Research Article

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Analysis of Drude model using fractional derivatives without singular kernels

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Abstract: We report study exploring the fractional Drude model in the time domain, using fractional derivatives without singular kernels, Caputo-Fabrizio (CF), and fractional derivatives with a stretched Mittag-Leffler function. It is shown that the velocity and current density of electrons moving through a metal depend on both the time and the fractional order $0 < \gamma \leq 1$. Due to non-singular fractional kernels, it is possible to consider complete memory effects in the model, which appear neither in the ordinary model, nor in the fractional Drude model with Caputo fractional derivative. A comparison is also made between these two representations of the fractional derivatives, resulting a considered difference when $\gamma < 0.8$.

Keywords: Fractional Calculus, Drude model, Caputo-Fabrizio derivative, Atangana-Baleanu derivative

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1 Introduction

Fractional calculus (FC), involving derivatives and integrals of non-integer order, is the natural generalization of classical calculus, which during recent decades has become a powerful and widely used tool for better modelling and control of processes in many areas of science and engineering [1–11]. The fractional derivatives are non-local operators because they are defined using integrals. Therefore, the fractional derivative in time contains information about the function at earlier points, thus it possesses a memory effect, and it includes non-local spatial effects. In other words, such derivatives consider the history and non-local distributed effects which are essential for better and more precise descriptions and understanding of complex and dynamic system behaviour. Due to the lack of a consistent geometric and physical interpretation of the fractional derivative, several definitions of fractional derivatives and integrals exist, see [12] for a review of definitions for fractional derivatives and integrals. These definitions include, Riemann-Liouville, Grunwald-Letnikov, Caputo, and Weyl, among others. The most used definitions are the Riemann-Liouville and the Caputo fractional derivatives. There are classical applications where FC has showed its great capabilities, such as the tautochrone problem [13], models based on memory mechanisms [14], fractional diffusion equations [15], new linear capacitor theory [16], the non-local description of quantum dynamics like Brownian motion and anomalous diffusion [17, 18]. Other interesting applications are given in viscoelastic materials [19–24], anomalous non-Gaussian transport [25] and dielectric materials [26–29].

The concept of a fractional curl operator and a fractional paradigm in electromagnetic theory was introduced in [30]. The application of the fractional curl operator to different electromagnetic problems is discussed in [31–33]. In [34, 35] it is shown that the electromagnetic fields and waves in a wide class of dielectric materials are described by fractional differential equations with derivatives of non-integer order with respect to time. The order of these derivatives is defined by exponentials of the universal response laws for frequency dependence of the dielectric susceptibility. In [36] proposed a systematic way to construct fractional differential equations and applied them to the propagation of electromagnetic waves in an infinitely extended homogeneous medium at rest.

Despite the accurate results obtained with the Riemann-Liouville and Caputo fractional derivatives, they have the disadvantage that their kernel has a singularity

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at the end point of the interval. To avoid this problem, [37] proposed the Capauto-Fabrizio (CF) derivative. This is a new fractional-order derivative that does not have any singularity. The main advantage of the new definition is that the singular power-law kernel, Caputo, is now replaced by a non-singular exponential kernel CF, which is easier to use in theoretical analysis, numerical calculations and real-world applications. Based on this new derivative, some interesting studies can be found in [38–44]. However, some researchers have concluded that the operator is not a derivative with fractional order, but instead a filter with fractional parameters. To correct this deficiency, two fractional derivatives in the Caputo and Riemann-Liouville sense were defined by Atangana-Baleanu [45], based on the generalized stretched Mittag-Leffler function. These new derivatives have been applied to different systems in [46–48].

Motivated by recent experimental results [49–51] showing that simple models are best described by fractional differential equations, in [52] the fractional Caputo derivative is applied to study of the Drude model. However, as was mentioned above, this definition of the derivative has a singular kernel and cannot accurately describe the full memory effect. To correct this disadvantage, in the present work we analyse the Drude model using a Caputo-Fabrizio fractional derivative [37], as well as the Atangana-Baleanu in the Caputo sense (ABC) derivative [45], for different sources. Numerical simulations of these models are given in order to compare them and evaluate their effectiveness.

This work is organized as follows: in section 2, some basic concepts about Caputo (C), Caputo-Fabrizio (CF) and ABC fractional derivatives are given; in section 3, the classical Drude model is reviewed; in section 4, a new fractional Drude model is proposed using the CF fractional derivative; in section 5, the ABC derivative is applied to the same problem; finally, in section 6, the obtained results are discussed and compared; the conclusion is presented in section 7.

