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# New exact solution of generalized biological population model

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## Abstract

In this study, a mathematical model of the generalized biological population model (GBPM) gets a new exact solution with a conformable derivative operator (CDO). The new exact solution of this model will be obtained by a new approximate analytic technique named three dimensional conformable reduced differential transform method (TCRDTM). By using this technique, it is possible to find new exact solution as well as closed analytical approximate solution of a partial differential equations (PDEs). Three numerical applications of GBPM are given to check the accuracy, effectiveness, and convergence of the TCRDTM. In these applications, obtained new exact solutions in conformable sense are compared with the exact solutions in Caputo sense in literature. The comparisons are illustrated in 3D graphics. The results show that when $\alpha \to 1$, the exact solutions in conformable and Caputo sense converge to each other. In other cases, exact solutions different from each other are obtained. 

Keywords: Numerical solution, biological populations model, reduced differential transform method, conformable derivative, partial differential equations.
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## 1. Introduction

Linear and non-linear fractional and non-fractional problems of differential equations play a major role in various fields such as biology, physics, chemistry, mathematics, astronomy, fluids mechanics, mathematics, and engineering. It is not always possible to find analytical solutions to these problems [1, 2, 4–19, 24, 25, 28, 34, 35, 37, 38, 40, 42, 48–54]. Therefore, it is very important to handle these problems appropriately and solve them or develop solutions. Recently, a new derivative called CDO was introduced and also by the help of this newly defined derivative, the behaviors of many problems have been studied and some solutions techniques have been developed [1, 3, 6, 7, 12, 14, 20, 21, 23, 29–31, 47]. In 2016, Acan et al. [6] introduced two dimensional conformable reduced differential transform method (CRDTM) based on RDTM and CDO for the PDEs. It is shown that CRDTM is an easy applicable analytical method

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and gives the exact solution for PDEs. We will extend this method, which is called TCRDTM, to three dimensions. This method is used to find new exact solution of the GBPM in conformable sense as described below.

Scientists in biology believe that migration or scattering is very important in regulating species populations. In a region $\Omega$, the diffusion of species is defined by the three position parameters $\vec{\xi} = (x, y)$ and time $t$ that are population density (PD) $\rho(\vec{\xi}, t)$, the population supply $\psi(\vec{\xi}, t)$, and diffusion velocity (DV) $\upsilon(\vec{\xi}, t)$ [27]. The $\rho(\vec{\xi}, t)$ gives the individual number per unit volume at time $t$ and position $\vec{\xi}$; over any sub-region $G$ of $\Omega$, the integral $\rho(\vec{\xi}, t)$ gives all population of the region $G$ at time $t$. $\psi(\vec{\xi}, t)$ presents the rate at which individuals are supplied, per unit volume, at position $\vec{\xi}$ by births and deaths. The DV $\upsilon(\vec{\xi}, t)$ shows the average velocity belonging to the individuals who occupy the position $\vec{\xi}$ at time $t$, and it reports the population flow from point to point. The entities $\rho(\vec{\xi}, t)$, $\upsilon(\vec{\xi}, t)$, and $\psi(\vec{\xi}, t)$ must obey the following population balance law, for any regular sub-region $G$ of $\Omega$ and for any time $t$

$$\frac{d^\alpha}{dt^\alpha} \int_G \rho dV + \int_{\partial G} \rho \vec{\upsilon} \cdot \hat{m} dA = \int_G \psi dV, \tag{1.1}$$

where $\hat{m}$ is the outward unit normal to the boundary $\partial G$ of $G$. In (1.1), the derivative has been taken in the conformable derivative sense. By the assumptions [33]

$$\psi = \psi(\rho) \text{ and } \upsilon = -\mu(\rho) \nabla \rho,$$

where $\nabla$ is the Laplace operator and $\mu(\rho) > 0$ for $\rho > 0$, two-dimensional non-linear degenerate parabolic PDEs for the PD $\rho$ can be obtained as

$$\frac{\partial^\alpha}{\partial t^\alpha} \rho(x, y, t) = \frac{\partial^2}{\partial x^2} \omega(\rho) + \frac{\partial^2}{\partial y^2} \omega(\rho) + \psi(\rho), \; x, y \in R, \; t \in [0, \infty), \; 0 < \alpha \leqslant 1. \tag{1.2}$$

The statement (1.2) is BPM. Gurney and Nisbet [26] employed a method, for a special case for the modeling of the animals' population. Model (1.2) with $\omega(\rho) = \rho^2$, is the following equation

$$\frac{\partial^\alpha}{\partial t^\alpha} \rho(x, y, t) = \frac{\partial^2}{\partial x^2} \rho^2 + \frac{\partial^2}{\partial y^2} \rho^2 + \psi(\rho), \; x, y \in R, \; t \in [0, \infty), \; 0 < \alpha \leqslant 1, \tag{1.3}$$

subject to the initial condition (IC) $\rho(x, y, 0)$. For $\alpha = 1$, (1.3) reduces to the standard biological population model

$$\frac{\partial}{\partial t} \rho(x, y, t) = \frac{\partial^2}{\partial x^2} \rho^2 + \frac{\partial^2}{\partial y^2} \rho^2 + \psi(\rho), \; x, y \in R, \; t \in [0, \infty).$$

Some properties of (1.3) such as holder estimates and its solutions have been studied in [33]. The constitutive equations for $\psi(\rho)$ can be given as

(i) $\psi(\rho) = k\rho$, $k$ constant, Malthusian law [27];
(ii) $\psi(\rho) = k_1 \rho - k_2 \rho^2$, $k_1, k_2$ positive constants, Verhulst law [33];
(iii) $\psi(\rho) = k\rho^\gamma$, $(k > 0, \; 0 < \gamma \leqslant 1)$, Porous media [36].

