SINGULARLY PERTURBED BURGERS-HUXLEY EQUATION BY A MESHLESS METHOD

by

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A meshless method based upon radial basis function is utilized to approximate the singularly perturbed Burgers-Huxley equation with the viscosity coefficient ε. The proposed method shows that the obtained solutions are reliable and accurate. Convergence analysis of method was analyzed in a numerical way for different small values of singularity parameter.

Key words: *singularly perturbed Burgers-Huxley equation, multiquadratic, radial basis function method*

Introduction

Non-linear partial differential equations usually arise in modeling of various phenomena in most of the engineering and physical science branches. In the spatially homogeneous media, behavior of bifurcations and periodic trawling waves in excitable media are different. This difference cause by the strongly non-linearity and singular characteristics of the local reaction kinetics play significant role. Indeed, singular perturbation theory utilize the mentioned characteristics of excitable media. We recall that KKP-Fisher [1] equation can be utilized successfully in modelling the diffusion phenomena which admits a traveling front solution connecting the two steady-states. Among possible generalizations of the Fisher equation, the Burgers-Huxley (BH) equation is most important one of the form:

$$-\frac{\partial^2 u}{\partial x^2}+\alpha u\frac{\partial u}{\partial x}+\frac{\partial u}{\partial t}-\beta u(1-u)(u-\gamma)=0 \tag{1}$$

It is well-known that a large class of physical phenomena such as the interaction between convection effects, reaction mechanism, and diffusion transports can be described by the BH equation. Moreover, the BH equations are successfully applied to describe some ecological models. Let us consider a population as breeding in a medium, then the corresponding dynamical system is:

$$n_t=-kn+\kappa m(n)n^2+D\Delta n$$

where $m(n)$ is the mass of food and n is the size of the population per unit volume [2]. When $\alpha=0$, the BH equation can be assumed as eq. (1) when the mass of food varies as

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$m(n) = m_0(1 - n/n_0)$. More appropriate parabolic type of eq. (1) which characterized by the singular perturbation parameter, is singularly perturbed Burgers-Huxley (SPBH) equation. Actually, many standard numerical approaches are not converges for these types of equations and there are only few type of numerical methods that are successful for these problems. The SPBH equation with the initial and boundary conditions is:

$$-\varepsilon \frac{\partial^2 u}{\partial x^2} + \alpha u \frac{\partial u}{\partial x} + \frac{\partial u}{\partial t} - \beta u(1-u)(u-\gamma) = 0 \tag{2}$$

$$(x,t) \in D \equiv \Omega_X \times \Omega_t \equiv (0,1) \times (0,T] \tag{3}$$

$$u(x,0) = u_0(x), \quad x \in \bar{\Omega}_x \tag{4}$$

$$u(0,t) = s_0(t), \quad u(1,t) = s_1(t), \quad t \in \bar{\Omega}_t \tag{5}$$

where α, β, and γ are the parameters that take the values $\alpha \geq 0$, $\beta \geq 0$, and $\gamma \in (0,1)$. Here $\varepsilon \in (0,1]$ denotes the singular perturbation parameter. When $\varepsilon \to 0$, an outflow boundary-layer is turned out in the neighborhood of right part of the lateral surface of the domain and the model can be investigated as a non-linear singularly perturbed problem. For the case $\alpha > 0$, the solution of SPBH eq. (2) with small value of singular perturbation parameter, ε, demonstrates an exponential outflow boundary-layer in $\Gamma_r = \{(x,t) : x = 1, t \in \bar{\Omega}_t\}$. We recall that most of the standard numerical approaches can approximate the solution of BH equation without small singular perturbation parameter, ε. The cumbersome procedure arising in the numerical treatment even of linear singularly perturbed equations is well known [3-5].

The BH equation has been investigated by many researchers in recent years. The approximate analytical solutions of BH equation were obtained by the homotopy analysis method [6]. Alos, some travelling wave solutions corresponding to the generalized BH equation are obtained within the first integral method in [7]. The variational iteration method is utilized in [8] to solve this equation without discretization. Liu *et al.* in [9] reported a class of multi-soliton solutions for the generalized BH equation. Kinks and periodic wave solutions were build in [10] by utilizing the tanh-coth method.