2 Some fractional derivatives

The usual Caputo fractional derivative of order γ is defined by [14]

$$\frac{d^{\gamma} f(t)}{dt^{\gamma}} = {}_{a}D_{t}^{\gamma} f(t) = \frac{1}{\Gamma(1-\gamma)} \int_{a}^{t} \frac{\dot{f}(\tau)}{(t-\tau)^{\gamma}} d\tau, \qquad (1)$$

with $0 < \gamma \leq 1$ and $a \in [-\infty, t]$, $f \in H^{1}(a, b)$, $b > a$. By changing the kernel $(t-\tau)^{-\gamma}$ with the function $e^{-\frac{\gamma}{1-\gamma} t}$ and $\frac{1}{\Gamma(1-\gamma)}$ with $\frac{M(\gamma)}{1-\gamma}$, the Caputo-Fabrizio fractional derivative [37] is obtained

$${}^{CF}{}_{a}D_{t}^{\gamma} f(t) = \frac{M(\gamma)}{(1-\gamma)} \int_{a}^{t} \dot{f}(\tau) \exp\left[-\frac{\gamma(t-\tau)}{1-\gamma}\right] d\tau, \qquad (2)$$

where $\frac{M(\gamma)}{1-\gamma}$ is a normalization function with the property $M(0) = M(1) = 1$. If $f(t)$ is a constant function, then the Caputo-Fabrizio derivative (2) is zero. However, in contrast to definition (1), the kernel in (2) does not have singularity for $t = \tau$. This property is of particular interest, because it can describe the full memory effect for a given system. The Laplace transform of the CF fractional derivative is given by

$$L[{}^{CF}{}_{a}D_{t}^{\gamma} f(t)] = \frac{sF(s) - f(0)}{s + \gamma(1-s)}. \qquad (3)$$

In [45], two new fractional derivatives appeared. We will apply one of them, defined as: $f \in H^{1}(a, b)$, $a < b$, $\gamma \in [0, 1]$, then the Atangana-Baleanu (AB) fractional derivative in the Caputo sense (ABC) is

$${}^{ABC}{}_{a}D_{t}^{\gamma} [f(t)] = \frac{B(\gamma)}{1-\gamma} \int_{a}^{t} f'(x) E_{\gamma}\left[-\gamma \frac{(t-x)^{\gamma}}{1-\gamma}\right] dx, \qquad (4)$$

where $E_{\gamma}(\cdot)$ and $E_{\beta,\gamma}(\cdot)$ are the one- and two-parameter Mittag-Leffler functions defined in [9]

$$E_{\gamma}(z) = \sum_{n=0}^{\infty} \frac{z^{n}}{\Gamma(\gamma n + 1)}, \qquad \gamma > 0, \qquad (5)$$

$$E_{\beta,\gamma}(z) = \sum_{n=0}^{\infty} \frac{z^{n}}{\Gamma(\beta n + \gamma)}, \qquad \beta, \gamma > 0. \qquad (6)$$

Expression (4) has a non-singular and non-local kernel. The Laplace transform for (4) is

$$L[{}^{ABC}{}_{a}D_{t}^{\gamma}] = \frac{B(\gamma)}{1-\gamma} \cdot \frac{s^{\gamma} F(s) - s^{\gamma-1} f(0)}{s^{\gamma} + \frac{\gamma}{1-\gamma}}, \quad 0 < \gamma \leq 1, \qquad (7)$$

where $B(\gamma)$ has the same properties as in the CF case. These formulae are necessary for our study.

3 Classical Drude models

The fundamental properties of materials play an extremely important role in developing new technologies in various areas. Among the important properties of various materials, the electromagnetic interaction with matter is of great importance.

The Drude model is known as the first realistic model to describe metals. Despite the fact that it is a very simple model, it can explain electrical conductivity, thermal conductivity, and optical properties of metals [53, 54]. This model regards metals as a classical gas of electrons executing diffusive motion. The one-dimensional equation, which describes the motion of a charged particle e with mass m affected by an external electric field $E(t)$, is given by [55]

$$m\frac{dv}{dt} + \frac{m}{\tau}v = -eE(t). \tag{8}$$

Regarding the left side, the first term express the acceleration of the charges induced by the electric field; the second term describes the damping factor due to electron scattering, where τ is the relaxation parameter.

For a constant electric field $E(t) = E_0$, the solution of (8) is given by

$$v(t) = -\frac{eE_0\tau}{m}\left[1 - \exp\left(-\frac{t}{\tau}\right)\right]. \tag{9}$$

If the electric field is defined by a pulse, like a delta Dirac distribution $E = E_0\delta(t)$ [56], then the solution of (8) is

$$v(t) = -\frac{eE_0}{m}\exp\left(-\frac{t}{\tau}\right). \tag{10}$$

In the case of an oscillating field $E(t) = E_0\cos(\omega t)$, the solution of (8) is given by

$$v(t) = -\frac{eE_0\tau}{m[1+(\omega\tau)^2]} \cdot \left[\cos(\omega t) + \omega\tau\sin(\omega t) - \exp\left(-\frac{t}{\tau}\right)\right]. \tag{11}$$

Typically, only the current density $\vec{j} = -eN\vec{v}$ is experimentally accessible [57]. Then, for the oscillating field the current density is

$$j(t) = \frac{\sigma_0}{1+(\omega\tau)^2} \cdot \left[\cos(\omega t) + \omega\tau\sin(\omega t) - \exp\left(-\frac{t}{\tau}\right)\right]E_0, \tag{12}$$

where $\sigma_0 = \frac{e^2N\tau}{m}$ is the static Drude electric conductivity.

