# STEADY-STATE HEAT CONDUCTION IN A MEDIUM WITH SPATIAL NON-SINGULAR FADING MEMORY
## Derivation of Caputo-Fabrizio Space-Fractional Derivative from Cattaneo Concept with Jeffrey's Kernel and Analytical Solutions

by

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Original scientific paper
https://doi.org/10.2298/TSCI160229115H

*Starting from the Cattaneo constitutive relation with a Jeffrey's kernel the derivation of a transient heat diffusion equation with relaxation term expressed through the Caputo-Fabrizio time fractional derivative has been developed. This approach allows seeing the physical background of the newly defined Caputo-Fabrizio time fractional derivative and demonstrates how other constitutive equations could be modified with non-singular fading memories.*

Key words: *non-linear diffusion, non-singular fading memory, Jeffrey kernel, Caputo-Fabrizio derivative, integral balance approach*

## Introduction

This article refers to a hot topic in modelling of dissipative phenomena [1-3] by application of fractional derivatives. As it is stated in the seminal works of Caputo and Fabrizio [1, 3] many classical constitutive equations (see the comments in [1] and [3] and the references therein) can not model the transport properties of new materials with advanced characteristics. To satisfy these requirements a new time-fractional derivative with a non-singular smooth exponential kernel was conceived by Caputo and Fabrizio [1]:

$$ {}_{CF}D_t^{\alpha} f(t) = \frac{M(\alpha)}{1-\alpha}\int_0^t \exp\left[-\frac{\alpha(t-s)}{1-\alpha}\right]\frac{\mathrm{d}f(t)}{\mathrm{d}t}\mathrm{d}s = \frac{1}{1-\alpha}\int_0^t \exp\left[-\frac{\alpha(t-s)}{1-\alpha}\right]\frac{\mathrm{d}f(t)}{\mathrm{d}t}\mathrm{d}s \quad \text{(1a,b)} $$

where $M(\alpha)$ in eq. (1a) is a normalization function such that $M(0) = M(1) = 1$. With   suggested for convenience in [1] we get the final definition of the Caputo-Fabrizio time-fractional derivative [1, 3] in the form of eq. (1b). The derivative of a constant is zero as in the classical Caputo derivative [2], but now the exponential (Jeffrey's) kernel has no singularity [1, 3, 4].

The Caputo-Fabrizio derivative [1] already has been applied to various practically relevant problems such as: elasticity [3], resistance and numerical modelling of fractional electric circuit [5, 6], the Keller-Segel model [7], Fisher's reaction-diffusion equation [8], coupled systems of time-fractional differential problems [9], mass-spring damped systems [10], groundwater flow [11], *etc*.

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In the previous note [12] it was demonstrated that starting with the general Cattaneo constitutive equation see eq. (4), and exponential Jeffrey's kernel it was possible to derive the model of 1-D transient heat condition in a homogeneous medium in terms of the Caputo-Fabrizio time fractional derivative (1), namely:

$$\frac{\partial T(x,t)}{\partial t} = a_1 \frac{\partial^2 T(x,t)}{\partial x^2} + a_2 (1-\alpha)_{\mathrm{CF}} \mathrm{D}_t^{\alpha} \frac{\partial^2 T(x,t)}{\partial x^2}, \quad t > 0 \tag{2}$$

Now, we refer to the extreme case of steady-state ($t \to \infty$) heat conduction in a medium with a spatial memory of the heat flux and consequently to a model in term of Caputo-Fabrizio space-fractional derivative with an exponential Kernel matching the construction of that in the time-fractional derivative (1).

## From the Cattaneo concept to time and space memory effects

Starting from the conservation law, the transient heat conduction is generally described as:

$$\rho C_p \frac{\partial T}{\partial t} = -\frac{\partial q}{\partial x}, \quad q(x,t) = -k \frac{\partial T(x,t)}{\partial x} \quad \rho C_p \frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2} \tag{3a-c}$$

The basic assumption (3b) states that the flux $q(x, t)$ is proportional to the temperature gradient, thus defining the thermal conductivity, $k$. Consequently the Fourier law (3c) can be easily derived but it defines unphysical infinite speed of flux's propagation.

The Cattaneo concept [13] is a generalization of the Fourier law through a linear superposition of the heat flux and its time derivative related to its history [14, 15] related to the time-delay, $s$:

$$q(x,t) = -\int_{-\infty}^{t} R(x,t)\, \nabla T(x, t-s)\, \mathrm{d}s \tag{4a}$$

Let express the flux in the form (4) with memories (delay with respect to both the time and space:

$$q(x+\lambda, t+\tau) = -k_1 \frac{\partial T(x,t)}{\partial x} \tag{4b}$$

The memory distance, $\lambda$, is finite (*i. e.* $\lambda$ = const.) and it is the length scale of spatial effects on the heat flux correlation to the temperature gradient. The first order approximation, in $\tau$ and $\lambda$, which can be simply developed by a conventional Taylor expansion, results in a modified Fourier law, namely:

$$q(x+\lambda, t+\tau) \approx q(x,t) + \tau \frac{\partial q(x,t)}{\partial t} + \lambda \frac{\partial q(x,t)}{\partial x} \tag{5}$$

### *Temporal memory only: the guiding example*

If $R(x, t)$ is space-independent and represented by the Jeffrey kernel [1, 16] as $R(t) = \exp[-(t-s)/\tau]$, then, the energy balance yields the Cattaneo equation [13]:

$$\frac{\partial T(x,t)}{\partial t} = \frac{a_2}{\tau} \int_0^t \exp\left(-\frac{t-s}{\tau}\right) \frac{\partial T(x,s)}{\partial x} \mathrm{d}s, \quad a_2 = \frac{k_2}{\rho C_p} \tag{6}$$

The first order approximation see eq. (5) with respect to $\tau$ results in a modified Fourier law is [16]:

$$q(x,t+\tau)=-k_1\frac{\partial T(x,t)}{\partial x},\quad q(x,t+\tau)\approx q(x,t)+\tau\frac{\partial q(x,t)}{\partial t} \tag{7}$$

