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On high order fractional integro-differential equations including the Caputo–Fabrizio derivative

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Abstract

By using the fractional Caputo–Fabrizio derivative, we introduce two types new high order derivations called CFD and DCF. Also, we study the existence of solutions for two such type high order fractional integro-differential equations. We illustrate our results by providing two examples.

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1 Introduction

Fractional integro-differential equations have been studied by many researchers from different points of view during the last decades (see for example, [5, 10] and [15–19]). In 2015, a new fractional derivation without singular kernel was introduced by Caputo and Fabrizio ([8]). Some researchers tried to use it for solving different equations (see, for example, [2, 9] and [14]). Recently, approximate solutions of some fractional differential equations have been reviewed (see, for example, [3, 4, 6, 12, 13] and [7]). Also, one is finding some new applications for fractional derivations (see, for example, [3]).

In this manuscript we consider $b > 0$, $x \in H^1(0, b)$ and $\alpha \in (0, 1)$. The expression of the Caputo–Fabrizio fractional derivative of order α for the function x has the form ${}^{CF}D^\alpha x(t) = \frac{B(\alpha)}{1-\alpha} \int_0^t \exp(\frac{-\alpha}{1-\alpha}(t-s))x'(s) ds$, where $t \geq 0$ ([1, 8] and [9]). $B(\alpha)$ is a normalization constant ($B(1) = B(0) = 1$). The fractional integral of order α for the function x is written as ([14]) ${}^{CF}I^\alpha x(t) = \frac{1-\alpha}{B(\alpha)}x(t) + \frac{\alpha}{B(\alpha)} \int_0^t x(s) ds$, whenever $0 < \alpha < 1$. If $n \geq 1$ and $\alpha \in [0, 1]$, then the fractional derivative ${}^{CF}D^{\alpha+n}$ of order $n + \alpha$ is defined by ${}^{CF}D^{\alpha+n}x := {}^{CF}D^\alpha(D^n x(t))$ ([6] and [8]). If the function x is such that $x^{(k)} = 0$ for $k = 1, 2, 3, \dots, n$, then ${}^{CF}D^\alpha(D^n x(t)) = D^n({}^{CF}D^\alpha x(t))$ ([8]). Here, D is the ordinary derivation.

Lemma 1.1 ([1] and [14]) *Let $0 < \alpha < 1$. Then the unique solution for the problem ${}^{CF}D^\alpha x(t) = y(t)$ is given by $x(t) = x(0) + \frac{1-\alpha}{B(\alpha)}y(t) + \frac{\alpha}{B(\alpha)} \int_0^t x(s) ds$.*

Theorem 1.2 ([11]) *Let (X, d) be a complete metric space and $F : X \rightarrow X$ be a mapping such that $\varphi(d(Fx, Fy)) \leq \varphi(d(x, y)) - \phi(d(x, y))$, for all $x, y \in X$, where $\varphi, \phi : [0, 1] \rightarrow [0, 1]$*

are continuous non-decreasing maps and $\varphi(t) = \phi(t) = 0$ if and only if $t = 0$. Then F has a unique fixed point.

2 Main result

Let n be a natural number, $\alpha \in (0, 1)$ and $x^{(n)} \in H^1(0, 1)$. Then the fractional CFD of order α and n is defined by

$${}^{CF}D^{\alpha+n}x(t) = {}^{CF}D^\alpha(D^n x(t)) = \frac{B(\alpha)}{1-\alpha} \int_0^t \exp\left(\frac{-\alpha}{1-\alpha}(t-s)\right) x^{(n+1)}(s) ds.$$

Also, the fractional DCF of order α and n is defined by

$$({}^{CF}D^\alpha)^{(n)}x(t) = D^n({}^{CF}D^\alpha x(t)) = \frac{B(\alpha)}{1-\alpha} \frac{d^n}{dt^n} \int_0^t \exp\left(\frac{-\alpha}{1-\alpha}(t-s)\right) x'(s) ds.$$

Here, D is the ordinary derivative.

Lemma 2.1 *Let n be a natural number and $\alpha \in (0, 1)$. Then*

$$({}^{CF}D^\alpha)^{(n)}x(t) = {}^{CF}D^{\alpha+n}x(t) + \exp\left(\frac{-\alpha}{1-\alpha}t\right)\sigma(\alpha, n, 0),$$

where $\sigma(\alpha, n, t) = \frac{B(\alpha)}{1-\alpha} \sum_{i=1}^n \left(\frac{-\alpha}{1-\alpha}\right)^{n-i} x^{(i)}(t)$.

Proof For each $k \geq 1$, we have

$$\begin{aligned} \int_0^t \exp\left(\frac{-\alpha}{1-\alpha}(t-s)\right) x^{(k)}(s) ds &= x^{(k-1)}(t) - \exp\left(\frac{-\alpha}{1-\alpha}t\right) x^{(k-1)}(0) \\ &\quad + \left(\frac{-\alpha}{1-\alpha}\right) \int_0^t \exp\left(\frac{-\alpha}{1-\alpha}(t-s)\right) x^{(k-1)}(s) ds. \end{aligned}$$

Now by using repetition of the last relation, we get

$$\begin{aligned} {}^{CF}D^{\alpha+n}x(t) &= {}^{CF}D^\alpha(D^n x(t)) = \frac{B(\alpha)}{1-\alpha} \int_0^t \exp\left(\frac{-\alpha}{1-\alpha}(t-s)\right) x^{(n+1)}(s) ds \\ &= \frac{B(\alpha)}{1-\alpha} \left(\frac{-\alpha}{1-\alpha}\right)^n \int_0^t \exp\left(\frac{-\alpha}{1-\alpha}(t-s)\right) x'(s) ds \\ &\quad + \frac{B(\alpha)}{1-\alpha} \sum_{i=1}^n \left(\frac{-\alpha}{1-\alpha}\right)^{n-i} x^{(i)}(t) \\ &\quad - \frac{B(\alpha)}{1-\alpha} \exp\left(\frac{-\alpha}{1-\alpha}t\right) \sum_{i=1}^n \left(\frac{-\alpha}{1-\alpha}\right)^{n-i} x^{(i)}(0) \\ &= \frac{B(\alpha)}{1-\alpha} \left(\frac{-\alpha}{1-\alpha}\right)^n \int_0^t \exp\left(\frac{-\alpha}{1-\alpha}(t-s)\right) x'(s) ds \\ &\quad + \sigma(\alpha, n, t) - \exp\left(\frac{-\alpha}{1-\alpha}t\right)\sigma(\alpha, n, 0) \\ &= \left(\frac{-\alpha}{1-\alpha}\right)^n {}^{CF}D^\alpha x(t) + \sigma(\alpha, n, t) - \exp\left(\frac{-\alpha}{1-\alpha}t\right)\sigma(\alpha, n, 0). \end{aligned}$$

Also, we have

$$\begin{aligned}
 ({}^{\text{CF}}D^\alpha)^{(n)}x(t) &= D^n(D^\alpha x(t)) = \left(\frac{-\alpha}{1-\alpha}\right)^n {}^{\text{CF}}D^\alpha x(t) + \frac{B(\alpha)}{1-\alpha} \sum_{i=1}^n \left(\frac{-\alpha}{1-\alpha}\right)^{n-i} x^{(i)}(t) \\
 &= \frac{B(\alpha)}{1-\alpha} \left(\frac{-\alpha}{1-\alpha}\right)^n \int_0^t \exp\left(\frac{-\alpha}{1-\alpha}(t-s)\right) x'(s) ds \\
 &\quad + \frac{B(\alpha)}{1-\alpha} \sum_{i=1}^n \left(\frac{-\alpha}{1-\alpha}\right)^{n-i} x^{(i)}(t) \\
 &= \left(\frac{-\alpha}{1-\alpha}\right)^n {}^{\text{CF}}D^\alpha x(t) + \sigma(\alpha, n, t).
 \end{aligned}$$

Hence $({}^{\text{CF}}D^\alpha)^{(n)}x(t) = {}^{\text{CF}}D^{\alpha+n}x(t) + \exp(\frac{-\alpha}{1-\alpha}t)\sigma(\alpha, n, 0)$. □

By using Lemma 2.1, we conclude that ${}^{\text{CF}}D^{\alpha+n}x(t) = ({}^{\text{CF}}D^\alpha)^{(n)}x(t)$ whenever $x^{(k)}(0) = 0$ for $0 \leq k \leq n$.

