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On solving fractional mobile/immobile equation

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Abstract

In this article, a numerical efficient method for fractional mobile/immobile equation is developed. The presented numerical technique is based on the compact finite difference method. The spatial and temporal derivatives are approximated based on two difference schemes of orders $O(\tau^{2-\alpha})$ and $O(h^4)$, respectively. The proposed method is unconditionally stable and the convergence is analyzed within Fourier analysis. Furthermore, the solvability of the compact finite difference approach is proved. The obtained results show the ability of the compact finite difference.

Keywords

Mobile/immobile equation, time fractional, compact finite difference, Fourier analysis, stability, convergence, solvability

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Introduction

The governing equation of transport was derived based on Fick's law and is commonly called the advection-dispersion equation (ADE).[1] The ADE will predict a breakthrough curve (BTC) that can be described by a Gaussian distribution function from an instantaneously releasing solute source.[1] The interested readers can find more details in previous studies.[2–7] Also, the mobile/immobile model is considered in previous studies.[8–12]

Here, the time fractional mobile/immobile equation is studied to the following form[12,13]

$$\frac{\partial u(x,t)}{\partial t}+\frac{\partial^{\alpha}u(x,t)}{\partial t^{\alpha}}=\frac{\partial^2 u(x,t)}{\partial x^2}-\frac{\partial u(x,t)}{\partial x}+f(x,t),\ (x,T)\in[0\ h]\times[0\ T],\quad 0<\alpha<1 \tag{1}$$

when the boundary conditions are

$$\begin{aligned} u(0,t)&=\phi_1(t) \\ u(h,t)&=\phi_2(t) \end{aligned}\qquad 0<t<T \tag{2}$$

and the initial condition is

$$u(x,0)=g(x),\quad 0<x<h \tag{3}$$

where $\partial^{\alpha}(\bullet)/\partial t^{\alpha}$ is the Caputo fractional derivative of order $0<\alpha<1$. For getting more information on fractional PDEs the interested readers can refer to.[21–25]

Some numerical methods have been developed for the solution of equation (1) such as finite difference (FD) method[12,13] and meshless method.[14] Also, the fractional equation is studied by several methods, for example, high-order FD scheme for modified anomalous fractional sub-diffusion equation[15,16] and FD method for a class of fractional sub-diffusion equations.[17]

The main aim of this article is to see the performances of the compact FD for the fractional mobile/

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immobile equation. The article is organized as follows. In section "Compact FD scheme," we develop a high-order FD scheme. Section "Stability analysis" presents the stability analysis for the proposed difference scheme. In section "Convergence analysis," the convergence analysis is studied. Some numerical results are presented in section "Numerical results." Finally, a brief conclusion is written in section "Conclusion."

Compact FD scheme

Let $h = L/M$ and $\tau = T/N$ be the step sizes of spatial and temporal variables, respectively. So, we can define spatial and temporal nodes as $x_j = jh$, for $j = 0, 1, 2, \ldots, M$, and $t_k = k\tau$, for $k = 0, 1, 2, \ldots, N$.

The exact and approximate solutions at the point (x_j, t_k) are denoted by u_j^k and U_j^k, respectively. We introduce the following notations

$$\delta_x u_j^k = \frac{u_{j+1}^k - u_{j-1}^k}{h}$$
$$\delta_x^2 u_j^k = \frac{u_{j+1}^k - 2u_j^k + u_{j-1}^k}{h^2}$$
$$\frac{\partial u_j^k}{\partial t} = \delta_t u_j^k = \frac{u_j^k - u_j^{k-1}}{\tau} + O(\tau)$$

where $u_j^k = u(x_j, t_k)$.

Lemma 1[26]. Assume the below equation

$$\gamma \frac{d^2 y(x)}{dx^2} - \beta \frac{dy(x)}{dx} = f(x), \quad x \in (0, L) \tag{4}$$

with

$$y(0) = y_0, \quad y(L) = y_L$$

A CFD scheme for it is as follows

$$\left(\gamma + \frac{\beta^2 h^2}{12\gamma}\right)\delta_x^2 y_j - \beta \delta_x y_j = f_j + \frac{h^2}{12}\left(\delta_x^2 f_j - \frac{\beta}{\gamma}\delta_x f_j\right) + O(h^4) \tag{5}$$

Let

$$w(x,t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial u(x,z)}{\partial z}(t-z)^{-\alpha} dz \tag{6}$$

so equation (1) can be written as follows

$$-\frac{\partial^2 u(x,t)}{\partial x^2} + \frac{\partial u(x,t)}{\partial x} + \frac{\partial u(x,t)}{\partial t} = f(x,t) - w(x,t) \tag{7}$$

we can write equation (7) as follows

$$-\left(1 + \frac{h^2}{12}\right)\delta_x^2 u_j^k + \delta_x u_j^k = \left(f_j^k - w_j^k - \frac{\partial u_j^k}{\partial t}\right) + (r_1)_j^k + \frac{h^2}{12}\left(\delta_x^2\left(f_j^k - w_j^k - \frac{\partial u_j^k}{\partial t}\right) - \delta_x\left(f_j^k - w_j^k - \frac{\partial u_j^k}{\partial t}\right)\right) \tag{8}$$

where there is a constant c_1 such that

$$\left|(r_1)_j^k\right| \le c_1(h^4) \tag{9}$$

Based on Lemma 2 in Sun and Wu[18] we have

$$W_j^k = \frac{1}{\Gamma(1-\alpha)} \int_0^{t_k} \frac{\partial u(x_j,t)}{\partial t}(t_k - t)^{-\alpha} dt = \frac{\tau^{-\alpha}}{\Gamma(2-\alpha)} \left[b_0 u_j^k - \sum_{m=1}^{k-1}(b_{k-m-1} - b_{k-m})u_j^m - b_{k-1}u_j^0\right] + (r_2)_j^k \tag{10}$$