This leads to a first order differential equation [17]:

$$\frac{1}{\tau}q(x,t)+\frac{\partial q(x,t)}{\partial t}=-\frac{k_1}{\tau}\frac{\partial T(x,t)}{\partial x} \tag{8}$$

Further, with a modified relaxation function [16, 17]:

$$R_{\mathrm{JP}}=k_1\delta_{\mathrm{t}}(s)+\frac{k_2}{\tau}\exp\left(-\frac{s}{\tau}\right),\quad \int_{0}^{0}\delta_t(s)\mathrm{d}s=1 \tag{9}$$

where $\delta_t(s)$ is Dirac delta function and $\int_{0}^{0}\delta_t(s)\mathrm{d}s=1$, we may define the flux as [16, 17]:

$$q(x,t)=-k_1\frac{\partial T(x,t)}{\partial x}-\frac{k_2}{\tau}\int_{-\infty}^{t}\mathrm{e}^{-\frac{t-s}{\tau}}\frac{\partial T(x,s)}{\partial x}\mathrm{d}s \tag{10}$$

Equation (10) defines the effective thermal conductivity, $k_1$, and the elastic conductivity, $k_2$. The energy conservation equation [18] results in the Jeffrey type integro-differential equation [17]:

$$\frac{\partial T(x,t)}{\partial t}=a_1\frac{\partial^2 T(x,t)}{\partial x^2}+a_2\beta\int_{-\infty}^{t}\mathrm{e}^{-\beta(t-s)}\frac{\partial^2 T(x,s)}{\partial x^2}\mathrm{d}s,\quad a_1=\frac{k_1}{\rho C_p},\quad a_2=\frac{k_2}{\rho C_p},\quad \beta=\frac{1}{\tau} \tag{11}$$

The integration by parts of the last term of eq. (11) (see [12] for details) leads to eq. (2).

### *Spatial memory: an analysis by analogy*

Now, for the steady-state heat conduction in a medium with space memory effects we may define by analogy that the heat flux is related to the temperature gradient as eq. (12a):

$$q(x)=-\int_{-\infty}^{x}R(x)\nabla T(x-u)\mathrm{d}u,\quad q(x)=-\int_{-\infty}^{x}\mathrm{e}^{-\frac{x-u}{\lambda}}\frac{\mathrm{d}T(u)}{\mathrm{d}x}\mathrm{d}u \tag{12a,b}$$

If the kernel is defined as $R(x)=\exp[-(x-u)/\lambda]$ we get eq. (12b). As mentioned earlier, $\lambda$ is the spatial memory length scale while its inverse $\gamma=1/\lambda$ is a space memory constant. The first order approximation of the heat flux with respect to $\lambda$ is:

$$q(x+\lambda)=-k_1\frac{\mathrm{d}T(x)}{\mathrm{d}x},\quad q(x+\lambda)\approx q(x)+\lambda\frac{\mathrm{d}q(x)}{\mathrm{d}x} \tag{13a,b}$$

Repeating, by analogy, the concept of a modified space-related memory function [17] we may define:

$$R^{*}(x) \quad k_1 \delta_x(u) \quad \frac{k_{2x}}{\lambda} \exp \quad \frac{u}{\lambda}, \quad \int_{0}^{0} \delta_x(u) \quad 1 \tag{14a}$$

Then, the flux can be defined:

$$q(x) \quad k_1 \frac{\mathrm{d}T(x)}{\mathrm{d}x} \quad \frac{k_{2x}}{\lambda} \int_{\infty}^{x} \mathrm{e}^{\frac{x \quad u}{\lambda}} \frac{\mathrm{d}T(u)}{\mathrm{d}x} \mathrm{d}u \tag{14b}$$

Equation (14b) defines the effective thermal conductivity, $k_1$, which is the same as that in eq. (8). The conductivity $k_{2x}$ can be defined as a structural elastic conductivity related to the spatial memory effects. If no memory effects exist, then:

$$q(x) \quad (k_1 \quad k_{2x}) \frac{\mathrm{d}T(x)}{\mathrm{d}x}, \quad k \quad k_1 \quad k_{2x} \tag{14c}$$

Now, after differentiation we may present eq. (14b) as:

$$\frac{\mathrm{d}}{\mathrm{d}x} q(x) \quad k_1 \frac{\mathrm{d}^2 T(x)}{\mathrm{d}x^2} \quad \frac{k_{2x}}{\lambda} \int_{\infty}^{x} \mathrm{e}^{\frac{x \quad u}{\lambda}} \frac{\mathrm{d}^2 T(u)}{\mathrm{d}x^2} \mathrm{d}u \tag{15}$$

Further, applying the basic relation (3a) we get integro-differential equation of Jeffrey's type:

$$\frac{\partial T}{\partial t} \quad a_1 \frac{\mathrm{d}^2 T(x)}{\mathrm{d}x^2} \quad a_{2x} \gamma \int_{\infty}^{x} \mathrm{e}^{\gamma(x \quad u)} \frac{\mathrm{d}^2 T(u)}{\mathrm{d}x^2} \mathrm{d}u, \quad a_1 \quad \frac{k_1}{\rho C_p}, \quad a_{2x} \quad \frac{k_{2x}}{\rho C_p}, \quad \gamma \quad \frac{1}{\lambda} \tag{16}$$

The integration by parts of the last term of eq. (16) and denoting $F(x) = \mathrm{d}T(x)/\mathrm{d}x$ we get:

$$\gamma \int_{\infty}^{x} \mathrm{e}^{y(x \quad s)} F(x) \mathrm{d}s \quad \mathrm{e}^{y(x \quad s)} \quad F(s) \quad F(x) \Big|_{\infty}^{x} \quad \gamma \int_{\infty}^{t} \mathrm{e}^{y(x \quad s)} [F(x) \quad F(s)] \mathrm{d}s \tag{17}$$

The second term in the RHS of eq. (17) one resembles the definition of the Caputo-Fabrizio fractional derivative [3] with a Jeffrey's kernel with respect to the space variable $x$, namely:

$${}_{CF}\mathrm{D}_x^{\mu} F(x) \quad \frac{\mu}{(1 \quad \mu)^2} \int_{4}^{x} f(x) \quad f_a(u) \exp \quad \frac{\mu}{1 \quad \mu}(x \quad u) \; \mathrm{d}u, \quad t \quad 0 \tag{18}$$

