

Research Article

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Analysis of a New Fractional Model for Damped Burgers' Equation

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Abstract: In this article, we present a fractional model of the damped Burgers' equation associated with the Caputo-Fabrizio fractional derivative. The numerical solution is derived by using the concept of an iterative method. The stability of the applied method is proved by employing the postulate of fixed point. To demonstrate the effectiveness of the used fractional derivative and the iterative method, numerical results are given for distinct values of the order of the fractional derivative.

Keywords: Time-fractional damped Burgers' equation; Nonlinear equation; Caputo-Fabrizio fractional derivative; Iterative method; Fixed-point theorem

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1 Introduction, Motivation and Preliminaries

Fractional calculus has been gaining more and more importance due to its extensive uses in the multiple aspects of science and engineering. Fractional derivatives and fractional integrals are key topics in fractional calculus and recently many related studies have been published by

researchers and scientists [1–8]. Tarasov [9] investigated the 3 D lattice equations pertaining to long-range interactions of Grünwald-Letnikov kind for fractional extension of gradient elasticity. Choudhary *et al.* [10] studied the fractional order differential equations occurring in fluid dynamics. Bulut *et al.* [11] reported the analytical study of differential equations of arbitrary order. Razminia *et al.* [12] analyzed the fractional diffusivity equation considering wellbore storage and skin effects. Atangana and Koca [13] studied the new fractional Baggs and Freedman models. Atangana [14] presented a fractional model of a nonlinear Fisher's reaction-diffusion equation. Singh *et al.* [15] discussed a fractional biological populations model. Singh *et al.* [16] reported the solution of fractional reaction-diffusion equations. Baleanu *et al.* [17] studied the fractional finite difference inclusion. In a recent work, Atangana *et al.* [18] analyzed the fractional Hunter-Saxton equation. Sequentially, Kurt *et al.* [19] derived the solution of Burgers' equation, Tasbozan *et al.* [20] presented new solutions for conformable fractional Boussinesq and combined KdV-mKdV equations, Atangana and Baleanu [21] introduced a novel fractional derivative having nonlocal and non-singular kernel, Alsaedi *et al.* [22] studied the coupled systems of time-fractional differential problems, Coronel-Escamilla *et al.* [23] investigated the formulation of Euler-Lagrange and Hamilton equations, Gomez-Aguilar *et al.* [24] obtained the analytical solutions of the electrical RLC circuit, Gomez-Aguilar *et al.* [25] examined a fractional Lienard type model of a pipeline, Doungmo Goufo [26] studied the Korteweg-de Vries-Burgers equation involving the Caputo-Fabrizio fractional derivative without singular kernel and many others. Consequently, numerous distinct definitions of fractional derivatives and fractional integrals have been used in literature for example the Riemann-Liouville definition and the Caputo definition. In a recent work, Caputo and Fabrizio proposed a novel derivative of arbitrary order without singular kernel [1, 2]. This new Caputo is an improvement over the old version because the full outcome of the memory can be predicted. In this paper, it is asserted that the Caputo-Fabrizio fractional derivative has additional stimulus effects in comparison to the older derivative.


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Definition 1. Assume that $\phi \in H^1(a, b)$, $b > a$, $\beta \in [0, 1]$ then the new fractional derivative discovered by Caputo and Fabrizio is explained and presented as:

$$D_t^\beta [\phi(t)] = \frac{M(\beta)}{1-\beta} \int_a^t \phi'(x) \exp \left[-\beta \frac{t-x}{1-\beta} \right] dx, \quad (1)$$

In the above equation $M(\beta)$ denotes the normalization function, which satisfies the property $M(0) = M(1) = 1$ [1]. The derivative can be reformulated as given below when $\phi \notin H^1(a, b)$

$$D_t^\beta [\phi(t)] = \frac{\beta M(\beta)}{1-\beta} \int_a^t (\phi(t) - \phi(x)) \exp \left[-\beta \frac{t-x}{1-\beta} \right] dx. \quad (2)$$

The Eq. (2) takes the below form

$$D_t^\beta [\phi(t)] = \frac{N(\sigma)}{\sigma} \int_a^t \phi'(x) \exp \left[-\frac{t-x}{\sigma} \right] dx, \quad N(0) = N(\infty) = 1, \quad (3)$$

if $\sigma = \frac{1-\beta}{\beta} \in [0, \infty)$, $\beta = \frac{1}{1+\sigma} \in [0, 1]$ as stated by the authors [14].