where there exists a constant c_2 such that

$$\left|(r_2)_j^k\right| \le c_2(\tau^{2-\alpha}) \tag{11}$$

Substituting equation (10) into equation (8) gives

$$-\left(1 + \frac{h^2}{12}\right)\delta_x^2 u_j^k + \delta_x u_j^k = \left(f_j^k - W_j^k - \delta_t u_j^k\right) + \frac{h^2}{12}\left(\delta_x^2\left(f_j^k - W_j^k - \delta_t u_j^k\right) - \delta_x\left(f_j^k - W_j^k - \delta_t u_j^k\right)\right) + R_j^k \tag{12}$$

where

$$\left|R_j^k\right| \le C(\tau + h^4) \tag{13}$$

Denoting $\mu = \tau^{-\alpha}/(\Gamma(2-\alpha))$, the proposed difference scheme is

$$\mathbf{A}\mathbf{U}^k = \frac{1}{\tau}\mathbf{B}\mathbf{U}^{k-1} + \mu \sum_{m=1}^{k-1}(b_{k-m-1} - b_{k-m})\mathbf{B}\mathbf{U}^m + \mu b_{k-1}\mathbf{B}\mathbf{U}^0 + \mathbf{B}\mathbf{F}^k \tag{14}$$

where

$$\mathbf{A} = tri\left[-\frac{1}{h^2}\left(1+\frac{h^2}{12}\right)-\frac{1}{2h}+\frac{\mu}{12}+\frac{1}{12\tau}+\frac{h\mu}{24}+\frac{h}{24\tau},\ \frac{2}{h^2}\left(1+\frac{h^2}{12}\right)+\frac{5\mu}{6}+\frac{5}{6\tau}\right.$$
$$\left.-\frac{1}{h^2}\left(1+\frac{h^2}{12}\right)+\frac{1}{2h}+\frac{\mu}{12}+\frac{1}{12\tau}-\frac{h\mu}{24}-\frac{h}{24\tau}\right]$$

$$B = \begin{bmatrix} \frac{5}{6} & \left(\frac{1}{12}-\frac{h}{24}\right) & & \\ \left(\frac{1}{12}+\frac{h}{24}\right) & \ddots & \ddots & \\ & \ddots & \ddots & \left(\frac{1}{12}-\frac{h}{24}\right) \\ & & \left(\frac{1}{12}+\frac{h}{24}\right) & \frac{5}{6} \end{bmatrix}$$

Lemma 2. The matrix A is non-singular.

Proof. For $j = 1, 2, \ldots, M-1$, the eigenvalues of A are

$$\lambda_j = \left(2\mu_1+\frac{5\mu}{6}+\frac{5}{6\tau}\right) + \sqrt{\left[\left(\frac{\mu}{12}-\mu_1+\frac{1}{12\tau}\right)+\left(\frac{h\mu}{24}+\frac{h}{24\tau}-\frac{1}{2h}\right)\right]} \times \sqrt{\left[\left(\frac{\mu}{12}-\mu_1+\frac{1}{12\tau}\right)-\left(\frac{h\mu}{24}+\frac{h}{24\tau}-\frac{1}{2h}\right)\right]} \cos\left(\frac{j\pi}{M}\right)$$

that is

$$\lambda_j = \left(2\mu_1+\frac{5\mu}{6}+\frac{5}{6\tau}\right) + 2\sqrt{\left[\left(\frac{\mu}{12}-\mu_1+\frac{1}{12\tau}\right)^2-\left(\frac{h\mu}{24}+\frac{h}{24\tau}-\frac{1}{2h}\right)^2\right]} \cos\left(\frac{j\pi}{M}\right)$$

In case of

$$\left(\frac{\mu}{12}-\mu_1+\frac{1}{12\tau}\right)^2-\left(\frac{h\mu}{24}+\frac{h}{24\tau}-\frac{1}{2h}\right)^2 \le 0$$

the eigenvalues of A can be written as $\lambda = a + bi$ with non-zero real part ($a \neq 0$). For the case

$$\left(\frac{\mu}{12}-\mu_1+\frac{1}{12\tau}\right)^2-\left(\frac{h\mu}{24}+\frac{h}{24\tau}-\frac{1}{2h}\right)^2 > 0$$

if $(\mu/12) - \mu_1 + (1/12\tau) \ge 0$ we have

$$\lambda_j \ge 2\mu_1+\frac{5\mu}{6}+\frac{5}{6\tau}-2\left(\frac{\mu}{12}-\mu_1+\frac{1}{12\tau}\right) = 4\mu_1+\frac{2\mu}{3}+\frac{2}{3\tau}>0$$

and if $(\mu/12) - \mu_1 + (1/12\tau) < 0$

$$\lambda_j \ge 2\mu_1+\frac{5\mu}{6}+\frac{5}{6\tau}+2\left(\frac{\mu}{12}-\mu_1+\frac{1}{12\tau}\right) = \mu+\frac{1}{\tau}>0$$

Since $\mu_1, \mu > 0$ thus A is non-singular matrix.

Theorem 1. For the appeared scheme in equation (14), there is a unique solution.

Proof. We must solve linear system of equations $AU^k = b$ at each $t_k = k\tau$ to obtain the numerical solution. Since for any μ and μ_1, the coefficient matrix A is invertible so the solution of scheme in equation (14) exists and is unique.