By analogy of the results from [12] a pro-Caputo (non-normalized) space-derivative denoted as ${}_{PC}\mathrm{D}_x^{\gamma}$, can be defined:

$${}_{PC}\mathrm{D}_x^{\gamma} F(x) \quad \gamma \int_{\infty}^{x} \mathrm{e}^{\gamma(x \quad u)} \; F(x) \quad F(u) \; \mathrm{d}u \quad \gamma \int_{\infty}^{x} \mathrm{e}^{\gamma(t \quad s)} \frac{\mathrm{d}F(u)}{\mathrm{d}x} \mathrm{d}u \tag{19}$$

Obviously, the space memory constant $\gamma$ in eq. (19) controls the kernel and $\gamma \quad (0, \infty)$. If we like to refine ${}_{PC}\mathrm{D}_x^{\gamma}$ as an integral operator controlled by a single parameter $\mu$ we have to satisfy the conditions: for $\mu \quad [0.1] \quad 1/\gamma \quad [0, \infty]$. With $\gamma(\mu) \quad \mu/(1-\mu)$ [1, 2] the desired properties are obtained, namely:

$$\frac{1}{\gamma} = \frac{1-\mu}{\mu} \in [0,\infty], \quad \mu = \frac{1}{1+\frac{1}{\mu}} = \frac{\gamma}{1+\gamma} \in [0,1], \quad \frac{\mu}{(1-\mu)^2} = \frac{\gamma}{(1-\mu)} \tag{20a-c}$$

Further, following the basic definition of the Caputo-Fabrizio derivative [1, 2] with Jeffrey's kernel, see eq. (1), and considering the lower limit of integral in eq. (15) at $x = 0$, *i. e.* $a = 0$, we have:

$$ {}_{CF}\mathrm{D}_x^{\mu}T(x) = \frac{N(\chi)}{\chi}\,{}_{PC}\mathrm{D}_x^{\gamma}T(x) = \frac{N(\chi)}{\chi}\int_0^x \mathrm{e}^{-\frac{\mu}{1-\mu}(x-u)}\frac{\mathrm{d}F(u)}{\mathrm{d}x}\mathrm{d}u = \frac{M(\mu)}{1-\mu}\int_0^x \mathrm{e}^{-\frac{\mu}{1-\mu}(x-u)}\frac{\mathrm{d}F(u)}{\mathrm{d}x}\mathrm{d}u \tag{21a-c}$$

In the terms used here $\chi = \gamma$, while $N(\chi)$ and $M(\mu)$ are normalization functions [1, 2]. The problem of definition of $M(\mu)$ is still open [1, 4] and in accordance with the general definition we should have $M(0) = M(1) = 1$. It was demonstrated that for a special case studied by Losada and Nieto [4] $M(\mu) = 2/(2-\mu)$ which in the present case works correctly for $\alpha = 0$ but gives $M(1) = 2$. Further, if $M(\mu) = 1/(2-\mu)$ then we have $M(0) = 1/2$ and $M(1) = 1$. It is obvious that both conditions can not be satisfied simultaneously by these simple expressions of $M(\mu)$. Then, for convenience, we define $M(\mu) = 1$ as it was suggested by Caputo and Fabrizio in [1]. Consequently, the form eq. (21c) reduces to Caputo-Fabrizio space-fractional derivative of $T(x)$ eq. (22a) defined by analogy of eq. (1b) and its Laplace transform is eq. (22b):

$$ {}_{CF}\mathrm{D}_x^{\mu}T(x) = \frac{1}{1-\mu}\int_0^x \mathrm{e}^{-\frac{\mu}{1-\mu}(x-u)}\frac{\mathrm{d}T(u)}{\mathrm{d}x}\mathrm{d}u, \quad L\left[{}_{\mathrm{CF}}\mathrm{D}_x^{\mu}T(x_{\lambda})\right] = \frac{pT(p)-T(0)}{p+\mu(1-p)} \tag{22a,b}$$

From the general rule of differentiation ${}_{CF}\mathrm{D}_x^{\mu}T(x)$ eq. (23a) [1] we have:

$$\mathrm{D}^{(n)}\left[{}_{CF}\mathrm{D}_x^{\mu}f(x)\right] = {}_{CF}\mathrm{D}_x^{\mu}\left[\mathrm{D}^{(n)}f(x)\right] \tag{23a}$$

For $n = 1$ we may derive equivalent expressions, eqs. 23b,c, [1], namely:

$$\mathrm{D}^{(1)}\left[{}_{CF}\mathrm{D}_x^{\mu}f(x)\right] = \frac{1}{1-\mu}\frac{\mathrm{d}f(x)}{\mathrm{d}x} - \frac{\mu}{1-\mu}\int_0^x \frac{\mathrm{d}f(x)}{\mathrm{d}x}\exp\left[-\frac{\mu(x-u)}{1-\mu}\right]\mathrm{d}u \tag{23b}$$

$$\mathrm{D}^{1}\left[{}_{CF}\mathrm{D}_x^{\mu}f(x)\right] = {}_{CF}\mathrm{D}_x^{\mu}\left[\frac{\mathrm{d}f(x)}{\mathrm{d}x}\right] \tag{23c}$$

It is worthnoting that from the definition of the exponential kernel the physical dimension of $\gamma$ is length [m]. However, the fractional order $\mu$ is dimensionless. Therefore, to avoid this conflict, we may present $\lambda = 1/\gamma = l_s[(1-\mu)/\mu]$ where the dimension of the factor $l_s$ is length [m]. Without loss of generality we may assume that $l_s = 1$. In this way, as it will be mentioned further in this work the ratio $(1-\mu)/\mu$ has a dimension of length but at the same time $\mu$ is dimensionless.