Let us deal with a more general form of $\psi(\rho)$ as $\psi(\rho) = h\rho^\eta (1 - \mu\rho^\tau)$ so that (1.3) becomes

$$\frac{\partial^\alpha}{\partial t^\alpha} \rho(x, y, t) = \frac{\partial^2}{\partial x^2} \rho^2 + \frac{\partial^2}{\partial y^2} \rho^2 + h\rho^\eta (1 - \mu\rho^\tau), x, y \in R, t \in [0, \infty), 0 < \alpha \leqslant 1, \tag{1.4}$$

where $h$, $\eta$, $\tau$, $\mu \in R$. If $h = k$, $\eta = 1$, $\mu = 0$ and $h = k_1$, $\eta = \tau = 1$, $\mu = \frac{k_2}{k_1}$, then (1.4) leads to Malthusian law and Verhulst law, respectively. The above mentioned models have been a research topic for many researchers and scientists. Different methods and derivative definitions have been used to explore these models [13, 22, 26, 27, 32, 33, 36, 39, 41, 43–46].

In this study, we propose a new exact solution for GBPM. TCRDTM, based on CDO and RDTM, is used to obtain this new solution. The obtained new exact solutions in conformable sense are compared with the exact solutions [46] in Caputo sense for some $\alpha$ values. The comparisons of these exact solutions are illustrated in 3D graphics. For this, in Section 2, we give basic definitions and important properties of CDO. In Section 3, two dimensions CRDTM has been extended to three dimensions CRDTM (TCRDTM). For this new method some definitions and theorems are given. In Section 4, some applications for GBPM are given. And in the final section, we give the conclusion.

## 2. On the conformable derivative operator

**Definition 2.1** ([1, 12, 30]). Given a function $f : [0, \infty) \to \mathbb{R}$, then the conformable derivative (CD) of f of order $\alpha$ is defined by:

$$(T_\alpha f)(t) = \frac{d^\alpha}{dt^\alpha} f(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}$$

for all $t > 0$, $\alpha \in (0, 1]$.

**Lemma 2.2** ([1, 12, 30]). *Let $f, g$ be $\alpha$-differentiable at a point $t > 0$ for $\alpha$. Then*

(i) $T_\alpha(af + bg) = a(T_\alpha f) + b(T_\alpha g)$ *for all* $a, b \in \mathbb{R}$ *and* $\alpha \in (0, 1]$;

(ii) $T_\alpha(f(t)) = 0$ *for constant function* $f(t) = \lambda$, $\alpha \in (0, 1]$;

(iii) $T_\alpha(fg) = f(T_\alpha g) + g(T_\alpha f)$, $\alpha \in (0, 1]$;

(iv) $T_\alpha(f/g) = \dfrac{g(T_\alpha f) - f(T_\alpha g)}{g^2}$, $\alpha \in (0, 1]$;

(v) *if* f *is* n *times differentiable at* t, *then* $T_\alpha(f(t)) = t^{\lceil\alpha\rceil-\alpha} f^{(\lceil\alpha\rceil)}(t)$, $\alpha \in (n, n+1]$, *where* $\lceil\alpha\rceil$ *is the smallest integer greater than or equal to* $\alpha$.

**Lemma 2.3** ([1]). *Suppose that* f *is infinitely* $\alpha$*-differentiable function for* $\alpha \in (0, 1]$ *at a neighborhood of a point* $t_0$. *Then* f *has the conformable power series expansion*

$$f(t) = \sum_{k=0}^{\infty} \frac{\left(T_\alpha^{(k)} f\right)(t_0)\,(t - t_0)^{\alpha k}}{\alpha^k k!}, \; t_0 < t < t_0 + R^{1/\alpha}, R > 0.$$

*Here* $\left(T_\alpha^{(k)} f\right)(t_0)$ *denotes the application of the CD for* k *times.*

## 3. Three dimensional conformable reduced differential transform method

In 2016, Acan et al. [6] introduced the two dimensional CRDTM. In this section, we will extend this method to three dimensions. Throughout this study, the lowercase $u(x, y, t)$ represents the original function while the uppercase $U_k^\alpha(x, y)$ stands for three dimensional conformable reduced differential transformed (TCRDT) function. The basic definitions of TCRDTM are presented as follows.

**Definition 3.1.** Assume that $u(x, y, t)$ is analytic and differentiated continuously with respect to three variables x, y, t in its domain. The TCFRDT of $u(x, y, t)$ is defined as

$$U_k^\alpha(x, y) = \frac{1}{\alpha^k k!} \left[\left(T_\alpha^{(k)} u\right)\right]_{t=t_0},$$

where $0 < \alpha \leqslant 1$, $\alpha$ is a parameter describing the order of conformable derivative,

$$T_\alpha^{(k)} u = \underbrace{(T_\alpha T_\alpha \cdots T_\alpha)}_{k \text{ times}} u(x, y, t)$$

and the t dimensional spectrum function $U_k^\alpha(x, y)$ is the TCRDT function.

**Definition 3.2.** Let $U_k^\alpha(x,y)$ be the TCRDT of $u(x,y,t)$. Inverse TCRDT of $U_k^\alpha(x,y)$ is defined as

$$u(x_1,x_2,\ldots,x_n,t)=\sum_{k=0}^{\infty}U_k^\alpha(x,y)(t-t_0)^{\alpha k}=\sum_{k=0}^{\infty}\frac{1}{\alpha^k k!}\left[T_\alpha^{(k)}u\right]_{t=t_0}(t-t_0)^{\alpha k}.$$

TCRDT of initial conditions for integer order derivatives are defined as

$$U_k^\alpha(x,y)=\begin{cases}\frac{1}{(\alpha k)!}\left[\frac{\partial^{\alpha k}}{\partial t^{\alpha k}}u(x,y,t)\right]_{t=t_0}, & \text{if } \alpha k\in\mathbb{Z}^+,\\ 0, & \text{if } \alpha k\notin\mathbb{Z}^+,\end{cases}\quad \text{for } k=0,1,2,\ldots,\left(\frac{m}{\alpha}-1\right),$$

where $m$ is the order of conformable PDE.