Stability analysis

Let $\tilde{U}_j^n$ be the approximated solution of equation (14). Consider

$$\rho_j^k = U_j^k - \tilde{U}_j^k, \quad j = 0, 1, \ldots, M, \quad k = 0, 1, \ldots, N$$

With this definition and regarding equation (14), we can obtain the round-off error equation as

$$\mathbf{A}\rho^k = \frac{1}{\tau}\mathbf{B}\rho^{k-1} + \mu\sum_{m=1}^{k-1}(b_{k-m-1}-b_{k-m})\mathbf{B}\rho^m + \mu b_{k-1}\mathbf{B}\rho^0 \tag{15}$$

where

$$\rho_0^k = \rho_0^k = 0$$

where

$$\rho^k = \left[\rho_1^k, \rho_2^k, \ldots, \rho_{M-1}^k\right]$$

Now we define the grid function[19]

$$\rho^k(x) = \begin{cases} \rho_j^k & x_j - \frac{h}{2} < x \le x_j + \frac{h}{2} \\ 0 & 0 \le x \le \frac{h}{2} \text{ or } L - \frac{h}{2} < x \le L \end{cases}$$

Then $\rho^k(x), k = 1, 2, \ldots, N,$ can be expanded in a Fourier series

$$\rho^k(x) = \sum_{l=-\infty}^{\infty} d_k(l)e^{i2\pi lx/L}$$

where

$$d_k(l) = \frac{1}{L}\int_0^L \rho^k(x)e^{-i2\pi lx/L}dx$$

Now introduce the following norm[19]

$$\|\rho^k\|_2 = \left(\sum_{j=1}^{M-1} h\left|\rho_j^k\right|^2\right)^{\frac{1}{2}} = \left[\int_0^L \left|\rho^k(x)\right|^2 dx\right]^{\frac{1}{2}}$$

Note that we can obtain

$$\|\rho^k\|_2^2 = \sum_{l=-\infty}^{\infty} |d_k(l)|^2 \tag{16}$$

by using Parseval equality

$$\int_0^L \left|\rho^k(x)\right|^2 dx = \sum_{l=-\infty}^{\infty} |d_k(l)|^2$$

We can assume the solution of equation (15) as

$$\rho_j^k = d_k e^{i\sigma jh}$$

where $\sigma = 2\pi l/L$. Now, substituting this expression into equation (15) yields

$$d_k = \frac{\left(1 - \frac{\sin^2\frac{\theta}{2}}{3} - \frac{ih\sin\theta}{12}\right)\left[\sum_{m=1}^{k-1}\mu(b_{k-m-1} - b_{k-m})d_m + \mu b_{k-1}d_0 + \frac{1}{\tau}d_{k-1}\right]}{4\mu_1\sin^2\frac{\theta}{2} + \mu - \frac{\mu}{3}\sin^2\frac{\theta}{2} + \frac{1}{\tau} - \frac{1}{3\tau}\sin^2\frac{\theta}{2} + \frac{i}{h}\sin\theta - \frac{ih\mu}{12}\sin\theta - \frac{ih}{12\tau}\sin\theta} \tag{17}$$

where

$$\mu_1 = \frac{1}{h^2}\left(1 + \frac{h^2}{12}\right), \quad \mu = \frac{1}{\tau^\alpha\Gamma(2-\alpha)}$$

Lemma 3. The below inequality holds

$$\left|\frac{\left(\mu - \frac{\mu}{3}\sin^2\frac{\theta}{2} - \frac{ih\mu}{12}\sin\theta\right) + \left(\frac{1}{\tau} - \frac{1}{3\tau}\sin^2\frac{\theta}{2} - \frac{ih}{12\tau}\sin\theta\right)}{4\mu_1\sin^2\frac{\theta}{2} + \mu - \frac{\mu}{3}\sin^2\frac{\theta}{2} + \frac{1}{\tau} - \frac{1}{3\tau}\sin^2\frac{\theta}{2} + \frac{i}{h}\sin\theta - \frac{ih\mu}{12}\sin\theta - \frac{ih}{12\tau}\sin\theta}\right| \le 1 \tag{18}$$

Proof. Because of $h \le 1$, $\tau \le 1$, and $\Gamma(2-\alpha) > 0$ we can see that $\mu > 0$. Furthermore $\mu_1 > 1$, too. So, it is easy to see that

$$\begin{aligned}
&\frac{8}{3}\mu\mu_1\sin^2\frac{\theta}{2} - \frac{8\mu\mu_1}{3}\sin^4\frac{\theta}{2} = \frac{8}{3}\mu\mu_1\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2} \ge 0\\
&\frac{8\mu_1}{3\tau}\sin^2\frac{\theta}{2} - \frac{8\mu_1}{3\tau}\sin^4\frac{\theta}{2} = \frac{8\mu_1}{3\tau}\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2} \ge 0\\
&\frac{2}{3\tau}\mu_1\sin^2\frac{\theta}{2} - \frac{2}{3\tau}\sin^2\frac{\theta}{2} = \frac{2}{3}\sin^2\frac{\theta}{2}(\mu_1 - 1) \ge 0\\
&\frac{2}{3}\mu\mu_1\sin^2\frac{\theta}{2} - \frac{2}{3}\mu\sin^2\frac{\theta}{2} = \frac{2}{3}\mu\sin^2\frac{\theta}{2}(\mu_1 - 1) \ge 0
\end{aligned} \tag{19}$$

The relation (18) holds if and only if

$$0 \le \left(4\mu_1\sin^2\frac{\theta}{2}\right)^2 + 2\left(4\mu_1\sin^2\frac{\theta}{2}\right)\left(\mu - \frac{\mu}{3}\sin^2\frac{\theta}{2} + \frac{1}{\tau} - \frac{1}{3\tau}\sin^2\frac{\theta}{2}\right) + \frac{1}{h^2}\sin^2\theta - 2\left(\frac{1}{h}\sin\theta\right)\left(\frac{h\mu}{12}\sin\theta + \frac{h}{12\tau}\sin\theta\right)$$

If we put

$$\begin{aligned}
\frac{\mu}{6}\sin^2\theta &= \frac{2}{3}\mu\sin^2\frac{\theta}{2} - \frac{2}{3}\mu\sin^4\frac{\theta}{2}\\
\frac{1}{6\tau}\sin^2\theta &= \frac{2}{3\tau}\sin^2\frac{\theta}{2} - \frac{2}{3\tau}\sin^4\frac{\theta}{2}
\end{aligned}$$

then relation (18) holds if and only if

$$\begin{aligned}
&\left(4\mu_1\sin^2\frac{\theta}{2}\right)^2 + \left(\frac{8}{3}\mu_1\mu\sin^2\frac{\theta}{2} - \frac{8}{3}\mu_1\mu\sin^4\frac{\theta}{2}\right) + \frac{14}{3}\mu_1\mu\sin^2\frac{\theta}{2}\\
&+ \left(\frac{8}{3\tau}\mu_1\sin^2\frac{\theta}{2} - \frac{8}{3\tau}\mu_1\sin^4\frac{\theta}{2}\right) + \frac{1}{h^2}\sin^2\frac{\theta}{2} + \frac{14}{3\tau}\mu_1\sin^2\frac{\theta}{2}\\
&+ \left(\frac{2}{3}\mu_1\mu\sin^2\frac{\theta}{2} - \frac{2}{3}\mu\sin^2\frac{\theta}{2}\right) + \frac{2}{3}\mu\sin^4\frac{\theta}{2} + \frac{2}{3\tau}\sin^4\frac{\theta}{2}
\end{aligned}$$

which completes the proof.