In the following sections, we analyse the Drude model from the point of view of the fractional derivatives of CF and ABC, to obtain the current density of electrons in metals for different sources.

4 Fractional Drude models with Caputo-Fabrizio derivative

To go from an ordinary differential equation to a fractional one, it is necessary to make it dimensionless. For this purpose, we make the redefinition [52]

$$u(t) = -\frac{m}{e\tau}v(t). \tag{13}$$

Substituting (13) in (8), we obtain the dimensionless differential equation

$$\frac{du}{d\bar{t}} + u(\bar{t}) = E(\bar{t}), \tag{14}$$

with dimensionless parameter $\bar{t} = t/\tau$. Therefore, we can proceed to apply a fractional derivative, either CF, ABC, or another fractional derivative.

We replace the Caputo-Fabrizio fractional derivative. Then, equation (14) takes the form

$$^{CF}{}_0D^{\gamma}_{\bar{t}}u(\bar{t}) + u(\bar{t}) = E(\bar{t}), \qquad 0 < \gamma \leq 1. \tag{15}$$

This fractional equation could give better experimental results for the current density in metals, due to the parameter $0 < \gamma \leq 1$ [49]-[51]. Recall that, in principle, we assume that the materials are homogeneous and isotropic, however, this is only the case under certain conditions. In the real world, such conditions are rarely fulfilled. We will study this fractional differential equation by means of three different sources:

First Case. The source is a constant, $E(\bar{t}) = E_0$. So, applying the Laplace transform (3)

$$L\left[^{CF}{}_0D^{\gamma}_{\bar{t}}u(\bar{t})\right] + L\left[u(\bar{t})\right] = L\left[E_0\right], \tag{16}$$

we have

$$\frac{\bar{s}U(\bar{s}) - u(0)}{\bar{s} + \gamma(1-\bar{s})} + U(\bar{s}) = \frac{E_0}{\bar{s}}, \tag{17}$$

where $\bar{s} = \tau s$. Taking the initial condition $u(0) = 0$, and applying the inverse Laplace transform in (17), we obtain

$$u(\bar{t}) = \left[1 - \frac{1}{2-\gamma}\exp\left(-\frac{\gamma}{2-\gamma}\bar{t}\right)\right]E_0. \tag{18}$$

Considering (13), we have the fractional electron velocity

$$v(t;\gamma) = -\frac{eE_0\tau}{m}\left[1 - \frac{1}{2-\gamma}\exp\left(-\frac{\gamma}{2-\gamma}\cdot\frac{t}{\tau}\right)\right], \quad 0 < \gamma \leq 1. \tag{19}$$

In the ordinary Drude model, the current density of electrons is given by $\vec{j} = -eN\vec{v}$; in the fractional case, it has the form

$$j_{CF}(t,\gamma) = \sigma_0\left[1 - \frac{1}{2-\gamma}\exp\left(-\frac{\gamma}{2-\gamma}\cdot\frac{t}{\tau}\right)\right]E_0, \quad 0 < \gamma \leq 1, \tag{20}$$

where $\sigma_0 = \frac{e^2 N\tau}{m}$ is the static Drude conductivity.

Second case. We have $E_0\delta(t)$ as a source, where $\delta(t)$ is the delta Dirac function. The corresponding fractional differential equation is

$$^{CF}{}_0D^{\gamma}_{\bar{t}}u(\bar{t}) + u(\bar{t}) = E_0\delta(\bar{t}). \quad (21)$$

The response to the impulse function was interesting, as any system can be conveniently characterized in the time domain using its impulse response, because it is the time domain version of the transfer function [56]. Applying the Laplace transform (3), we obtain

$$\frac{\bar{s}U(\bar{s}) - u(0)}{\bar{s} + \gamma(1-\bar{s})} + U(\bar{s}) = E_0. \quad (22)$$

As above, we consider the initial condition $u(0) = 0$ and $\bar{s} = \tau s$. Then,

$$u(\bar{t}) = \left[\frac{1-\gamma}{2-\gamma}\delta(\bar{t}) + \frac{\gamma}{(2-\gamma)^2}\exp\left(-\frac{\gamma}{2-\gamma}\bar{t}\right)\right]E_0. \quad (23)$$

