# ANALYSIS OF FRACTIONAL NON-LINEAR DIFFUSION BEHAVIORS BASED ON ADOMIAN POLYNOMIALS

by

***Guo-Cheng WU[a], Dumitru BALEANU[b,c]*, and Wei-Hua LUO[a]***

[a] College of Mathematics and Information Science, Neijiang Normal University, Neijiang, China
[b] Department of Mathematics, Cankaya University, Balgat, Ankara, Turkey
[c] Institute of Space Sciences, Magurele-Bucharest, Romania

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*A time-fractional non-linear diffusion equation of two orders is considered to investigate strong non-linearity through porous media. An equivalent integral equation is established and Adomian polynomials are adopted to linearize non-linear terms. With the Taylor expansion of fractional order, recurrence formulae are proposed and novel numerical solutions are obtained to depict the diffusion behaviors more accurately. The result shows that the method is suitable for numerical simulation of the fractional diffusion equations of multi-orders.*

Key words: *fractional calculus, two fractional terms, numerical solutions, Adomian decomposition method, Taylor series of fractional order*

## Introduction

As is well known, porous media hold rich nanoscale pores, the governing equations based on continue media mechanics and differential or partial differential equations often become invalid. As a result, fractal geometry and fractional calculus are becoming the two effective tools. They have been considered into the application of permeability and heat diffusion [1-5]. Due to memory effects of fractional calculus and other operator properties, the fractional partial differential equation methods are often used to depict anomalous diffusion equation in discrete media such like soil, salt rock, nanomaterials, and so on.

Recently, fractional diffusion equations of multi-orders were proposed. More fractional parameters were included in the models which can depict more complicated diffusion behaviors. Several numerical methods have been developed [6-16]. As a popular analytical method, Adomian decomposition method was applied to various fractional models. In this paper, a numerical method based on Adomian polynomials is given and diffusion behaviors are discussed for various fractional orders.

## Problems

Let's revisit the definitions of the fractional calculus [14]. The fractional integral is defined by:

* Corresponding author, e-mail: dumitru@cankaya.edu.tr

$$ {}_aI_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-\tau)^{-\alpha} f(\tau)\,\mathrm{d}\tau, \quad 0<t, \quad 0<\alpha<1 \tag{1} $$

where $a$ is an initial point.

The famous Caputo derivative of $\alpha$ order is defined:

$$ {}_a^C\mathrm{D}_t^{\alpha} f(t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-\tau)^{-\alpha} u'(\tau)\,\mathrm{d}\tau, \quad 0<t, \quad 0<\alpha<1 \tag{2} $$

From previous definition, we can see the fractional derivative has a memory effect for $\alpha \neq 1$. That's the main reason the fractional derivative has been used in diffusion issue as well as the space is non-locality [10, 15-16] and often appeared as a fractional partial differential equation method. For example, the time fractional diffusion equation reads:

$$ {}_a^C\mathrm{D}_t^{\alpha} u = Ku_{xx}, \quad 0<\alpha<1 \tag{3} $$

where $K$ is a diffusion coefficient. In this paper, we consider a complicated case with two fractional terms:

$$ {}_a^C\mathrm{D}_t^{\alpha} u + \kappa\, {}_a^C\mathrm{D}_t^{\beta} u = Ku_{xx} - \frac{1}{\mathrm{R}} uu_x, \quad 0<\beta<\alpha<1 \tag{4} $$

subjected to conditions:

$$ u(a,x) = \sin(\pi x), \quad u(t,0) = u(t,1) = 0 $$

We can see the diffusion depends on the past statues through two memory terms ${}_0^C\mathrm{D}_t^{\alpha} u$ and ${}_0^C\mathrm{D}_t^{\beta} u$. But this also results in difficulty to find solutions, particularly the calculations for engineering researchers. In the next section, we adopt new Adomian polynomials [17-19] and give a simple and efficient numerical method.