**Theorem 3.3.** *Let $a$ and $b$ be constants. If $u(x,y,t)=av(x,y,t)\pm bw(x,y,t)$, then $U_k^\alpha(x,y)=aV_k^\alpha(x,y)\pm bW_k^\alpha(x,y)$.*

*Proof.* CFRDT of $v(x,y,t)$ and $w(x,y,t)$ can be written as

$$V_k^\alpha(x,y)=\frac{1}{\alpha^k k!}\left[T_\alpha^{(k)}v\right]_{t=t_0},\quad W_k^\alpha(x,y)=\frac{1}{\alpha^k k!}\left[T_\alpha^{(k)}w\right]_{t=t_0}.$$

Because of Lemma 2.2 (i),

$$U_k^\alpha(x,y)=\frac{1}{\alpha^k k!}\left[T_\alpha^{(k)}(av\pm bw)\right]_{t=t_0}=\frac{a}{\alpha^k k!}\left[T_\alpha^{(k)}v\right]_{t=t_0}\pm\frac{b}{\alpha^k k!}\left[T_\alpha^{(k)}w\right]_{t=t_0}=aV_k^\alpha(x,y)\pm bW_k^\alpha(x,y).$$

The proof is completed. □

**Theorem 3.4.** *If $u(x,y,t)=v(x,y,t)w(x,y,t)$, then $U_k^\alpha(x,y)=\sum_{s=0}^{k}V_s^\alpha(x,y)W_{k-s}^\alpha(x,y)$.*

*Proof.* By the help of Definition 3.2, $v(x,y,t)$ and $w(x,y,t)$ can be written as

$$v(x,y,t)=\sum_{k=0}^{\infty}V_k^\alpha(x,y)(t-t_0)^{\alpha k},\quad w(x,y,t)=\sum_{k=0}^{\infty}W_k^\alpha(x,y)(t-t_0)^{\alpha k}.$$

Then, $u(x,t)$ is obtained as

$$\begin{aligned}U_k^\alpha(x,y)&=\sum_{k=0}^{\infty}V_k^\alpha(x,y)(t-t_0)^{\alpha k}\sum_{k=0}^{\infty}W_k^\alpha(x,y)(t-t_0)^{\alpha k}\\
&=\left(V_0^\alpha(x,y)+V_1^\alpha(x,y)(t-t_0)^\alpha+V_2^\alpha(x,y)(t-t_0)^{2\alpha}+\cdots+V_n^\alpha(x,y)(t-t_0)^{n\alpha}+\cdots\right)\\
&\quad\times\left(W_0^\alpha(x,y)+W_1^\alpha(x,y)(t-t_0)^\alpha+W_2^\alpha(x,y)(t-t_0)^{2\alpha}+\cdots+W_n^\alpha(x,y)(t-t_0)^{n\alpha}+\cdots\right)\\
&=V_0^\alpha(x,y)W_0^\alpha(x,y)+\left(V_0^\alpha(x,y)W_1^\alpha(x,y)+V_1^\alpha(x,y)W_0^\alpha(x,y)\right)(t-t_0)^\alpha\\
&\quad+\left(V_0^\alpha(x,y)W_2^\alpha(x,y)+V_1^\alpha(x,y)W_1^\alpha(x,y)+V_2^\alpha(x,y)W_0^\alpha(x,y)\right)(t-t_0)^{2\alpha}+\cdots\\
&=\sum_{k=0}^{\infty}\sum_{s=0}^{k}V_s^\alpha(x,y)W_{k-s}^\alpha(x,y)(t-t_0)^{k\alpha}.\end{aligned}$$

Hence, $U_k^\alpha(x,y)$ is found as

$$U_k^\alpha(x,y)=\sum_{s=0}^{k}V_s^\alpha(x,y)\,W_{k-s}^\alpha(x,y).$$

The proof is completed. □

**Theorem 3.5.** *If* $u(x,y,t)=T_\alpha v(x,y,t)$*, then* $U_k^\alpha(x,y)=\alpha(k+1)V_{k+1}^\alpha(x,y)$.

*Proof.* TCRDT of $v(x,y,t)$ can be written as

$$V_k^\alpha(x,y)=\frac{1}{\alpha^k k!}\left[T_\alpha^{(k)}v\right]_{t=t_0}$$

for $u(x,y,t)=T_\alpha v(x,y,t)$,

$$\begin{aligned}U_k^\alpha(x,y)=\frac{1}{\alpha^k k!}\left[T_\alpha^{(k)}(T_\alpha v)\right]_{t=t_0}&=\frac{1}{\alpha^k k!}\left[T_\alpha^{(k+1)}v\right]_{t=t_0}\\&=\alpha(k+1)\frac{1}{\alpha^{k+1}(k+1)!}\left[T_\alpha^{(k+1)}v\right]_{t=t_0}=\alpha(k+1)V_k^\alpha(x,y).\end{aligned}$$

The proof is completed. □

**Theorem 3.6.** *If* $u(x,y,t)=x^p y^q(t-t_0)^m$*, then* $U_k^\alpha(x,y)=x^p y^q\delta\left(k-\frac{m}{\alpha}\right)$*, where* $\delta(k)=\begin{cases}1, & \textit{if } k=0,\\ 0, & \textit{if } k\neq 0.\end{cases}$