Theorem 2. Let d_k be the solutions of equation (17), so

$$|d_k| \le |d_0|, \quad k = 1, 2, \ldots, N \tag{20}$$

Proof. In case of $k = 1$ according to equation (17), we have

$$d_1 = \frac{\left(1 - \frac{\sin^2\frac{\theta}{2}}{3} - \frac{ih\sin\theta}{12}\right)\left(\mu + \frac{1}{\tau}\right)d_0}{4\mu_1\sin^2\frac{\theta}{2} + \mu - \frac{\mu}{3}\sin^2\frac{\theta}{2} + \frac{1}{\tau} - \frac{1}{3\tau}\sin^2\frac{\theta}{2} + \frac{i}{h}\sin\theta - \frac{ih\mu}{12}\sin\theta - \frac{ih}{12\tau}\sin\theta}$$

or

$$d_1 = \frac{\left(1 - \frac{\sin^2\frac{\theta}{2}}{3} - \frac{ih\sin\theta}{12}\right)\left(\mu + \frac{1}{\tau}\right)d_0}{\left(4\mu_1\sin^2\frac{\theta}{2} + \mu\left(1 - \frac{\sin^2\frac{\theta}{2}}{3}\right) + \frac{1}{\tau}\left(1 - \frac{\sin^2\frac{\theta}{2}}{3}\right)\right) + i\left(\frac{\sin\theta}{h} - \frac{h\mu\sin\theta}{12} - \frac{h\sin\theta}{12\tau}\right)}$$

Also, we can write

$$|d_1| = \frac{\sqrt{\left(1 - \frac{1}{3}\sin^2\frac{\theta}{2}\right)^2 + \left(\frac{h\sin\theta}{12}\right)^2}\left(\mu + \frac{1}{\tau}\right)|d_0|}{\sqrt{\left(4\mu_1\sin^2\frac{\theta}{2} + \mu\left(1 - \frac{\sin^2\frac{\theta}{2}}{3}\right) + \frac{1}{\tau}\left(1 - \frac{\sin^2\frac{\theta}{2}}{3}\right)\right)^2 + \left(\frac{\sin\theta}{h} - \frac{h\mu\sin\theta}{12} - \frac{h\sin\theta}{12\tau}\right)^2}}$$

Now, Lemma 3 yields

$$|d_1| \le |d_0|$$

Suppose

$$|d_n| \le |d_0|, \quad n = 1, 2, \ldots, k-1$$

Now, equation (17) yields

$$|d_{k-1}| \le \frac{\left|1 - \frac{\sin^2\frac{\theta}{2}}{3} - \frac{ih\sin\theta}{12}\right|\left(\mu(b_0 - b_{k-1})|d_0| + \mu b_{k-1}|d_0| + \frac{1}{\tau}|d_0|\right)}{\left|4\mu_1\sin^2\frac{\theta}{2} + \mu - \frac{\mu}{3}\sin^2\frac{\theta}{2} + \frac{1}{\tau} - \frac{1}{3\tau}\sin^2\frac{\theta}{2} + \frac{i}{h}\sin\theta - \frac{ih\mu}{12}\sin\theta - \frac{ih}{12\tau}\sin\theta\right|}$$

so

$$|d_k| \le |d_0|$$

which completes the proof.

Theorem 3. The CFD scheme (14) is unconditionally stable.

Proof. Using inequality (20) in equation (16) yields

$$\|\rho^k\|_2^2 = \sum_{l=\infty}^{\infty} |d_k(l)|^2 \le \sum_{l=\infty}^{\infty} |d_0(l)|^2 = \|\rho^0\|_2^2$$

so we have

$$\|\rho^k\|_2 \le \|\rho^0\|_2$$

Convergence analysis

Here, we analyzed the compact difference scheme (14) in terms of convergence. We will show the relation (14) has the spatial accuracy of fourth order. For this end, we need some lemmas and theorems that will be expressed. First, we consider definitions of $e^k(x)$ and $R^k(x)$ in Chen et al.[20] Thus, for $k = 0, 1, \ldots, N$, $e^k(x)$ and $R^k(x)$ are

$$e^k(x) = \sum_{l=-\infty}^{\infty} \eta_k(l) e^{\frac{i2\pi lx}{L}}$$

$$R^k(x) = \sum_{l=-\infty}^{\infty} \xi_k(l) e^{\frac{i2\pi lx}{L}}$$

where

$$\eta_k(l) = \frac{1}{L}\int_0^L e^k(x)e^{\frac{-i2\pi lx}{L}}dx$$

$$\xi_k(l) = \frac{1}{L}\int_0^L R^k(x)e^{\frac{-i2\pi lx}{L}}dx$$

Now, we define the following notations[20]

$$e_j^k = u(x_j, t_k) - U_j^k \tag{21}$$

where

$$e^k = \left[e_1^k, e_2^k, \ldots, e_{M-1}^k\right], \quad R^k = \left[R_1^k, R_2^k, \ldots, R_{M-1}^k\right]$$

and introduce the following norms

$$\|e^k\|_2 = \left(\sum_{j=1}^{M-1} h\left|e_j^k\right|^2\right)^{\frac{1}{2}} = \left[\int_0^L \left|e^k(x)\right|^2 dx\right]^{\frac{1}{2}} \tag{22}$$

$$\|R^k\|_2 = \left(\sum_{j=1}^{M-1} h\left|R_j^k\right|^2\right)^{\frac{1}{2}} = \left[\int_0^L \left|R^k(x)\right|^2 dx\right]^{\frac{1}{2}} \tag{23}$$