Moreover,

$$\lim_{\sigma \rightarrow 0} \frac{1}{\sigma} \exp \left[-\frac{t-x}{\sigma} \right] = \delta(x-t). \quad (4)$$

The authors of [2] proposed the associated integral of the new Caputo derivative of arbitrary order as described in the following manner.

Definition 2. The fractional order integral of the function $\phi(t)$ of order β , $0 < \beta < 1$ is expressed as [2]

$$I_\beta^t [\phi(t)] = \frac{2(1-\beta)}{(2-\beta)M(\beta)} \phi(t) + \frac{2\beta}{(2-\beta)M(\beta)} \int_0^t \phi(s) ds, \quad t \geq 0. \quad (5)$$

Eq. (5) further yields

$$\frac{2(1-\beta)}{(2-\beta)M(\beta)} + \frac{2\beta}{(2-\beta)M(\beta)} = 1. \quad (6)$$

The above result gives an explicit formula for

$$M(\beta) = \frac{2}{(2-\beta)}, \quad 0 \leq \beta \leq 1. \quad (7)$$

Further, the authors of [2] suggested that the new fractional derivative of order $0 < \beta < 1$ defined by Caputo and

Fabrizio can be redefined in view of the above discussed relation as

$$D_t^\beta [\phi(t)] = \frac{1}{1-\beta} \int_a^t \phi'(x) \exp \left[-\beta \frac{t-x}{1-\beta} \right] dx. \quad (8)$$

Here we give some important theorems and important properties of new fractional derivative that will be employed in the present article.

Theorem 1. [1] Consider the function $\phi(t)$ given by $\phi^s(a) = 0$, $s = 1, 2, \dots, n$ then for the new Caputo-Fabrizio derivative of fractional order we have,

$${}_0^{CF} D_t^\beta \left({}_0^{CF} D_t^n [\phi(t)] \right) = {}_0^{CF} D_t^n \left({}_0^{CF} D_t^\beta [\phi(t)] \right).$$

For proof see [1].

Theorem 2. [14] The fractional derivative defined by Caputo-Fabrizio with the following ordinary differential equation

$${}_0^{CF} D_t^\beta [\phi(t)] = \phi(t), \quad \phi(0) \neq 0, \quad (9)$$

gives a non-trivial solution for $0 < \beta < 1$. For proof see [14].

Theorem 3. [1] The Laplace transform of the new fractional derivative of a function $\phi(t)$ defined by Caputo and Fabrizio is expressed as

$$L \left({}_0^{CF} D_t^\beta [\phi(t)] \right) = M(\beta) \frac{s\bar{\phi}(s) - \phi(0)}{s + \beta(1-s)},$$

where $\bar{\phi}(s)$ indicates the Laplace transform of the function $\phi(t)$. For proof of this theorem see [1].

2 Fractional model of the damped Bergers' equation with Caputo-Fabrizio derivative

The damped Bergers' equation arises in fluid mechanics, gas dynamics, traffic flow and nonlinear acoustics among other fields. Along with the initial condition, it is presented as

$$\frac{\partial u}{\partial t} + uu_x - u_{xx} + \lambda u = 0, \quad (10)$$

$$u(x, 0) = \psi(x) \quad (11)$$

where $u(x, t)$ is a function of two variables x and t indicating the displacement; x is a space variable, t is a time variable and λ is a constant. The damped Burgers' equations has been studied by several researchers such as Babolian and Saeidian [27], Fakhari et al. [28], Inc [29], Song and Zhang [30], Peng and Chen [31] and others. The fractional generalization of the damped Bergers' equations

is investigated by Esen *et al.* [32]. A recent study carried out by Hristov [33] showed that the Cattaneo constitutive equation with Jeffrey's fading memory naturally results in a heat conduction equation having a relaxation term interpreted by the Caputo-Fabrizio time derivative of fractional order. This paved the way to see the physical background of the newly defined Caputo-Fabrizio derivative having non-singular kernel in scientific fields. Therefore, we replace the first order time-derivative by the newly introduced derivative of fractional order in Eq. (10), this converts it into the nonlinear fractional damped Burgers' equation written as