*Proof.* TCRDT of $u(x,y,t)=x^p y^q(t-t_0)^m$ is

$$U_k^\alpha(x,y)=\frac{1}{\alpha^k k!}\left[T_\alpha^{(k)}\left(x^p y^q(t-t_0)^m\right)\right]_{t=t_0}.$$

If the conformable derivative of $u(x,y,t)=x^p y^q(t-t_0)^m$, with respect to $t$, is calculated for $k$ times, where $\alpha\in(0,1]$, then

$$U_k^\alpha(x,y)=\frac{1}{\alpha^k k!}\left[x^p y^q\left(m(m-\alpha)\cdots(m-(k-1)\alpha)(t-t_0)^{m-k\alpha}\right)\right]_{t=t_0}$$

is obtained. If $k=\frac{m}{\alpha}$, then

$$\begin{aligned}U_k^\alpha(x,y)&=\frac{1}{\alpha^{\frac{m}{\alpha}}\left(\frac{m}{\alpha}\right)!}x^p y^q\left(m(m-\alpha)\cdots\left(m-\left(\frac{m}{\alpha}-1\right)\alpha\right)(t-t_0)^{m-k\alpha}\right)\\&=\frac{1}{\alpha^{\frac{m}{\alpha}}\left(\frac{m}{\alpha}\right)!}x^p y^q\left(m(m-\alpha)\cdots(\alpha)\right)=\frac{\alpha^{\frac{m}{\alpha}}}{\alpha^{\frac{m}{\alpha}}\left(\frac{m}{\alpha}\right)!}x^p y^q\left(\frac{m}{\alpha}\left(\frac{m}{\alpha}-1\right)\cdots(1)\right)=x^p y^q.\end{aligned}$$

Otherwise, for $t=t_0$,

$$U_k^\alpha(x,y)=0,$$

hence

$$U_k^\alpha(x,y)=x^p y^q\delta\left(k-\frac{m}{\alpha}\right)$$

is obtained. The proof is completed. □

## 4. Numerical applications

Three applications are considered in this section. All of the results are calculated by MAPLE program.

**Example 4.1.** First, we consider the following non-linear BPM [46]:

$$\frac{\partial^\alpha}{\partial t^\alpha}\rho(x,y,t)=\frac{\partial^2}{\partial x^2}\rho^2+\frac{\partial^2}{\partial y^2}\rho^2+h\rho,\ 0<\alpha\leqslant 1 \tag{4.1}$$

subject to the IC

$$\rho(x,y,0)=\sqrt{xy}. \tag{4.2}$$

The exact solution in Caputo sense of the non-linear BPM (4.1) is given as [46]:

$$\rho(x,y,t)=\sqrt{xy}E_\alpha\left(ht^\alpha\right), \tag{4.3}$$

where $E_\alpha\left(ht^\alpha\right)$ is the Mittag-Leffler function, defined as $E_\alpha\left(\zeta\right)=\sum\limits_{k=0}^{\infty}\frac{\zeta^k}{\Gamma(1+k\alpha)}$.

Now solve this problem by using TCRDTM. By taking the TCRDT of (4.1), it can be obtained that

$$\alpha\left(k+1\right)P_{k+1}^\alpha\left(x,y\right)=\frac{\partial^2}{\partial x^2}\left[\sum_{s=0}^{k}P_s^\alpha\left(x,y\right)P_{k-s}^\alpha\left(x,y\right)\right]+\frac{\partial^2}{\partial y^2}\left[\sum_{s=0}^{k}P_s^\alpha\left(x,y\right)P_{k-s}^\alpha\left(x,y\right)\right]+hP_k^\alpha\left(x,y\right), \tag{4.4}$$

where $P_k^\alpha\left(x,y\right)$ is the TCRDT function. From IC (4.2) we write

$$P_0^\alpha\left(x,y\right)=\sqrt{xy}. \tag{4.5}$$

From (4.4) and (4.5), we can obtain the following $P_k^\alpha\left(x,y\right)$ values

$$P_1^\alpha\left(x,y\right)=\sqrt{xy}\frac{h}{\alpha},\quad P_2^\alpha\left(x,y\right)=\sqrt{xy}\frac{h^2}{2!\alpha^2},\cdots,P_m^\alpha\left(x,y\right)=\sqrt{xy}\frac{h^m}{m!\alpha^m},\cdots. \tag{4.6}$$

Then, from (4.6), the set of values $\left\{P_k^\alpha\left(x\right)\right\}_{k=0}^m$ gives us the following approximate result

$$\tilde{\rho}_m(x,y,t)=\sum_{k=0}^{m}P_k^\alpha\left(x,y\right)t^{k\alpha}=\sum_{k=0}^{m}\sqrt{xy}\frac{h^k}{k!\alpha^k}t^{k\alpha}. \tag{4.7}$$

From (4.7) we obtain

$$\rho(x,y,t)=\lim_{m\to\infty}\tilde{\rho}_m(x,y,t)=\sqrt{xy}e^{h\frac{t^\alpha}{\alpha}}. \tag{4.8}$$

The obtained analytical approximate solution given in (4.8) is a new exact solution in conformable sense for the non-linear BPM in (4.1).

*Remark* 4.2. If we take $\alpha=1$, then Example 4.1 is reduced to standard biological population model

$$\frac{\partial}{\partial t}\rho(x,y,t)=\frac{\partial^2}{\partial x^2}\rho^2+\frac{\partial^2}{\partial y^2}\rho^2+h\rho$$

with IC

$$\rho(x,y,0)=\sqrt{xy}.$$

Our new exact solution (4.8) in conformable sense and the exact solution (4.3) in Caputo sense imply

$$\rho(x,y,t)=\sqrt{xy}e^{ht}.$$

This result is the exact solution of the standard problem in the literature.