Now, expressing eq. (15) in terms of $T(x)$ we get the complete heat transfer equation expressed by the space-fractional Caputo-Fabrizio derivative:

$$\frac{\partial T}{\partial t} \quad a_1 \frac{\mathrm{d}^2 T(x)}{\mathrm{d}x^2} \quad a_{2x}(1 \quad \mu)_{CF}\, \mathrm{D}_x^{\mu} T(x) \quad a_1 \quad \frac{k_1}{\rho C_p}, \quad a_{2x} \quad \frac{k_2}{\rho C_p} \tag{24}$$

Now, after differentiation using eqs. (23a-c) we may present eq. (14b) in three equivalent forms:

$$\frac{\mathrm{d}}{\mathrm{d}x} q(x) \quad k_1 \frac{\mathrm{d}^2 T(x)}{\mathrm{d}x^2} \quad \frac{k_{2x}}{\lambda} \,{}_{\infty}^{x}\, \mathrm{e}^{-\frac{x-u}{\lambda}} \frac{d^2 T(u)}{\mathrm{d}x^2} \mathrm{d}u \tag{25a}$$

$$\frac{\partial T}{\partial t} \quad a_1 \frac{\mathrm{d}^2 T(x,t)}{\mathrm{d}x^2} \quad a_{2x}(1 \quad \mu)_{CF}\, \mathrm{D}_x^{\mu} \frac{\mathrm{d}T(x)}{\mathrm{d}x} \tag{25b}$$

$$\frac{\partial T}{\partial t} \quad a_1 \frac{\mathrm{d}^2 T(x,t)}{\mathrm{d}x^2} \quad a_{2x}(1 \quad \mu) \frac{\mathrm{d}}{\mathrm{d}x}[{}_{CF}\, \mathrm{D}_x^{\mu} T(x)] \tag{25c}$$

The steady-state condition at $t \quad \infty$ yields three equivalent expressions:

$$0 \quad a_1 \frac{\mathrm{d}^2 T(x)}{\mathrm{d}x^2} \quad a_{2x}\gamma \,{}_{0}^{x}\, \mathrm{e}^{\ \gamma(x \quad u)} \frac{\mathrm{d}^2 T(u)}{\mathrm{d}x^2} \mathrm{d}u \tag{26a}$$

$$0 \quad a_1 \frac{\mathrm{d}^2 T(x,t)}{\mathrm{d}x^2} \quad a_{2x}(1 \quad \mu)_{CF}\, \mathrm{D}_x^{\mu} \frac{\mathrm{d}T(x)}{\mathrm{d}x} \tag{26b}$$

$$0 \quad a_1 \frac{\mathrm{d}^2 T(x,t)}{\mathrm{d}x^2} \quad a_{2x}(1 \quad \mu) \frac{\mathrm{d}}{\mathrm{d}x}\left[{}_{CF}\, \mathrm{D}_x^{\mu} T(x)\right] \tag{26c}$$

## Short and long range governing equations: analyses and solutions

Let us consider a material with a spatial memory arranged as a long bar of length, $L$, with thermally insulated surface and subjected to a thermal loads at $x = 0$ and $x = L$, that is:

$$T(0) \quad T_0, \quad T(L) \quad T_L, \quad {}_{CF}\, \mathrm{D}_x^{\mu} T_0 \quad {}_{CF}\, \mathrm{D}_x^{\mu} T_L \quad 0, \quad T_0 \quad T_L \tag{27a,b}$$

Without loss of generality, we may assume that $T_{\mathrm{L}} = 0$ (this is only a shift in the temperature scale), that will simplify the calculations. Moreover, the condition (27b) follows from the definition of the Caputo-Fabrizio derivative [1].

### *Short-range memory effects: Space memory only*

If the Cattaneo kernel is only taken into account, that is the modified relaxation function $R^*(x)$ is omitted, then we have to use eq. (12) which is a spatial analogue of eq. (6). Now, we may write in terms of Caputo-Fabrizio spatial derivative the flux relation:

$$q_2 \quad k_{2x}(1 \quad \mu)_{CF}\, \mathrm{D}_x^{\mu} T(x) \tag{28}$$

This is the equation used by Yang *et al.* [19] despite the fact that the fractional derivative used by these authors (it is of pseudo-Riemann-Liouville type) does not match exactly (see more details in [19]). Physically, eq. (28) means that the macroscopic temperature field is ignored and the zonal temperature distribution is considered only. Further, eq. (28) accounts only the spatial structural memory affect on the heat diffusion and it is valid only within a zone defined as $x_0 < x < x_0 + \lambda$ where $0 < x_0 < L$ and $0 < \lambda < L$. The space-memory related thermal diffusivity, $a_{2x}$, accounts only short-distance structural memory effects. For $\mu = 1$ we get the transient (29a) and the steady-state (29b) versions, namely:

$$\frac{\partial T(x,t)}{\partial t} \quad a_{2x}\frac{\mathrm{d}^2T(x)}{\mathrm{d}x^2}, \quad 0 \quad a_{2x}\frac{\mathrm{d}^2T(x)}{\mathrm{d}x^2}, t \quad \infty \tag{29a,b}$$

Assuming $T(x_0) > T(x_\lambda) > T(\lambda)$ where eq. (28) is expressed in the local co-ordinate system with origin at $x = x_0$ and an axis extending up to $x = x_\lambda = x_0 = \lambda$. Applying the Laplace transform to eq. (28) we get:

$$\frac{q_\lambda}{1 \quad \mu}\frac{1}{p} \quad \frac{pT(p) \quad T(x_0)}{p \quad \mu(1 \quad p)} \qquad T(p) \quad [q_\lambda(1 \quad \mu) \quad T_{x0}]\frac{1}{p} \quad q_\lambda\mu\frac{1}{p^2}, \qquad q_\lambda \quad \frac{q_2}{k_{2x}} \tag{30a, b}$$

The inverses Laplace transform $L^{-1}T$ yields:

$$T(x_\lambda) \quad [q_\lambda(1 \quad \mu) \quad T(x_\lambda) \quad q_\lambda\mu x_\lambda \qquad \frac{T_{x0} \quad T(x_\lambda)}{q_\lambda} \quad (1 \quad \mu) \quad \mu x_\lambda \tag{31a,b}$$

$$x_0 \quad x_\lambda \quad x_0 \quad \lambda \quad \text{and} \quad T_{x0} \quad T(x_\lambda) \quad T_{(x_0 \quad \lambda)} \tag{31c}$$