Using the Parseval equality

$$\int_0^L \left|e^k(x)\right|^2 dx = \sum_{l=-\infty}^{\infty} |\eta_k(l)|^2 \tag{24}$$

and

$$\int_0^L \left|R^k(x)\right|^2 dx = \sum_{l=-\infty}^{\infty} |\xi_k(l)|^2 \tag{25}$$

Also, we have

$$\|e^k\|_2^2 = \sum_{l=-\infty}^{\infty} |\eta_k(l)|^2 \tag{26}$$

$$\|R^k\|_2^2 = \sum_{l=-\infty}^{\infty} |\xi_k(l)|^2 \tag{27}$$

We can obtain

$$\left|R_j^k\right| \le \mathcal{C}(\tau + h^4) \tag{28}$$

Subtracting equation (14) from equation (12), we obtain

$$\mathbf{A}\mathbf{e}^k = \frac{1}{\tau}\mathbf{B}\mathbf{e}^{k-1} + \mu\sum_{m=1}^{k-1}(b_{k-m-1} - b_{k-m})\mathbf{B}\mathbf{e}^m + \mathbf{R}^k \tag{29}$$

with

$$e_0^k = e_M^k = 0, \quad k = 1, 2, \ldots, N-1$$
$$e_j^0 = 0, \quad j = 1, 2, \ldots, M-1$$

Now, assume that e_j^k and R_j^k are as follows

$$e_j^k = \eta_k e^{i(\sigma jh)}$$
$$R_j^k = \xi_k e^{i(\sigma jh)}$$

where $\sigma = 2l\pi/L$. By substituting the obtained relations into equation (29), we yield

$$\eta_k = \frac{\left(1 - \frac{\sin^2\frac{\theta}{2}}{3} - \frac{ih\sin\theta}{12}\right)\left[\sum_{m=1}^{k-1}\mu(b_{k-m-1} - b_{k-m})\eta_m + \frac{1}{\tau}\eta_{k-1}\right] + \xi_k}{4\mu_1\sin^2\frac{\theta}{2} + \mu - \frac{\mu}{3}\sin^2\frac{\theta}{2} + \frac{1}{\tau} - \frac{1}{3\tau}\sin^2\frac{\theta}{2} + \frac{i}{h}\sin\theta + -\frac{ih\mu}{12}\sin\theta - \frac{ih}{12\tau}\sin\theta} \tag{30}$$

where $\theta = \sigma h$. Because of $e^0 = 0$, we can write

$$\eta_0 \equiv \eta_0(l) = 0$$

Furthermore, the first equality in equation (23) and inequality of equation (28) yield

$$\|R^k\|_2 \le \sqrt{Mh}C_1\left(\tau + h^4\right) = C_1\sqrt{L}\left(\tau + h^4\right) \tag{31}$$

Also there is a positive constant C_2 as[20]

$$|\xi_k| \equiv |\xi_k(n)| \le C_2\tau|\xi_1| \equiv C_2\tau|\xi_1(n)|, \quad k = 1, 2, \ldots, N \tag{32}$$

Lemma 4. The below inequality holds

$$\left|\frac{1}{4\mu_1\sin^2\frac{\theta}{2} + \mu - \frac{\mu}{3}\sin^2\frac{\theta}{2} + \frac{1}{\tau} - \frac{1}{3\tau}\sin^2\frac{\theta}{2} + \frac{i}{h}\sin\theta + -\frac{ih\mu}{12}\sin\theta - \frac{ih}{12\tau}\sin\theta}\right| \le 1$$

Proof. Since $0<\alpha,\tau\leq 1$, we can obtain $\mu = 1/(\tau^{\alpha}\Gamma(2-\alpha))>1$. Now the above relation holds if and only if

$$\frac{1}{\left(4\mu_1\sin^2\frac{\theta}{2}+\mu-\frac{\mu}{3}\sin^2\frac{\theta}{2}+\frac{1}{\tau}-\frac{1}{3\tau}\sin^2\frac{\theta}{2}\right)^2+\left(\frac{1}{h}\sin\theta-\frac{h\mu}{12}\sin\theta-\frac{h}{12\tau}\sin\theta\right)^2}\leq 1$$

Now, we have

$$\begin{aligned}1&\leq\left(2\mu+2\frac{1}{\tau}\right)^2=9\left(\frac{2\mu}{3}+\frac{2}{3\tau}\right)^2\\&=9\left(\mu-\frac{\mu}{3}+\frac{1}{\tau}-\frac{1}{3\tau}\right)^2\\&\leq 9\left(\mu-\frac{\mu}{3}\sin^2\frac{\theta}{2}+\frac{1}{\tau}-\frac{1}{3\tau}\sin^2\frac{\theta}{2}\right)^2\\&\leq 9\left(\mu-\frac{\mu}{3}\sin^2\frac{\theta}{2}+\frac{1}{\tau}-\frac{1}{3\tau}\sin^2\frac{\theta}{2}\right)^2\\&\quad+9\left(\frac{1}{h}\sin\theta-\frac{h\mu}{12}\sin\theta-\frac{h}{12\tau}\sin\theta\right)^2\end{aligned}$$

This completes the proof.