Now, we can compare the new exact solution in conformable sense and the exact solution [46] in Caputo sense for the non-linear BPM (4.1). For some $\alpha$ values, the comparison of exact solutions in conformable and Caputo sense are illustrated by 3D graphics in Figs. 1 and 2. The MAPLE codes for graph drawings are given in Table 1.

Table 1: The MAPLE codes for graphics.

```
restart:
with(plots):
Sol-Con:=P1(x,y,z):  #The solution of conformable sense (Enter P1(x,y,z).)
Sol-Cap:=P2(x,y,z):  #The solution of Caputo sense (Enter P2(x,y,z).)
alpha:=m:  #Order of fractional derivative (Enter a number for m.)
y:=n:  #Enter a number for n.
h:=1:  #If it is required for the solution, please use.
mu:=1:  #If it is required for the solution, please use.
G1:=plot3d(Sol-Con,x=0..1,t=0..1,color=blue,style=point,linestyle=solid,
symbol=circle,axes=box):
G2:=plot3d(Sol-Cap,x=0..1,t=0..1,color=red,style=line,linestyle=spacedash,
symbol=cross,axes=box):
display(G1,G2);
```

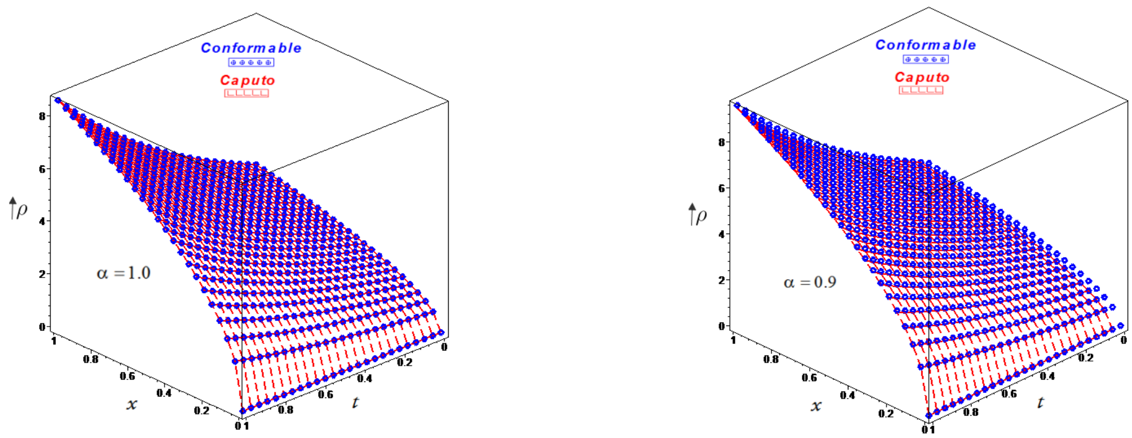


Figure 1: The plots show the comparison of exact solutions of (4.1) in conformable and Caputo sense [46] for $y = 10$, $h = 1$, and some $\alpha$ values.

**Example 4.3.** Second, let us consider the following non-linear BPM [46]

$$\frac{\partial^\alpha}{\partial t^\alpha}\rho(x,y,t) = \frac{\partial^2}{\partial x^2}\rho^2 + \frac{\partial^2}{\partial y^2}\rho^2 + \rho, \ 0 < \alpha \leqslant 1 \tag{4.9}$$

subject to the IC

$$\rho(x,y,0) = \sqrt{\sin x \sinh y}. \tag{4.10}$$

The exact solution in Caputo sense of non-linear BPM (4.9) is given as [46]

$$\rho(x,y,t) = \sqrt{\sin x \sinh y} E_\alpha\left(t^\alpha\right), \tag{4.11}$$

where $E_\alpha\left(t^\alpha\right)$ is the Mittag-Leffler function, defined as $E_\alpha(\zeta) = \sum\limits_{k=0}^{\infty} \frac{\zeta^k}{\Gamma(1+k\alpha)}$.

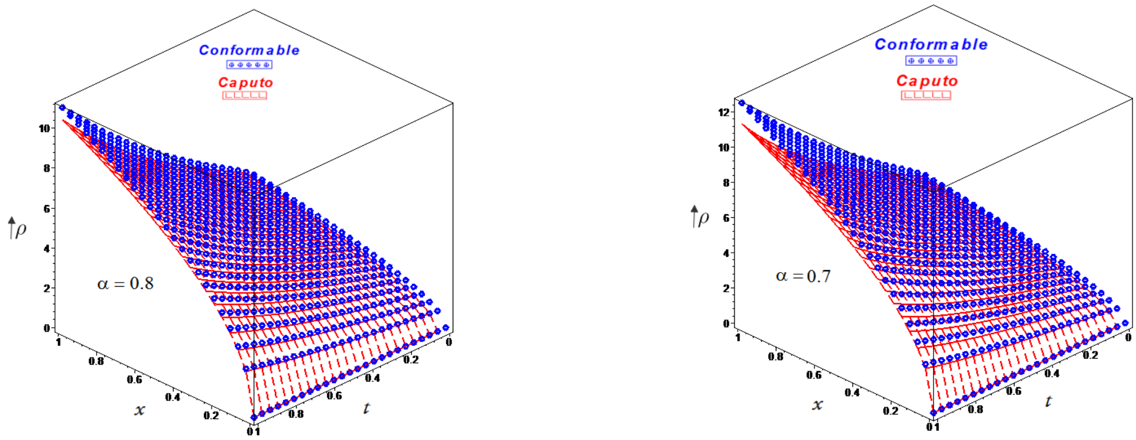


Figure 2: The plots show the comparison of exact solutions of (4.1) in conformable and Caputo sense [46] for $y = 10$, $h = 1$, and some $\alpha$ values.