The linear temperature profile eqs. (31b) or (31c), confirms the solution of Yang *et al.* [19] irrespective of the differences in the definition of the space-fractional derivatives used. For $\mu = 1$ eqs. (28b) and (38c) reduces to the classical Fourier solution. Futher, for $x_\lambda = x_0$ and $x_\lambda = x_0 + \lambda$ we have see eq. (31a) correspondingly:

$$T_\lambda \quad [q_\lambda(1 \quad \mu) \quad T_{x0}] \quad q_\lambda\mu(x_0 \quad \lambda) \tag{32a}$$

$$T_{x0} \quad [q_\lambda(1 \quad \mu) \quad T_{x0}] \quad q_\lambda\mu x_0 \tag{32b}$$

Subtracting eq. (32b) from eq. (32a) we obtain:

$$T_{x0} \quad T_\lambda \quad q_\lambda\mu\lambda \qquad q_\lambda \quad \frac{T_{x0} \quad T_\lambda}{\mu\lambda} \tag{33c}$$

The relation (33c) is the linear Fourier law across the memory zone of thickness $\lambda$. Replacing $q_\lambda$ in eq. (31a) by eq. (32c) and with $\lambda = (1 - \mu)/\mu$, the dimensionless profile in the zonal co-ordinate system is:

$$\frac{T_{x0} \quad T(x_\lambda)}{T_{x0} \quad T_\lambda} \quad 1 \quad \frac{x_\lambda}{\lambda} \quad 1 \quad \frac{\mu}{1 \quad \mu}x_\lambda \tag{34a}$$

The transition to the global co-ordinate system defined by $0 < x < L$, a simple shift $x_\lambda = x - \lambda$ results in:

$$\frac{T_{x0} \quad T(x)}{T_{x0} \quad T_\lambda} \quad \frac{x}{\lambda} \quad \frac{\mu}{1 \quad \mu}z, \; 0 \quad z_\lambda \quad 1 \tag{34b}$$

Therefore, across the memory zone we have a linear temperature profile controlled by the fractional order $\mu$. Figures 1(a) and (b) presents the solutions (31c) and (34b) for several values of $\mu$.

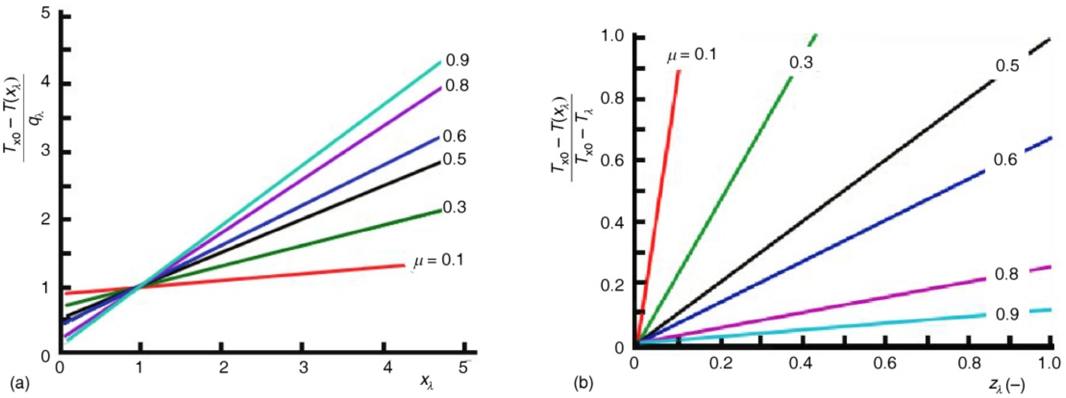


**Figure 1. Zonal temperature distribution, space memory only: (a) dimensional and (b) dimensionless profiles**

### *Space memory with extended relaxation function*

With the complete relaxation function and the rules of differentiation of ${}_{CF}\mathrm{D}_x^\mu T(x)$ we have:

$$\frac{\partial T(x,t)}{\partial t} \quad a_1 \frac{\mathrm{d}^2 T(x)}{\mathrm{d}x^2} \quad a_{2x} \frac{\mathrm{d}^2 T(x)}{\mathrm{d}x^2} \quad (1 \quad \mu)\frac{\mathrm{d}}{\mathrm{d}x}{}_{CF}\mathrm{D}_x^\mu T(x) \tag{35}$$

Two steady-state equivalent versions of eq. (35) can be expressed:

$$0 \quad (a_1 \quad a_{2x})\frac{\mathrm{d}^2 T(x)}{\mathrm{d}x^2} \quad a_{2x}(1 \quad \mu)\frac{\mathrm{d}}{\mathrm{d}x}{}_{CF}\mathrm{D}_x^\mu T(x) \tag{36}$$

$$0 \quad \frac{\mathrm{d}^2 T(x)}{\mathrm{d}x^2} \quad m(1 \quad \mu)\frac{\mathrm{d}}{\mathrm{d}x}{}_{CF}\mathrm{D}_x^\mu T(x), \quad m \quad \frac{a_{2x}}{a_1 \quad a_{2x}} \tag{37}$$

For $\mu = 1$ we have that the thermal conductivity $k_0 = k_1 + k_{2x}$ ($a_0 = a_1 + a_{2x}$) as in the case of a temporal memory [17]. The partition coefficient $0 < m < 1$ accounts the contribution of the space memory effect (the elastic structural thermal diffusivity $a_{2x}$) to the total thermal diffusivity $a_0$. For $\mu = 1$, we get $m = 0$.

#### *Laplace transform solution*

Te integration of eq. (37) yields:

$$C_3 \quad \frac{\mathrm{d}T(x)}{\mathrm{d}x} \quad m(1 \quad \mu)_{\mathrm{CF}}\mathrm{D}_x^\mu T(x) \tag{38}$$

This step in the solution is provoked by the fact that $L[f_{xx}(x)] = p^2 f(p) - pf_x(0) - f(0)$. If we associate $f(x)$ with $T(x)$ then from eq. (35), for example, we get a new problem since $T_x(0)$ is unknown and the Laplace transform can not be completed.

