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# Exact solutions of fractional mBBM equation and coupled system of fractional Boussinesq-Burgers

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ABSTRACT

The new exact solutions of nonlinear fractional partial differential equations (FPDEs) are established by adopting first integral method (FIM). The Riemann-Liouville (R-L) derivative and the local conformable derivative definitions are used to deal with the fractional order derivatives. The proposed method is applied to get exact solutions for space-time fractional modified Benjamin-Bona-Mahony (mBBM) equation and coupled time-fractional Boussinesq-Burgers equation. The suggested technique is easily applicable and effectual which can be implemented successfully to obtain the solutions for different types of nonlinear FPDEs.

## 1. Introduction

Nowadays the use of nonlinear equations is extensive as the nonlinearity in the world exists everywhere. Nonlinear FPDEs is a special case of nonlinear equations that have been attracted great interest due to their frequent appearance in many applications such as in biology, physics, chemistry, electromagnetic, polymeric materials, neutron point kinetic model, control and vibration, image and signal processing, acoustics and fluid dynamics [1–4]. Researchers devoted significant efforts for the study of numerical as well as explicit solutions of nonlinear FPDEs due to their potential applications [5]. To attain exact and approximate solutions of FPDEs various powerful techniques are presented such as hyperbolic function method [6], extended hyperbolic tangent method [7,8], the sub-equation method [9], homotopy perturbation technique [10,11], exponential rational function method [12] and homotopy analysis method [13].

Feng has proposed a very effectual and reliable method to investigate travelling wave solutions for nonlinear partial differential equations also named as Feng's first integral method (FIM) [14]. The FIM is eminently used by many researchers to interpret results for various kinds of nonlinear problems [15–19]. The basic scheme of FIM is to establish a first integral of explicit form with polynomial coefficients by applying division theorem [14]. In contrast with other methods, the proposed technique has many advantages, such as, it avoids complex and tedious computations, and provides us exact and explicit solutions. Based on these facts, we implemented the proposed method to some nonlinear FPDEs to obtain their exact solutions using two definitions of fractional derivative, namely, the R-L derivative [2,20] and the conformable derivative [21].

The paper is arranged as follows. Section 2 comprises of the basic definitions and properties for R-L and conformable derivative. Section 3 illustrates the main steps of FIM. In Section 4, the exact solutions of fractional mBBM equation and fractional Boussinesq-Burgers equation are given as an application of FIM. Finally, Section 5 consists of conclusions and remarks.

## 2. Preliminaries

### 2.1. Riemann-Liouville derivative

Riemann-Liouville introduce the following definition [2]:

**Definition 1.** Let there be a continuous function $g$ such that $g: R \to R, t \to g(t)$. Then its R-L derivative of fractional order $\alpha$ is expressed below:

$$D_a^\alpha = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x \frac{g(t)}{(x-t)^\alpha}dt, \quad 0 < \alpha < 1, \tag{1a}$$

$$D_a^\alpha = \frac{1}{\Gamma(m-\alpha)}\frac{d^m}{dx^m}\int_a^x \frac{g(t)}{(x-t)^{\alpha-m+1}}dt, \quad m-1 \leqslant \alpha \leqslant m, \ m \in \mathbb{N}. \tag{1b}$$

From the above definition (1), we have

$$D^\alpha t^m = \frac{\Gamma(1+m)}{\Gamma(1+m-\alpha)}t^{m-\alpha}, \quad m > -1, \ 0 < \alpha < 1. \tag{2}$$

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### 2.2. Conformable derivative

Recently, Khalil et al. presented a new simple definition of derivative of fractional order which is called conformable fractional derivative [21]. The definition depends just on the basic limit definition of the derivative, so first let us recall the basic limit definition.

Supppose $g$: $[0,\infty) \to R$ and $x > 0$. The definition for derivative of function $g$ at any point $x$ is $\frac{dg}{dx} = \lim_{\varepsilon\to 0}\frac{g(x+\varepsilon)-g(x)}{\varepsilon}$. By this definition, we have $\frac{dx^n}{dx} = nx^{n-1}$. Based on this definition, Khalil et al. presented the new definition for the fractional derivatives [21].

**Definition 2.** Let $g$: $[0,\infty) \to R$ be a function. Then its fractional conformable derivative which is of $\alpha$th order is as follows,

$$T_\alpha(g)(x) = \lim_{\varepsilon\to 0}\frac{g(x+\varepsilon x^{1-\alpha}) - g(x)}{\varepsilon}, \tag{3}$$

where $\alpha \in (0,1)$ and it holds for all $x > 0$. If the function $g$ is $\alpha$-differentiable in $(0,l)$, for $l > 0$ and further $\lim_{x\to 0^+} g^{(\alpha)}(x)$ exists, then the conformable derivative at 0 is defined as $g^{(\alpha)}(0) = \lim_{x\to 0^+} g^{(\alpha)}(x)$.

Also conformable integral of function $g$ is defined as:

$$I_\alpha^l(g)(x) = \int_l^x \frac{g(t)}{t^{1-\alpha}}dt, \tag{4}$$

where $l \geqslant 0$, and $\alpha \in (0,1]$. Here the integral in Eq. (4) represents usual Riemann improper integral.