Now solve this problem by using TCRDTM. By taking the TCRDT of (4.9), it can be obtained that

$$\alpha (k+1) P_{k+1}^{\alpha}(x,y) = \frac{\partial^2}{\partial x^2}\left[\sum_{s=0}^{k} P_s^{\alpha}(x,y) P_{k-s}^{\alpha}(x,y)\right] + \frac{\partial^2}{\partial y^2}\left[\sum_{s=0}^{k} P_s^{\alpha}(x,y) P_{k-s}^{\alpha}(x,y)\right] + P_k^{\alpha}(x,y), \quad (4.12)$$

where $P_k^{\alpha}(x,y)$ is the TCRDT function. From IC (4.10) we write

$$P_0^{\alpha}(x,y) = \sqrt{\sin x \sinh y}. \quad (4.13)$$

From (4.12) and (4.13), we can obtain the following $P_k^{\alpha}(x,y)$ values

$$\begin{aligned} P_1^{\alpha}(x,y) &= \sqrt{\sin x \sinh y}\frac{1}{\alpha}, \\ P_2^{\alpha}(x,y) &= \sqrt{\sin x \sinh y}\frac{1}{2!\alpha^2}, \\ P_3^{\alpha}(x,y) &= \sqrt{\sin x \sinh y}\frac{1}{3!\alpha^3}, \\ &\vdots \\ P_m^{\alpha}(x,y) &= \sqrt{\sin x \sinh y}\frac{1}{m!\alpha^m}, \cdots . \end{aligned} \quad (4.14)$$

Then, from (4.14), the set of values $\left\{P_k^{\alpha}(x)\right\}_{k=0}^{m}$ gives us the following approximate result

$$\tilde{\rho}_m(x,y,t) = \sum_{k=0}^{m} P_k^{\alpha}(x,y)\, t^{k\alpha} = \sum_{k=0}^{m} \sqrt{\sin x \sinh y}\frac{1}{k!\alpha^k} t^{k\alpha}. \quad (4.15)$$

From (4.15) we obtain

$$\rho(x,y,t) = \lim_{m\to\infty} \tilde{\rho}_m(x,y,t) = \sqrt{\sin x \sinh y}\, e^{\frac{t^{\alpha}}{\alpha}}. \quad (4.16)$$

The obtained analytical approximate solution given in (4.16) is a new exact solution in conformable sense for the non-linear BPM in (4.9).

*Remark* 4.4. If we take $\alpha = 1$, then Example 4.3 is reduced to standard biological population model

$$\frac{\partial}{\partial t}\rho(x,y,t) = \frac{\partial^2}{\partial x^2}\rho^2 + \frac{\partial^2}{\partial y^2}\rho^2 + \rho$$

with IC

$$\rho(x, y, 0) = \sqrt{\sin x \sinh y}.$$

Our new exact solution (4.16) in conformable sense and the exact solution (4.11) in Caputo sense imply

$$\rho(x, y, t) = \sqrt{\sin x \sinh y} e^t.$$

This result is the exact solution of the standard problem in the literature.

Now, we can compare the new exact solution in conformable sense and the exact solution [46] in Caputo sense for the non-linear BPM (4.9). For some $\alpha$ values, the comparison of exact solutions in conformable and Caputo sense are illustrated by 3D graphics in Figs. 3 and 4. The MAPLE codes for graph drawings are given in Table 1.

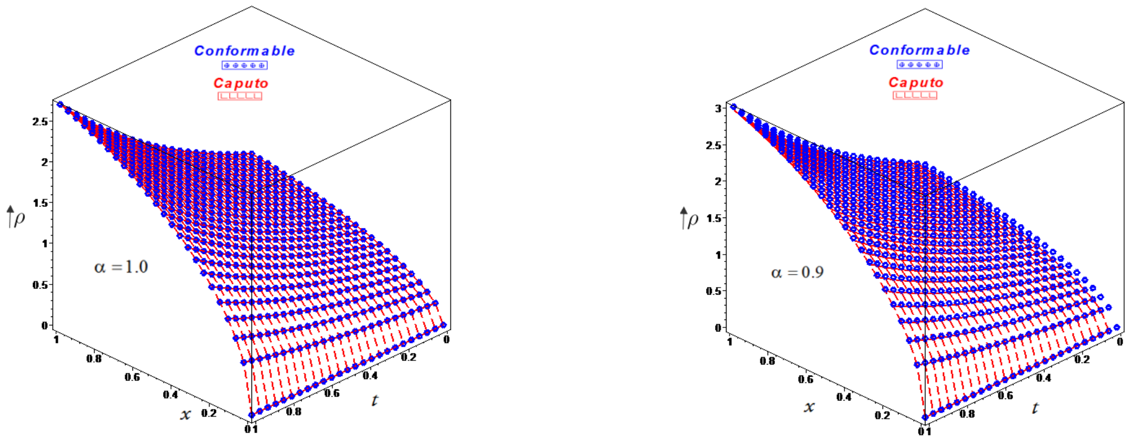


Figure 3: The plots show the comparison of exact solutions of (4.9) in conformable and Caputo sense [46] for $y = 1$ and some $\alpha$ values.

