Adaptive pinning synchronization in fractional–order uncertain complex dynamical networks with delay, Physica A 444 (2016) 49–62.

[8] R. P. Agarwal, D. O'Regan, S. Hristova, Stability of Caputo fractional differential equations by Lyapunov functions, Appl. Math. 60 (2015) 653–676.

[9] W. Y. Ma, C.P. Li, Y.J. Wu, Y.Q. Wu, Adaptive synchronization of fractional neural networks with unknown parameters and time delays, Entropy, 16 (2014) 6286–6299.

[10] G.J. Li, H. Liu, Stability analysis and synchronization for a class of fractional–order neural networks, Entropy 18 (2016) 55.

[11] F. M. Atici, P. W. Eloe, Initial value problems in discrete fractional calculus, Proc. Americ. Math. Soc. 137 (2007) 981–989.

[12] Y. Y. Pan, Z.L. Han, S.R. Sun, C.X. Hou, The existence of solutions to a class of boundary value problems with fractional difference equations, Adv. Diff. Equa., 2013 (2013) Article. ID. 275, 20pp.

[13] C. S. Goodrich, Systems of discrete fractional boundary value problems with nonlinearities satisfying no growth conditions, J. Diff. Equa. Appl., 21 (2015) 437–453.

[14] C. S. Goodrich, On discrete sequential fractional boundary value problems, J. Math. Anal. Appl. 385 (2012) 111–124.

[15] W. D. Lv, J.S. Feng, Nonlinear discrete fractional mixed type sum–difference equation boundary value problems in Banach spaces, Adv. Diff. Equa., 2014 (2014) Article. ID. 184, 12pp.

[16] F. Atici, S. Sengul, Modeling with fractional difference equations, J. Math. Anal. Appl. 369 (2010) 1–9.

[17] M. T. Holm, The theory of discrete fractional calculus: Development and application. Ph.D. dissertation, University of Nebraska–Lincoln, (2011).

[18] R. Abu–Saris, Q. Al–Mdallal, On the asymptotic stability of linear system of fractional–order difference equations, Frac. Calc. Appl. Anal. 16 (2013) 613–629.

[19] J. Cermak, I. Gyori, L. Nechvatal, On explicit stability conditions for a linear fractional difference system, Fract. Calc. Appl. Anal., 18 (2015) 651–672.

[20] F. L. Chen, Fixed points and asymptotic stability of nonlinear fractional difference equations, Elect. J. Qualit. Theor. Diff. Equ., 39 (2011) 1–18.

[21] F. L. Chen, A review of existence and stability results for discrete fractional equations, J. Comput. Complex. Appl. 1 (2015) 22–53.

[22] F. L. Chen, Z.G. Liu, Asymptotic stability results for nonlinear fractional difference equations, J. Appl. Math. 2012 (2012) Article. ID. 879657, 14pp.

[23] B. G. Jia, The asymptotic behavior of Caputo delta fractional equations, Math. Method Appl. Sci. 39 (2016) 5355–5364.

[24] M. Wyrwas, E. Pawluszewicz, E. Girejko, Stability of nonlinear h–difference systems with N fractional orders, Kybernetika, 51 (2015) 112–136.

[25] B. G. Jia, L. Erbe, A. Peterson, Comparison theorems and asymptotic behavior of solutions of discrete fractional equations, Elect. J. Qualt. Theor. Diff. Equa. 89 (2015) 1–18.

[26] G. C. Wu, D. Baleanu, H. P. Xie, F.L. Chen, Chaos synchronization of fractional chaotic maps based on the stability condition, Physica A 460 (2016) 374–383.

[27] G. C. Wu, D. Baleanu, Z. G. Deng, S.D. Zeng, Lattice fractional diffusion equation in terms of a Riesz–Caputo difference, Physica A, 438 (2015) 335–339.

[28] G. C. Wu, D. Baleanu, S. D. Zeng, Z.G. Deng, Discrete fractional diffusion equation, Nonlinear Dyn. 80 (2015) 281–286.

[29] G. C. Wu, D. Baleanu, S. D. Zeng, Several fractional differences and their applications to discrete maps, J. Appl. Nonlinear Dyn. 4 (2015) 339–348.

[30] F. Wu, J.F. Liu, Discrete fractional creep model of salt rock, J. Comput. Complex. Appl. 2 (2016) 1–6.

[31] D. Mozyrska, M. Wyrwas, The $Z-$transform method and Delta type fractional difference operators, Discrete Dyn. Nat. Soc. ID. 852734, 2015.