From (13), and taking into account $\vec{j} = -en\vec{v}$, we have

$$j_{CF}(t;\gamma) = \frac{Ne^2}{m(2-\gamma)} \cdot \left[(1-\gamma)\delta(t) + \frac{\gamma}{2-\gamma}\exp\left(-\frac{\gamma}{2-\gamma}\cdot\frac{t}{\tau}\right)\right]E_0. \quad (24)$$

Third case. An oscillatory source $E(t) = E_0\cos(\omega t)$. We can write it as

$$^{CF}{}_0D^{\gamma}_{\bar{t}}u(\bar{t}) + u(\bar{t}) = E_0\cos(\omega\tau\bar{t}). \quad (25)$$

Applying the Laplace transform (3) with the same initial condition, we have

$$U(\bar{s}) = \frac{\bar{s}[\bar{s} + \gamma(1-\bar{s})]}{[(\bar{s})^2 + (\omega\tau)^2][2\bar{s} + \gamma(1-\bar{s})]}E_0, \quad (26)$$

where $\bar{s} = \tau s$. Taking the inverse Laplace transform and considering (13), the current density is

$$j_{CF}(t;\gamma) = \frac{(2-\gamma)\sigma_0}{\gamma^2 + (\omega\tau)^2(2-\gamma)^2} \cdot \left[A\cos(\omega t) + B\sin(\omega t) + C\exp\left(-\frac{\gamma}{2-\gamma}\cdot\frac{t}{\tau}\right)\right] \quad (27)$$

where

$$A = (\omega\tau)^2(1-\gamma) + \frac{\gamma^2}{2-\gamma}, \quad (28)$$
$$B = \gamma(\omega\tau) - \frac{\gamma(1-\gamma)}{\omega\tau},$$
$$C = \frac{\gamma^2}{2-\gamma}\left(1 - \gamma - \frac{1}{2-\gamma}\right).$$

We can see that, in the particular case $\gamma = 1$, these expressions become $A = 1; B = \omega\tau$ and $C = -1$, which is just the ordinary case.

5 Drude models with fractional derivatives with non-singular Mittag-Leffler kernel

Now, we will consider the same equation (14) and the same sources, but taking into account the Atangana-Baleanu fractional derivative in the Caputo sense (4). That is

$$^{ABC}{}_0D^{\gamma}_{\bar{t}}u(\bar{t}) + u(\bar{t}) = E(\bar{t}). \quad (29)$$

First case. A constant source $E(t) = E_0$. Then, applying the Laplace transform (7) in (29), we have

$$U(\bar{s}) = \frac{\bar{s}^{\gamma}(1-\gamma) + \gamma}{\bar{s}[(2-\gamma)\bar{s}^{\gamma} + \gamma]}E_0. \quad (30)$$

From here, we obtain the inverse Laplace transform $u(t)$, and using expression (13), the current density is given as

$$j_{ABC}(t;\gamma) = \sigma_0\left(1 - \frac{1}{2-\gamma}E_{\gamma}\left[-\frac{\gamma}{2-\gamma}\left(\frac{t}{\tau}\right)^{\gamma}\right]\right)E_0, \quad (31)$$

where $E_{\gamma}(\cdot)$ is the Mittag-Leffler function (5).

Second case. Delta Dirac as a source, $E_0\delta(t)$. In this case, we have

$$^{ABC}{}_0D^{\gamma}_{\bar{t}}u(\bar{t}) + u(\bar{t}) = E_0\delta(\bar{t}). \quad (32)$$

Applying the direct Laplace transform (7), we have

$$U(\bar{s}) = \frac{(1-\gamma)\bar{s}^{\gamma} + \gamma}{(2-\gamma)\bar{s}^{\gamma} + \gamma}E_0, \quad (33)$$

taking its inverse Laplace transform and considering equation (13), we have the current density

$$j_{ABC}(t;\gamma) = \frac{Ne^2}{m(2-\gamma)}\left((1-\gamma)\delta(t) + \frac{\gamma}{2-\gamma}\left(\frac{t}{\tau}\right)^{\gamma-1}E_{\gamma,\gamma}\left[-\frac{\gamma}{2-\gamma}\left(\frac{t}{\tau}\right)^{\gamma}\right]\right)E_0, \quad (34)$$

where $E_{\gamma,\gamma}$ is the two parametric Mitag-Leffler function (6).