In account of the definition in Eq. (3), Khalil et al. presented following theorem [21], which provides some useful properties satisfied by the conformable derivative.

**Theorem.** *Suppose the functions u and v are $\alpha$-differentiable at any point $x > 0$, for $\alpha \in (0,1]$. Then*

(1) $T_\alpha(au + bv) = aT_\alpha(u) + bT_\alpha(v)\ \forall\ a,b \in \mathbb{R}$.
(2) $T_\alpha(x^m) = mx^{m-\alpha}\ \forall\ m \in \mathbb{R}$.
(3) $T_\alpha(C) = 0\ \forall\ u(x) = C$ (Constant functions).
(4) $T_\alpha(uv) = uT_\alpha(v) + vT_\alpha(u)$.
(5) $T_\alpha(\frac{u}{v}) = \frac{vT_\alpha(u) - uT_\alpha(v)}{v^2}$.
(6) *Additionally, if the function u is differentiable, then* $T_\alpha(u)(x) = x^{1-\alpha}\frac{du}{dx}$.

This new definition has gained significant attention these days due to its simplicity, so a remarkable work has been done on it by many scientists. For instance, Abdeljawad used conformable derivative to express chain rule, integration by parts, exponential functions, Taylor power series expansion, Gronwall's inequality, and Laplace transform [22]. Conformable time-scale calculus is introduced by Benkhettoua et al. [23]. Many scientists used this new derivative in some physical applications due to its convenience, simplicity and usefulness [24–26]. Chung discussed conformable Newtonian mechanics using this new definition [27]. Hammad and Khalil interpret the results for the conformable heat equation [28].

## 3. First integral method

Here, a brief exposition of the FIM is presented:

**Step 1** First, we take into account a nonlinear FPDE of the following form:

$$F\left(\frac{\partial^\alpha u}{\partial t^\alpha},\frac{\partial^\alpha u}{\partial x_1^\alpha},\frac{\partial^\alpha u}{\partial x_2^\alpha},\ldots,\frac{\partial^\alpha u}{\partial x_m^\alpha},\frac{\partial^{2\alpha} u}{\partial t^{2\alpha}},\frac{\partial^{2\alpha} u}{\partial x_1^\alpha\partial x_1^\alpha},\frac{\partial^{2\alpha} u}{\partial x_2^\alpha\partial x_2^\alpha}\ldots\right) = 0. \tag{5}$$

**Step 2** Then, the following transformation is applied,

$$u(x_1,x_2,\ldots,x_m,t) = U(\xi). \tag{6}$$

In order to apply R-L derivative, we have

$$\xi = \frac{k_1x_1^\alpha + k_2x_2^\alpha + \ldots + k_mx_m^\alpha \pm ct^\alpha}{\Gamma(1+\alpha)}. \tag{7}$$

In order to use conformable derivative, we have

$$\xi = \frac{k_1x_1^\alpha + k_2x_2^\alpha + \cdots + k_mx_m^\alpha \pm ct^\alpha}{\alpha}. \tag{8}$$

Using these transformations given in Eq. (7) and Eq. (8), we reduce the FPDE into an integer order nonlinear ODE as follows:

$$H(U(\xi),U'(\xi),U''(\xi),\ldots) = 0, \tag{9}$$

where $U'(\xi) = \frac{dU(\xi)}{d\xi}$ and $\xi$ is a new transformed variable.

**Step 3** Afterwards, introducing some new independent variables, we get

$$U(\xi) = X(\xi),\quad U_\xi(\xi) = Y(\xi). \tag{10}$$

Then a new system of nonlinear ODE is generated which is given as follows:

$$\frac{\partial X}{\partial \xi} = Y(\xi),\quad \frac{\partial Y}{\partial \xi} = G(X(\xi),Y(\xi)). \tag{11}$$

**Step 4** According to the qualitative theory of ODEs, the general solutions of Eq. (11) can be directly obtained if one can find integrals to Eq. (11). Generally it is very difficult to obtain even one first integral, because there is no systematic or logical procedure to find first integrals for plane independent system. The Division theorem presents us the idea to find first integrals. One first integral to Eq. (11) is then obtained by applying the Division theorem, which reduces our nonlinear ODE to an integrable first order ODE. Finally, we obtain exact solutions of the problem after solving this system.

The division theorem is stated below which is defined in $\mathbb{C}$ for two variables.

**Division Theorem**: Assume there are two polynomials $P(x,y)$ and $Q(x,y)$ in complex domain $\mathbb{C}(x,y)$, such that $P(x,y)$ is an irreducible polynomial in $\mathbb{C}(x,y)$. If at all the zero points of $P(x,y)$, the polynomial $Q(x,y)$ vanishes, then a polynomial $R(x,y)$ exists in $\mathbb{C}(x,y)$ so that following equality holds

$$Q(x,y) = P(x,y)R(x,y). \tag{12}$$

## 4. Applications

In this section, FIM is applied to solve space-time fractional mBBM equation and time-fractional Boussinesq-Burgers equation. FIM is an alternative method other than homotopy perturbation method, Adomian decomposition method, and homotopy analysis method to obtain solutions of different types of FPDEs. FIM is better than other existing analytical and numerical techniques in different ways, for example, FIM reduces successive applications of integration for higher order problems as compared to homotopy perturbation technique and Adomian decomposition method. FIM has not any restrictive assumptions and expansion of small parameters like in perturbation techniques. Moreover, FIM is a direct and concise and provides explicit solutions of FPDEs. Therefore, FIM is a reliable and efficient method to establish solutions of non linear FPDEs arising in mathematical physics [19].