Wang *et al.* [11] obtained the solitary wave solution of generalized BH equation and multi-soliton solutions of this equation have been studied by Liu *et al.* [9]. Pseudospectral method and Darvishi's preconditioning are utilized to approximate this equation by Javidi [12]. Khattak [13] applied the radial basis function (RBF) method to the generalized BH equation without singular perturbation parameter and Rathish Kumar *et al.* [14] approximated the SPBH equation with three-step Taylor-Galerkin method. Xie and Li [15] applied the combination of multiquadric (MQ)-RBF and high-order temporal approximation for the Burgers equation. Some recent papers in RBF are listed in [16-21].

Radial basis function approximation

In recent years, many researchers have investigated physical problems with RBF method. Published papers in this field and comparisons with other methods shows that RBF method is a powerful, reliable, and convergent method for most of the physical and engineering problems. In this paper we apply this method to approximate a singular problem which is of parabolic type and many approaches can not approximate this problem.

Let us to write the unknown function $u(X)$ as linear combination of M radial functions:

$$u(X) = \sum_{j=1}^{M} \lambda_j \phi_j(X) + \Psi(X), \quad X \in \Omega \subset R^d \tag{6}$$

where $X = (x_1, x_2, \ldots, x_d)^T$ and d is the dimension of the problems domain. Moreover, in eq. (6), $\lambda_j, j = 1, \ldots, M$ are unknown coefficients to be determined, φ_j is RBF, and Ψ is an polynomial which impose to the problem in order to reduce the condition number of condition number of discretized system. Two major class of RBF are exist, *i. e.* infinitely smooth RBF and other ones which are not infinitely smooth at centers. The MQ, Gaussian (GA), inverse multiquadric (IMQ), inverse quadric (IQ) are in the first class and thin plate spline, cubic, linear are in the second one. In the current paper we use the MQ functions defined by:

$$\phi_j(X) = \sqrt{r_j^2 + c_j^2} \tag{7}$$

where $r_j = \left\| X - X_j \right\|$ is the Euclidian norm and c_j is the free shape parameter.

If L_q^d denotes the space of at most q order of d-variate polynomials, and supposing $L_q^d = span\{p_1, \ldots, p_m\}$ concludes:

$$\Psi(X) = \sum_{i=1}^{m} \xi_i p_i(X) \tag{8}$$

where $m = (q - 1 + d)!/[d!(q-1)!]$.

So, to determine the unknown vector of coefficients $(\lambda_1, \ldots, \lambda_M)$ and $(\xi_1, \ldots, \xi_m)$, the collocation method can be used. Obviously, in addition to the M equations extracted from collocating eq. (6), we need to m equations to determine these unknown coefficients. We impose the m conditions for eq. (6) as:

$$\sum_{j=1}^{M} \xi_i p_i(X_j) = 0, \ i = 1, \ldots, m \tag{9}$$

It can be noted that, for any partial differential operator $\wp$ of linear type, we have:

$$\wp u(X) = \sum_{j=1}^{M} \lambda_j \wp \phi_j(X) + \wp \Psi(X) \tag{10}$$

Imposing this equality into the original equation helps our to determine the unknown coefficients.

Numerical approximation schemes

In order to approximate the SPBH equation, a new numerical scheme based upon a compact form of second-order finite difference method for time approximation and MQ-RBF for a spatial approximation, is constructed (scheme I). Notice that this scheme will be utilized in the numerical examples of this paper. Another approximate method (scheme II) based upon the method of line [22, 23] is also represented in detail for comparison.