Now, applying the Laplace transform solution to eq. (38) we have:

$$T(x) = T_0 + C_3 \left[\frac{a}{b}x + \frac{b-a}{b^2}(1 - \mathrm{e}^{-bx})\right], \quad a = \frac{\mu}{1-\mu}, \quad b = m + a \tag{39}$$

Now, the constant $C_3$ should be defined. We continue with eq. (38) integrating both sides, namely:

$$C_3 x = T(x) - m(1-\mu)\int [{}_{CF}\mathrm{D}_x^{\mu} T(x)]\mathrm{d}x + C_4 \tag{40}$$

With the boundary conditions (27a) and the relations (27b) we get:

$$\int \left[{}_{CF}\mathrm{D}_x^{\mu} T(0)\right]\mathrm{d}x = \int \left[{}_{CF}\mathrm{D}_x^{\mu} T(L)\right]\mathrm{d}x = 0 \tag{41a}$$

$$C_3 = \frac{T_L - T_0}{L} \text{ and } C_4 = T_0 \tag{41b}$$

Hence, in terms in the process variables the solution (39) can be presented:

$$\theta_{\mu} = \frac{T_0 - T(x)}{T_0 - T_L} = \frac{a}{b}\frac{x}{L} + \frac{1}{L}\frac{a-b}{b^2}\left[1 - \exp(-bx)\right] - \frac{b-a}{b^2} \tag{42a}$$

Alternatively, if assume $T_L = 0$, which means only a shift of the temperature scale, we get:

$$\theta_{\mu(T_L = 0)} = \frac{T(x)}{T_0} = 1 - \frac{a}{b}\frac{x}{L} + \frac{1}{L}\frac{a-b}{b^2}\left[1 - \exp(-bx)\right] - \frac{b-a}{b^2} \tag{42b}$$

The equivalent forms of eqs. (42a) and (42b) are:

$$\theta_{\mu} = \frac{T_0 - T(x)}{T_0 - T_L} = \frac{\mu}{\mu + m(1-\mu)}\frac{x}{L} - \frac{1}{L}\frac{m(1-\mu)^2}{[\mu + m(1-\mu)]^2}\left\{1 - \exp\left[-\left(m + \frac{\mu}{1-\mu}\right)x\right]\right\} \tag{43a}$$

$$\theta_{\mu(T_L = 0)} = \frac{T(x)}{T_0} = 1 - \frac{\mu}{\mu + m(1-\mu)}\frac{x}{L} + \frac{1}{L}\frac{m(1-\mu)^2}{[\mu + m(1-\mu)]^2}\left\{1 - \exp\left[-\left(m + \frac{\mu}{1-\mu}\right)x\right]\right\} \tag{43b}$$

Now, bearing in mind that $\gamma = 1/\lambda = \mu/(1 - \mu)$ we may express eq. (43a) in a dimensionless form:

$$\theta_{\mu} = \frac{\mu}{\mu + m(1-\mu)}z - \frac{\lambda}{L}\frac{m\mu(1-\mu)}{\left[\mu + m(1-\mu)\right]^2}[1 - \mathrm{e}^{-(\gamma + m)x}] \tag{44}$$

The ratio $\lambda/L$ and the product $(\gamma + m)x \approx x/\lambda$ are dimensionless. Obviously, the contribution of the non-linear term depends on the ratio $\lambda/L$.

The linear approximation of the exponential term in eq. (44) as a series within the range where the space memory takes place (that is for $0 < x < \lambda$ and $\lambda \ll L$, as well as $x \ll L \Rightarrow x/L \ll 1$) is: $\exp[-(\gamma + m)x] \approx 1 - (\gamma + m)(x/\lambda) + O[(x/\lambda)^2]$. Then, we may approximate eq. (44) as:

$$\theta_{\mu} \approx \frac{\mu}{\mu + m(1-\mu)}\frac{x}{L} - \frac{1-\mu}{\mu}\frac{m\mu(1-\mu)}{\left[\mu + m(1-\mu)\right]^2}\frac{(\gamma + m)(x/\lambda)}{L} \tag{45}$$

Now, with $(x-\lambda \;\; L \ll$ we may see that the first term of eq. (45) dominates. However, if we look only at the range $x_0 < x < \lambda$ we may neglect the first term in eq. (45), that is assuming $x/L$ because $x/L < \lambda/L \ll 1$ and eq. (45) reduces to two equivalent forms: (46a) and (46b).

$$\theta_{\mu(x \;\; \lambda)} \quad \frac{1 \quad \mu}{\mu} \frac{m\mu(1 \quad \mu)}{[\mu \quad m(1 \quad \mu)]^2} \frac{(\gamma \quad m)(x \quad \lambda)}{L}, \quad \theta_{\mu(x \;\; \lambda)} \quad \frac{\lambda}{L} \frac{m\mu(1 \quad \mu)}{[\mu \quad m(1 \quad \mu)]^2}(\gamma \quad m)(x \quad \lambda) \tag{46a,b}$$

The approximations (46a,b) represent a zonal solution (within the range defined by $\lambda$ with space memory effects) but with a temperature field defined by the thermal loads at the macroscopic boundaries $x = 0$ and $x = L$. However, $\lambda/L$ may reach values of order of magnitude of unity as it shown further in this article and therefore, the results of (46a,b) are valid for cases with small memory effects.