Theorem 4. If $\eta_k(k=1,2,\ldots,N)$ be the solutions of equation (30), then a positive constant C_2 exists such that

$$|\eta_k|\leq C_2(1+3\tau)^k|\xi_1|,\quad k=1,2,\ldots,N$$

Proof. Applying equations (30), (32), and Lemma 6, we can write

$$\begin{aligned}|\eta_1|&=\frac{|\xi_1|}{\left|4\mu_1\sin^2\frac{\theta}{2}+\mu-\frac{\mu}{3}\sin^2\frac{\theta}{2}+\frac{1}{\tau}-\frac{1}{3\tau}\sin^2\frac{\theta}{2}+\frac{i}{h}\sin\theta+-\frac{ih\mu}{12}\sin\theta-\frac{ih}{12\tau}\sin\theta\right|}\\&\leq 3\tau C_2|\xi_1|\leq(1+3\tau)C_2|\xi_1|\end{aligned}$$

Now, suppose that

$$|\eta_n|\leq(1+3\tau)^nC_2|\xi_1|,\quad n=1,2,\ldots,k-1$$

From Lemmas 3 and 4

$$\begin{aligned}|\eta_k|&=\frac{\left|1-\frac{\sin^2\frac{\theta}{2}}{3}-\frac{ih\sin\theta}{12}\right|\left[\sum_{m=1}^{k-1}\mu(b_{k-m-1}-b_{k-m})|\eta_m|+\frac{1}{\tau}|\eta_{k-1}|\right]+|\xi_k|}{\left|4\mu_1\sin^2\frac{\theta}{2}+\mu-\frac{\mu}{3}\sin^2\frac{\theta}{2}+\frac{1}{\tau}-\frac{1}{3\tau}\sin^2\frac{\theta}{2}+\frac{i}{h}\sin\theta+-\frac{ih\mu}{12}\sin\theta-\frac{ih}{12\tau}\sin\theta\right|}\\&\leq\frac{\left|1-\frac{\sin^2\frac{\theta}{2}}{3}-\frac{ih\sin\theta}{12}\right|(1+3\tau)^{k-1}|\xi_1|C_2\left[\mu+\frac{1}{\tau}\right]+3\tau|\xi_1|C_2}{\left|4\mu_1\sin^2\frac{\theta}{2}+\mu-\frac{\mu}{3}\sin^2\frac{\theta}{2}+\frac{1}{\tau}-\frac{1}{3\tau}\sin^2\frac{\theta}{2}+\frac{i}{h}\sin\theta+-\frac{ih\mu}{12}\sin\theta-\frac{ih}{12\tau}\sin\theta\right|}\\&\leq\left((1+3\tau)^{k-1}+3\tau\right)|\xi_1|C_2\\&\leq(1+3\tau)^k|\xi_1|C_2\end{aligned}$$

Theorem 5. Suppose $u(x,t)$ is the solution of equation (1), then the compact FD scheme described in equation (14) is L_2-convergent and its convergence order is $\mathcal{O}(\tau+h^4)$.

Proof. According to Theorem 4, equations (26), (27), and (31), we have

$$\|e^k\|_2\leq(1+3\tau)^kC_2\|R^1\|_2\leq C_1\sqrt{L}C_2e^{3k\tau}(\tau+h^4)$$

Since $k\tau\leq T$, we have

$$\|e^k\|_2\leq C(\tau+h^4)$$

where

$$C=C_1C_2\sqrt{L}e^{3T}$$

Numerical results

Here, we present some test problem to verify the developed method. To demonstrate the accuracy of this method, we use L_∞ norm for computing errors. Also, the computational orders of the presented method (denoted by C-Order) is calculated using the following formula

Table 1. Evaluated computational orders and errors with $\tau = 0.02$ and $\gamma = 1/30$, for Test problem 1.

h	$\alpha = 0.25$		$\alpha = 0.75$		CPU time (s)
	L_∞	C-order	L_∞	C-order	
1/4	7.1511×10^{-1}	–	7.0721×10^{-1}	–	00.3280
1/8	1.3344×10^{-2}	5.7439	1.3344×10^{-2}	5.7279	00.5000
1/16	7.0041×10^{-4}	4.2518	7.0052×10^{-4}	4.2516	00.8280
1/32	4.2169×10^{-5}	4.0539	4.2177×10^{-5}	4.0539	01.5310
1/64	2.6181×10^{-6}	4.0096	2.6123×10^{-6}	4.0131	02.7040
1/128	1.6394×10^{-7}	3.9972	1.6398×10^{-7}	3.9937	05.2030
1/256	1.0241×10^{-8}	4.0480	1.0244×10^{-8}	4.0007	10.2500
1/512	6.3905×10^{-10}	4.0023	6.4120×10^{-10}	3.9978	10.2500

CPU: central processing unit.