Third case. An oscillatory source $E(t) = E_0\cos(\omega t)$. Then,

$$^{ABC}{}_0D^{\gamma}_{\bar{t}}u(\bar{t}) + u(\bar{t}) = E_0\cos(\omega\tau\bar{t}). \quad (35)$$

Applying the Laplace transform in (7), we obtain

$$U(\bar{s}) = \frac{\bar{s}[(1-\gamma)\bar{s}^{\gamma} + \gamma]}{[\bar{s}^2 + (\omega\tau)^2][(2-\gamma)\bar{s}^{\gamma} + \gamma]}, \quad (36)$$

where $\bar{s} = \tau s$. We take the highest power of $\bar{s}$ as a common factor from the denominator and then expanding it in an alternating geometric series [59], we have

$$u(t;\gamma) = \frac{1-\gamma}{2-\gamma}\sum_{n,m=0}^{\infty}\frac{(-1)^{n+m}(\omega\tau)^{2n}\gamma^m}{(2-\gamma)^m\Gamma[2n + m\gamma + 1]}\left(\frac{t}{\tau}\right)^{(2n+m\gamma)} \quad (37)$$

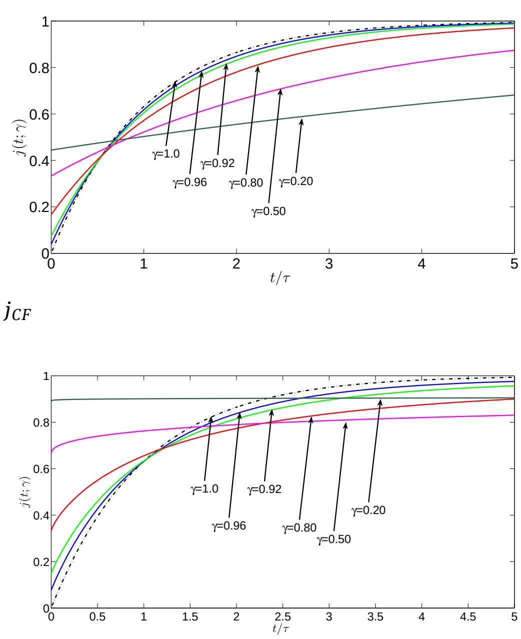
$$+\frac{\gamma}{2-\gamma}\sum_{n,m=0}^{\infty}\frac{(-1)^{n+m}(\omega\tau)^{2n}\gamma^m}{(2-\gamma)^m\Gamma[(m+1)\gamma+2n+1]}\cdot\left(\frac{t}{\tau}\right)^{([m+1]\gamma+2n)},$$

then, from (13), the current density due to an oscillating field is

$$j_{ABC}(t;\gamma)=\sigma_0 u(t;\gamma)E_0. \qquad (38)$$

6 Results and discussion

Figure 1, show the current density $j(t;\gamma)$ for an constant electric field E_0, with some values of γ. The asymptotic approximations for expression (20) are: when $t\to 0$, then $j_{CF}(t;\gamma)\to\sigma_0(1-\frac{1}{2-\gamma})E_0$, and if $t\to\infty$, then $j_{CF}(t;\gamma)\to\sigma_0E_0$, which is in full agreement with the ordinary case Figure 1 (a).



(a) j_{CF}

(b) j_{ABC}

Figure 1: These plots correspond to equations (20) and (31), respectively, for a range of values of γ. If $\gamma=1$, the CF and ABC derivatives give the same curve.

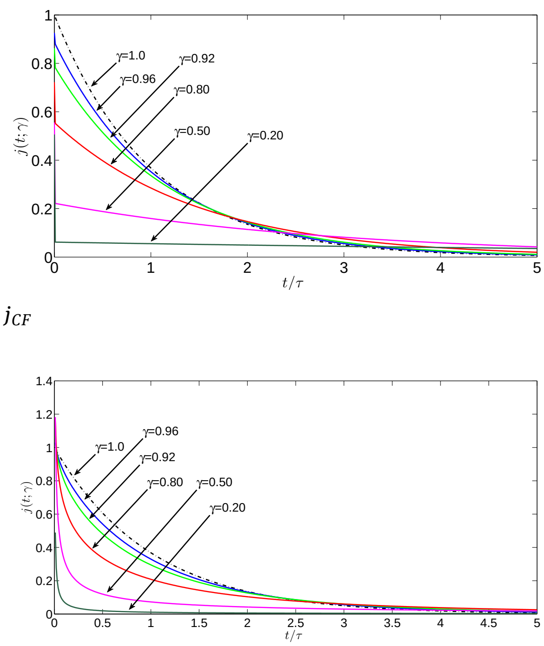
In the case $j_{ABC}(t;\gamma)$, the asymptotic approximations to the Mittag-Leffler function for small $t\to 0$ and larger $t\to\infty$ times, in first approximation, are [58]

$$E_\gamma(-t^\gamma)\sim e^{-\frac{t^\gamma}{\Gamma(1-\gamma)}}, \qquad t\to 0, \qquad (39)$$

$$E_\gamma(-t^\gamma)\sim\frac{t^{-\gamma}}{\Gamma(1-\gamma)}, \qquad t\to\infty. \qquad (40)$$

As a consequence, the Mittag-Leffler function interpolates for intermediate time t between the stretched exponential and the negative power law. The stretched exponential models the very fast decay for small time t, whereas, the asymptotic power law is due to very slow decay for large time t, as it can be seen in Figure 1 (b), showing the behaviour of (31) for different values of γ.