### *Direct integration approach and problems thereof*

From the intermediate results (40) and (41a,b) we get:

$$\frac{x}{L} \quad \frac{T(x) \quad T_0}{T_L \quad T_0} \quad \frac{1}{T_L \quad T_0} m(1 \quad \mu) \;\; [{}_{CF}\mathrm{D}_x^{\mu} T(x)]\mathrm{d}x \tag{47}$$

Futher, from the linearity of ${}_{CF}\mathrm{D}_x^{\mu} T(x)$ we have: ${}_{CF}\mathrm{D}_x^{\mu}[T(x) \quad c] \;\; {}_{CF}\mathrm{D}_x^{\mu} T(x)$, as well as ${}_{CF}\mathrm{D}_x^{\mu}[T(x)/c] \;\; [{}_{CF}\mathrm{D}_x^{\mu} T(x)]/c$, where $c$ is a constant. Hence, eq. (47) can be expressed:

$$\theta \quad z \quad m \; \left[(1 \quad \mu)_{CF}\mathrm{D}_x^{\mu}\theta\right]\mathrm{d}x, \qquad \theta \quad \frac{T_0 \quad T(x)}{T_0 \quad T_L}, \qquad z \quad \frac{x}{L} \tag{48}$$

The second term in eq. (48) is integer-order integral from the Caputo-Fabrizio fractional derivative. To avoid its evaluation, we differentiate eq. (48) and this operation leads to a fractional differential equation.

$${}_{CF}\mathrm{D}_x^{\mu}\theta \quad \frac{1}{m(1 \quad \mu)} \frac{\mathrm{d}\theta}{\mathrm{d}x} \quad \frac{1}{L_m} \frac{1}{(1 \quad \mu)} \quad 0, \quad \theta(z \quad 1) \;\; 1, \quad \theta(z \quad 0) \;\; 0 \tag{49}$$

The Laplace transform of eq. (49) yields:

$$\theta(p) \quad \frac{1}{L} \frac{1}{p^2} \frac{p \quad \gamma}{p \quad (m \quad \gamma)}, \quad \theta(z) \quad \frac{\gamma}{m \quad \gamma} \; z \quad \frac{1}{L} \quad \frac{1}{m \quad \gamma} \exp[\;\; (m \quad \gamma)x] \quad \frac{1}{m \quad \gamma} \tag{50a,b}$$

In terms of $z = x/L$ only, we may express eq. (50b) as:

$$\theta(z) \quad \frac{\gamma}{m \quad \gamma} \; z \quad \frac{1}{L} \quad \frac{1}{m \quad \gamma} \exp\left[\;\; L(m \quad \gamma)z\right] \quad \frac{1}{m \quad \gamma} \tag{50c}$$

Equation (50b) is an analogue of the solutions (42a) and (43a). In addition, the coefficient of the fist term in the RHS of eq. (44) in terms of $\gamma$ is $\mu/(m \quad \mu) \quad \gamma/[\gamma \quad m(1 \quad \gamma)]$. Moreover, $\gamma \qquad \lambda$ is [$\mathrm{m}^{-1}$] and therefore the product $(m + \gamma x$ as well as $L(m \quad \gamma) \quad L\gamma \quad L/\lambda \quad (\lambda/L)^{1}$ are dimensionless. Physically, $L/\lambda$ is the ratio of the macroscopic length scale $L$ to the spatial memory length scale $\lambda$ and we have to define its relationship to the factional order $\mu$.

***The ratio $\lambda$ L and the fractional order $\mu$***

The ratio $\lambda/L$ defines the rage of the total range of interest with a length $L$ by the zone of length$\lambda$ where the spatial memory takes place. Obviously, $\lambda/L$ 1 an we have to relate this ratio to the fractional order because by the definition the space fractional derivative (see 23a) we have $\lambda$ $(1 - \mu$ $\mu$. Denoting $\lambda$ $L = K_L$ we have there equivalent relations:

$$\frac{1\ \ \mu}{\mu}\frac{1}{L} \quad K_L,$$

$$\mu \quad \frac{1}{1\ \ K_L}, \quad K_L \quad \frac{1\ \ \mu}{\mu}, \quad L \quad 1 \quad 0 \quad K_L \quad 1 \tag{51a,b,c}$$

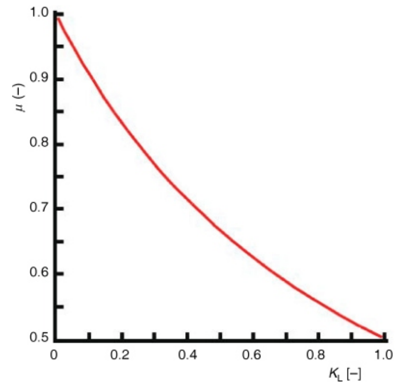


**Figure 2. Relationship between the fractional order m and the ratio $K_L = \lambda/L$**

Since, the global length, $L$, always can be defined as $L = 1$ irrespective of the real physical dimension and the units used for it this lead to eqs. (51b) and (51c).Then, from known $K_L$ we may define, the fractional order $\mu$ or at least to estimate the rage where it could vary. The relationship $\mu$ $f(K_L)$ expressed by (51b) is shown in fig. 2. Since 0 $K_L = \lambda$ $L$ 1, it follows that 0.5 $\mu$ 1. The lower boundary $\mu$ 0.5 corresponds to $\lambda$ $L$ where the entire area is covered by the memory zone, while $\mu = 1$ means $\lambda = 0$ (no memory effects). The decrease in size of $\lambda$ increases the fractional order $\mu$ and reduces the memory effect on the global transport process. The reasonable question is: What happens of $\mu < 0.5$? Simply if $0 < \mu < 0.5$ it follows that $\lambda$ $L$ which violates the physician meaning of a memory zone embedded in a large homogeneous area of length $L$.

***The redistribution coefficient m and the fractional order $\mu$***

The developed solutions defined the redistribution coefficient $m = a_{2x}/a_0$ and it is natural to ask what is the relationship between $m$ and fractional order $\mu$. From the physics we have that when $\mu$ 1 consequently $m$ 0, and *vice versa*. Hence, if we define the relation $m = (1 -$ $\mu$ $^N$ the limits are obeyed. From the analysis of the heat wave [17], which is used as a template, we have that the slower relations effects, the larger the value of $k_2$ and *vice versa*. Replacement of $m$ by $(1 - \mu$ $^N$ in the solutions makes the fractional order $\mu$ the only parameter controlling the process. The tests with different $N$ of the coefficient of the linear terms in the solution (43a and 43b) $M =$ $\mu/[\mu + m$ $(1 - \mu)]$ and $m = (1 - \mu$ $^N$ are shown in fig. 3. Since the conditions $K_m = 1$ and $m = 0$ are satisfied simultaneously only for $N = 1$, the numerical simulations demonstrated next use $N = 1$.