Lemma 2.2 *Let n be a natural number, $\alpha \in (0, 1)$ and $y \in H^1(0, 1)$. Then the solution of the problem ${}^{\text{CF}}D^{\alpha+n}x(t) = y(t)$ is given by*

$$x(t) = \frac{1-\alpha}{B(\alpha)} J^n y(t) + \frac{\alpha}{B(\alpha)} J^{n+1} y(t) + x(0) + tx'(0) + t^2 \frac{x''(0)}{2!} + \dots + t^n \frac{x^{(n)}(0)}{n!}.$$

Proof By using Lemma 1.1 for the equation ${}^{\text{CF}}D^{\alpha+n}x(t) = {}^{\text{CF}}D^\alpha x^{(n)}(t) = y(t)$, we get $x^{(n)}(t) = x^{(n)}(0) + \frac{1-\alpha}{B(\alpha)} y(t) + \frac{\alpha}{B(\alpha)} \int_0^t y(s) ds$. By using an integration, we obtain

$$x^{(n-1)}(t) = x^{(n-1)}(0) + tx^{(n)}(0) + \frac{1-\alpha}{B(\alpha)} \int_0^t y(s) ds + \frac{\alpha}{B(\alpha)} \int_0^t \int_0^s y(r) dr ds.$$

By repeating this method, we deduce that

$$\begin{aligned}
 x^{(n-2)}(t) &= x^{(n-2)}(0) + tx^{(n-1)}(0) + \frac{t^2}{2} x^{(n)}(0) \\
 &\quad + \frac{1-\alpha}{B(\alpha)} \int_0^t \int_0^s y(r) dr ds + \frac{\alpha}{B(\alpha)} \int_0^t \int_0^s \int_0^r y(k) dk dr ds.
 \end{aligned}$$

By continuing the process, we conclude that

$$x(t) = \frac{1-\alpha}{B(\alpha)} J^n y(t) + \frac{\alpha}{B(\alpha)} J^{n+1} y(t) + x(0) + tx'(0) + t^2 \frac{x''(0)}{2!} + \dots + t^n \frac{x^{(n)}(0)}{n!}.$$

On the other hand, by using some calculation, one can find that the given map $x(t)$ is a solution for the problem ${}^{\text{CF}}D^{\alpha+n}x(t) = y(t)$. □

Lemma 2.3 *Let n be a natural number, $\alpha \in (0, 1)$ and $y \in H^1(0, 1)$. Then the solution of the problem $({}^{CF}D^\alpha)^{(n)}x(t) = y(t)$ is given by*

$$x(t) = \frac{1-\alpha}{B(\alpha)}J^n y(t) + \frac{\alpha}{B(\alpha)}J^{n+1}y(t) + x(0) + tx'(0) + \dots + t^n \frac{x^{(n)}(0)}{n!} - \frac{t^n}{n!} \sum_{i=1}^n \left(\frac{-\alpha}{1-\alpha}\right)^{n-i} x^{(i)}(0).$$

Proof By using Lemma 2.2 for $({}^{CF}D^\alpha)^{(n)}x(t) = {}^{CF}D^{\alpha+n}x(t) + \exp(\frac{-\alpha}{1-\alpha}t)\sigma(\alpha, n, 0)$, we get

$$x(t) = \frac{1-\alpha}{B(\alpha)}J^n \left(y(t) - \exp\left(\frac{-\alpha}{1-\alpha}t\right)\sigma(\alpha, n, 0) \right) + \frac{\alpha}{B(\alpha)}J^{n+1} \left(y(t) - \exp\left(\frac{-\alpha}{1-\alpha}t\right)\sigma(\alpha, n, 0) \right) + x(0) + tx'(0) + t^2 \frac{x''(0)}{2!} + \dots + t^n \frac{x^{(n)}(0)}{n!}$$

or equivalently

$$\begin{aligned} x(t) &= \frac{1-\alpha}{B(\alpha)}J^n y(t) - \frac{1-\alpha}{B(\alpha)}\sigma(\alpha, n, 0)J^n \exp\left(\frac{-\alpha}{1-\alpha}t\right) + \frac{\alpha}{B(\alpha)}J^{n+1}y(t) \\ &\quad - \frac{\alpha}{B(\alpha)}\sigma(\alpha, n, 0)J^{n+1} \exp\left(\frac{-\alpha}{1-\alpha}t\right) + x(0) + tx'(0) + t^2 \frac{x''(0)}{2!} + \dots + t^n \frac{x^{(n)}(0)}{n!} \\ &= \frac{1-\alpha}{B(\alpha)}J^n y(t) + \frac{\alpha}{B(\alpha)}J^{n+1}y(t) - \sigma(\alpha, n, 0)J^n \left[\frac{1-\alpha}{B(\alpha)} \exp\left(\frac{-\alpha}{1-\alpha}t\right) \right. \\ &\quad \left. + \frac{\alpha}{B(\alpha)}J^1 \exp\left(\frac{-\alpha}{1-\alpha}t\right) \right] + x(0) + tx'(0) + t^2 \frac{x''(0)}{2!} + \dots + t^n \frac{x^{(n)}(0)}{n!} \\ &= \frac{1-\alpha}{B(\alpha)}J^n y(t) + \frac{\alpha}{B(\alpha)}J^{n+1}y(t) - \sigma(\alpha, n, 0)J^n \left[\frac{1-\alpha}{B(\alpha)} \exp\left(\frac{-\alpha}{1-\alpha}t\right) \right. \\ &\quad \left. + \frac{1-\alpha}{B(\alpha)} \left(1 - \exp\left(\frac{-\alpha}{1-\alpha}t\right) \right) \right] + x(0) + tx'(0) + t^2 \frac{x''(0)}{2!} + \dots + t^n \frac{x^{(n)}(0)}{n!} \\ &= \frac{1-\alpha}{B(\alpha)}J^n y(t) + \frac{\alpha}{B(\alpha)}J^{n+1}y(t) + x(0) + tx'(0) + t^2 \frac{x''(0)}{2!} + \dots + t^n \frac{x^{(n)}(0)}{n!} \\ &\quad - \frac{t^n}{n!} \sum_{i=1}^n \left(\frac{-\alpha}{1-\alpha}\right)^{n-i} x^{(i)}(0). \end{aligned}$$

□

Lemma 2.4 *Let $\alpha \in (0, 1)$, $2 < q = 2 + \alpha < 3$ and $y \in H^1(0, 1)$. The fractional differential equation ${}^{CF}D^q x(t) = y(t)$ with boundary conditions $x(0) = 0$, $x'(1) + x'(0) = 0$ and $x''(0) = 0$ has the unique solution of the form $x(t) = \int_0^1 G(t,s)y(s) ds$, where $G(t,s) = \frac{-(1-\alpha)t}{2B(\alpha)} - \frac{\alpha t}{2B(\alpha)}$ whenever $0 < t \leq s < 1$ and $G(t,s) = \frac{1-\alpha}{B(\alpha)}(t-s) + \frac{\alpha}{2B(\alpha)}(t-s)^2 - \frac{(1-\alpha)t}{2B(\alpha)} - \frac{\alpha t}{2B(\alpha)}(t-s)$ whenever $0 < s \leq t < 1$.*

Proof By using Lemma 2.2, we get $x(t) = \frac{1-\alpha}{B(\alpha)}J^2 y(t) + \frac{\alpha}{B(\alpha)}J^3 y(t) + tx'(0)$. Hence, we obtain $x'(t) = \frac{1-\alpha}{B(\alpha)}J^1 y(t) + \frac{\alpha}{B(\alpha)}J^2 y(t) + x'(0)$. By using the boundary conditions $x'(1) + x'(0) = 0$ and $x'(1) = \frac{1-\alpha}{B(\alpha)}J^1 y(1) + \frac{\alpha}{B(\alpha)}J^2 y(1) + x'(0)$, we have $x(t) = \frac{1-\alpha}{B(\alpha)}J^2 y(t) + \frac{\alpha}{B(\alpha)}J^3 y(t) - \frac{(1-\alpha)t}{2B(\alpha)}J^1 y(1) -$

$\frac{\alpha t}{2B(\alpha)} J^2 y(1)$. Thus, $x(t) = \frac{1-\alpha}{B(\alpha)} \int_0^t y(s)(t-s) ds + \frac{\alpha}{2B(\alpha)} \int_0^t y(s)(t-s)^2 ds - \frac{(1-\alpha)t}{2B(\alpha)} \int_0^1 y(s) ds - \frac{\alpha t}{2B(\alpha)} \int_0^1 y(s)(t-s) ds = \int_0^1 G(t,s)y(s) ds$. Note that ${}^{\text{CF}}D^\alpha x(t) = 0$ if and only if $x(t) = 0$. This implies that the given map $x(t)$ is a unique solution. \square

Note that $|G(t,s)| \leq |\frac{1-\alpha}{B(\alpha)}| + |\frac{\alpha}{2B(\alpha)}| + |\frac{-\alpha t}{2B(\alpha)}| + |\frac{-(1-\alpha)t}{2B(\alpha)}| < \frac{3}{2B(\alpha)}$, for $t \in [0, 1]$. Let $\mu, \mu_1, \mu_2, k_1, k_2 \in C^1[0, 1]$, m_1, m_2, h and g be bounded continuous functions on $I := [0, 1]$ with $M_1 = \sup_{t \in I} |\mu(t)| < \infty, M_2 = \sup_{t \in I} |\mu_1(t)| < \infty, M_3 = \sup_{t \in I} |\mu_2(t)| < \infty, M_4 = \sup_{t \in I} |k_1(t)| < \infty, M_5 = \sup_{t \in I} |k_2(t)| < \infty, M_6 = \sup_{t \in I} |m_1(t)| < \infty, M_7 = \sup_{t \in I} |m_2(t)| < \infty, M_8 = \sup_{t \in I} |h(t)| < \infty, M_9 = \sup_{t \in I} |g(t)| < \infty, N_1 = \sup_{t \in I} |\mu'(t)| < \infty, N_2 = \sup_{t \in I} |\mu_1'(t)| < \infty, N_3 = \sup_{t \in I} |\mu_2'(t)| < \infty, N_4 = \sup_{t \in I} |K_1'(t)| < \infty$ and $N_5 = \sup_{t \in I} |K_2'(t)| < \infty$. Let $\alpha \in (0, 1)$ and $2 < q = 2 + \alpha < 3$. Now, we investigate the CFD fractional integro-differential problem

$${}^{\text{CF}}D^q x(t) = \mu(t)x(t) + \mu_1(t)x'(t) + \mu_2(t)x''(t) + k_1(t) {}^{\text{CF}}D^{\beta_1} x(t) + k_2(t) {}^{\text{CF}}D^{\beta_2} x(t) + \int_0^t f(s, x(s), m_1(s)x'(s), m_2(s)x''(s), h(s) {}^{\text{CF}}D^\gamma x(s), g(s) {}^{\text{CF}}D^\nu x(s)) ds, \tag{1}$$

with boundary conditions $x(0) = 0, x'(1) + x'(0) = 0$ and $x''(0) = 0$, where $1 < \beta_1 < 2 < \beta_2 < 3$ and $1 < \gamma < 2 < \nu < 3$.