Scheme I

Let us to discretize the time derivative in eq. (2) using the first-order forward difference:

$$-\varepsilon u_{xx}^n + \alpha u^n u_x^n + \frac{u^{n+1}-u^n}{\tau} - \beta u^n u^n + \beta\gamma u^n - \beta(1+\gamma)(u^n)^2 + \beta(u^n)^3 - e^n = 0, \quad n \geq 0 \quad (11)$$

where τ is the time step, $u^n = u(x, n*\tau)$, and e^n is the truncation error given by:

$$e^n = \frac{\tau}{2} u_{tt}^n + O(\tau^2) \quad (12)$$

Time differentiating of eq. (2) with respect to time variable, leads to a higher order compact finite difference scheme. Therefore, we have:

$$u_{tt} = \varepsilon u_{xxt} - \alpha u_t u_x - \alpha u u_{xt} + 2\beta(1+\gamma) u u_t - \beta\gamma u_t - 3\beta u_t u^2 = 0 \quad (13)$$

and time discretizing of (13), we obtain:

$$u_{tt}^n = \varepsilon\left[\frac{u_{xx}^{n+1}-u_{xx}^n}{\tau}\right] - \alpha u_x^n\left[\frac{u^{n+1}-u^n}{\tau}\right] - \alpha u^n\left[\frac{u_x^{n+1}-u_x^n}{\tau}\right] + 2\beta(1+\gamma)u^n\left[\frac{u^{n+1}-u^n}{\tau}\right] -$$
$$-\beta\gamma\left[\frac{u^{n+1}-u^n}{\tau}\right] - 3\beta(u^n)^2\left[\frac{u^{n+1}-u^n}{\tau}\right] + O(\tau) \quad (14)$$

Substituting eq. (14) into eq. (12) concludes:

$$e^n = \frac{\varepsilon}{2}\left[u_{xx}^{n+1}-u_{xx}^n\right] - \frac{\alpha}{2}u_x^n\left[u^{n+1}-u^n\right] - \frac{\alpha}{2}u^n\left[u_x^{n+1}-u_x^n\right] + \beta(1+\gamma)u^n\left[u^{n+1}-u^n\right] -$$
$$-\frac{\beta\gamma}{2}\left[u^{n+1}-u^n\right] - \frac{3}{2}\beta(u^n)^2\left[u^{n+1}-u^n\right] + O(\tau^2) \quad (15)$$

After substituting eq. (15) into eq. (11) we obtain our numerical scheme:

$$\left[1 + \frac{\alpha\tau}{2}u_x^n - \beta\tau u^n + \frac{\beta\gamma\tau}{2} + \frac{3\beta\tau}{2}(u^n)^2 - \beta\gamma\tau u^n\right]u^{n+1} - \frac{\varepsilon\tau}{2}u_{xx}^{n+1} + \frac{\alpha\tau}{2}u^n u_x^{n+1} =$$
$$= \left[1 - \frac{\beta\gamma\tau}{2} + \frac{\beta\tau}{2}(u^n)^2\right]u^n + \frac{\varepsilon\tau}{2}u_{xx}^n \quad (16)$$

Now, at each time level n, we approximate u^n by the modified MQ method introduced by Kansa [24]:

$$u^n(x) = \sum_{j=0}^{M}\lambda_j^n\sqrt{(x-x_j)^2 + c_j^2} + \lambda_{M+1}^n x + \lambda_{M+2} \quad (17)$$

where $x_j = j/M$, $j = 0,1,\ldots,M$. The c_j are shape parameters which affect the accuracy of solutions.

In order to determine the $(M+3)$ unknown coefficients λ_j^{n+1}, $j = 0,1,\ldots,M+2$, in the $(n+1)^{th}$ time level, firstly two boundary conditions (11) are used:

$$u^{n+1}(x_0) = s_0(t), \; u^{n+1}(x_1) = s_1(t) \quad (18)$$

and then $(M+1)$ distinct points $\hat{x}_j = j/(M+2)$ in (0,1) using eq. (16).