*4.1. Exact solutions of the space-time fractional mBBM equation*

Let us consider the mBBM equation fractional in space and time [29]

$$\frac{\partial^\alpha u}{\partial t^\alpha}+\frac{\partial^\alpha u}{\partial x^\alpha}-vu^2\frac{\partial^\alpha u}{\partial t^\alpha}+\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{\partial^\alpha}{\partial x^\alpha}\left(\frac{\partial^\alpha u}{\partial x^\alpha}\right)\right)=0,\quad x>0,\ t>0, \tag{13}$$

where $\alpha\in(0,1)$. Eq. (13) describes propagation of surface long waves in nonlinear dispersive channel. The model (c.f. Eq. (13)) has been solved using modified Kudryashov method and by generalized Kudryashov method [30,31].

Firstly, we apply R-L definition of fractional derivative. The following transformation is introduced:

$$\xi=\frac{kx^\alpha}{\Gamma(1+\alpha)}+\frac{ct^\alpha}{\Gamma(1+\alpha)},$$
$$u(\xi)=u(x,t), \tag{14}$$

where $\xi$ is the transformation variable and $k,c$ be the constants. As a result of this transformation, we obtain:

$$\frac{\partial^\alpha(.)}{\partial t^\alpha}=c\frac{d(.)}{d\xi},\frac{\partial^\alpha(.)}{\partial x^\alpha}=k\frac{d(.)}{d\xi},\frac{\partial^{3\alpha}(.)}{\partial x^{3\alpha}}=k^3\frac{d^3(.)}{d\xi^3}. \tag{15}$$

Surrogating Eq. (14) and Eq. (15) into Eq. (13), we convert our problem into an ODE:

$$c\frac{du}{d\xi}+k\frac{du}{d\xi}-vku^2\frac{du}{d\xi}+k^3\frac{d^3u}{d\xi^3}=0. \tag{16}$$

Integration of Eq. (16) with respect to $\xi$ results in

$$cu+ku-\frac{1}{3}vku^3+k^3\frac{d^2u}{d\xi^2}=0. \tag{17}$$

Then using Eq. (10), the 2-D autonomous system is attained

$$\frac{dX}{d\xi}=Y, \tag{18a}$$

$$\frac{dY}{d\xi}=-\frac{1}{k^2}X-\frac{c}{k^3}X+\frac{v}{3k^2}X^3. \tag{18b}$$

Now, to find the first integral of Eq. (18) we implement the division theorem. In accordance with the FIM, it is assumed that non-trivial solutions of the above system (c.f. Eq. (18)) are $X$ and $Y$ respectively. Thus, irreducible polynomial $Q(X,Y)=\sum_{j=0}^{n}a_j(X)Y^j$ exists in $\mathbb{C}[X,Y]$ and following holds

$$Q(X(\xi),Y(\xi))=\sum_{j=0}^{n}a_j(X(\xi))Y(\xi)^j=0, \tag{19}$$

for $j=0,1,\ldots,n$, and $a_n(X)\neq0$. Now a polynomial $r(X)+s(X)Y$ exists in $\mathbb{C}[X,Y]$ so

$$\frac{\partial Q}{\partial\xi}=\frac{\partial Q}{\partial X}\frac{\partial X}{\partial\xi}+\frac{\partial Q}{\partial Y}\frac{\partial Y}{\partial\xi}=(r(X)+s(X)Y)\left(\sum_{j=0}^{n}a_j(X)Y^j\right). \tag{20}$$

Suppose $n=1$. On equating coefficients of $Y^j(j=0,1)$ in Eq. (20) on both sides, we get:

$$a_1'(X)=a_1(X)s(X), \tag{21}$$

$$a_0'(X)=r(X)a_1(X)+s(X)a_0(X), \tag{22}$$

$$r(X)a_0(X)=a_1(X)\left(-\frac{1}{k^2}X-\frac{c}{k^3}X+\frac{v}{3k^2}X^3\right). \tag{23}$$

As $a_j(X)$ are polynomials of $X$, then from Eq. (21) we come to know that the polynomial $a_1(X)$ is constant in nature, therefore $s(X)=0$. Let us consider $a_1(X)=1$, for convenience. After substituting these values we balance the degrees of the functions $r(X)$ and $a_0(X)$ and deduce the $deg(r(X))$ equal to 0 or 1. Assume that $r(X)=A_1X+A_0$, therefore Eq. (22) gives,

$$a_0(X)=\frac{1}{2}A_1X^2+A_0X+B. \tag{24}$$

Here, B is the integration constant.