Theorem 2.5 *Let $\xi_1, \xi_2, \xi_3, \xi_4$ and ξ_5 be nonnegative real numbers, $f : [0, 1] \times \mathbb{R}^5 \rightarrow \mathbb{R}$ an integrable function such that*

$$|f(t, x, y, z, v, w) - f(t, x', y', z', v', w')| \leq \xi_1 |x - x'| + \xi_2 |y - y'| + \xi_3 |z - z'| + \xi_4 |v - v'| + \xi_5 |w - w'|,$$

for all real numbers $x, y, z, v, w, x', y', z', v', w' \in \mathbb{R}$ and $t \in I$. If $\Delta < \frac{1}{2}$, then the problem (1) has a unique solution, where $\Delta := \max\{\Delta_1, \Delta_2, \Delta_3, \Delta_4\}$, $\Delta_1 = \frac{3}{2B(\alpha)} [M_1 + M_2 + M_3 + \frac{M_4 B(\beta_1 - 1)}{2 - \beta_1} + \frac{M_5 B(\beta_2 - 2)}{3 - \beta_2} + \xi_1 + \xi_2 M_6 + \xi_3 M_7 + \xi_4 \frac{M_8 B(\gamma - 1)}{2 - \gamma} + \xi_5 \frac{M_9 B(\nu - 2)}{3 - \nu}]$, $\Delta_2 = \frac{3}{2} [\frac{3 + 4\alpha}{2B(\alpha)}] [M_1 + M_2 + M_3 + \frac{M_4 B(\beta_1 - 1)}{2 - \beta_1} + \frac{M_5 B(\beta_2 - 2)}{3 - \beta_2} + \xi_1 + \xi_2 M_6 + \xi_3 M_7 + \xi_4 \frac{M_8 B(\gamma - 1)}{2 - \gamma} + \xi_5 \frac{M_9 B(\nu - 2)}{3 - \nu}]$, $\Delta_3 = \frac{1 + \alpha}{B(\alpha)} [M_1 + M_2 + M_3 + \frac{M_4 B(\beta_1 - 1)}{2 - \beta_1} + \frac{M_5 B(\beta_2 - 2)}{3 - \beta_2} + \xi_1 + \xi_2 M_6 + \xi_3 M_7 + \xi_4 \frac{M_8 B(\gamma - 1)}{2 - \gamma} + \xi_5 \frac{M_9 B(\nu - 2)}{3 - \nu}]$ and $\Delta_4 = \frac{\alpha}{B(\alpha)} [M_1 + M_2 + M_3 + \frac{M_4 B(\beta_1 - 1)}{2 - \beta_1} + \frac{M_5 B(\beta_2 - 2)}{3 - \beta_2} + \xi_1 + \xi_2 M_6 + \xi_3 M_7 + \xi_4 \frac{M_8 B(\gamma - 1)}{2 - \gamma} + \xi_5 \frac{M_9 B(\nu - 2)}{3 - \nu}] + \frac{1 - \alpha}{B(\alpha)} [N_1 + M_1 + N_2 + M_2 + N_3 + M_3 + B(\beta_1 - 1) [\frac{|1 - \beta_1| M_4}{(2 - \beta_1)^2} + \frac{N_4 + M_4}{2 - \beta_1}] + B(\beta_2 - 2) [\frac{|2 - \beta_2| M_5}{(3 - \beta_2)^2} + \frac{M_5 + N_5}{3 - \beta_2}] + \xi_1 + \xi_2 M_6 + \xi_3 M_7 + \xi_4 \frac{M_8 B(\gamma - 1)}{2 - \gamma} + \xi_5 \frac{M_9 B(\nu - 1)}{3 - \nu}]$.

Proof Consider the Banach space $C_{\mathbb{R}}^3[0, 1]$ equipped with the norm $\|x\| = \max_{t \in I} |x(t)| + \max_{t \in I} |x'(t)| + \max_{t \in I} |x''(t)| + \max_{t \in I} |x'''(t)|$. Define the map $F : C_{\mathbb{R}}^3[0, 1] \rightarrow C_{\mathbb{R}}^3[0, 1]$ by

$$\begin{aligned} Fx(t) &= \int_0^1 G(t,s)R(s) ds \\ &= \frac{1-\alpha}{B(\alpha)} \int_0^t R(s)(t-s) ds + \frac{\alpha}{2B(\alpha)} \int_0^t R(s)(t-s)^2 ds \\ &\quad - \frac{(1-\alpha)t}{2B(\alpha)} \int_0^1 R(s) ds - \frac{\alpha t}{2B(\alpha)} \int_0^1 R(s)(t-s) ds, \end{aligned}$$

where

$$(Rx)(t) = \mu(t)x(t) + \mu_1(t)x'(t) + \mu_2(t)x''(t) + k_1(t)^{CF}D^{\beta_1}x(t) + k_2(t)^{CF}D^{\beta_2}x(t) + \int_0^t f(s, x(s), m_1(s)x'(s), m_2(s)x''(s), h(s)^{CF}D^\gamma x(s), g(s)^{CF}D^\nu x(s)) ds$$

and

$$(R'x)(t) = \mu(t)x'(t) + \mu'(t)x(t) + \mu'_1(t)x'(t) + \mu_1x''(t) + \mu'_2(t)x''(t) + \mu_2(t)x'''(t) + k'_1(t)^{CF}D^{\beta_1}x(t) + k_1(t) \left[\frac{1 - \beta_1}{2 - \beta_1} {}^{CF}D^{\beta_1}x(t) + \frac{B(\beta_1 - 1)}{2 - \beta_1} x''(t) \right] + k_2(t) \left[\frac{2 - \beta_2}{3 - \beta_2} {}^{CF}D^{\beta_2}x(t) + \frac{B(\beta_2 - 2)}{3 - \beta_2} x'''(t) \right] + k'_2(t)^{CF}D^{\beta_2}x(t) + f(t, x(t), m_1(t)x'(t), m_2(t)x''(t), h(t)^{CF}D^\gamma x(t), g(t)^{CF}D^\nu x(t)).$$

By using Lemma 2.4, x_0 is a solution for the problem (1) if and only if x_0 is a fixed point of the operator F . Note that

$$\begin{aligned} & |(Rx)(t) - (Ry)(t)| \\ & \leq \left| \mu(t)x(t) + \mu_1(t)x'(t) + \mu_2(t)x''(t) + k_1(t)^{CF}D^{\beta_1}x(t) + k_2(t)^{CF}D^{\beta_2}x(t) + \int_0^t f(s, x(s), m_1(s)x'(s), m_2(s)x''(s), h(s)^{CF}D^\gamma x(s), g(s)^{CF}D^\nu x(s)) ds \right. \\ & \quad \left. - \left(\mu(t)y(t) + \mu_1(t)y'(t) + \mu_2(t)y''(t) + k_1(t)^{CF}D^{\beta_1}y(t) + k_2(t)^{CF}D^{\beta_2}y(t) + \int_0^t f(s, y(s), m_1(s)y'(s), m_2(s)y''(s), h(s)^{CF}D^\gamma y(s), g(s)^{CF}D^\nu y(s)) ds \right) \right| \\ & \leq |\mu(t)| |x(t) - y(t)| + |\mu_1(t)| |x'(t) - y'(t)| + |\mu_2(t)| |x''(t) - y''(t)| \\ & \quad + |k_1(t)|^{CF}D^{\beta_1} |x(t) - y(t)| + |k_2(t)|^{CF}D^{\beta_2} |x(t) - y(t)| \\ & \quad + \int_0^t |f(s, x(s), m_1(s)x'(s), m_2(s)x''(s), h(s)^{CF}D^\gamma x(s), g(s)^{CF}D^\nu x(s)) - f(s, y(s), m_1(s)y'(s), m_2(s)y''(s), h(s)^{CF}D^\gamma y(s), g(s)^{CF}D^\nu y(s))| ds \\ & \leq M_1 \|x - y\| + M_2 \|x - y\| + M_3 \|x - y\| \\ & \quad + \frac{M_4 B(\beta_1 - 1)}{2 - \beta_1} \|x - y\| + \frac{M_5 B(\beta_2 - 2)}{3 - \beta_2} \|x - y\| \\ & \quad + \left[\xi_1 + \xi_2 M_6 + \xi_3 M_7 + \xi_4 \frac{M_8 B(\gamma - 1)}{2 - \gamma} + \xi_5 \frac{M_9 B(\nu - 2)}{3 - \nu} \right] \|x - y\| \\ & \leq \left[M_1 + M_2 + M_3 + \frac{M_4 B(\beta_1 - 1)}{2 - \beta_1} + \frac{M_5 B(\beta_2 - 2)}{3 - \beta_2} \right. \\ & \quad \left. + \xi_1 + \xi_2 M_6 + \xi_3 M_7 + \xi_4 \frac{M_8 B(\gamma - 1)}{2 - \gamma} + \xi_5 \frac{M_9 B(\nu - 2)}{3 - \nu} \right] \|x - y\| \end{aligned}$$