$${}_0^CF_t^\beta u(x, t) = -u(x, t) \frac{\partial u(x, t)}{\partial x} + \frac{\partial^2 u(x, t)}{\partial x^2} - \lambda u(x, t), \quad (12)$$

with the initial condition

$$u(x, 0) = \psi(x). \quad (13)$$

3 Solution of the fractional model of the nonlinear damped Burgers' equation by an iterative scheme

In this section, we find the solution of Eq. (12), with the help of an iterative approach. Employing the Laplace transform (LT) on Eq. (12) yields

$$M(\beta) \frac{s\bar{u}(x, s) - u(x, 0)}{s + \beta(1 - s)} = L \left[-u \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} - \lambda u \right]. \quad (14)$$

On simplifying, we get

$$\bar{u}(x, s) = \frac{u(x, 0)}{s} + \frac{s + \beta(1 - s)}{M(\beta)s} L \left[-u \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} - \lambda u \right]. \quad (15)$$

Next, employing the inverse LT on Eq. (15), it yields

$$u(x, t) = u(x, 0) + L^{-1} \left[\frac{s + \beta(1 - s)}{M(\beta)s} L \left[-u \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} - \lambda u \right] \right]. \quad (16)$$

Then, the recursive formula is expressed as

$$u_0(x, t) = u(x, 0), \quad (17)$$

and

$$u_{n+1}(x, t) = u_n(x, t) + L^{-1} \left[\frac{s + \beta(1 - s)}{M(\beta)s} L \left[-u_n \frac{\partial u_n}{\partial x} + \frac{\partial^2 u_n}{\partial x^2} - \lambda u_n \right] \right]. \quad (18)$$

Thus, the solution of Eq. (12) is presented in the following manner

$$u(x, t) = \lim_{n \rightarrow \infty} u_n(x, t). \quad (19)$$

4 Application of fixed-point theorem for stability analysis of the iterative scheme

Let H be a self-map of X where $(X, \|\cdot\|)$ stands for a Banach space and a particular recursive procedure assumed by $y_{n+1} = f(H, y_n)$. Suppose at least one element and that converges to a point $p \in F(H)$ where $F(H)$ is the fixed-point set of H . Consider the relation $e_n = \|x_{n+1} - f(H, x_n)\|$ where $\{x_n\} \subseteq X$ is to be assumed. The iteration scheme $y_{n+1} = f(H, y_n)$ is called H -stable if $\lim_{n \rightarrow \infty} e^n = 0$ implies that $\lim_{n \rightarrow \infty} x^n = p$. It must be assumed that the sequence $\{x_n\}$ has an upper boundary without any loss of generality otherwise the possibility of convergence cannot be expected. If all these restrictions are fulfilled for $y_{n+1} = f(H, y_n)$ this is called Picard's iteration, hence the iteration is known as H -stable.

Theorem 4. [34] Suppose that H is a self-map of X where $(X, \|\cdot\|)$ is a Banach space verifying

$$\|H_x - H_y\| \leq C \|x - H_x\| + c \|x - y\|,$$

For all values of x, y in X , where the values of C and c are $0 \leq C$ and $0 \leq c \leq 1$. Assume that H satisfies Picard's H -stability.

For the fractional model of the damped Burgers' equation, consider the following succession as

$$u_{n+1}(x, t) = u_n(x, t) + L^{-1} \left[\frac{s + \beta(1 - s)}{M(\beta)s} L \left[-\tilde{u}_n \frac{\partial u_n}{\partial x} + \frac{\partial^2 u_n}{\partial x^2} - \lambda u_n \right] \right], \quad (20)$$

where \tilde{u}_n indicates a restricted variation satisfying $\delta \tilde{u}_n = 0$ and $\frac{s + \beta(1 - s)}{M(\beta)s}$ denotes the fractional Lagrange's multiplier. Next, we establish the following result.