Scheme II

In this approximation method, spatial derivatives are firstly approximated using the RBF method, and the governing equation is reduced to a system of non-linear ODE. Then, the

resulting system of ODE have been solved using the fourth-order Runge-Kutta (RK4). Moreover, a function $u(x,t)$ can be approximated by:

$$u^M(x,t) = \sum_{j=1}^{M} \lambda_j \phi_j(x) = \Phi^T(x)\lambda \tag{19}$$

where M is the total number of distinct points $x_j, j = 1,2,\ldots,M$ in [a, b], and:

$$\Phi(x) = [\phi_1(x), \phi_2(x), ..., \phi_M(x)]^T, \quad \lambda = [\lambda_1, \lambda_2, ..., \lambda_M]^T \tag{20}$$

Let $u^M(x_i,t) = u_i(t) = u(x_i,t)$, then eq. (19) becomes:

$$A\lambda = U \tag{21}$$

where $U = [u_1(t), u_2(t), ..., u_M(t)]^T$, and the coefficient matrix:

$$A = \begin{bmatrix} \Phi^T(x_1) \\ \Phi^T(x_2) \\ \vdots \\ \Phi^T(x_M) \end{bmatrix} = \begin{bmatrix} \phi_1(x_1) & \phi_2(x_1) & \cdots & \phi_M(x_1) \\ \phi_1(x_2) & \phi_2(x_2) & \cdots & \phi_M(x_2) \\ \vdots & \vdots & \ddots & \vdots \\ \phi_1(x_M) & \phi_2(x_M) & \cdots & \phi_M(x_M) \end{bmatrix}$$

From eq. (19) and eq. (21) we can write:

$$u^M(x,t) = \Phi^T(x)A^{-1}U = \Lambda(x)U \tag{22}$$

where $\Lambda(x) = \Phi^T(x)A^{-1} = [\Lambda_1(x), \Lambda_2(x), ..., \Lambda_M(x)]$. From the RBF approximation we obtain:

$$u(x,t) \approx u^M(x,t) = \sum_{j=1}^{M} \lambda_j \varphi_j(x) = \Phi^T(x)\lambda = \Phi^T(x)A^{-1}U = \Lambda(x)U \tag{23}$$

Applying eq. (23) to eq. (2), and collocating at the node x_i, we have:

$$\frac{\mathrm{d}u_i}{\mathrm{d}t} - \varepsilon \frac{\partial^2 \Lambda}{\partial x^2}(x_i) + \alpha u_i \frac{\partial \Lambda}{\partial x}(x_i) - \beta u_i(1-u_i)(u_i - \gamma) = 0, \quad i = 1,2,\ldots,M \tag{24}$$

This system of equations can be written in the compact form:

$$\frac{\mathrm{d}U}{\mathrm{d}t} - \varepsilon(\Lambda_{xx}U) + \alpha U^*(\Lambda_x U) + \beta\gamma U - \beta(1+\gamma)U^2 + \beta U^3 = 0 \tag{25}$$

where * denotes the two vectors component-by-component multiplication. Equation (25) can also be written:

$$\frac{\mathrm{d}U}{\mathrm{d}t} = \Xi(U) \tag{26}$$

where

$$\Xi(U) = \varepsilon(\Lambda_{xx}U) - \alpha U^*(\Lambda_x U) - \beta\gamma U + \beta(1+\gamma)U^2 - \beta U^3 \tag{27}$$

Related initial condition is $U^0 = [g_0(x_1), g_0(x_2), \cdots, g_0(x_M)]^T$, and from eq. (5), we can write:

$$u_1(t) = f_1(t), \quad u_M(t) = f_2(t) \tag{28}$$

Equation (26) can be solved using the RK4. Also, solving this system of non-linear ODE, concludes the unknown coefficients λ by using the relationship shown in eq. (21).

Numerical examples

Here we present two examples with different initial and boundary conditions to illustrate the power and convergence of two explained schemes in the previous section.