## Numerical solutions

We take the fractional integral eq. (1) to both sides of eq. (4) such that an integral equation is obtained:

$$ u(t,x) = u(a,x) - \kappa\, {}_aI_t^{\alpha-\beta}\left[u - u(a,x)\right] - \frac{1}{\mathrm{R}}\, {}_aI_t^{\alpha} uu_x + K\, {}_0I_t^{\alpha} u_{xx} \tag{5} $$

Using finite difference method, replace the second order partial derivative $u_{xx}$ by:

$$ u_{xx} = \frac{u_{i+1} - 2u_i + u_{i-1}}{h^2}, \quad 1 \le i \le N-1 $$

where $h = 1/N$, $u_i = u_i(t) = u(t, ih)$, and $u_i(0) = \sin(ih)$.

Now we can obtain an integral system:

$$ u_i = u_i(a) + u_i(a)\frac{\kappa(t-a)^{\alpha-\beta}}{\Gamma(1+\alpha-\beta)} - \kappa\, {}_aI_t^{\alpha-\beta} u_i + \frac{K}{h^2}\, {}_aI_t^{\alpha}(u_{i+1} - 2u_i + u_{i-1}) - $$
$$ - \frac{1}{\mathrm{R}h^2}\, {}_aI_t^{\alpha} u_i(u_i - u_{i-1}) $$

According to the successive iteration method, we can have an analytical iteration solution:

$$u_{i,j}=u_{i,0}(a)+u_{i,0}(a)\frac{\kappa(t-a)^{\alpha-\beta}}{\Gamma(1+\alpha-\beta)}-\kappa\,{}_aI_t^{\alpha-\beta}u_{i,j-1}+$$
$$+\frac{K}{h^2}\,{}_aI_t^{\alpha}(u_{i+1,j-1}-2u_{i,j-1}+u_{i-1,j-1})-\frac{1}{\mathrm{R}h^2}\,{}_aI_t^{\alpha}u_{i,j-1}(u_{i,j-1}-u_{i-1,j-1}),\quad 1\le j$$

For simplicity, we consider:

$$\alpha=2\beta=2\lambda$$

In order to find the solution, the main problem is to deal with the non-linear terms. Here we adopt new Adomian polynomials [17-19] of the *m*-variable is calculated by:

$$A_n=\frac{1}{n}\sum_{i=1}^{N-1}\sum_{k=0}^{n-1}(k+1)u_{i,k+1}\frac{\partial A_{n-1-k}}{\partial u_{i,0}}$$

Using a Taylor expansion of $u_i$ as:

$$u_i=\sum_{n=0}^{\infty}u_{i,n},u_{i,n}=\sum_{j=0}^{n}c_{i,j}(t-t_0)^{j\lambda} \tag{6}$$

we can successively have:

$$n=0,\quad c_{i,0}=u_0(a)$$
$$n=1,\quad c_{i,0}=0$$
$$n=j,\quad c_{i,j}=-K\frac{\Gamma[1+(j-1)\lambda]}{\Gamma(1+j\lambda)}c_{i,j-1}+\frac{1}{\mathrm{R}h^2}\frac{\Gamma[1+(j-2)\lambda]}{\Gamma(1+j\lambda)}(c_{i+1,j-2}-2c_{i,j-2}+c_{i-1,j-2})-$$
$$-\frac{1}{h}\frac{1}{\Gamma(1+n\lambda)}A_{i,j-2},\quad 2\le j$$

Substituting the results into eq. (6), we can obtain the analytical solution $u_i\approx\phi_n(t,t_0,c_{i,0})$. For $t=t_0=a$, we have $u_i(t_0)=u_i(a)=c_{i,0}$. According to the multi-steps idea in [17], let $t_1=t_0+h$. Obviously, we derive $u_i(t_1)=\phi_n(t_1,t_0,c_{i,0})$ and successively other numerical values $u_i(t_j)$, $t_j=t_0+jh$, $2\le j$.

Let $K=0.1$, $n=10$ and $N=20$. We can vary the fractional orders and plot the solutions in figs.1-3.