Replacing the values of $a_0,a_1,r$ and $s$ in Eq. (23), we get a nonlinear system of algebraic equations by putting all coefficients equal to zero for same powers of X. After some calculations, we get:

**Case 1.**

$$A_0=0,A_1=\sqrt{\frac{2v}{3k^2}},B=-\sqrt{\frac{3k^2}{2v}}\left(\frac{c+k}{k^3}\right). \tag{25}$$

Applying the conditions given in Eq. (25) and Eq. (24) in Eq. (19), we have

$$Y(\xi)=-\frac{1}{2}A_1X^2-B. \tag{26}$$

Combining Eq. (26) with Eq. (18a), the solution of fractional mBBM equation with R-L derivative is obtained as:

$$u_1(x,t)=-\sqrt{\frac{-3(k+c)}{vk}}\tan\left[\sqrt{\frac{-(k+c)}{2k^3}}\left(\frac{kx^\alpha+ct^\alpha}{\Gamma(1+\alpha)}\right)+\sqrt{\frac{-(k+c)}{2k^3}}\gamma\right]. \tag{27}$$

**Case 2.**

$$A_0=0,A_1=-\sqrt{\frac{2v}{3k^2}},B=\sqrt{\frac{3k^2}{2v}}\left(\frac{c+k}{k^3}\right). \tag{28}$$

Applying the conditions given in Eq. (28) and Eq. (24) in Eq. (19), we have

$$Y(\xi)=-\frac{1}{2}A_1X^2-B. \tag{29}$$

Combining Eq. (29) with Eq. (18a), the solution of fractional mBBM equation with R-L derivative is obtained as:

$$u_2(x,t)=\sqrt{\frac{-3(k+c)}{vk}}\tan\left[\sqrt{\frac{-(k+c)}{2k^3}}\left(\frac{kx^\alpha+ct^\alpha}{\Gamma(1+\alpha)}\right)+\sqrt{\frac{-(k+c)}{2k^3}}\gamma\right]. \tag{30}$$

Now we apply conformable definition of fractional derivative. The following transformation is introduced:

$$\xi=\frac{kx^\alpha}{\alpha}+\frac{ct^\alpha}{\alpha},$$
$$u(\xi)=u(x,t), \tag{31}$$

where $\xi$ is the transformation variable and $k,c$ be the constants. As a result of this transformation, we obtain:

$$\frac{\partial^\alpha(.)}{\partial t^\alpha}=c\frac{d(.)}{d\xi},\frac{\partial^\alpha(.)}{\partial x^\alpha}=k\frac{d(.)}{d\xi},\frac{\partial^{3\alpha}(.)}{\partial x^{3\alpha}}=k^3\frac{d^3(.)}{d\xi^3}. \tag{32}$$

which is the same as in Eq. (15). Afterwards, adopting the same procedure given from Eq. (16) to Eq. (24), we get two different solutions:

**Case 1**: For $A_0=0,A_1=\sqrt{\frac{2v}{3k^2}},B=-\sqrt{\frac{3k^2}{2v}}\left(\frac{c+k}{k^3}\right)$,

$$u_3(x,t)=-\sqrt{\frac{-3(k+c)}{vk}}\tan\left[\sqrt{\frac{-(k+c)}{2k^3}}\left(\frac{kx^\alpha}{\alpha}+\frac{ct^\alpha}{\alpha}\right)+\sqrt{\frac{-(k+c)}{2k^3}}\gamma\right]. \tag{33}$$

**Case 2**: For $A_0=0,A_1=-\sqrt{\frac{2v}{3k^2}},B=\sqrt{\frac{3k^2}{2v}}\left(\frac{c+k}{k^3}\right)$,

$$u_4(x,t)=\sqrt{\frac{-3(k+c)}{vk}}\tan\left[\sqrt{\frac{-(k+c)}{2k^3}}\left(\frac{kx^\alpha}{\alpha}+\frac{ct^\alpha}{\alpha}\right)+\sqrt{\frac{-(k+c)}{2k^3}}\gamma\right]. \tag{34}$$

It is important to note that the solutions $u_1,u_2$ are acquired by using R-L derivative and $u_3,u_4$ are obtained by using conformable derivative. In Figs. 1 and 2, graphs of exact solutions of fractional mBBM equation are

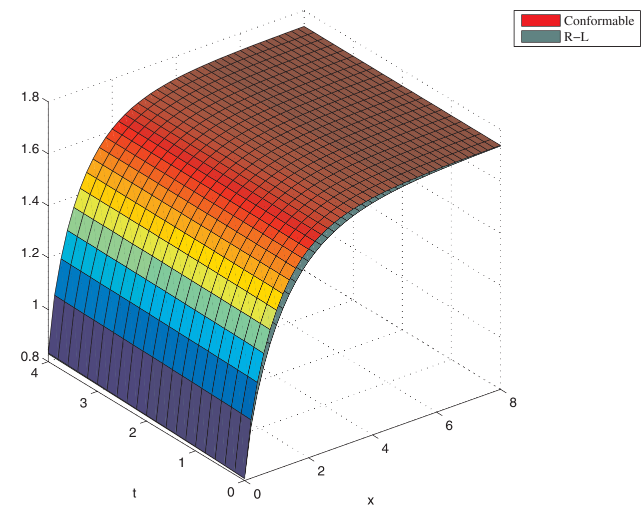


**Fig. 1.** Exact solutions $u_1(x,t)$,$u_3(x,t)$ of fractional mBBM equation at $v = 0.5$,$k = 1$,$c = 0.01$,$\gamma = 1$,$\alpha = 0.8$.