Figure 2 shows plots of the current density $j(t;\gamma)$ when electrons are excited by a pulsed electric field. The plots include different values of γ. Plot (a) corresponds to $j_{CF}(t;\gamma)$ (24) and plot (b) to $j_{ABC}(t;\gamma)$ (34).



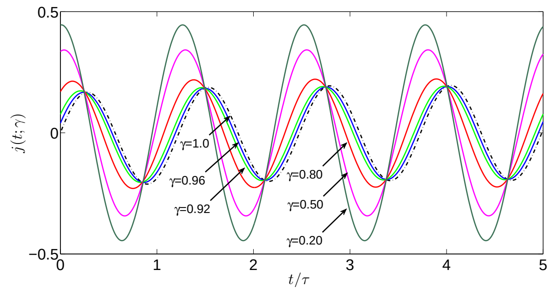
(a) j_{CF}

(b) j_{ABC}

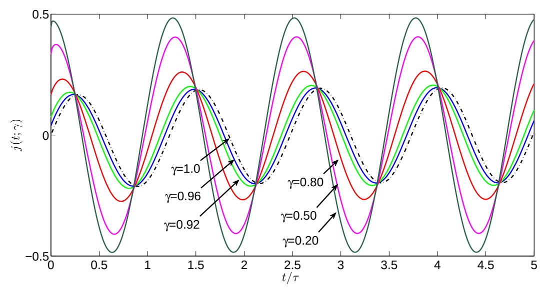
Figure 2: These plots correspond to equations (24) and (34), respectively, for different values of γ.

Figure 3 shows graphs of the current density $j(t;\gamma)$ when the electrons are excited by an oscillating electric field. The graphs have been plotted for different values of γ. In the case when the electric field is an oscillating field, the current density is an oscillatory signal affected by the fractional order γ. Two characteristics are influenced by γ: the phase and the amplitude, which are similar for the cases j_{CF} and j_{ABC}.

The numerical solutions of the current density with constant E_0, impulse $\delta(t)$ and alternating $\cos\omega t$ field, given in Figures 1, 2 and 3, show that the current density solution of fractional derivative models in the Caputo-Fabrizio sense exhibits an exponential decay similar to the classical integer-order model (9) and (20). It is clear that



(a) j_{CF}



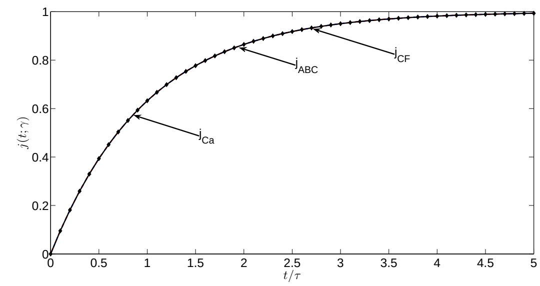
(b) j_{ABC}

Figure 3: These plots correspond to equations (27) and (38), respectively, for different values of γ. In this case $\omega\tau = 5.0$.

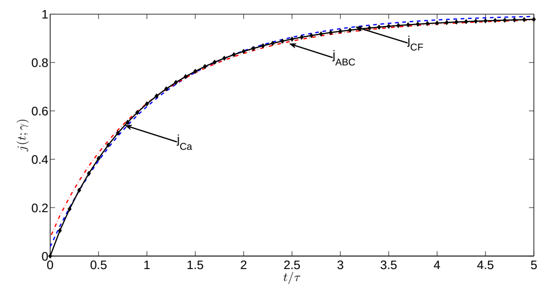
the CF derivative has limitations in characterizing anomalous diffusion with non-exponential nature [43]. On the other hand, using the Atangana-Baleanu definition, the solution is given by a Mittag-Leffler function, which interpolates for intermediate time t between the stretched exponential and the negative power law. The stretched exponential (39) models the very fast decay at small t, whereas the asymptotic power law (40) is due to very slow decay at large t. Notable differences between CF and ABC occur when $\gamma < 0.8$.

In the following figures we have taken the current densities j_{Ca} and Eqs. (13) and (24) from [52], where a Caputo derivative was used, to compare the Caputo-Fabrizio and Atangana-Baleanu derivatives for the specified values of γ.