Example 1

Here we discuss the following BH equation without the singularity parameter:

$$\frac{\partial u}{\partial t}+\alpha u\frac{\partial u}{\partial x}-\frac{\partial^2 u}{\partial x^2}-\beta u(1-u)(u-\gamma)=0 \tag{29}$$

with conditions:

$$u(x,0)=\frac{\gamma}{2}+\frac{\gamma}{2}\tanh(A_1 x),\ x\in\bar{\Omega}_x \tag{30}$$

and:

$$u(0,t)=\frac{\gamma}{2}+\frac{\gamma}{2}\tanh(-A_1A_2t),\ u(1,t)=\frac{\gamma}{2}+\frac{\gamma}{2}\tanh\left[A_1(1-A_2t)\right],\ t\in\bar{\Omega}_t \tag{31}$$

The corresponding exact solitary wave solution [25, 26] is given by:

$$u(x,t):=\frac{\gamma}{2}+\frac{\gamma}{2}\tanh\left[A_1(x-A_2t)\right] \tag{32}$$

where

$$A_1=\frac{-\alpha+\sqrt{\alpha^2+8\beta}}{8},\quad A_2=\frac{\gamma\alpha}{2}-\frac{(2-\gamma)\left(-\alpha+\sqrt{\alpha^2+8\beta}\right)}{4} \tag{33}$$

Table 1. Comparison of numerical and exact solutions with $M = 10$, $T = 1$, and $\tau = 0.1$ for example 1 with uniform mesh at various mesh points

x_i	Exact solution	Scheme I	Scheme II
0.1	6.370e–004	6.355e–004	5.148e–004
0.2	6.506e–004	6.436e–004	5.296e–004
0.3	6.640e–004	6.530e–004	5.444e–004
0.4	6.771e–004	6.637e–004	5.590e–004
0.5	6.889e–004	6.757e–004	5.736e–004
0.6	7.024e–004	6.890e–004	5.880e–004
0.7	7.147e–004	7.035e–004	6.023e–004
0.8	7.266e–004	7.192e–004	6.164e–004
0.9	7.382e–004	7.360e–004	6.301e–004

Numerical results with the parameters $\alpha = 0.5$, $\beta = 1$, and $\gamma = 0.001$ are reported in tab. 1. It can be seen from this table that results of scheme I are more satisfactory than the second one.

Example 2

This example is an SPBH equation defined by:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}-\varepsilon\frac{\partial^2 u}{\partial x^2}=(1-u)(u-0.5)u \tag{34}$$

with sinusoidal initial condition:

$$u(x,0)=\sin(\pi x),\ x\in\bar{\Omega}_x \tag{35}$$

and:

$$u(0,t)=0,\quad u(1,t)=0,\quad t\in\bar{\Omega}_t \tag{36}$$

In this example we utilize the major method of this paper (scheme I) to approximate the solution of problem. Let us to take $M = 8$, $T = 1$, and $\tau = 0.1$ and we multiply M by 2. The exact solution of SPBH equation for small values of the viscosity coefficient, ε, is not available. Hence, to show the performance of this scheme at low singular perturbation parameter, ε, we use to estimate the pointwise error:

$$e_{\varepsilon}^{M,\tau} = | u^{M}(x_i, t^n) - u^{2M}(x_i, t^n) | \tag{37}$$

Moreover, the maximum nodal error for each ε, have the following form:

$$E_{\varepsilon}^{M,\tau} = \max_{i,n} e_{\varepsilon}^{M,\tau} \tag{38}$$

In tab. 2, maximum nodal errors for different values of ε up to 2^{-8}, and different knots have been demonstrated. Indeed, this table shows that maximum nodal error decreases when point numbers increase which this fact demonstrate the numerical stability of current method. Also, tab. 3 shows the comparison of obtained results with current method and monotone finite difference scheme [27] for different values of the viscosity coefficient. Results of our method are more beater than the reported results in reference [28], especially for small values of singularity parameter. Finally, figs. 1-4 demonstrate approximate solutions of SPBH equation in example 2 with $\varepsilon = 2^{0}$, 2^{-6}, 2^{-12}, and 2^{-24}, respectively. Figures of approximate solutions in x-direction and different time values are plotted beside of each approximate solution in 3-D to show the behave of singularity parameter in the problem.