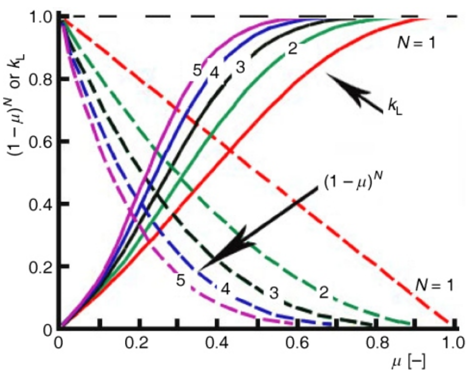


**Figure 3. Effect of the fractional order $\mu$ and the exponent $N$ on the functional relationship $m = f(\mu) = (1 - \mu)^N$ and the coefficient $K_m$ of the first term of the solution (43)**

***Temperature profiles***

The temperature profile accounting the memory effects are shown in figs. 4 and 5 we especially separated the profiles corresponding to the two terms of

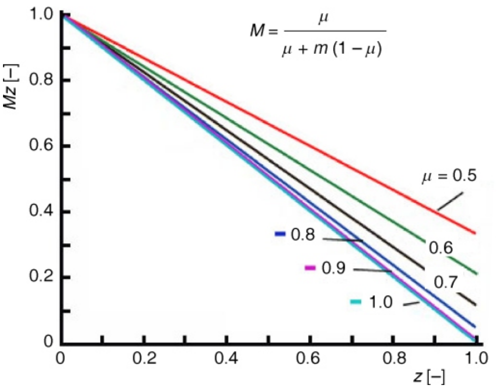


**Figure 4. Profiles determined by the first term of the solution (43) in the range 0 $\mu$ corresponding to the range 0 $\lambda$ *L* 1**

the solution (43) in order to demonstrate how they are affected by the value of the fractional order $\mu$ see fig. 5. It is clear that with low $\mu$ 0.5 the zone with memory effects will cover the area of interest and the ratio $\lambda$ $L$ approaches unity. The increase in reduces the non-linearity of the temperature profiles which is clearly demonstrated by the profiles corresponding to the exponential term (the lower parts of figs. 5a-d) and the first term (see fig. 4) where $z = x/L$. With increase in $\mu$ the contribution of the non-linearity decreases, especially that of the exponential term and the complete profile approaches the linear Fourier solution.

## Conclusions

The article demonstrates how a space-fractional derivative in the Caputo-Fabrizio sense

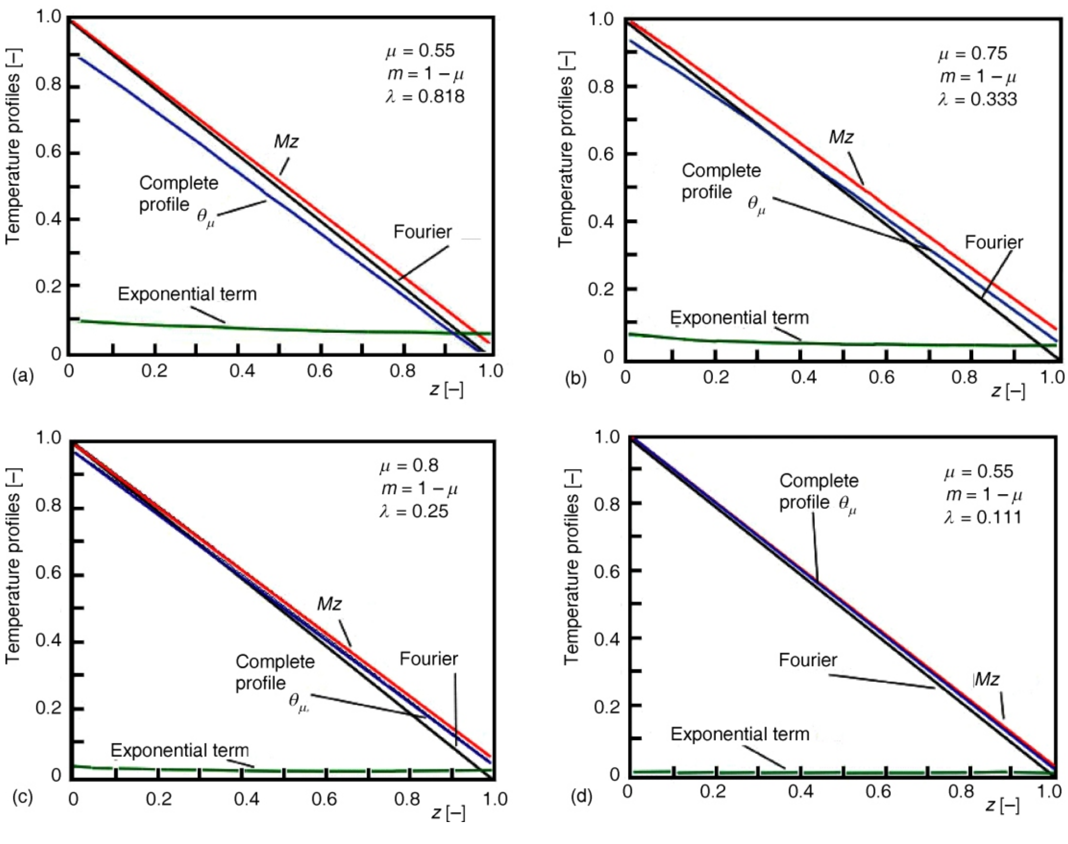


**Figure 5. Temperature profiles accounting memory effects for various values of the fractional order $\mu$**

can be developed starting from an analogue of the Cattaneo relation with a space-dependent fading memory and especially using the exponential non-singular kernel of Jeffrey's type. The solutions developed stress the attention on the steady-state problem in the zonal frame (space memory effects only) and the entire macroscopic zone with a memory zone inclusion. It was demonstrated that the Yang's solution [19] is valid only within the memory zone and it is related only to the first term of the complete solution.

The definition of space fractional Caputo-Fabrizio with exponential kernel strongly relates the fractional order $\mu$ with the ratio of the memory length scale $\lambda$ to the macroscopic length scale $L$: that is the range 1 $\lambda$ $L$ 0 corresponds to 0 $\mu$ 1.

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Paper submitted: February 29, 2016
Paper revised: May 4, 2016
Paper accepted: May 4, 2016