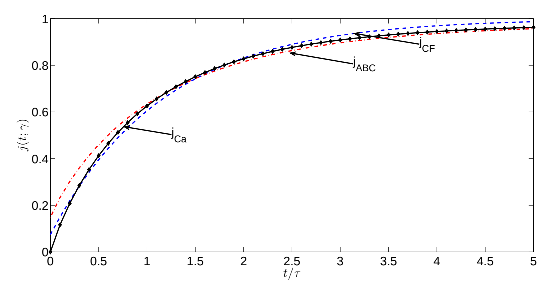
From Figures 4, 5 and 6, we can observe that, when $\gamma = 1$, the three results obtained using the derivatives of Caputo, Caputo-Fabrizio and Atangana-Baleanu are exactly the same as the ordinary case. However, as gamma takes values smaller than one, the results obtained become a little different, with notable differences when $\gamma < 0.8$. This is due to the kernel in the definitions of the fractional derivatives. The difference between the results obtained for current density using the Caputo and Atangana-Baleanu fractional derivatives for very short and very large time is the term $\frac{\gamma}{2-\gamma}$.



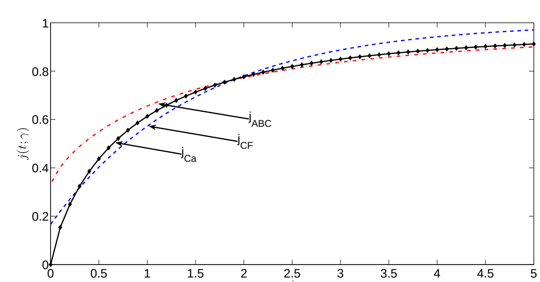
(a) $\gamma = 1$



(b) $\gamma = 0.96$



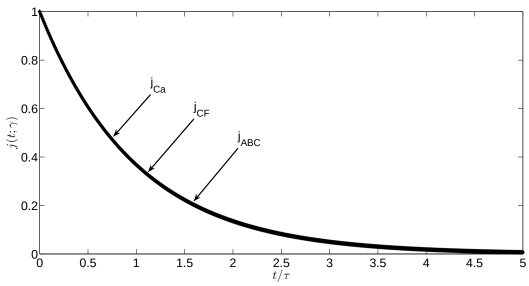
(c) $\gamma = 0.92$



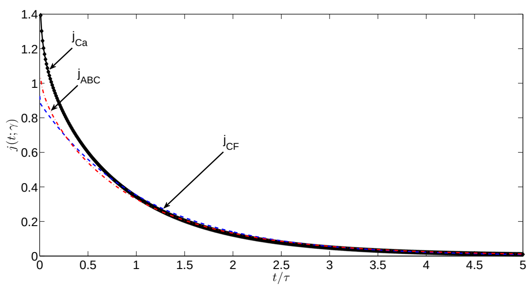
(d) $\gamma = 0.80$

Figure 4: Variation of current density $j(t;\gamma)$ with a constant electric field using: Caputo, CF and ABC fractionals derivatives with the specified values of γ.

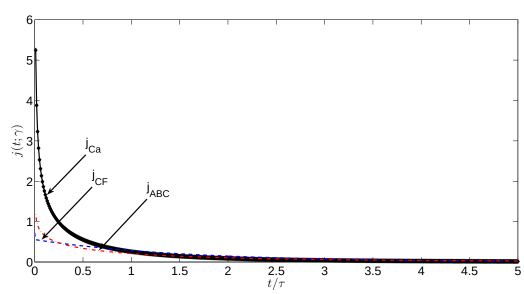
The graphs in Figure 7 represent the differences between the different fractional derivatives used here. We can see again that the difference becomes notable for $\gamma < 0.8$. It can also be observed that for large times, there is