Theorem 5. Suppose T be a self-map described in the following way

$$T(u_n(x, t)) = u_{n+1}(x, t) = u_n(x, t) + L^{-1} \left[\frac{s + \beta(1 - s)}{M(\beta)s} L \left[-u_n \frac{\partial u_n}{\partial x} + \frac{\partial^2 u_n}{\partial x^2} - \lambda u_n \right] \right], \quad (21)$$

then the iteration is T -Stable in $L^1(a, b)$ if the following condition holds

$$\left[1 + \frac{\beta_1}{2} (A + B) f(\gamma) + \beta_2^2 g(\gamma) + \lambda h(\gamma) \right] < 1,$$

where f, g and h are functions arising from $L^{-1} \left[\frac{s+\beta(1-s)}{M(\beta)s} L \right]$.

Proof. Firstly we show that T has a fixed point. In order to derive this result, we assess the succession for $(n, m) \in N \times N$.

$$\begin{aligned}
 & T(u_n(x, t)) - T(u_m(x, t)) = u_n(x, t) - u_m(x, t) \\
 & + L^{-1} \left[\frac{s + \beta(1-s)}{M(\beta)s} L \left[- \left(u_n \frac{\partial u_n}{\partial x} - u_m \frac{\partial u_m}{\partial x} \right) \right. \right. \\
 & \quad \left. \left. + \frac{\partial^2}{\partial x^2} (u_n - u_m) - \lambda(u_n - u_m) \right] \right]. \tag{22} \\
 & = u_n(x, t) - u_m(x, t) + L^{-1} \left[\frac{s + \beta(1-s)}{M(\beta)s} \right. \\
 & \quad \cdot L \left[-\frac{1}{2} \left(\frac{\partial(u_n^2)}{\partial x} - \frac{\partial(u_m^2)}{\partial x} \right) \right. \\
 & \quad \left. \left. + \frac{\partial^2}{\partial x^2} (u_n - u_m) - \lambda(u_n - u_m) \right] \right]. \tag{23}
 \end{aligned}$$

Now applying the norm on both sides of Eq. (23) and without loss of generality, we get

$$\begin{aligned}
 & \|T(u_n(x, t)) - T(u_m(x, t))\| \\
 & \leq \|u_n(x, t) - u_m(x, t)\| + \left\| L^{-1} \left[\frac{s + \beta(1-s)}{M(\beta)s} \right. \right. \\
 & \quad \cdot L \left[-\frac{1}{2} \left(\frac{\partial(u_n^2)}{\partial x} - \frac{\partial(u_m^2)}{\partial x} \right) \right. \\
 & \quad \left. \left. + \frac{\partial^2}{\partial x^2} (u_n - u_m) - \lambda(u_n - u_m) \right] \right\|. \tag{24}
 \end{aligned}$$

Using the property of the norm in particular, the triangular inequality, the R.H.S. of equation (24) is converted to

$$\begin{aligned}
 & \|T(u_n(x, t)) - T(u_m(x, t))\| \leq \|u_n(x, t) - u_m(x, t)\| \\
 & + L^{-1} \left[\frac{s + \beta(1-s)}{M(\beta)s} L \left(\left\| -\frac{1}{2} \left(\frac{\partial(u_n^2)}{\partial x} - \frac{\partial(u_m^2)}{\partial x} \right) \right\| \right) \right] \\
 & + L^{-1} \left[\frac{s + \beta(1-s)}{M(\beta)s} L \left(\left\| \frac{\partial^2}{\partial x^2} (u_n - u_m) \right\| \right) \right] \\
 & + L^{-1} \left[\frac{s + \beta(1-s)}{M(\beta)s} L \left(\left\| -\lambda(u_n - u_m) \right\| \right) \right] \\
 & \leq \|u_n(x, t) - u_m(x, t)\| \\
 & + L^{-1} \left[\frac{s + \beta(1-s)}{M(\beta)s} L \left(\left\| \frac{1}{2} \beta_1 (u_n^2 - u_m^2) \right\| \right) \right]
 \end{aligned}$$

$$\begin{aligned}
 & + L^{-1} \left[\frac{s + \beta(1-s)}{M(\beta)s} L \left(\left\| \beta_2^2 (u_n - u_m) \right\| \right) \right] \\
 & + L^{-1} \left[\frac{s + \beta(1-s)}{M(\beta)s} L \left(\left\| -\lambda(u_n - u_m) \right\| \right) \right] \tag{25}
 \end{aligned}$$