and

$$\begin{aligned}
 & |R'x(t) - R'y(t)| \\
 & \leq |\mu'(t)| |x(t) - y(t)| + |\mu(t) + \mu_1'(t)| |x'(t) - y'(t)| \\
 & \quad + |\mu_1(t) + \mu_2'(t)| |x''(t) - y''(t)| + |\mu_2(t)| |x'''(t) - y'''(t)| \\
 & \quad + \left| \frac{1 - \beta_1}{2 - \beta_1} k_1(t) + k_1'(t) \right| {}^{\text{CF}}D^{\beta_1} |x(t) - y(t)| + \frac{|k_1(t)B(\beta_1 - 1)|}{2 - \beta_1} |x''(t) - y''(t)| \\
 & \quad + \left| \frac{2 - \beta_2}{3 - \beta_2} k_2(t) + k_2'(t) \right| {}^{\text{CF}}D^{\beta_2} |x(t) - y(t)| + \frac{|k_2'(t)B(\beta_2 - 2)|}{3 - \beta_2} |x'''(t) - y'''(t)| \\
 & \quad + |f(t, x(t), m_1(t)x'(t), m_2(t)x''(t), h(t) {}^{\text{CF}}D^\nu x(t), g(t) {}^{\text{CF}}D^\nu x(t)) \\
 & \quad - f(t, y(t), m_1(t)y'(t), m_2(t)y''(t), h(t) {}^{\text{CF}}D^\nu y(t), g(t) {}^{\text{CF}}D^\nu y(t))|.
 \end{aligned}$$

Hence, we get

$$\begin{aligned}
 & |R'x(t) - R'y(t)| \\
 & \leq \left(N_1 + M_1 + N_2 + M_2 + N_3 + M_3 + B(\beta_1 - 1) \left[\frac{|1 - \beta_1|M_4}{(2 - \beta_1)^2} + \frac{N_4 + M_4}{2 - \beta_1} \right] \right. \\
 & \quad \left. + B(\beta_2 - 2) \left[\frac{|2 - \beta_2|M_5}{(3 - \beta_2)^2} + \frac{M_5 + N_5}{3 - \beta_2} \right] + \xi_1 + \xi_2 M_6 + \xi_3 M_7 \right. \\
 & \quad \left. + \xi_4 \frac{M_8 B(\gamma - 1)}{2 - \gamma} + \xi_5 \frac{M_9 B(\nu - 1)}{3 - \nu} \right) \|x - y\|.
 \end{aligned}$$

On the other hand, we have

$$\begin{aligned}
 |Fx(t) - Fy(t)| & \leq \frac{3}{2B(\alpha)} \left[M_1 + M_2 + M_3 + \frac{M_4 B(\beta_1 - 1)}{2 - \beta_1} + \frac{M_5 B(\beta_2 - 2)}{3 - \beta_2} \right. \\
 & \quad \left. + \xi_1 + \xi_2 M_6 + \xi_3 M_7 + \xi_4 \frac{M_8 B(\gamma - 1)}{2 - \gamma} + \xi_5 \frac{M_9 B(\nu - 2)}{3 - \nu} \right] \|x - y\| \\
 & = \Delta_1 \|x - y\|
 \end{aligned}$$

and

$$\begin{aligned}
 F'x(t) - F'y(t) & = \left[\frac{1 - \alpha}{B(\alpha)} + \frac{\alpha}{B(\alpha)} t \right] \int_0^t (Rx(s) - Ry(s)) ds - \frac{\alpha}{B(\alpha)} \int_0^t s(Rx(s) - Ry(s)) ds \\
 & \quad - \frac{(1 - \alpha)}{2B(\alpha)} \int_0^1 (Rx(s) - Ry(s)) ds - \frac{\alpha t}{B(\alpha)} \int_0^1 (Rx(s) - Ry(s)) ds \\
 & \quad + \frac{\alpha}{2B(\alpha)} \int_0^1 s(Rx(s) - Ry(s)) ds \\
 & \leq \frac{3}{2} \left[\frac{3 + 4\alpha}{2B(\alpha)} \right] \left[M_1 + M_2 + M_3 + \frac{M_4 B(\beta_1 - 1)}{2 - \beta_1} + \frac{M_5 B(\beta_2 - 2)}{3 - \beta_2} \right. \\
 & \quad \left. + \xi_1 + \xi_2 M_6 + \xi_3 M_7 + \xi_4 \frac{M_8 B(\gamma - 1)}{2 - \gamma} + \xi_5 \frac{M_9 B(\nu - 2)}{3 - \nu} \right] \|x - y\| \\
 & = \Delta_2 \|x - y\|
 \end{aligned}$$

and so $|F'x(t) - F'y(t)| \leq \Delta_2 \|x - y\|$. Also, we have

$$\begin{aligned} |F''x(t) - F''y(t)| &\leq \frac{\alpha}{B(\alpha)} \int_0^t |Rx(s) - Ry(s)| ds + \frac{1-\alpha}{B(\alpha)} |Rx(t) - Ry(t)| \\ &\quad + \frac{\alpha}{B(\alpha)} \int_0^1 |Rx(s) - Ry(s)| ds \\ &\leq \frac{1+\alpha}{B(\alpha)} \left[M_1 + M_2 + M_3 + \frac{M_4 B(\beta_1 - 1)}{2 - \beta_1} + \frac{M_5 B(\beta_2 - 2)}{3 - \beta_2} \right. \\ &\quad \left. + \xi_1 + \xi_2 M_6 + \xi_3 M_7 + \xi_4 \frac{M_8 B(\gamma - 1)}{2 - \gamma} + \xi_5 \frac{M_9 B(\nu - 2)}{3 - \nu} \right] \\ &= \Delta_3 \|x - y\| \end{aligned}$$

and

$$\begin{aligned} |F'''x(t) - F'''y(t)| &= \frac{\alpha}{B(\alpha)} |Rx(t) - Ry(t)| + \frac{1-\alpha}{B(\alpha)} |R'x(t) - R'y(t)| \\ &\leq \left(\frac{\alpha}{B(\alpha)} \left[M_1 + M_2 + M_3 + \frac{M_4 B(\beta_1 - 1)}{2 - \beta_1} + \frac{M_5 B(\beta_2 - 2)}{3 - \beta_2} + \xi_1 + \xi_2 M_6 + \xi_3 M_7 \right. \right. \\ &\quad \left. \left. + \xi_4 \frac{M_8 B(\gamma - 1)}{2 - \gamma} + \xi_5 \frac{M_9 B(\nu - 2)}{3 - \nu} \right] + \frac{1-\alpha}{B(\alpha)} \left[N_1 + M_1 + N_2 + M_2 + N_3 + M_3 \right. \right. \\ &\quad \left. \left. + B(\beta_1 - 1) \left[\frac{|1 - \beta_1| M_4}{(2 - \beta_1)^2} + \frac{N_4 + M_4}{2 - \beta_1} \right] + B(\beta_2 - 2) \left[\frac{|2 - \beta_2| M_5}{(3 - \beta_2)^2} + \frac{M_5 + N_5}{3 - \beta_2} \right] \right. \right. \\ &\quad \left. \left. + \xi_1 + \xi_2 M_6 + \xi_3 M_7 + \xi_4 \frac{M_8 B(\gamma - 1)}{2 - \gamma} + \xi_5 \frac{M_9 B(\nu - 1)}{3 - \nu} \right] \right) \|x - y\| \\ &= \Delta_4 \|x - y\|. \end{aligned}$$

Thus, $\|Fx - Fy\| \leq \Delta \|x - y\|$ for all $x, y \in C^3_{\mathbb{R}}[0, 1]$. Put $\varphi(t) = 2t$ and $\phi(t) = t$ for all t . Now by using Theorem 1.2, F has a unique fixed point which is the unique solution for the problem (1). □

Lemma 2.6 *Let $\alpha \in (0, 1)$ and $y \in H^1(0, 1)$. Then the fractional differential equation ${}^{CF}D^{\alpha(2)} x(t) = y(t)$ with boundary conditions $x(0) = 0$, $x'(1) + x'(0) = 0$ and $x''(0) = 0$ has the unique solution $x(t) = \int_0^1 G(t, s)y(s) ds$, where $G(t, s) = \frac{-(1-\alpha)t}{(2-\alpha)B(\alpha)} + \frac{-\alpha t}{(2-\alpha)B(\alpha)}(t - s)$ whenever $0 < t \leq s < 1$ and $G(t, s) = \frac{1-\alpha}{B(\alpha)}(t - s) + \frac{\alpha}{2B(\alpha)}(t - s)^2 - \frac{(1-\alpha)t}{B(\alpha)(2-\alpha)} - \frac{\alpha t}{B(\alpha)(2-\alpha)}(t - s)$ whenever $0 < s \leq t < 1$.*