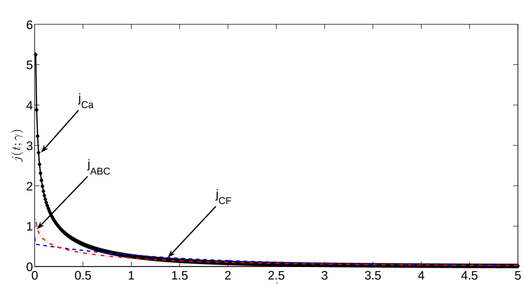
(a) $\gamma = 1$



(b) $\gamma = 0.96$

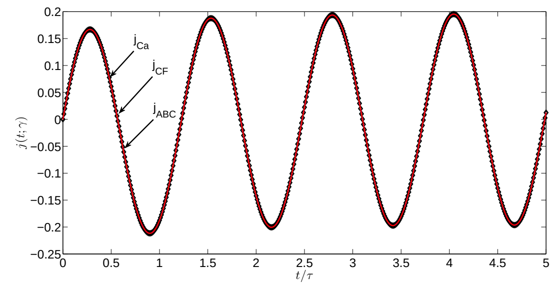


(c) $\gamma = 0.92$

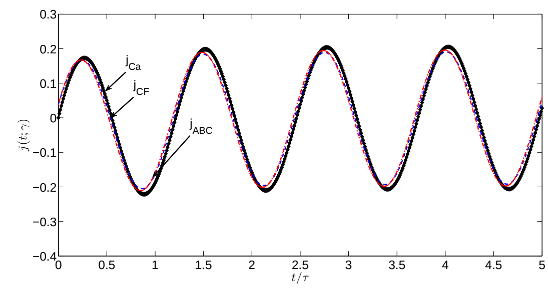


(d) $\gamma = 0.80$

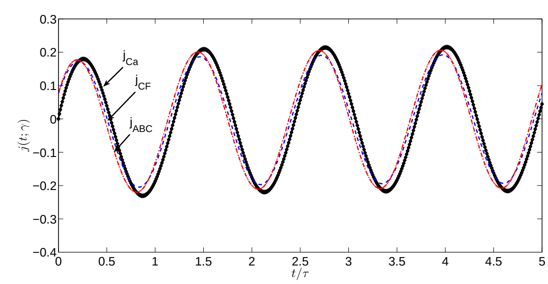
Figure 5: Variation of current density $j(t;\gamma)$ when the electric field is an impulse given by $\delta(t)$, using Caputo, CF and ABC fractionals derivatives and the specified values of γ.



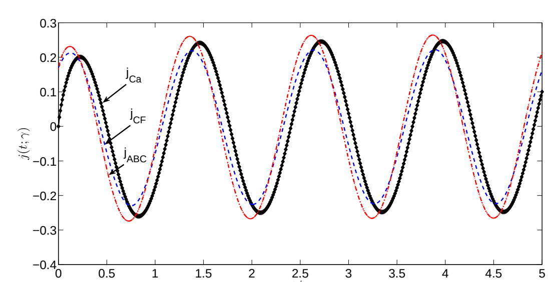
(a) $\gamma = 1$



(b) $\gamma = 0.96$



(c) $\gamma = 0.92$



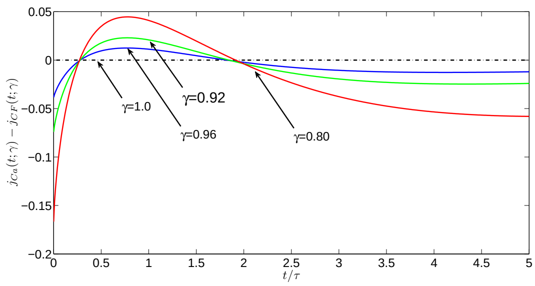
(d) $\gamma = 0.80$

Figure 6: Variation of the current density $j(t;\gamma)$ when the electric field is oscillating using Caputo, CF and ABC fractionals derivatives and various values of γ.

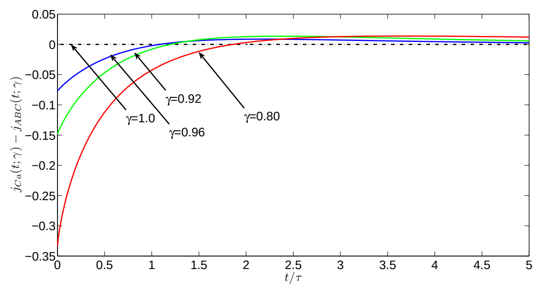
no practical difference between the fractional derivatives of Caputo and ABC.

7 Conclusion

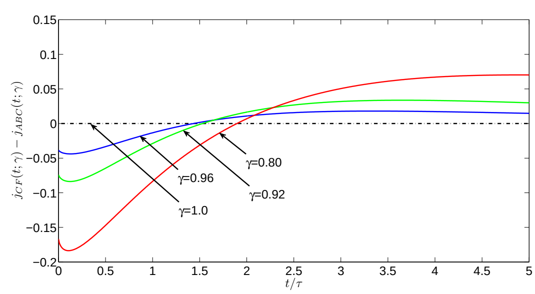
An analysis of the Drude model in the time domain was realized using fractional derivatives without singular ker-



(a) $j_{Ca} - j_{CF}$



(b) $j_{Ca} - j_{ABC}$



(c) $j_{CF} - j_{ABC}$

Figure 7: Graphs showing differences between the three fractional solutions and a DC electric field using several values of γ.

nels. The fractional derivatives were: Caputo-Fabrizio (CF) and Atangana-Baleanu in the Caputo sense (ABC). We considered the cases when electrons are induced by a constant electric field, a pulse described by a delta Dirac distribution, and an oscillating electric field. The obtained results in the interval $\gamma \in (0, 1)$ show that the current density depends not only on the electric field or the fractional order derivative, but also on the definition of the fractional derivative. Also, based on the current densities j_{Ca}, Eqs. (13) and (24) of [52], we have compared fractional derivatives with singular and non-singular kernels. We conclude that the models described by fractional Atangana-Baleanu derivatives could describe a large class of complex physical problems, particularly those with non-exponential decay.

The study of these models in the frequency domain is of great interest, in order to analyse their optical properties. This work is under investigation and will be presented in the future.

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