Since $u_n(x, t), u_m(x, t)$ are bounded functions, two distinct constants can be obtained $A, B > 0$ such that $\forall t$

$$\|u_n(x, t)\| \leq A, \|u_m(x, t)\| \leq B. \tag{26}$$

Next, using Eq. (26) in Eq. (25), we arrive at the following result

$$\begin{aligned}
 & \|T(u_n(x, t)) - T(u_m(x, t))\| \\
 & \leq \left[1 + \left(\frac{1}{2} \beta_1 (A + B) \right) f(\gamma) + \beta_2^2 g(\gamma) + \lambda h(\gamma) \right] \\
 & \cdot \|u_n(x, t) - u_m(x, t)\|, \tag{27}
 \end{aligned}$$

where f, g and h are functions of $L^{-1} \left[\frac{s+\beta(1-s)}{M(\beta)s} L \right]$.

For $\left[1 + \left(\frac{1}{2} \beta_1 (A + B) \right) f(\gamma) + \beta_2^2 g(\gamma) + \lambda h(\gamma) \right] < 1$.

Therefore, the nonlinear T -self mapping attains a fixed point. Now, we verify that T fulfills the conditions in Theorem 4. Suppose Eq. (27) be held, hence substituting

$$c = 0, C = 1 + \left(\frac{1}{2} \beta_1 (A + B) \right) f(\gamma) + \beta_2^2 g(\gamma) + \lambda h(\gamma). \tag{28}$$

Then Eq. (28) demonstrates that for the nonlinear mapping T the inequality of Theorem 4 holds. Thus, considering the nonlinear mapping T along with the satisfaction of all the conditions in Theorem 4, T is Picard's T -stable. Hence, we proved the Theorem 5.

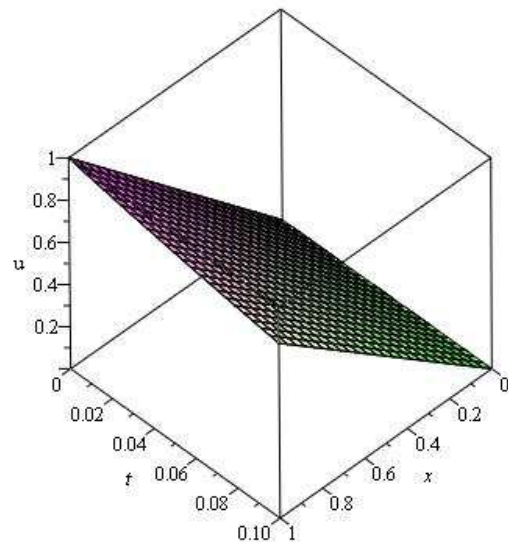


Figure 1: The plots of the solution $u(x, t)$ w.r. to space x and time t are found at $\beta = 1$ and $\lambda = 1$.

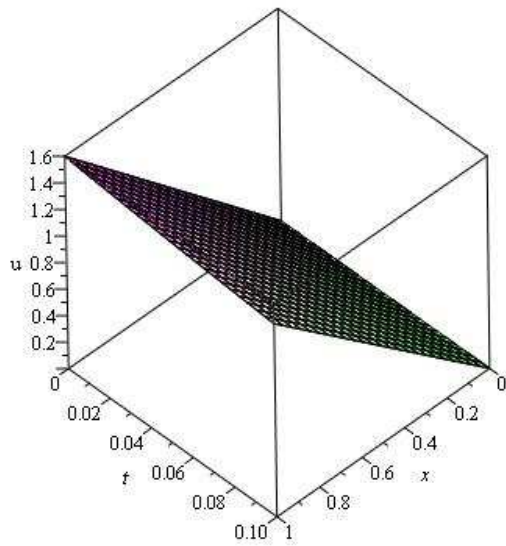


Figure 2: The plots of the solution $u(x, t)$ w.r. to space x and time t are derived if $\beta = 0.85$ and $\lambda = 1$.