Table 2. Maximum nodal errors with $T = 1$ and $\tau = 0.1$ for Example 2 with uniform mesh at various singular perturbation parameters

ε	$M = 4$	$M = 6$	$M = 8$
2^{0}	1.901e–05	3.767e–06	1.017e–07
2^{-2}	2.360e–04	1.110e–04	3.124e–05
2^{-4}	1.370e–02	1.710e–03	1.22e–04
2^{-6}	2.098e–01	1.987e–02	8.081e–03
2^{-8}	5.474e–01	2.294e–01	1.725e–02

Table 3. Comparison of maximum pointwise errors for Example 2 with the parameters $\alpha = 1$, $\beta = 1$, and $\gamma = 0.5$ on uniform mesh

ε	$M = 16$ (Our method)	$M = 16$ [28]	$M = 32$ (Our method)	$M = 32$ [28]
2^{0}	5.3220e–08	6.8456e–06	1.6764e–09	4.0455e–07
2^{-2}	4.7534e–06	1.0777e–03	5.7923e–07	5.8450e–04
2^{-4}	3.5695e–05	5.4069e–03	2.5473e–06	1.3862e–03
2^{-6}	6.6293e–04	3.7340e–01	3.0004e–05	7.3680e–02
2^{-8}	2.3230e–03	1.3865	9.3234e–04	1.1618
2^{-10}	1.9313e–02	1.6736	8.3640e–03	1.9856

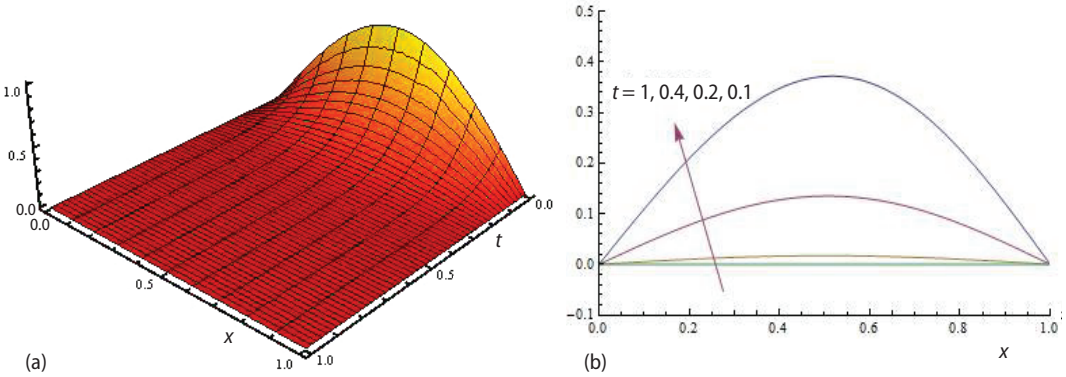

Figure 1. Approximate profile of Example 2 with $M = 10$, $\tau = 0.01$, and $\varepsilon = 2^{0}$
(*for color image see journal web site*)

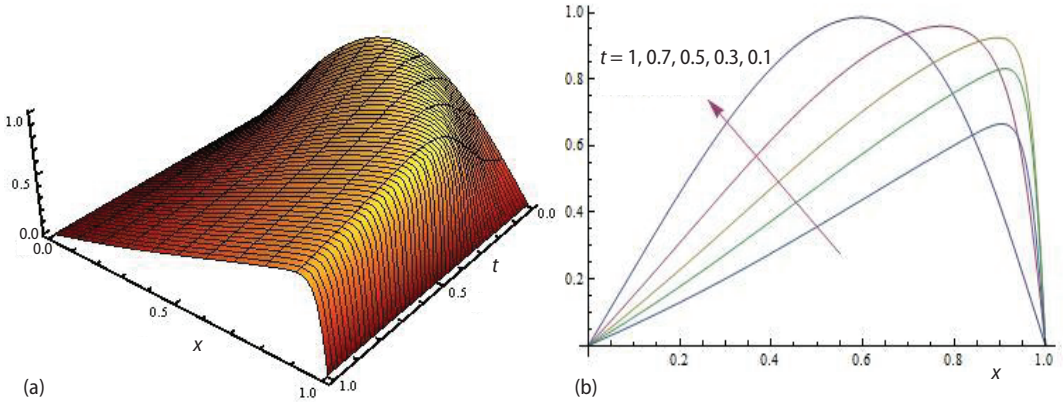

Figure 2. Approximate profile of Example 2 with $M = 10$, $\tau = 0.01$, and $\varepsilon = 2^{-6}$
(*for color image see journal web site*)