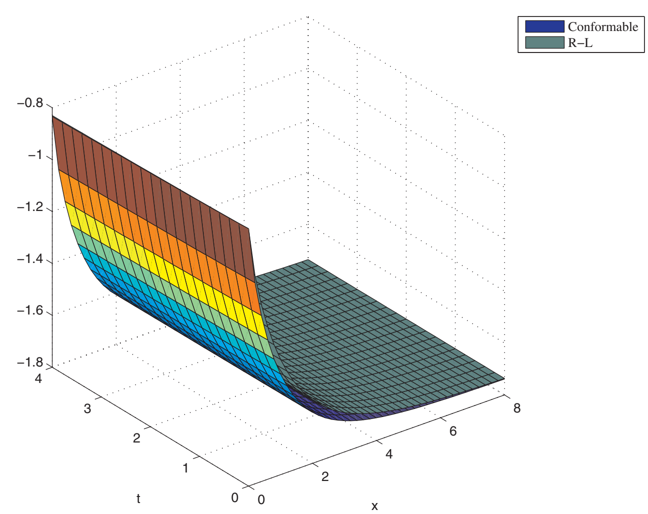


**Fig. 2.** Exact solutions $u_2(x,t)$,$u_4(x,t)$ of fractional mBBM equation at $v = 0.5$,$k = 1$,$c = 0.01$,$\gamma = 1$,$\alpha = 0.8$.

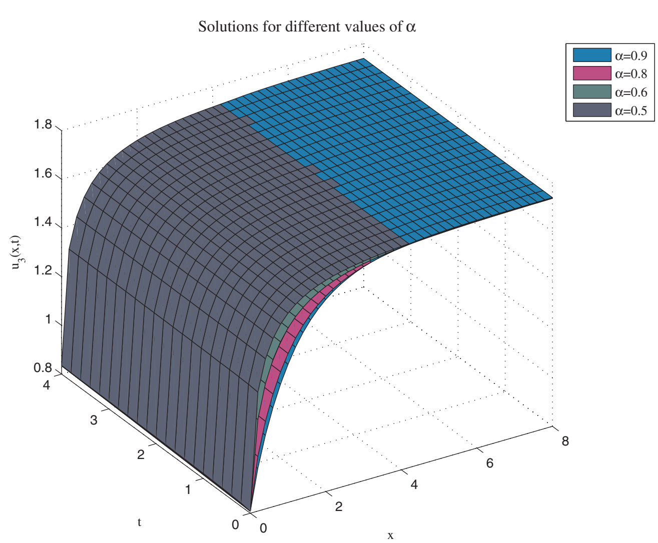


**Fig. 3.** Exact solution $u_3(x,t)$ of fractional mBBM equation at $\alpha = 0.9$, 0.8, 0.6, 0.5, $v = 0.5$,$k = 1$,$c = 0.01$,$\gamma = 1$.

presented by using R-L and conformable derivatives. Fig. 3 shows the effects of $\alpha$ on the solutions $u_3(x,t)$ using the conformable definition.

### 4.2. Exact solutions of the time-fractional Boussinesq-Burgers equation

The Boussinesq-Burgers equation plays an important role for the investigation of fluids flow in a dynamic system. The Boussinesq-Burgers model also illustrates the proliferation of waves in shallow water. Using residual power series method (RPSM), Sunil et al. investigated the numerical solutions for nonlinear time-fractional coupled Boussinesq-Burgers equations [32] and accurate results were found as compared with the result attained from modified homotopy analysis transform technique. Mostafa et al. seek the exact solutions for time-fractional coupled Boussinesq-Burgers equation using the generalized Kudryashov method [33].

Let us consider the time fractional coupled Boussinesq-Burgers equation [32]

$$\frac{\partial^\alpha u}{\partial t^\alpha} - \frac{1}{2}\frac{\partial v}{\partial x} + 2u\frac{\partial u}{\partial x} = 0 \tag{35a}$$

$$\frac{\partial^\alpha v}{\partial t^\alpha} - \frac{1}{2}\frac{\partial^3 u}{\partial x^3} + 2\frac{\partial}{\partial x}(uv) = 0 \tag{35b}$$

Here, $x$ and $t$ represent the normalized space and time variables, $u(x,t)$ corresponds to velocity field along the horizontal, the water surface height from horizontal level is represented by $v(x,t)$. In Eq. (1) fractional derivative is $\alpha$th order and $\alpha \in (0,1)$.