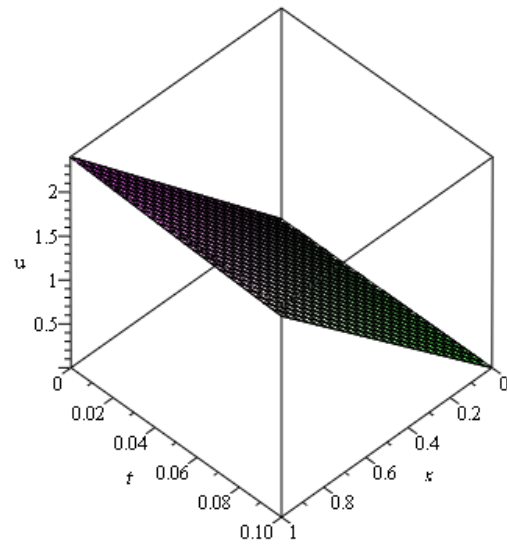


Figure 4: The plots of the solution $u(x, t)$ w.r. to space x and time t are derived at $\beta = 0.65$ and $\lambda = 1$.

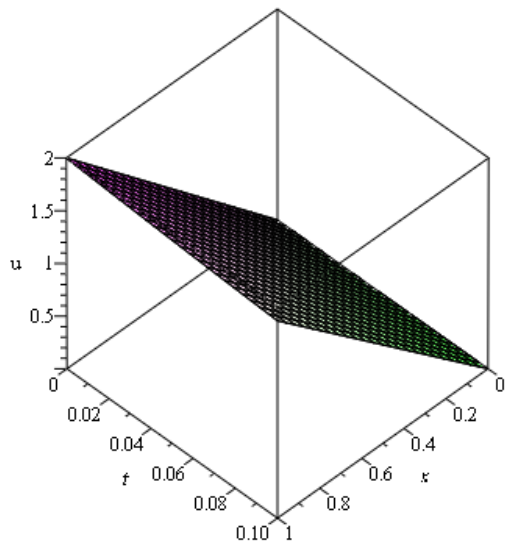


Figure 3: The plots of the solution $u(x, t)$ w.r. to space x and time t are found at $\beta = 0.75$ and $\lambda = 1$.

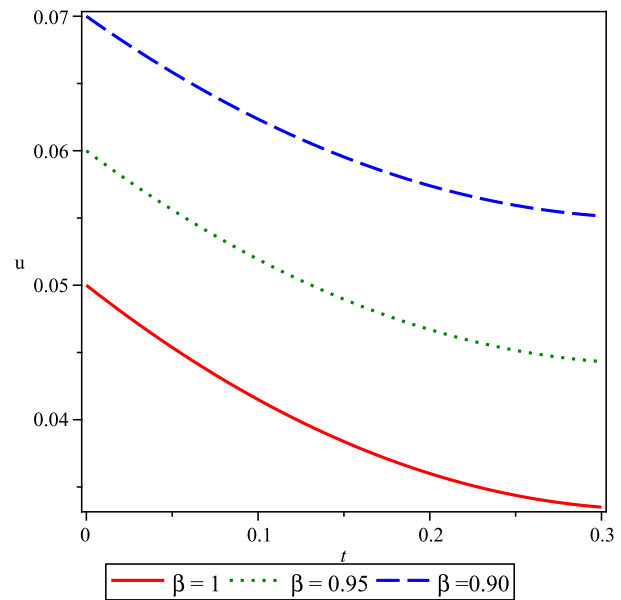


Figure 5: The plots of the solution $u(x, t)$ vs. t for distinct values of β at $x = 0.05$ and $\lambda = 1$.