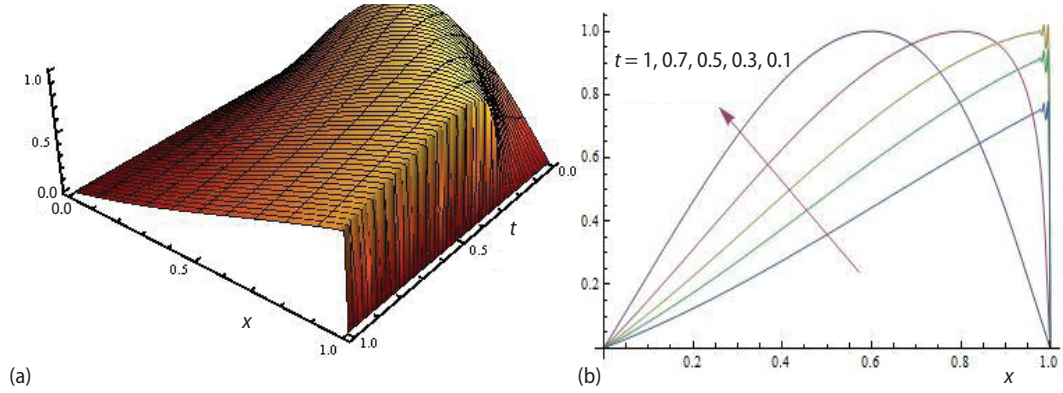

Figure 3. Approximate profile of Example 2 with $M = 10$, $\tau = 0.01$, and $\varepsilon = 2^{-12}$
(*for color image see journal web site*)

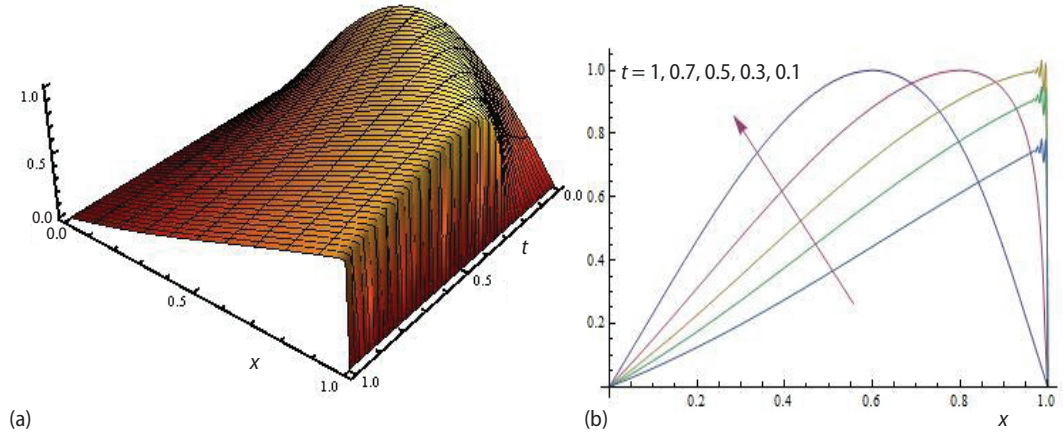

Figure 4. Approximate profile of Example 2 with $M = 10$, $\tau = 0.01$, and $\varepsilon = 2^{-24}$
(*for color image see journal web site*)

Conclusion

In this manuscript we suggest a powerful and efficient method to solve the SPBH equation using a second-order compact finite difference scheme for time discretization and MQ-RBF for spatial approximation. Convergence of the proposed method was demonstrated for different values of singularity parameter. High accuracy and efficiency of the method were demonstrated by data presented in our tables and figures.

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