Proof By using Lemma 2.3, we get $x(t) = \frac{1-\alpha}{B(\alpha)} J^2 y(t) + \frac{\alpha}{B(\alpha)} J^3 y(t) + x'(0)t + \frac{\alpha}{1-\alpha} x'(0)t$. Hence, $x'(t) = \frac{1-\alpha}{B(\alpha)} J^1 y(t) + \frac{\alpha}{B(\alpha)} J^2 y(t) + \frac{1}{1-\alpha} x'(0)$. By using the boundary conditions $x'(1) + x'(0) = 0$ and $x'(1) = \frac{1-\alpha}{B(\alpha)} J^1 y(1) + \frac{\alpha}{B(\alpha)} J^2 y(1) + \frac{1}{1-\alpha} x'(0)$, we obtain $x(t) = \frac{1-\alpha}{B(\alpha)} J^2 y(t) + \frac{\alpha}{B(\alpha)} J^3 y(t) - \frac{(1-\alpha)t}{(2-\alpha)B(\alpha)} J^1 y(1) - \frac{\alpha t}{(2-\alpha)B(\alpha)} J^2 y(1)$. Thus, $x(t) = \frac{1-\alpha}{B(\alpha)} \int_0^t y(s)(t - s) ds + \frac{\alpha}{2B(\alpha)} \int_0^t y(s)(t - s)^2 ds - \frac{(1-\alpha)t}{(2-\alpha)B(\alpha)} \int_0^1 y(s) ds - \frac{\alpha t}{(2-\alpha)B(\alpha)} \int_0^1 y(s)(t - s) ds = \int_0^1 G(t, s)y(s) ds$. Note that $({}^{CF}D^{\alpha(2)})^2 x(t) = 0$ if and only if $x(t) = 0$. This implies that the given map $x(t)$ is a unique solution. □

Note that $|G(t, s)| \leq \frac{1-\alpha}{B(\alpha)} + \frac{\alpha}{2B(\alpha)} + \frac{-\alpha t}{B(\alpha)} + \frac{-(1-\alpha)t}{B(\alpha)} < \frac{2}{B(\alpha)}$, for $t \in [0, 1]$. Let $\alpha, \beta_1, \beta_2, \gamma, \nu \in (0, 1)$. Now, we investigate the DCF fractional integro-differential problem

$$\begin{aligned}
 ({}^{\text{CF}}D^\alpha)^{(2)}x(t) &= \mu(t)x(t) + \mu_1(t)x'(t) + \mu_2(t)x''(t) \\
 &\quad + \lambda_1 k_1(t)({}^{\text{CF}}D^{\beta_1})^{(1)}x(t) + \lambda_2 k_2(t)({}^{\text{CF}}D^{\beta_2})^{(2)}x(t) \\
 &\quad + \int_0^t f(s, x(s), m_1(s)x'(s), m_2(s)x''(s), \\
 &\quad h(s)({}^{\text{CF}}D^\nu)^{(1)}x(s), g(s)({}^{\text{CF}}D^\nu)^{(2)}x(s)) ds
 \end{aligned} \tag{2}$$

with boundary conditions $x(0) = 0, x'(1) + x'(0) = 0$ and $x''(0) = 0$.

Theorem 2.7 *Let $\xi_1, \xi_2, \xi_3, \xi_4$, and ξ_5 be nonnegative real numbers, $f : [0, 1] \times \mathbb{R}^5 \rightarrow \mathbb{R}$ an integrable function such that*

$$\begin{aligned}
 &|f(t, x, y, z, v, w) - f(t, x', y', z', v', w')| \\
 &\leq \xi_1|x - x'| + \xi_2|y - y'| + \xi_3|z - z'| + \xi_4|v - v'| + \xi_5|w - w'|
 \end{aligned}$$

for all real numbers $x, y, z, v, w, x', y', z', v', w'$ and $t \in I$. If $\Delta < \frac{1}{2}$, then the problem (2) has a unique solution, where $\Delta := \max\{\Delta_1, \Delta_2, \Delta_3, \Delta_4\}$, $\Delta_1 = \frac{2}{(2-\alpha)B(\alpha)}[M_1 + M_2 + M_3 + \frac{M_4B(\beta_1)}{(1-\beta_1)^2} + \frac{(\beta_2^2-\beta_2+1)M_5B(\beta_2)}{(1-\beta_2)^3} + \xi_1 + \xi_2M_6 + \xi_3M_7 + \xi_4 \frac{M_8B(\gamma)}{(1-\gamma)^2} + \xi_5 \frac{(v^2-\nu+1)M_9B(\nu)}{(1-\nu)^3}]$, $\Delta_2 = \frac{3+\alpha}{B(\alpha)(2-\alpha)}[M_1 + M_2 + M_3 + \frac{M_4B(\beta_1)}{(1-\beta_1)^2} + \frac{(\beta_2^2-\beta_2+1)M_5B(\beta_2)}{(1-\beta_2)^3} + \xi_1 + \xi_2M_6 + \xi_3M_7 + \xi_4 \frac{M_8B(\gamma)}{(1-\gamma)^2} + \xi_5 \frac{(v^2-\nu+1)M_9B(\nu)}{(1-\nu)^3}]$, $\Delta_3 = \frac{2+\alpha}{B(\alpha)(2-\alpha)}[M_1 + M_2 + M_3 + \frac{M_4B(\beta_1)}{(1-\beta_1)^2} + \frac{(\beta_2^2-\beta_2+1)M_5B(\beta_2)}{(1-\beta_2)^3} + \xi_1 + \xi_2M_6 + \xi_3M_7 + \xi_4 \frac{M_8B(\gamma)}{(1-\gamma)^2} + \xi_5 \frac{(v^2-\nu+1)M_9B(\nu)}{(1-\nu)^3}]$ and $\Delta_4 = \frac{\alpha}{B(\alpha)}[M_1 + M_2 + M_3 + \frac{M_4B(\beta_1)}{(1-\beta_1)^2} + \frac{(\beta_2^2-\beta_2+1)M_5B(\beta_2)}{(1-\beta_2)^3} + \xi_1 + \xi_2M_6 + \xi_3M_7 + \xi_4 \frac{M_8B(\gamma)}{(1-\gamma)^2} + \xi_5 \frac{(v^2-\nu+1)M_9B(\nu)}{(1-\nu)^3}] + \frac{1-\alpha}{B(\alpha)}[N_1 + M_1 + N_2 + M_2 + N_3 + M_3 + M_4B(\beta_1) \frac{(\beta_1^2-\beta_1+1)}{(1-\beta_1)^3} + \frac{B(\beta_1)N_4}{(1-\beta_1)^2} + M_5B(\beta_2) \frac{(2\beta_2^2-2\beta_2+1)}{(1-\beta_2)^4} + N_5B(\beta_2) \frac{(\beta_2^2-\beta_2+1)}{(1-\beta_2)^3} + \xi_1 + \xi_2M_6 + \xi_3M_7 + \frac{\xi_4M_8B(\gamma)}{(1-\gamma)^2} + \frac{(v^2-\nu+1)M_9B(\nu)}{(1-\nu)^3}]$.

Proof Consider the Banach space $C^3_{\mathbb{R}}[0, 1]$ equipped with the norm $\|x\| = \max_{t \in I} |x(t)| + \max_{t \in I} |x'(t)| + \max_{t \in I} |x''(t)| + \max_{t \in I} |x'''(t)|$. Define the map $F : C^3_{\mathbb{R}}[0, 1] \rightarrow C^3_{\mathbb{R}}[0, 1]$ by

$$\begin{aligned}
 Fx(t) &= \int_0^1 G(t, s)R(s) ds \\
 &= \frac{1-\alpha}{B(\alpha)} \int_0^t R(s)(t-s) ds + \frac{\alpha}{2B(\alpha)} \int_0^t R(s)(t-s)^2 ds \\
 &\quad - \frac{(1-\alpha)t}{B(\alpha)(2-\alpha)} \int_0^1 R(s) ds - \frac{\alpha t}{B(\alpha)(2-\alpha)} \int_0^1 R(s)(t-s) ds,
 \end{aligned}$$

where

$$\begin{aligned}
 (Rx)(t) &= \mu(t)x(t) + \mu_1(t)x'(t) + \mu_2(t)x''(t) \\
 &\quad + \lambda_1 k_1(t)({}^{\text{CF}}D^{\beta_1})^{(1)}x(t) + \lambda_2 k_2(t)({}^{\text{CF}}D^{\beta_2})^{(2)}x(t) \\
 &\quad + \int_0^t f(s, x(s), m_1(s)x'(s), m_2(s)x''(s), h(s)({}^{\text{CF}}D^\nu)^{(1)}x(s), g(s)({}^{\text{CF}}D^\nu)^{(2)}x(s)) ds
 \end{aligned}$$

and

$$\begin{aligned}
 (R'x)(t) &= \mu(t)x'(t) + \mu'(t)x(t) + \mu_1'(t)x'(t) + \mu_1x''(t) \\
 &\quad + \mu_2'(t)x''(t) + \mu_2(t)x'''(t) + k_1'(t)({}^{\text{CF}}D^{\beta_1})^{(1)}x(t) \\
 &\quad + k_1(t)B(\beta_1) \left[\left(\frac{-\beta_1}{1-\beta_1} \right)^2 {}^{\text{CF}}D^{\beta_1}x(t) - \frac{\beta_1}{(1-\beta_1)^2}x'(t) + \frac{1}{1-\beta_1}x''(t) \right] \\
 &\quad + k_2(t)B(\beta_2) \left[\left(\frac{-\beta_2}{1-\beta_2} \right)^3 {}^{\text{CF}}D^{\beta_2}x(t) + \frac{\beta_2^2}{(1-\beta_2)^3}x'(t) \right. \\
 &\quad \left. - \frac{\beta_2}{(1-\beta_2)^2}x''(t) + \frac{x'''(t)}{1-\beta_2} \right] + k_2'(t)({}^{\text{CF}}D^{\beta_2})^{(2)}x(t) \\
 &\quad + f(t, x(t), m_1(t)x'(t), m_2(t)x''(t), h(t)({}^{\text{CF}}D^\gamma)^{(1)}x(t), g(t)({}^{\text{CF}}D^\nu)^{(2)}x(t)).
 \end{aligned}$$