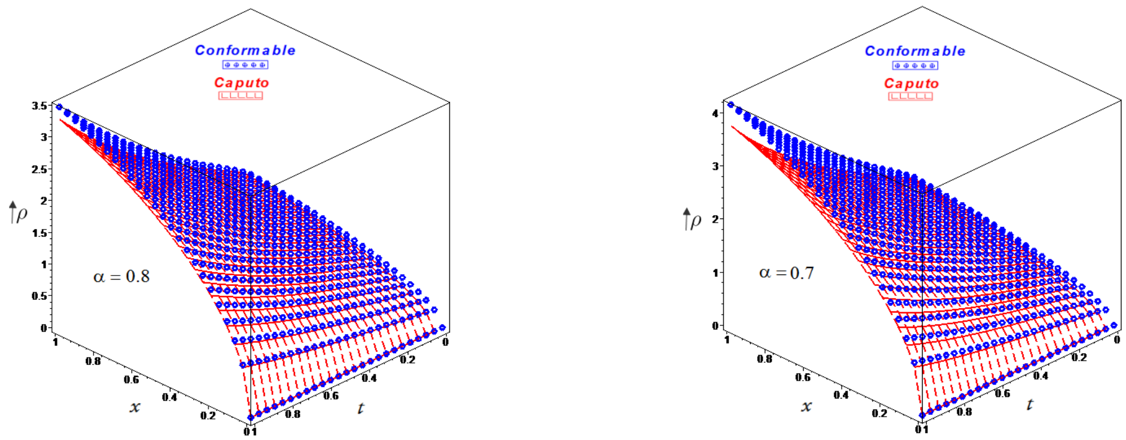


Figure 4: The plots show the comparison of exact solutions of (4.9) in conformable and Caputo sense [46] for $y = 1$ and some $\alpha$ values.

**Example 4.5.** In the final, we discuss the following non-linear GBPM [46]:

$$\frac{\partial^\alpha}{\partial t^\alpha}\rho(x, y, t) = \frac{\partial^2}{\partial x^2}\rho + \frac{\partial^2}{\partial y^2}\rho - \mu\rho^2 + \rho, \quad 0 < \alpha \leqslant 1 \tag{4.17}$$

subject to the IC

$$\rho(x,y,0) = e^{\frac{\sqrt{\mu}}{2\sqrt{2}}(x+y)}. \tag{4.18}$$

The exact solution in Caputo sense of non-linear GBPM (4.17) is given as [46]:

$$\rho(x,y,t) = e^{\frac{\sqrt{\mu}}{2\sqrt{2}}(x+y)} E_{\alpha}\left(t^{\alpha}\right), \tag{4.19}$$

where $E_{\alpha}\left(t^{\alpha}\right)$ is the Mittag-Leffler function, defined as $E_{\alpha}(\zeta) = \sum_{k=0}^{\infty} \frac{\zeta^k}{\Gamma(1+k\alpha)}$.

Now solve this problem by using TCRDTM. By taking the TCRDT of (4.17), it can be obtained that

$$\begin{aligned} \alpha(k+1) P_{k+1}^{\alpha}(x,y) = \frac{\partial^2}{\partial x^2}\left[\sum_{s=0}^{k} P_s^{\alpha}(x,y) P_{k-s}^{\alpha}(x,y)\right] + \frac{\partial^2}{\partial y^2}\left[\sum_{s=0}^{k} P_s^{\alpha}(x,y) P_{k-s}^{\alpha}(x,y)\right] \\ - \mu\left[\sum_{s=0}^{k} P_s^{\alpha}(x,y) P_{k-s}^{\alpha}(x,y)\right] + P_k^{\alpha}(x,y), \end{aligned} \tag{4.20}$$

where $P_k^{\alpha}(x,y)$ is the TCRDT function. From the IC (4.18) we write

$$P_0^{\alpha}(x,y) = e^{\frac{\sqrt{\mu}}{2\sqrt{2}}(x+y)}. \tag{4.21}$$

From (4.20) and (4.21), we can obtain the following $P_k^{\alpha}(x,y)$ values

$$\begin{aligned} P_1^{\alpha}(x,y) &= e^{\frac{\sqrt{\mu}}{2\sqrt{2}}(x+y)} \frac{1}{\alpha}, \\ P_2^{\alpha}(x,y) &= e^{\frac{\sqrt{\mu}}{2\sqrt{2}}(x+y)} \frac{1}{2!\alpha^2}, \\ P_3^{\alpha}(x,y) &= e^{\frac{\sqrt{\mu}}{2\sqrt{2}}(x+y)} \frac{1}{3!\alpha^3}, \\ &\vdots \\ P_m^{\alpha}(x,y) &= e^{\frac{\sqrt{\mu}}{2\sqrt{2}}(x+y)} \frac{1}{m!\alpha^m}, \cdots . \end{aligned} \tag{4.22}$$

Then, from (4.22), the set of values $\left\{P_k^{\alpha}(x)\right\}_{k=0}^{m}$ gives us the following approximate result

$$\tilde{\rho}_m(x,y,t) = \sum_{k=0}^{m} P_k^{\alpha}(x,y) t^{k\alpha} = \sum_{k=0}^{m} e^{\frac{\sqrt{\mu}}{2\sqrt{2}}(x+y)} \frac{1}{k!\alpha^k} t^{k\alpha}. \tag{4.23}$$

From (4.23) we obtain

$$\rho(x,y,t) = \lim_{m\to\infty} \tilde{\rho}_m(x,y,t) = e^{\frac{\sqrt{\mu}}{2\sqrt{2}}(x+y) + \frac{t^{\alpha}}{\alpha}}. \tag{4.24}$$

The obtained analytical approximate solution given in (4.24) is a new exact solution in conformable sense for non-linear GBPM in (4.17).