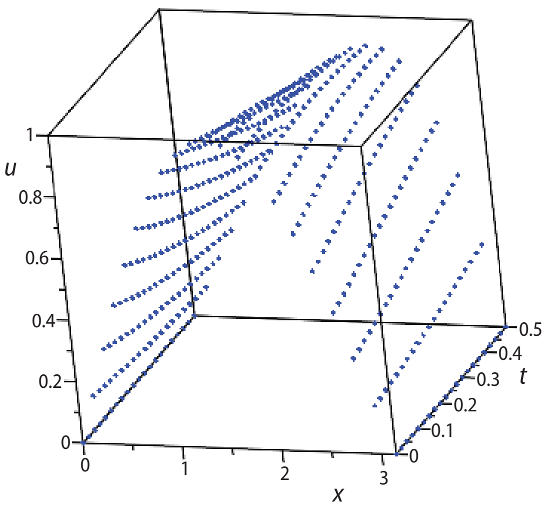


**Figure 1. Diffusion behaviors for $\alpha=0.9,\beta=45,K=0.1$, and $\kappa=1.8$**

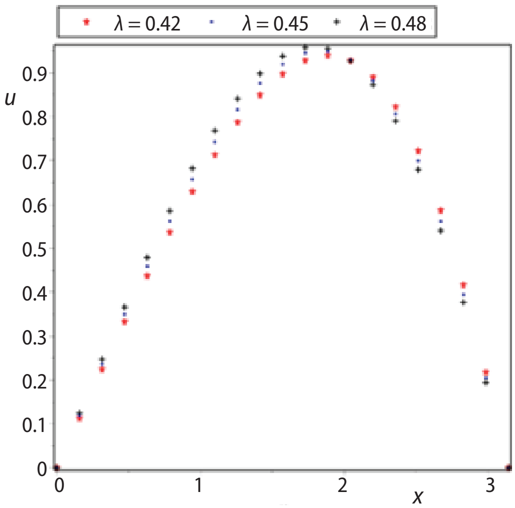


**Figure 2. Diffusion behaviors for different fractional orders at $t = 0.25$**

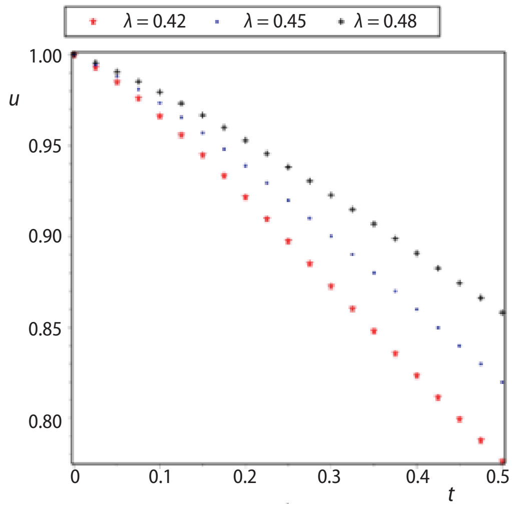


**Figure 3. Diffusion behaviors for different orders at $x = \pi/2$**

## Conclusions

This paper numerically investigates fractional non-linear diffusion equation with two time fractional terms. New Adomian polynomials are adopted to derive analytical solutions. Then numerical formulae are obtained with the help of Taylor series of fractional order. The numerical solutions for diffusion behaviors are illustrated which show a perspective of the method.

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## Nomenclature

$a$ – initial time, [s]
${}^{C}_{a}D^{\nu}_{t}$ – Caputo derivative, [–]
${}_{a}I_{t}^{-\nu}$ – fractional integral, [–]
$j$ – integer, [–]
$K$ – diffusion coefficient, [m$^2$s$^{-1}$]
$N$ – integer set, [–]
$n$ – integer, [–]
R – universal gas constant, [J mol$^{-1}$ K$^{-1}$]
$t$ – time, [s]
$u$ – concentration, [mol cm$^{-3}$]
$x$ – displacement, [cm]

*Greek symbols*

$\alpha$, $\beta$ – fractional order, [–]
$\Gamma$ – gamma function, [–]

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