By using Lemma 2.6, x_0 is a solution for the problem (2) if and only if x_0 is a fixed point of the operator F . Note that

$$\begin{aligned}
 (Rx)(t) - (Ry)(t) &\leq \left[\mu(t)x(t) + \mu_1(t)x'(t) + \mu_2(t)x''(t) + \lambda_1k_1(t)({}^{\text{CF}}D^{\beta_1})^{(1)}x(t) + k_2(t)({}^{\text{CF}}D^{\beta_2})^{(2)}x(t) \right. \\
 &\quad \left. + \int_0^t f(s, x(s), m_1(t)x'(s), m_2(s)x''(s), h(s)({}^{\text{CF}}D^\gamma)^{(1)}x(s), g(s)({}^{\text{CF}}D^\nu)^{(2)}x(s)) ds \right] \\
 &\quad - \left[\mu(t)y(t) + \mu_1(t)y'(t) + \mu_2(t)y''(t) + k_1(t)({}^{\text{CF}}D^{\beta_1})^{(1)}y(t) + k_2(t)({}^{\text{CF}}D^{\beta_2})^{(2)}y(t) \right. \\
 &\quad \left. + \int_0^t f(s, y(s), m_1(t)y'(s), m_2(s)y''(s), h(s)({}^{\text{CF}}D^\gamma)^{(1)}y(s), g(s)({}^{\text{CF}}D^\nu)^{(2)}y(s)) ds \right] \\
 &\leq |\mu(t)||x(t) - y(t)| + |\mu_1(t)||x'(t) - y'(t)| + |\mu_2(t)||x''(t) - y''(t)| \\
 &\quad + |k_1(t)|({}^{\text{CF}}D^{\beta_1})^{(1)}|x(t) - y(t)| + |k_2(t)|({}^{\text{CF}}D^{\beta_2})^{(2)}|x(t) - y(t)| \\
 &\quad + \int_0^t |f(s, x(s), m_1(t)x'(s), m_2(s)x''(s), h(s)({}^{\text{CF}}D^\gamma)^{(1)}x(s), g(s)({}^{\text{CF}}D^\nu)^{(2)}x(s)) - f(s, y(s), m_1(t)y'(s), m_2(s)y''(s), h(s)({}^{\text{CF}}D^\gamma)^{(1)}y(s), g(s)({}^{\text{CF}}D^\nu)^{(2)}y(s))| ds \\
 &\leq M_1\|x - y\| + M_2\|x - y\| + M_3\|x - y\| \\
 &\quad + \frac{M_4B(\beta_1)}{(1-\beta_1)^2}\|x - y\| + \frac{(\beta_2^2 - \beta_2 + 1)M_5B(\beta_2)}{(1-\beta_2)^3}\|x - y\| \\
 &\quad + \left[\xi_1 + \xi_2M_6 + \xi_3M_7 + \xi_4 \frac{M_8B(\gamma)}{(1-\gamma)^2} + \xi_5 \frac{(\nu^2 - \nu + 1)M_9B(\nu)}{(1-\nu)^3} \right] \|x - y\| \\
 &\leq \left[M_1 + M_2 + M_3 + \frac{M_4B(\beta_1)}{(1-\beta_1)^2} + \frac{(\beta_2^2 - \beta_2 + 1)M_5B(\beta_2)}{(1-\beta_2)^3} \right. \\
 &\quad \left. + \xi_1 + \xi_2M_6 + \xi_3M_7 + \xi_4 \frac{M_8B(\gamma)}{(1-\gamma)^2} + \xi_5 \frac{(\nu^2 - \nu + 1)M_9B(\nu)}{(1-\nu)^3} \right] \|x - y\|
 \end{aligned}$$

and

$$\begin{aligned}
 &|R'x(t) - R'y(t)| \\
 &\leq \left(N_1 + M_1 + N_2 + M_2 + N_3 + M_3 + M_4B(\beta_1) \frac{(\beta_1^2 - \beta_1 + 1)}{(1 - \beta_1)^3} + \frac{B(\beta_1)N_4}{(1 - \beta_1)^2} \right. \\
 &\quad + M_5B(\beta_2) \frac{(2\beta_2^2 - 2\beta_2 + 1)}{(1 - \beta_2)^4} + N_5B(\beta_2) \frac{(\beta_2^2 - \beta_2 + 1)}{(1 - \beta_2)^3} + \xi_1 + \xi_2M_6 + \xi_3M_7 \\
 &\quad \left. + \frac{\xi_4M_8B(\gamma)}{(1 - \gamma)^2} + \frac{(\nu^2 - \nu + 1)M_9B(\nu)}{(1 - \nu)^3} \right) \|x - y\|.
 \end{aligned}$$

Thus,

$$\begin{aligned}
 |Fx(t) - Fy(t)| &\leq \frac{2}{(2 - \alpha)B(\alpha)} \left[M_1 + M_2 + M_3 + \frac{M_4B(\beta_1)}{(1 - \beta_1)^2} + \frac{(\beta_2^2 - \beta_2 + 1)M_5B(\beta_2)}{(1 - \beta_2)^3} \right. \\
 &\quad \left. + \xi_1 + \xi_2M_6 + \xi_3M_7 + \xi_4 \frac{M_8B(\gamma)}{(1 - \gamma)^2} + \xi_5 \frac{(\nu^2 - \nu + 1)M_9B(\nu)}{(1 - \nu)^3} \right] \|x - y\| \\
 &= \Delta_1 \|x - y\|
 \end{aligned}$$

and

$$\begin{aligned}
 |F'x(t) - F'y(t)| &\leq \left[\frac{3 + \alpha}{B(\alpha)(2 - \alpha)} \right] \left[M_1 + M_2 + M_3 + \frac{M_4B(\beta_1)}{(1 - \beta_1)^2} + \frac{(\beta_2^2 - \beta_2 + 1)M_5B(\beta_2)}{(1 - \beta_2)^3} \right. \\
 &\quad \left. + \xi_1 + \xi_2M_6 + \xi_3M_7 + \xi_4 \frac{M_8B(\gamma)}{(1 - \gamma)^2} + \xi_5 \frac{(\nu^2 - \nu + 1)M_9B(\nu)}{(1 - \nu)^3} \right] \|x - y\| \\
 &= \Delta_2 \|x - y\|.
 \end{aligned}$$

Also, we have

$$\begin{aligned}
 |F''x(t) - F''y(t)| &\leq \frac{2 + \alpha}{B(\alpha)(2 - \alpha)} \left[M_1 + M_2 + M_3 + \frac{M_4B(\beta_1)}{(1 - \beta_1)^2} \right. \\
 &\quad + \frac{(\beta_2^2 - \beta_2 + 1)M_5B(\beta_2)}{(1 - \beta_2)^3} + \xi_1 + \xi_2M_6 + \xi_3M_7 \\
 &\quad \left. + \xi_4 \frac{M_8B(\gamma)}{(1 - \gamma)^2} + \xi_5 \frac{(\nu^2 - \nu + 1)M_9B(\nu)}{(1 - \nu)^3} \right] \\
 &= \Delta_3 \|x - y\|
 \end{aligned}$$

and

$$\begin{aligned}
 &|F'''x(t) - F'''y(t)| \\
 &\leq \frac{\alpha}{B(\alpha)} |Rx(t) - Ry(t)| + \frac{1 - \alpha}{B(\alpha)} |R'x(t) - R'y(t)| \\
 &\leq \left(\frac{\alpha}{B(\alpha)} \left[M_1 + M_2 + M_3 + \frac{M_4B(\beta_1)}{(1 - \beta_1)^2} + \frac{(\beta_2^2 - \beta_2 + 1)M_5B(\beta_2)}{(1 - \beta_2)^3} \right. \right. \\
 &\quad \left. \left. + \xi_1 + \xi_2M_6 + \xi_3M_7 + \xi_4 \frac{M_8B(\gamma)}{(1 - \gamma)^2} + \xi_5 \frac{(\nu^2 - \nu + 1)M_9B(\nu)}{(1 - \nu)^3} \right] \right) \|x - y\|
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{1-\alpha}{B(\alpha)} \left[N_1 + M_1 + N_2 + M_2 + N_3 + M_3 + M_4 B(\beta_1) \frac{(\beta_1^2 - \beta_1 + 1)}{(1-\beta_1)^3} \right. \\
 & + \frac{B(\beta_1)N_4}{(1-\beta_1)^2} + M_5 B(\beta_2) \frac{(2\beta_2^2 - 2\beta_2 + 1)}{(1-\beta_2)^4} + N_5 B(\beta_2) \frac{(\beta_2^2 - \beta_2 + 1)}{(1-\beta_2)^3} \\
 & \left. + \xi_1 + \xi_2 M_6 + \xi_3 M_7 + \frac{\xi_4 M_8 B(\gamma)}{(1-\gamma)^2} + \frac{(v^2 - v + 1)M_9 B(v)}{(1-v)^3} \right] \|x - y\| \\
 & = \Delta_4 \|x - y\|.
 \end{aligned}$$