5 Numerical simulations

In this section, we investigate the numerical simulations of the special solution of Eq. (12) as function of time and space for $u(x, 0) = \lambda x$, $\lambda = 1$ and distinct values of β . Figs. 1–5 represent simulations of the solution. The graphical representations demonstrate that the model depends

notably on the fractional order. Figures 1-4 show the clear difference at $\beta = 1$, $\beta = 0.85$, $\beta = 0.75$ and $\beta = 0.65$. The model narrates a new characteristic at $\beta = 0.85$, $\beta = 0.75$ and $\beta = 0.65$ that was invisible when modeling at $\beta = 1$. From Figs. 1-4 we can see that the displacement $u(x, t)$ increases with increasing the value of x whereas the displacement $u(x, t)$ decreases with increasing the value of

t. Fig. 5 describes the displacement $u(x, t)$ for distinct values of β . It can be noticed from Figs. 1–5, that the order of derivative significantly affects the displacement.

6 Conclusions

The Caputo-Fabrizio fractional derivative has many important qualities. For instance, at distinct scales it can illustrate matter diversities and configurations, where in local theories these clearly cannot be controlled. We exerted this new derivative to adapt the damped Burgers' equation. By using an iterative scheme we have derived the solution of the equation. In order to show the stability of the iterative technique we applied the theory of T -stable mapping and the postulate of fixed-point. We presented some interesting numerical simulations for distinct values of β and $\lambda = 1$. The outcomes demonstrate that the new fractional order derivative can be used to model various scientific and engineering problems.

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References

- [1] Caputo M., Fabrizio M., A new definition of fractional derivative without singular kernel, *Progr. Fract. Differ. Appl.*, 2015, 1, 73-85.
- [2] Losada J., Nieto J.J., Properties of the new fractional derivative without singular kernel, *Progr. Fract. Differ. Appl.*, 2015, 1, 87-92.
- [3] Odibat Z.M., Momani, S., Application of variational iteration method to nonlinear differential equation of fractional order, *Int. J. Nonlinear Sci. Numer. Simul.*, 2006, 7(1), 27-34.
- [4] Kilbas A.A., Srivastava H.M., Trujillo J.J., *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, The Netherlands, 2006.
- [5] Caputo M., Linear models of dissipation whose Q is almost frequency independent, part II, *Geophys. J. Int.*, 1967, 13(5), 529-539.
- [6] Podlubny L., *Fractional Differential Equations*, Academic Press, London, 1999.
- [7] Magin R.L., *Fractional Calculus in Bioengineering*, Begell House, Connecticut, 2006.
- [8] Baleanu D., Gvenc Z.B., Machado J.A.T.(Ed.), *New Trends in Nanotechnology and Fractional Calculus Applications*, Springer Dordrecht Heidelberg, London New York, 2010.
- [9] Tarasov V.E., Three-dimensional lattice models with long-range interactions of Grünwald-Letnikov type for fractional generalization of gradient elasticity, *Meccanica*, 2016, 51(1), 125-138.
- [10] Choudhary A., Kumar D., Singh J., Analytical solution of fractional differential equations arising in fluid mechanics by using sumudu transform method, *Nonlinear Eng.*, 2014, 3(3), 133-139.
- [11] Bulut, H., Baskonus H.M., Belgacem F.B.M., The analytical solutions of some fractional ordinary differential equations by sumudu transform method, *Abst. Appl. Anal.*, 2013, Article ID 203875, 6 pages.
- [12] Razminia K., Razminia A., Machado J.A.T., Analytical solution of fractional order diffusivity equation with wellbore storage and skin effects, *J. Comput. Nonlinear Dyn.*, 2016, 11(1), doi: 10.1115/1.4030534.
- [13] Atangana A., Koca I., On the new fractional derivative and application to nonlinear Baggs and Freedman Model, *J. Nonlinear Sci. Appl.*, 2016, 9, 2467-2480.
- [14] Atangana A., On the new fractional derivative and application to nonlinear Fisher's reaction- diffusion equation, *Appl. Math. Comput.*, 2016, 273 948-956.
- [15] Singh J., Kumar D., Kilichman A., Numerical solutions of nonlinear fractional partial differential equations arising in spatial diffusion of biological populations, *Abst. Appl. Anal.*, 2014, Article ID 535793, 12 pages.
- [16] Singh J., Kumar D., Rathore S., On the solutions of fractional reaction-diffusion equations, *Le Matematiche*, 2013, 68(1), 23-32.
- [17] Baleanu D., Rezapour S., Salehi S., A fractional finite difference inclusion, *J. Comput. Anal. Appl.*, 2016, 20(5), 834-842.
- [18] Atangana A., Baleanu D., Alsaedi A., Analysis of time-fractional Hunter-Saxton equation: a model of neumatic liquid crystal, *Open Phys.*, 2016, 14, 145-149.
- [19] Kurt A., Çenesiz Y., Tasbozan O., On the Solution of Burgers' Equation with the new fractional derivative, *Open Phys.*, 2015, 13 (1), 355-360.
- [20] Tasbozan O., Çenesiz Y., Kurt A., New solutions for conformable fractional Boussinesq and combined KdV-mKdV equations using Jacobi elliptic function expansion method, *Eur. Phys. J. Plus*, 2016 131 (7), 244.
- [21] Atangana A., Baleanu D., New fractional derivatives with non-local and non-singular kernel, Theory and application to heat transfer model, *Thermal Science*, 2016, 20(2), 763-769.
- [22] Alsaedi A., Baleanu D., Etemad S., Rezapour, S., On coupled systems of time-fractional differential problems by using a new fractional derivative, *Journal of Function Spaces*, 2016, Article Number: 4626940, doi.org/10.1155/2016/4626940
- [23] Coronel-Escamilla A., Gomez-Aguilar J.F., Baleanu D., Escobar-Jiménez R.F., Olivares-Peregrino V.H., Abundez-Pliego A., Formulation of Euler-Lagrange and Hamilton equations involving fractional operators with regular kernel, *Adv. Difference Equ.*, 2016, Article Number: 283, doi: 10.1186/s13662-016-1001-5.
- [24] Gomez-Aguilar J.F., Morales-Delgado V.F., Taneco-Hernandez M.A., Baleanu D., Escobar-Jiménez, R.F., Al Qurashi M.M., Analytical solutions of the electrical RLC circuit via Liouville-Caputo operators with local and non-local kernels, *Entropy*, 2016, 18(8), Article Number: 402.
- [25] Gomez-Aguilar J. F., Torres L., Yopez-Martinez H., Baleanu D., Reyes J. M., Sosa, I. O., Fractional Lienard type model of a pipeline within the fractional derivative without singular kernel. 2016, *Adv. Difference Equ.*, Article Number: 173, doi:10.1186/s13662-016-0908-1.