*Remark* 4.6. If we take $\alpha = 1$, then Example 4.5 is reduced to generalized standard biological population model

$$\frac{\partial}{\partial t}\rho(x,y,t) = \frac{\partial^2}{\partial x^2}\rho + \frac{\partial^2}{\partial y^2}\rho - \mu\rho^2 + \rho$$

with IC

$$\rho(x,y,0) = e^{\frac{\sqrt{\mu}}{2\sqrt{2}}(x+y)}.$$

Our new exact solution (4.24) in conformable sense and the exact solution (4.19) in Caputo sense imply

$$\rho(x, y, t) = e^{\frac{\sqrt{\mu}}{2\sqrt{2}}(x+y)+t}.$$

This result is the exact solution of the standard problem in the literature.

Now, we can compare the new exact solution in conformable sense and the exact solution [46] in Caputo sense for the non-linear GBPM (4.17). For some $\alpha$ values, the comparison of exact solutions in conformable and Caputo sense are illustrated by 3D graphics in Figs. 5 and 6. The MAPLE codes for graph drawings are given in Table 1.

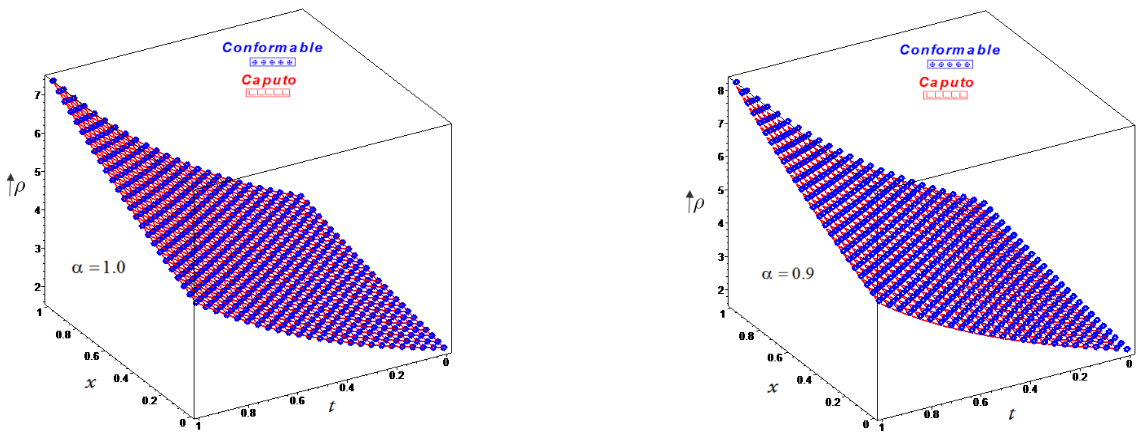


Figure 5: The plots show the comparison of exact solutions of (4.17) in conformable and Caputo sense [46] for $y = 1$, $\mu = 2$, and some $\alpha$ values.

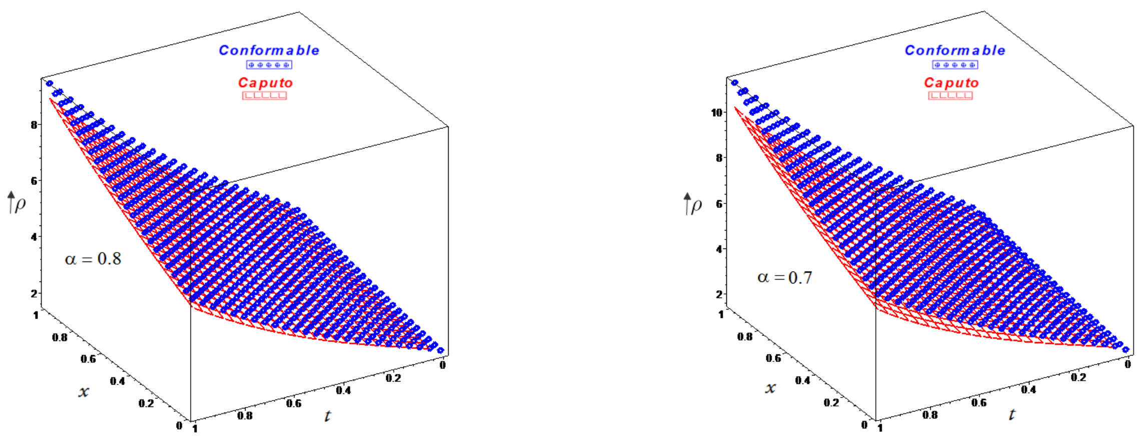


Figure 6: The plots show the comparison of exact solutions of (4.17) in conformable and Caputo sense [46] for $y = 1$, $\mu = 2$, and some $\alpha$ values.

## 5. Conclusion

In this paper, the TCRDTM is implemented in degenerate parabolic PDEs arising in the spatial diffusion biological populations. The solution obtained by the TCRDTM is an infinite power series for the appropriate IC which finds the solution without any discretization, perturbation, transformation or restrictive conditions. Three numerical examples are also illustrated considering the situations of non-linear phenomenon of GBPM to study the effectiveness and accurateness of the method. For each of the applications, new exact solutions with conformable sense are obtained. These new exact solutions obtained in conformable sense is compared with the exact solutions [46] in Caputo sense. For some values of $\alpha$, these comparisons are illustrated in 3D graphics. As a result of these processes, the following points on solutions can be noted as

(i) the conformable derivative produces exponential functions, while the Caputo derivative gives the Mittag Leffler functions;
(ii) when $\alpha = 1$, Mittag Leffler and exponential functions in the solutions are equal to each other. When $\alpha \neq 1$, the solutions are different from each other.

Hence, when $\alpha \to 1$, the exact solutions in conformable and Caputo sense converge to the exact results of the problems in classical sense. In other cases, the solutions exhibit similar behavior, but they are not equal to each other. As it offers new conformable solutions different from Caputo solutions for these models, we hope that it gives new solutions to many problems in the branches of sciences such as physics, chemistry, biology, mathematics, and engineering.

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