Hence, $\|Fx - Fy\| \leq \Delta \|x - y\|$ for all $x, y \in C^3_{\mathbb{R}}[0, 1]$. Put $\varphi(t) = 2t$ and $\phi(t) = t$ for all t . By using Theorem 1.2, F has a unique fixed point, which is the desired solution for the problem. \square

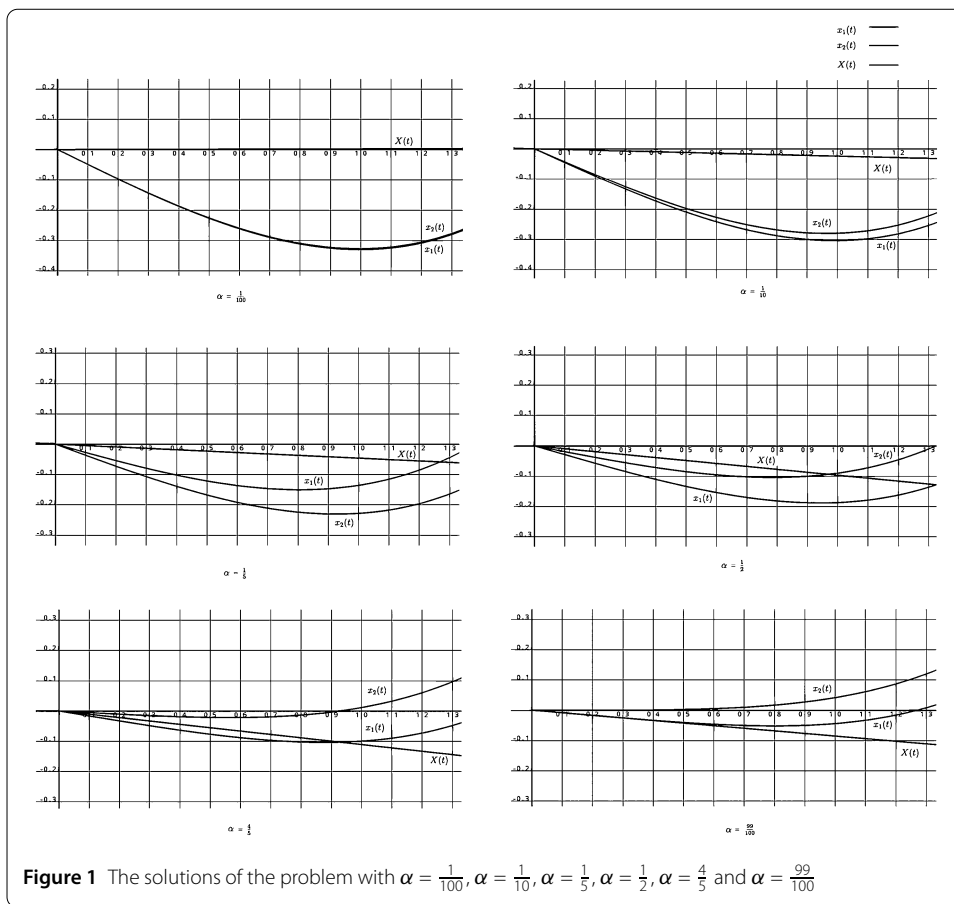
Here, we provide three examples to illustrate our main results. Consider the bounded continuous functions $\mu(t) = \frac{1}{100} \sin(t)$, $\mu_1(t) = \frac{3t-1}{20t+162}$, $\mu_2(t) = \frac{1}{100} e^{-6t}$, $k_1(t) = \frac{1}{300} t^3 + \frac{1}{100} t + \frac{1}{50}$, $k_2(t) = \frac{1}{800} \cos(t)$, $m_1(t) = e^{2t}$, $m_2(t) = \frac{Ln(t+2)}{20}$, $h(t) = 0$ and $g(t) = \frac{1}{t-900}$ for all $t \in I = [0, 1]$. Note that $M_1 = \sup_{t \in I} |\mu(t)| = \frac{1}{100}$, $M_2 = \sup_{t \in I} |\mu_1(t)| = \frac{1}{91}$, $M_3 = \sup_{t \in I} |\mu_2(t)| = \frac{1}{100e^6}$, $M_4 = \sup_{t \in I} |k_1(t)| = \frac{1}{30}$, $M_5 = \sup_{t \in I} |k_2(t)| = \frac{1}{800}$, $M_6 = \sup_{t \in I} |m_1(t)| = e^2$, $M_7 = \sup_{t \in I} |m_2(t)| = \frac{Ln(3)}{1200}$, $M_8 = \sup_{t \in I} |h(t)| = 0$ and $M_9 = \sup_{t \in I} |g(t)| = \frac{1}{900}$. Also, $N_1 = \sup_{t \in I} |\mu'(t)| = \frac{1}{100}$, $N_2 = \sup_{t \in I} |\mu'_1(t)| = \frac{506}{(162)^2}$, $N_3 = \sup_{t \in I} |\mu'_2(t)| = \frac{6}{100e^6}$, $N_4 = \sup_{t \in I} |k'_1(t)| = \frac{1}{50}$ and $N_5 = \sup_{t \in I} |k'_2(t)| = \frac{1}{800}$. Also, consider the function $B(\alpha) = 1$ for $\alpha \in (0, 1)$.

Example 2.1 Let $\alpha \in (0, 1)$. By using Lemma 2.4, the fractional differential equation ${}^{CF}D^{2+\alpha} x_1(t) = t$ with boundary conditions $x_1(0) = 0$, $x'_1(1) + x'_1(0) = 0$ and $x''_1(0) = 0$ has the unique solution $x_1(t)$. Also by using Lemma 2.6, the fractional differential equation $({}^{CF}D^\alpha)^{(2)} x_2(t) = t$ with boundary conditions $x_2(0) = 0$, $x'_2(1) + x'_2(0) = 0$ and $x''_2(0) = 0$ has the unique solution $x_2(t)$. For $\alpha = \frac{1}{100}$, $\alpha = \frac{1}{10}$, $\alpha = \frac{1}{5}$, $\alpha = \frac{1}{2}$, $\alpha = \frac{4}{5}$ and $\alpha = \frac{99}{100}$ we compare the solutions $x_1(t)$, $x_2(t)$ and $X(t) = x_1(t) - x_2(t)$ in Fig. 1.

Example 2.2 Consider the CFD fractional integro-differential problem

$$\begin{aligned}
 {}^{CF}D^{\frac{12}{5}} x(t) &= \frac{\sin(t)}{100} x(t) + \frac{3t-1}{20t+162} x'(t) + \frac{e^{-6t}}{100} x''(t) \\
 &+ \left(\frac{t^3 + 3t + 6}{300} \right) {}^{CF}D^{\frac{3}{2}} x(t) + \frac{\cos(t)}{800} {}^{CF}D^{\frac{5}{2}} x(t) \\
 &+ \int_0^t f \left(s, x(s), e^{2t} x'(s), \frac{Ln(t+2)}{20} x''(s), 0, \frac{1}{t-900} {}^{CF}D^{\frac{8}{3}} x(s) \right) ds \tag{3}
 \end{aligned}$$

with boundary conditions $x(0) = 0$, $x'(1) + x'(0) = 0$ and $x''(0) = 0$, where $1 < \beta_1 = \frac{3}{2} < 2 < \beta_2 = \frac{5}{2} < 3$ and $1 < \gamma = \frac{4}{3} < 2 < v = \frac{8}{3} < 3$. Put $f(t, x, y, z, v, w) = \frac{2}{91} t + \frac{3}{604} x + \frac{1}{200} y + \frac{1}{80} z + \frac{1}{e^{18}} w + 2v$. Note that $\Delta_1 = \frac{3}{2} [M_1 + M_2 + M_3 + \frac{M_4}{2-\beta_1} + \frac{M_5}{3-\beta_2} + \xi_1 + \xi_2 M_6 + \xi_3 M_7 + \xi_4 \frac{M_8}{2-\gamma} + \xi_5 \frac{M_9}{3-v}] = 0.02$, $\Delta_2 = \frac{3}{2} [\frac{3+4\alpha}{2}] [M_1 + M_2 + M_3 + \frac{M_4}{2-\beta_1} + \frac{M_5}{3-\beta_2} + \xi_1 + \xi_2 M_6 + \xi_3 M_7 + \xi_4 \frac{M_8}{2-\gamma} + \xi_5 \frac{M_9}{3-v}] = 0.46$, $\Delta_3 = (1 + \alpha) [M_1 + M_2 + M_3 + \frac{M_4}{2-\beta_1} + \frac{M_5}{3-\beta_2} + \xi_1 + \xi_2 M_6 + \xi_3 M_7 + \xi_4 \frac{M_8}{2-\gamma} + \xi_5 \frac{M_9}{3-v}] = 0.32$ and $\Delta_4 = \alpha [M_1 + M_2 + M_3 + \frac{M_4}{2-\beta_1} + \frac{M_5}{3-\beta_2} + \xi_1 + \xi_2 M_6 + \xi_3 M_7 + \xi_4 \frac{M_8}{2-\gamma} + \xi_5 \frac{M_9}{3-v}] + (1 - \alpha) [N_1 + M_1 + N_2 + M_2 + N_3 + M_3 + \frac{|1-\beta_1|M_4}{(2-\beta_1)^2} + \frac{N_4+M_4}{2-\beta_1} + \frac{|2-\beta_2|M_5}{(3-\beta_2)^2} + \frac{M_5+N_5}{3-\beta_2} + \xi_1 + \xi_2 M_6 + \xi_3 M_7 + \xi_4 \frac{M_8}{2-\gamma} + \xi_5 \frac{M_6}{3-v}] = 0.166$. By using Theorem 2.5, the problem (3) has a unique solution.