- [26] Doungmo Goufo E.F., Application of the Caputo-Fabrizio fractional derivative without singular kernel to Korteweg-de Vries-Burgers equation, *Mathematical Modelling and Analysis*, 2016, 21(2), 188-198.
- [27] Babolian E., Saeidian J., Analytic approximate solutions to Burgers, Fisher, Huxley equations and two combined forms of these equations, *Commun. Nonlinear. Sci. Numer. Simulat.*, 2009, 14, 1984-1992.
- [28] Fakhari A., Domairry G., Ebrahimpour, Approximate explicit solutions of nonlinear BBMB equations by homotopy analysis method and comparison with the exact solution, *Phys. Lett. A*, 2007, 368, 64-68.
- [29] Inc M., On numerical solution of Burgers equation by homotopy analysis method, *Phys. Lett. A*, 2008, 372, 356-360.
- [30] Song L., Zhang H.Q., Application of homotopy analysis method to fractional KdV-Burgers-Kuramoto equation, *Phys. Lett. A*, 2007, 367, 88-94.
- [31] Peng Y., Chen W., A new similarity solution of the Burgers equation with linear damping, *Czech. J. Phys.*, 2008, 56, 317-428.
- [32] Esen A., Yagmurlu N.M., Tasbozan O., Approximate Analytical Solution to Time- Fractional Damped Burger and Cahn- Allen Equations, *Appl. Math. Inf. Sci.*, 2013, 7(5), 1951-1956.
- [33] Hristov J., Transient heat diffusion with a non-singular fading memory: from the Cattaneo constitutive equation with Jeffrey's kernel to the Caputo-Fabrizio time-fractional derivative, *Thermal Science*, 2016, 20, 757-762.
- [34] Qing Y., Rhoades, B.E., T-stability of Picard iteration in metric spaces, *Fixed Point Theory and Applications*, 2008, Article ID 418971, 4 pages.