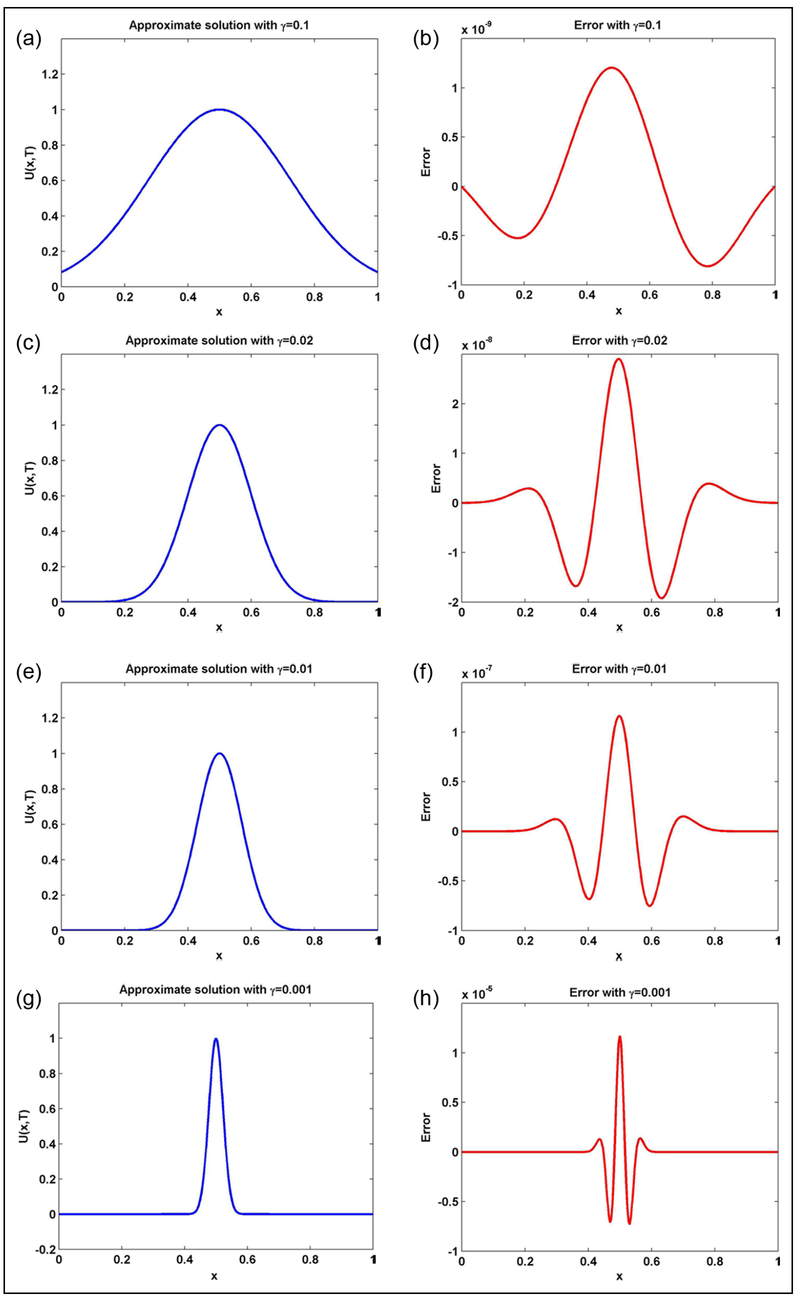

Figure 1. Approximated solutions and obtained errors for different values of γ using presented method from (a) to (h) for Test problem 1.

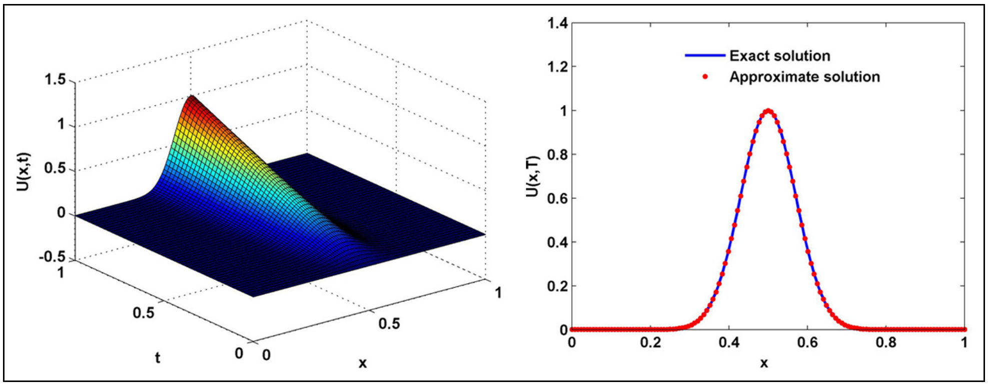

Figure 2. (Left) Surface plot of approximated solution; (right) exact and approximated solution with $\alpha = 0.65$, $\gamma = 0.001$, $\tau = 1/50$, and $h = 1/128$ for Test problem 1.

Table 2. Test problem 1 error values obtained for different parameters.

$h = \tau$	$\gamma = 1/10$		$\gamma = 1/100$	
	$\alpha = 0.4$	$\alpha = 0.9$	$\alpha = 0.4$	$\alpha = 0.9$
1/0	5.2339×10^{-4}	5.2262×10^{-4}	1.0742×10^{-1}	1.0727×10^{-1}
1/15	1.0297×10^{-4}	1.0280×10^{-4}	7.3552×10^{-3}	7.3558×10^{-3}
1/25	1.3323×10^{-5}	1.3300×10^{-5}	1.1490×10^{-3}	1.1491×10^{-3}
1/50	8.2954×10^{-7}	8.2808×10^{-7}	8.1048×10^{-5}	8.1056×10^{-5}
1/100	5.1797×10^{-8}	5.1705×10^{-8}	5.0049×10^{-6}	8.0054×10^{-6}

Table 3. Comparison of numerical solutions and errors obtained with $h = \tau = 1/100$ for Test problem 1.

x	Exact solution	Method of Zhang et al.[12]		Present method	
		Numerical solution	L_∞	Numerical solution	L_∞
0.1	0.162000	0.161843	1.5629×10^{-4}	0.161999	1.1560×10^{-9}
0.2	0.512000	0.510599	1.4006×10^{-3}	0.511999	2.1029×10^{-9}
0.3	0.882000	0.879024	2.9752×10^{-3}	0.881999	2.8325×10^{-9}
0.4	1.152000	1.147702	4.2977×10^{-3}	1.151999	3.3312×10^{-9}
0.5	1.250000	1.245027	4.9722×10^{-3}	1.249999	3.5813×10^{-9}
0.6	1.152000	1.147196	4.8034×10^{-3}	1.151999	3.5579×10^{-9}
0.7	0.882000	0.878184	3.8152×10^{-3}	0.881999	3.2303×10^{-9}
0.8	0.512000	0.509725	2.2746×10^{-3}	0.511999	2.5610×10^{-9}
0.9	0.162000	0.161279	7.2075×10^{-4}	0.161999	1.5031×10^{-9}

Table 4. Evaluated computational orders and error values with $\tau = 1/50$ for Test problem 2.

h	$\alpha = 0.1$		$\alpha = 0.8$		CPU time (s)
	L_∞	C-order	L_∞	C-order	
1/4	1.3792×10^{-3}	–	1.4064×10^{-3}	–	00.0982
1/8	8.6206×10^{-5}	3.9999	8.7903×10^{-5}	3.9999	00.3910
1/16	5.4151×10^{-6}	3.9937	5.5184×10^{-6}	3.9936	00.7500
1/32	3.3877×10^{-7}	3.9976	3.4545×10^{-7}	3.9977	01.5620
1/64	2.1176×10^{-8}	3.9998	2.1594×10^{-8}	3.9998	02.9840
1/128	1.3236×10^{-9}	3.9999	1.3495×10^{-9}	4.0001	05.3750
1/256	8.2376×10^{-11}	4.0061	8.1819×10^{-11}	4.0438	10.7960

CPU: central processing unit.