Firstly, we apply R-L definition of fractional derivative. The following transformation is introduced:

$$\xi = x - \frac{ct^\alpha}{\Gamma(1+\alpha)},\quad u(x,t) = u(\xi),\quad v(x,t) = v(\xi), \tag{36}$$

where $\xi$ is the transformation variable and $c$ be the constant. As a result of this transformation, we obtain:

$$\frac{\partial^\alpha(.)}{\partial t^\alpha} = -c\frac{d(.)}{d\xi}, \frac{\partial(.)}{\partial x} = \frac{d(.)}{d\xi}, \frac{\partial^2(.)}{\partial x^2} = \frac{d^2(.)}{d\xi^2}, \frac{\partial^3(.)}{\partial x^3} = \frac{d^3(.)}{d\xi^3}. \tag{37}$$

Surrogating Eq. (37) and Eq. (36) into Eq. (1), we convert our problem into an ODE:

$$-c\frac{du}{d\xi} - \frac{1}{2}\frac{dv}{d\xi} + 2u\frac{du}{d\xi} = 0, \tag{38a}$$

$$-c\frac{dv}{d\xi} - \frac{1}{2}\frac{d^3u}{d\xi^3} + 2\frac{d}{d\xi}(uv) = 0. \tag{38b}$$

Integrating Eq. (38) w.r.t $\xi$ and taking integration constant equal zero, we get the following equations:

$$-cu - \frac{1}{2}v + u^2 = 0, \tag{39a}$$

$$-cv - \frac{1}{2}\frac{d^2u}{d\xi^2} + 2uv = 0. \tag{39b}$$

From Eq. (39a), we get

$$v = 2(u^2 - cu). \tag{40}$$

Surrogating Eq. (40) in Eq. (39b)

$$-\frac{1}{2}\frac{d^2u}{d\xi^2} + 4u^3 - 6cu^2 + 2c^2u = 0. \tag{41}$$

Then using Eq. (10), the 2-D autonomous system is attained

$$\frac{dX}{d\xi} = Y, \tag{42a}$$

$$\frac{dY}{d\xi} = 8X^3 - 12cX^2 + 4c^2X. \tag{42b}$$

Now, to find the first integral to Eq. (42) we implement the division theorem. In accordance with the FIM, it is assumed that non-trivial solutions of the above system (c.f. Eq. (42)) are $X$ and $Y$ respectively. Thus, irreducible polynomial $Q(X,Y) = \sum_{j=0}^{n} a_j(X)Y^j$ exists in $\mathbb{C}[X,Y]$ and following holds

$$Q(X(\xi),Y(\xi)) = \sum_{j=0}^{n} a_j(X(\xi))Y(\xi)^j = 0, \tag{43}$$

for $j = 0,1,...,n$, and $a_n(X) \neq 0$. Now a polynomial $r(X) + s(X)Y$ exists in $\mathbb{C}[X,Y]$ so

$$\frac{\partial Q}{\partial \xi} = \frac{\partial Q}{\partial X}\frac{\partial X}{\partial \xi} + \frac{\partial Q}{\partial Y}\frac{\partial Y}{\partial \xi} = (r(X) + s(X)Y)\left(\sum_{j=0}^{n} a_j(X)Y^j\right). \tag{44}$$

Suppose $n = 1$. On equating coefficients of $Y^j (j = 0,1)$ in Eq. (44) on both sides, we get:

$$a_1'(X) = a_1(X)s(X), \tag{45}$$

$$a_0'(X) = r(X)a_1(X) + s(X)a_0(X), \tag{46}$$

$$r(X)a_0(X) = a_1(X)(8X^3 - 12cX^2 + 4c^2X). \tag{47}$$

As $a_j(X)$ are polynomials of $X$, then from Eq. (45) we come to know that the polynomial $a_1(X)$ is constant in nature, therefore $s(X) = 0$. Let us consider $a_1(X) = 1$, for convenience. After substituting these values we balance the degrees of the functions $r(X)$ and $a_0(X)$ and deduce the $deg(r(X))$ equal to 0 or 1. Assume that $r(X) = A_1X + A_0$, therefore Eq. (46) gives

$$a_0(X) = \frac{1}{2}A_1X^2 + A_0X + B. \tag{48}$$

Here, B is the integration constant.

Replacing the values of $a_0$,$a_1$,$r$ and $s$ in Eq. (47), we get a nonlinear system of algebraic equations by putting all coefficients equal to zero for same powers of X. After some calculations, we get:

**Case 1**:

$$A_1 = 4, A_0 = -2c, B = 0. \tag{49}$$

Applying the conditions given in Eq. (49) and Eq. (48) in Eq. (43), we have

$$Y_1(\xi) = -2X^2 + 2cX. \tag{50}$$

Combining Eq. (50) with Eq. (42a), the solutions of fractional Boussinesq-Burgers equation with R-L derivative are obtained as:

$$u_1(x,t) = \frac{c}{1 + c\gamma e^{-2c\left(x - \frac{ct^\alpha}{\Gamma(1+\alpha)}\right)}}. \tag{51}$$

$$v_1(x,t) = \frac{-2c^3\gamma e^{-2c\left(x - \frac{ct^\alpha}{\Gamma(1+\alpha)}\right)}}{\left(1 + c\gamma e^{-2c\left(x - \frac{ct^\alpha}{\Gamma(1+\alpha)}\right)}\right)^2}. \tag{52}$$

**Case 2**:

$$A_1 = -4, A_0 = 2c, B = 0. \tag{53}$$

Applying the conditions given in Eq. (53) and Eq. (48) in Eq. (43), we have

$$Y_2(\xi) = 2X^2 - 2cX. \tag{54}$$

Combining Eq. (54) with Eq. (42a), the solutions of fractional Boussinesq-Burgers equation with R-L derivative are obtained as:

$$u_2(x,t) = \frac{c}{1 + c\gamma e^{2c\left(x - \frac{ct^\alpha}{\Gamma(1+\alpha)}\right)}}. \tag{55}$$

$$v_2(x,t) = \frac{-2c^3\gamma e^{2c\left(x - \frac{ct^\alpha}{\Gamma(1+\alpha)}\right)}}{\left(1 + c\gamma e^{2c\left(x - \frac{ct^\alpha}{\Gamma(1+\alpha)}\right)}\right)^2}. \tag{56}$$