[32] M. Wyrwas, D. Mozyrska, E. Girejko, On solutions to fractional discrete systems with sequential h-differences, Abstr. Appl. Anal. 2013 (2013) Article ID. 475350, 11pp.

[33] N. R. O. Bastos, R. A. C. Ferreira, D. F. M. Torres, Discrete–time fractional variational problems, Sign. Proc. 91 (2011) 513–524.

[34] G. A. Anastassiou, About discrete fractional calculus with inequalities, in: Intelligent Mathematics: Computational Analysis, Springer, (2011) 575–585.

[35] T. Abdeljawad, On Riemann and Caputo fractional differences, Comput. Math. Appl. 62 (2011) 1602–1611.

[36] D. Mozyrska, E. Girejko, Overview of Fractional $h-$difference Operators, Advances in Harmonic Analysis and Operator Theory, Springer Basel, (2013) 253–268.

[37] F. Jarad, T. Abdeljawad, D. Baleanu, K. Bicen, On the stability of some discrete fractional nonautonomous systems, Abstr. Appl. Anal. 2012 (2012) Article ID. 476581, 9pp.

[38] K. Diethelm, N. J. Ford, A. D. Freed, A predictor–corrector approach for the numerical solution of fractional differential equations, Nonlinear Dyn. 29 (2002) 3–22.

[39] C. P. Li, G. J. Peng, Chaos in Chen's system with a fractional order, Chaos, Solit. Fract. 22 (2004) 443–450.

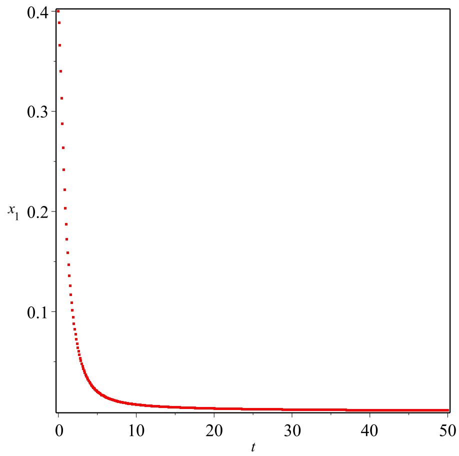


Figure 1: Asymptotic stability of x_1 for $\nu = 0.9$.

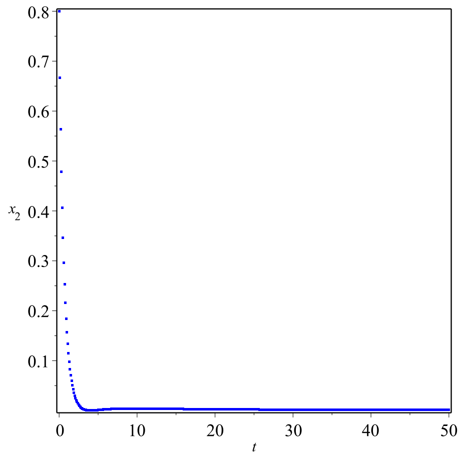


Figure 2: Asymptotic stability of x_2 for $\nu = 0.9$.

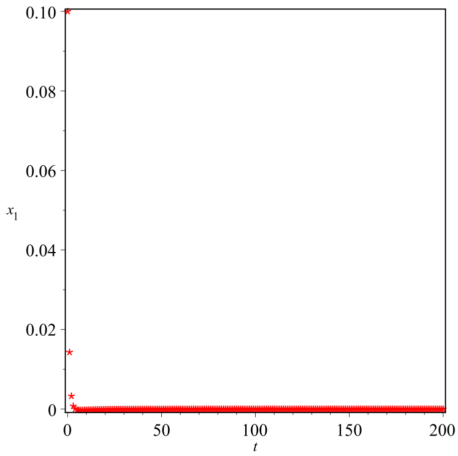


Figure 3: Asymptotic stability of x_1 for $\nu = 0.6$.

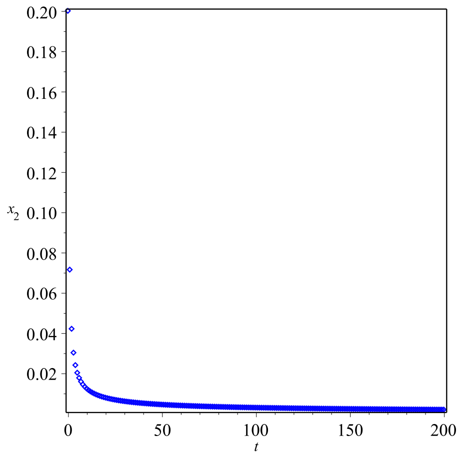


Figure 4: Asymptotic stability of x_2 for $\nu = 0.6$.