Table 5. Error values obtained with different values of α and τ for Test problem 4.

$h=\tau$	$\alpha=0.25$	$\alpha=0.45$	$\alpha=0.65$	$\alpha=.85$
1/10	3.5303×10^{-5}	3.5414×10^{-5}	3.5646×10^{-5}	3.6012×10^{-5}
1/15	7.0195×10^{-6}	7.0435×10^{-6}	7.0917×10^{-6}	7.1674×10^{-6}
1/25	9.0879×10^{-7}	9.1207×10^{-7}	9.1849×10^{-7}	9.2857×10^{-7}
1/50	5.8161×10^{-8}	5.7111×10^{-8}	5.7520×10^{-8}	5.6898×10^{-8}

$$\frac{\log\left(\frac{E_1}{E_2}\right)}{\log\left(\frac{h_1}{h_2}\right)}$$

where E_i is the error value that corresponds to grid with mesh size h_i.

Test problem 1

We consider the time fractional mobile/immobile equation with the form described in equation (1), where

$$f(x,t) = \left[\frac{t^{1-\alpha}}{\Gamma(2-\alpha)} + 1 - \frac{2(x-0.5)}{\gamma} + \frac{2}{\gamma} - \frac{4(x-0.5)^2}{\gamma^2}\right] \exp\left(-\frac{(x-0.5)^2}{\gamma}\right)$$

where exact solution is a Gaussian pulse with t height centered at $x=0.5$, that is

$$u(x,t) = t\exp\left(-\frac{(x-0.5)^2}{\gamma}\right)$$

boundary and initial conditions can be obtained from the exact solution.

Table 1 demonstrates the L_∞ error, computational order and total central processing unit (CPU) time (in second). The computational order of Table 1 is closed to the theoretical results. Figure 1 is based on $\alpha=0.35$, $h=1/256$, and $\tau=1/10$. Also, the used parameter in Figure 2 is $\alpha=0.65$, $\gamma=0.001$, $h=1/128$, and $\tau=1/50$.

Also, in Table 2, we can see error obtained with $h=\tau$ and different values of α for this problem.

Test problem 2

Again, we consider the time fractional mobile/immobile equation with the form described in equation (1) with source term

$$f(x,t) = 10x^2(1-x)^2\left(1+\frac{t^{1-\alpha}}{\Gamma(2-\alpha)}\right) + 10(t+1)\left(-2+14x-18x^2+4x^3\right)$$

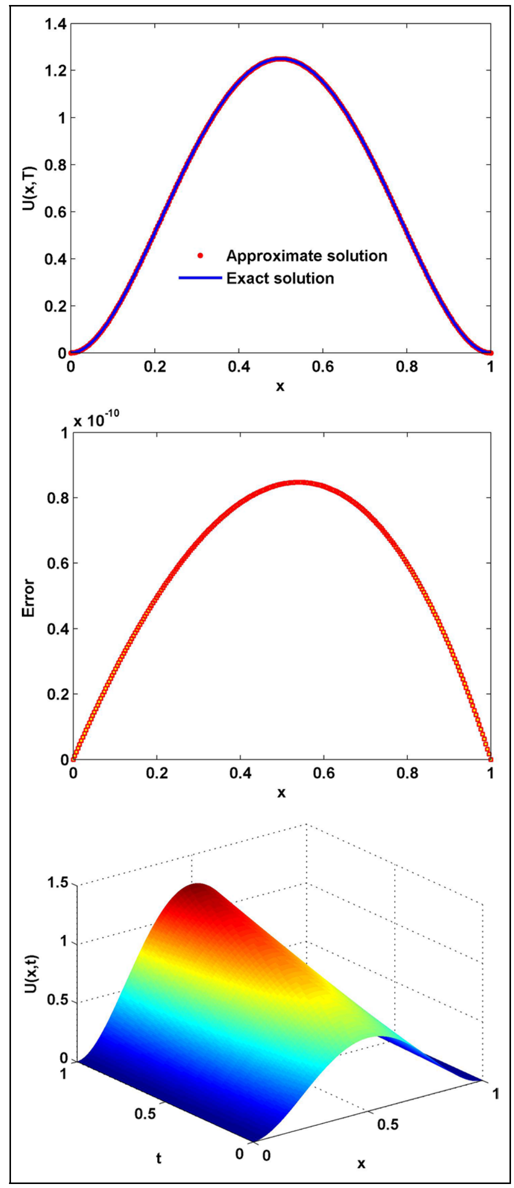

Figure 3. Graphs of exact and approximate solution, absolute error, and surface plot of approximated solution with parameters $\alpha=0.45$, $\tau=0.025$, and $h=1/256$, for Test problem 2.

when boundary and initial conditions are of the following form

$$u(0,t) = u(1,t) = 0$$
$$u(x,0) = 10x^2(1-x)^2$$

In this case the exact solution is as follows (Figure 3)

$$u(x, t) = 10(1 + t)x^2(1 - x)^2$$

In Table 3, we compare the errors obtained with the method of this article and method of Zhang et al.[12] where $\alpha = 1 - 0.5\exp(-x)$. We see that the present method has good results in comparison with the method of Zhang et al.[12] Computational order, L_∞ error and CPU time (in seconds) are shown in Table 4. Also, Table 5 presents the error of numerical results for this problem with different values of parameters α and τ. Figure 3 demonstrates graphs of exact and approximate solution, absolute error, and surface plot of approximated solution with parameters $\alpha = 0.45$, $\tau = 0.025$, and $h = 1/256$, for Test problem 2.

Conclusion

In this article, we built a compact difference scheme for the solution of time fractional mobile/immobile equation. We proved the unconditional stability property and convergence by Fourier analysis. It was shown that the numerical simulations obey the theoretical results. Examples are given and when the results obtained using this method with exact solutions are compared, this method shows applicability and efficiency.

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