Now we apply conformable definition of fractional derivative. The following transformation is introduced:

$$\xi = x - \frac{ct^\alpha}{\alpha},$$
$$u(x,t) = u(\xi),$$
$$v(x,t) = v(\xi), \tag{57}$$

where $\xi$ is the transformation variable and $c$ be the constant. As a result of this transformation, we obtain:

$$\frac{\partial^\alpha(.)}{\partial t^\alpha} = -c\frac{d(.)}{d\xi}, \frac{\partial(.)}{\partial x} = \frac{d(.)}{d\xi}, \frac{\partial^2(.)}{\partial x^2} = \frac{d^2(.)}{d\xi^2}, \frac{\partial^3(.)}{\partial x^3} = \frac{d^3(.)}{d\xi^3}. \tag{58}$$

which is the same as in Eq. (37). Afterwards, adopting the same procedure given from Eq. (38) to Eq. (48), we get two different solutions:

**Case 1**: For $A_1 = 4, A_0 = -2c, B = 0$,

$$u_3(x,t) = \frac{c}{1 + c\gamma e^{-2c\left(x - \frac{ct^\alpha}{\alpha}\right)}}. \tag{59}$$

$$v_3(x,t) = \frac{-2c^3\gamma e^{-2c\left(x - \frac{ct^\alpha}{\alpha}\right)}}{\left(1 + c\gamma e^{-2c\left(x - \frac{ct^\alpha}{\alpha}\right)}\right)^2}. \tag{60}$$

**Case 2**: For $A_1 = -4, A_0 = 2c, B = 0$,

$$u_4(x,t) = \frac{c}{1 + c\gamma e^{2c\left(x - \frac{ct^\alpha}{\alpha}\right)}}. \tag{61}$$

$$v_4(x,t) = \frac{-2c^3\gamma e^{2c\left(x - \frac{ct^\alpha}{\alpha}\right)}}{\left(1 + c\gamma e^{2c\left(x - \frac{ct^\alpha}{\alpha}\right)}\right)^2}. \tag{62}$$

It is important to note that the solutions $u_1$,$v_1$,$u_2$,$v_2$ are acquired by using R-L derivative and $u_3$,$v_3$,$u_4$,$v_4$ are obtained by using conformable derivative. In Figs. 4–7, exact solutions of fractional Boussinesq-Burgers equation are presented by using R-L and conformable derivatives. Fig. 8 shows the effects of $\alpha$ on the solution $u_1(x,t)$ using R-L definition.

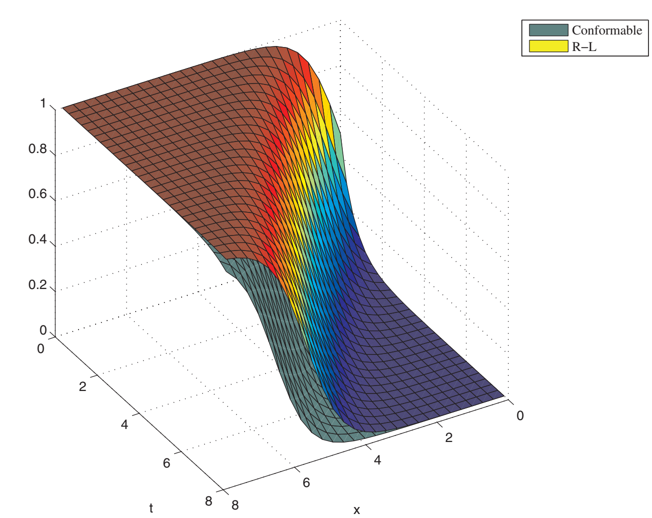


**Fig. 4.** Exact solutions $u_1(x,t)$,$u_3(x,t)$ of fractional Boussinesq-Burgers equation at $c = 1$,$\gamma = 1$,$\alpha = 0.8$.

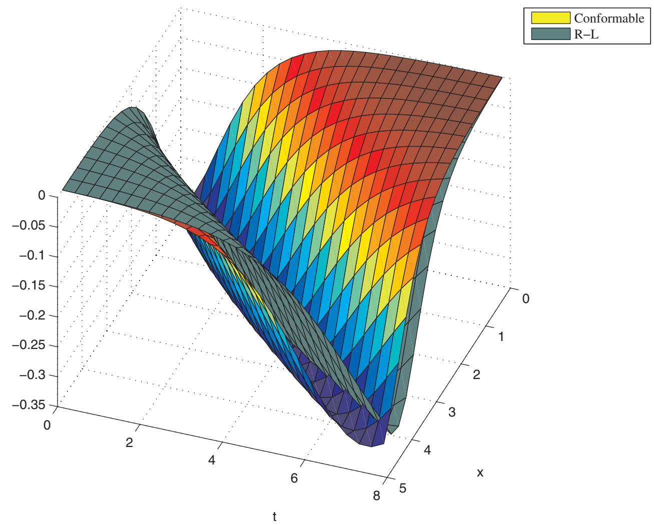


**Fig. 5.** Exact solutions $v_1(x,t)$,$v_3(x,t)$ of fractional Boussinesq-Burgers equation at $c = 0.8$,$\gamma = 1$,$\alpha = 0.8$.