Example 2.3 Consider the DCF fractional integro-differential problem

$$\begin{aligned}
 &({}^{\text{CF}}D^{\frac{2}{5}})^{(2)}x(t) \\
 &= \frac{\sin(t)}{100}x(t) + \frac{3t-1}{20t+162}x'(t) + \frac{e^{-6t}}{100}x''(t) \\
 &+ \left(\frac{t^3+3t+6}{300}\right)({}^{\text{CF}}D^{\frac{1}{2}})^{(1)}x(t) + \frac{\cos(t)}{800}({}^{\text{CF}}D^{\frac{2}{3}})^{(2)}x(t) \\
 &+ \int_0^t f\left(s, x(s), e^{-40t}x'(s), \frac{\text{Ln}(t+2)}{20}x''(s), 0, \frac{1}{t-900}({}^{\text{CF}}D^{\frac{1}{5}})^{(2)}x(s)\right) ds, \tag{4}
 \end{aligned}$$

with boundary conditions $x(0) = 0$, $x'(1) + x'(0) = 0$ and $x''(0) = 0$, where $\alpha = \frac{2}{5}$, $\beta_1 = \frac{1}{2}$, $\beta_2 = \frac{2}{3}$, $\gamma = \frac{1}{3}$ and $\nu = \frac{1}{5}$. Put $f(t, x, y, z, v, w) = \frac{2}{91}t + \frac{3}{604}x + \frac{1}{200}y + \frac{1}{80}z + \frac{1}{e^{18}}w + 2v$. Note that $\Delta_1 = \frac{2}{2-\alpha} [M_1 + M_2 + M_3 + \frac{M_4}{(1-\beta_1)^2} + M_5 \frac{(\beta_2^2 - \beta_2 + 1)}{(1-\beta_2)^3} + \xi_1 + \xi_2 M_6 + \xi_3 M_7 + \xi_4 \frac{M_8}{(1-\gamma)^2} + \xi_5 \frac{(v^2 - \nu + 1)M_9}{(1-\nu)^3}] < 0.391$, $\Delta_2 = [\frac{3+\alpha}{2-\alpha}][M_1 + M_2 + M_3 + \frac{M_4}{(1-\beta_1)^2} + \frac{(\beta_2^2 - \beta_2 + 1)M_5}{(1-\beta_2)^3} + \xi_1 + \xi_2 M_6 + \xi_3 M_7 + \xi_4 \frac{M_8}{(1-\gamma)^2} + \xi_5 \frac{(v^2 - \nu + 1)M_9}{(1-\nu)^3}] < 0.225$, $\Delta_3 = \frac{2+\alpha}{2-\alpha} [M_1 + M_2 + M_3 + \frac{M_4}{(1-\beta_1)^2} + \frac{(\beta_2^2 - \beta_2 + 1)M_5}{(1-\beta_2)^3} + \xi_1 + \xi_2 M_6 + \xi_3 M_7 + \xi_4 \frac{M_8}{(1-\gamma)^2} + \xi_5 \frac{(v^2 - \nu + 1)M_9}{(1-\nu)^3}] < 0.132$ and $\Delta_4 = \alpha [M_1 + M_2 + M_3 + \frac{M_4}{(1-\beta_1)^2} + M_5 \frac{(\beta_2^2 - \beta_2 + 1)}{(1-\beta_2)^3} + \xi_1 + \xi_2 M_6 + \xi_3 M_7 + \xi_4 \frac{M_8}{(1-\gamma)^2} + \xi_5 \frac{(v^2 - \nu + 1)M_9}{(1-\nu)^3}] + (1-\alpha) [N_1 + M_1 + N_2 + M_2 + N_3 + M_3 + M_4 \frac{(\beta_1^2 - \beta_1 + 1)}{(1-\beta_1)^3} + \frac{N_4}{(1-\beta_1)^2} + M_5 \frac{(2\beta_2^2 - 2\beta_2 + 1)}{(1-\beta_2)^4} + \frac{N_5(\beta_2^2 - \beta_2 + 1)}{(1-\beta_2)^3} + \xi_1 + \xi_2 M_6 + \xi_3 M_7 + \frac{\xi_4 M_8}{(1-\gamma)^2} + \xi_5 \frac{(v^2 - \nu + 1)M_9}{(1-\nu)^3}] < 0.493$. By using Theorem 2.7, the problem (4) has a unique solution.

3 Conclusion

It is important that researchers have some methods available enabling them to review some high order fractional integro-differential equations. In this manuscript, we introduce two types of new fractional derivatives entitled CFD and DCF and by using those we investigate the existence of solutions for two high order fractional integro-differential equations of such a type including the new derivatives.

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Abbreviations

DFC, Caputo–Fabrizio derivation followed by a differentiation; CFD, Differentiation followed by Caputo–Fabrizio derivation.

Availability of data and materials

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

Competing interests

The authors declare that they have no competing interests.

Authors' contributions

The authors declare that the study was realized in collaboration with equal responsibility. All authors read and approved the final manuscript.

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References

- Abdeljawad, T., Baleanu, D.: On fractional derivatives with exponential kernel and their discrete versions. *Rep. Math. Phys.* **80**(1), 11–27 (2017)
- Alsaedi, A., Baleanu, D., Etemad, S., Rezapour, Sh.: On coupled systems of time-fractional differential problems by using a new fractional derivative. *J. Funct. Spaces* **2016**, Article ID 4626940 (2016)
- Area, I., Losada, J., Nieto, J.J.: A note on the fractional logistic equation. *Phys. A, Stat. Mech. Appl.* **444**, 182–187 (2016)
- Aydogan, S.M., Baleanu, D., Mousalou, A., Rezapour, Sh.: On approximate solutions for two higher-order Caputo–Fabrizio fractional integro-differential equations. *Adv. Differ. Equ.* **2017**, 221 (2017)
- Baleanu, D., Hedayati, V., Rezapour, Sh., Al Qurashi, M.M.: On two fractional differential inclusions. *SpringerPlus* **5**, 882 (2016)
- Baleanu, D., Mousalou, A., Rezapour, Sh.: A new method for investigating some fractional integro-differential equations involving the Caputo–Fabrizio derivative. *Adv. Differ. Equ.* **2017**, 51 (2017)
- Baleanu, D., Mousalou, A., Rezapour, Sh.: On the existence of solutions for some infinite coefficient-symmetric Caputo–Fabrizio fractional integro-differential equations. *Bound. Value Probl.* **2017**, 147 (2017)
- Caputo, M., Fabrizio, M.: A new definition of fractional derivative without singular kernel. *Prog. Fract. Differ. Appl.* **1**(2), 73–85 (2015)
- Caputo, M., Fabrizio, M.: Applications of new time and spatial fractional derivatives with exponential kernels. *Prog. Fract. Differ. Appl.* **2**(1), 1–11 (2016)
- De La Sen, M., Hedayati, V., Atani, Y.G., Rezapour, Sh.: The existence and numerical solution for a k-dimensional system of multi-term fractional integro-differential equations. *Nonlinear Anal., Model. Control* **22**(2), 188–209 (2017)
- Dutta, P.N., Choudary, B.S.: A generalization of contraction principle in metric spaces. *Fixed Point Theory Appl.* **2008**, Article ID 406386 (2008)
- Kojabad, E.A., Rezapour, Sh.: Approximate solutions of a sum-type fractional integro-differential equation by using Chebyshev and Legendre polynomials. *Adv. Differ. Equ.* **2017**, 351 (2017)
- Kojabad, E.A., Rezapour, Sh.: Approximate solutions of a fractional integro-differential equation by using Chebyshev and Legendre polynomials. *J. Adv. Math. Stud.* **11**(1), 80–102 (2018)
- Losada, J., Nieto, J.J.: Properties of a new fractional derivative without singular kernel. *Prog. Fract. Differ. Appl.* **1**(2), 87–92 (2015)

15. Rezapour, Sh., Hedayati, V.: On a Caputo fractional differential inclusion with integral boundary condition for convex-compact and nonconvex-compact valued multifunctions. *Kragujev. J. Math.* **41**(1), 143–158 (2017)
16. Rezapour, Sh., Shabibi, M.: A singular fractional differential equation with Riemann–Liouville integral boundary condition. *J. Adv. Math. Stud.* **8**(1), 80–88 (2015)
17. Shabibi, M., Postolache, M., Rezapour, Sh.: Positive solutions for a singular sum fractional differential system. *Int. J. Anal. Appl.* **13**(1), 108–118 (2017)
18. Shabibi, M., Postolache, M., Rezapour, Sh., Vaezpour, S.M.: Investigation of a multi-singular pointwise defined fractional integro-differential equation. *J. Math. Anal.* **7**(5), 61–77 (2016)
19. Shabibi, M., Rezapour, Sh., Vaezpour, S.M.: A singular fractional integro-differential equation. *UPB Sci. Bull., Ser. A* **79**(1), 109–118 (2017)

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