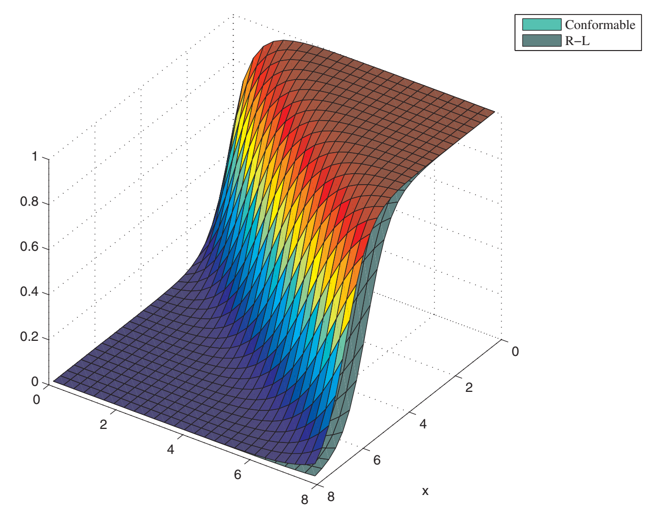


**Fig. 6.** Exact solutions $u_2(x,t)$,$u_4(x,t)$ of fractional Boussinesq-Burgers equation at $c = 1$,$\gamma = 1$,$\alpha = 0.8$.

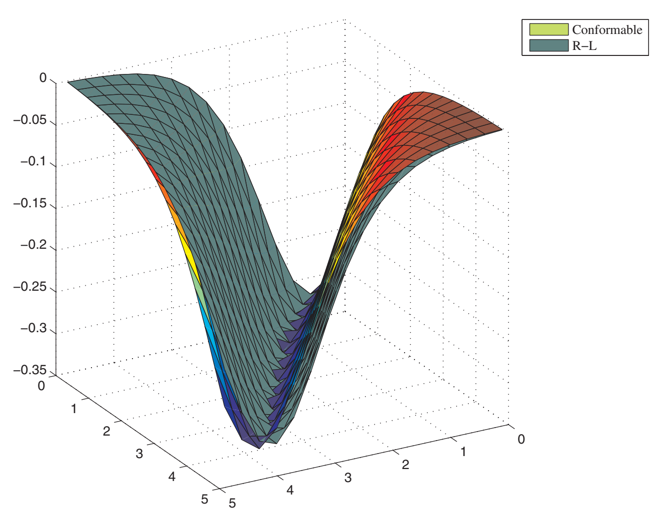


**Fig. 7.** Exact solutions $v_2(x,t)$,$v_4(x,t)$ of fractional Boussinesq-Burgers equation at $c = 0.8$,$\gamma = 0.5$,$\alpha = 0.8$.

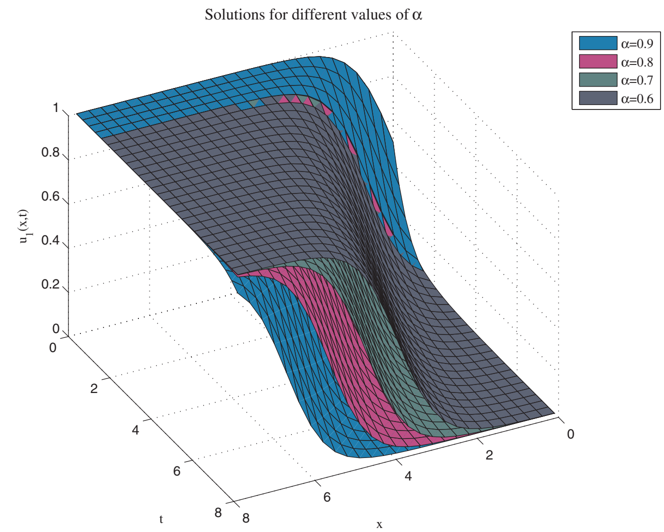


**Fig. 8.** Exact solution $u_1(x,t)$ of fractional Boussinesq-Burgers equation at $\alpha = 0.9,\ 0.8,\ 0.7,\ 0.6$, $c = 1$,$\gamma = 1$.

## 5. Conclusion

In this article, new exact solutions of FPDEs, namely, space-time fractional mBBM equation and time-fractional Boussinesq-Burgers equation were obtained using the FIM. R-L derivative and conformable derivative definitions were used to deal with the fractionl terms in FPDEs. The procedure indicates that the FIM is a direct and concise. The proposed technique permits us to perform tedious and complicated algebraic calculations easily using a computer. Therefore, FIM is very effective and reliable to obtain new exact solutions to system of nonlinear fractional order problems emerging in the areas of engineering and mathematical physics.

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## Appendix A. Supplementary data

Supplementary data associated with this article can be found, in the online version, athttp://dx.doi.org/10.1016/j.rinp.2018.04.026.

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