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Hardy-type inequalities within fractional derivatives without singular kernel

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Abstract
In this manuscript, we developed the Hardy-type inequality within the Caputo–Fabrizio fractional derivative. We presented some illustrative examples to confirm our work.

MSC: 26D10; 26D15; 26A33; 26A40; 26A42; 26A51

Keywords: Hardy-type inequality; Caputo–Fabrizio fractional derivative

1 Introduction

In 1920, Hardy [1] showed that, for $p_1 > 1$, $g \in L^{p_1}(0,\infty)$ being a non-negative function, the following inequality holds:

$$\int_0^\infty \left(\frac{1}{x}\int_0^x g(t)\,dt\right)^{p_1} dx \le \left(\frac{p_1}{p_1-1}\right)^{p_1} \int_0^\infty \bigl(g(x)\bigr)^{p_1}\,dx, \quad p_1 > 1, \tag{1.1}$$

As is well known, the inequality (1.1) is today called to as classical Hardy's integral inequality in the literature. It has many applications in analysis and in the theory of differential equations (see, e.g., [2, 3] and [4]). This inequality has been generalized and developed by many mathematicians. Various mathematicians studied new Hardy-type inequality for different fractional derivatives and integrals; see [4–9] and the references therein.

In 1964, Levinson [10] showed that inequality (1.1) holds for parameters a_1 and b_1. That is, for $0 < a_1 < b_1 < \infty$ the following inequality is valid:

$$\int_{a_1}^{b_1} \left(\frac{1}{x}\int_0^x g(t)\,dt\right)^{p_1} dx \le \left(\frac{p_1}{p_1-1}\right)^{p_1} \int_0^\infty \bigl(g(x)\bigr)^{p_1}\,dx, \quad p_1 > 1. \tag{1.2}$$

In 2010, Iqbal et al. [7] obtained new fractional inequalities within fractional derivatives and integrals of Riemann–Liouville type. In 2011, 2013 and 2014, they proved some new inequalities involving Riemann–Liouville fractional integrals, Caputo fractional derivative and other fractional derivatives; see [11–14].

In 2017, Iqbal et al. [15] presented the Hardy-type inequalities for Hilfer fractional derivative. Also, they obtained the Hardy-type inequality for generalized fractional integral within Mittag–Leffler function in its kernel utilizing convex and increasing functions. In the same year, Iqbal et al. [16] obtained Hardy-type inequalities for a generalized fractional integral operator within the Mittag–Leffler function in its kernel. Also, they set up a

Hilfer fractional derivative utilizing convex and monotone convex function. In 2017, Nasibullin [17] proved new Hardy-type inequalities with fractional integrals and derivatives of Riemann–Liouville.

Recently, a new type of fractional derivative was introduced by Caputo and Fabrizio in [18]. The reason of introducing this new type of derivative was to search for fractional derivative with nonsingular kernel and without the Gamma function. Since then many researchers discussed this and applied this new fractional derivative to several real world phenomena and excellent results were reported [19–35]. On the other side the discrete version of fractional derivative is one of the interesting topics nowadays [19, 36–40]. Some of the applications of the discrete fractional Caputo derivative can be found in [41, 42].

The organization of this paper is given below. In Sect. 1, we given introduction. In Sect. 2, basic definitions and theorems are introduced. Motivated by [12, 26, 36] several Hardy-type inequalities for the new left Riemann fractional derivative are established in Sect. 3. Several examples are given for our results in Sect. 4.

2 Basic definitions and theorems

In this section, we present the following definitions and theorems, which are useful in proofs of our results.

Definition 2.1 ([43]) Let I_1 be an interval, and let ψ be a function $I_1 \to \mathbb{R}$. ψ is called convex if

$$\psi\bigl(\beta x + (1-\beta)t\bigr) \leq \beta\psi(x) + (1-\beta)\psi(t) \tag{2.1}$$

for all points x and t in I_1 and all $0 \leq \beta \leq 1$. ψ is strictly convex if (2.1) holds strictly whenever x and t are distinct points and $0 < \beta < 1$.

Let $(\sum_1, \Omega_1, \mu_1)$ and $(\sum_2, \Omega_2, \mu_2)$ be measure spaces with positive σ-finite measures. Also, let $U_1(g)$ be the class of functions $h : \Omega_1 \to \mathbb{R}$ defined as

$$h(x) := \int_{\Omega_2} k_1(x,t) g(t)\, d\mu_2(t),$$

and let A_{k_1} be an integral operator defined as

$$A_{k_1} g(x) := \frac{h(x)}{K_1(x)} = \frac{1}{K_1(x)} \int_{\Omega_2} k_1(x,t) g(t)\, d\mu_2(t)$$

such that $k_1 : \Omega_1 \times \Omega_2 \to \mathbb{R}$ denotes a non-negative measurable function, $g : \Omega_2 \to \mathbb{R}$ represents a measurable function and

$$K_1(x) := \int_{\Omega_2} k_1(x,t)\, d\mu_2(t) > 0, \quad x \in \Omega_1. \tag{2.2}$$

Theorem 2.1 ([7]) *Let v be a weight function on Ω_1, and $k_1 : \Omega_1 \times \Omega_2 \to \mathbb{R}$ be a non-negative measurable function. Also, let K_1 be defined on Ω_1 by (2.2). Suppose that $x \mapsto v(x)\frac{k_1(x,t)}{K_1(x)}$ is an integrable function on Ω_1 for each fixed $t \in \Omega_2$. Define u on Ω_2 as*

$$u(t) := \int_{\Omega_1} v(x) \frac{k_1(x,t)}{K_1(x)}\, d\mu_1(x) < \infty.$$

If the function $\psi : (0,\infty) \to \mathbb{R}$ denotes a convex and increasing function, then the inequality

$$\int_{\Omega_1} v(x)\psi\left(\left|\frac{h(x)}{K_1(x)}\right|\right) d\mu_1(x) \leq \int_{\Omega_2} u(t)\psi\left(\left|g(t)\right|\right) d\mu_2(t) \tag{2.3}$$

holds for all measurable functions $g : \Omega_2 \to \mathbb{R}$.

In Theorem 2.1, by replacing $k_1(x,t)$ by $k_1(x,t)g_2(t)$ and g by $\frac{g_1}{g_2}$, where the functions $g_j : \Omega_2 \to \mathbb{R}$ are measurable for $j = 1, 2$, the following result is obtained (see [11]).

Theorem 2.2 *Let $g_j : \Omega_2 \to \mathbb{R}$ be measurable functions, $h_j \in U_1(g_j)$ $(j = 1,2)$, with $h_2(x) > 0$ for all $x \in \Omega_1$. Also, let v be a weight function on Ω_1 and $k_1 : \Omega_1 \times \Omega_2 \to \mathbb{R}$ be a non-negative measurable function. Suppose that $x \mapsto v(x)\frac{g_2(t)k_1(x,t)}{h_2(x)}$ is an integrable function on Ω_1 for each fixed $t \in \Omega_2$. Define u on Ω_2 by*

$$u(t) := g_2(t)\int_{\Omega_1} \frac{v(x)k_1(x,t)}{h_2(x)}\, d\mu_1(x) < \infty.$$

If the function $\psi : (0,\infty) \to \mathbb{R}$ denotes a convex and increasing function, then the following inequality holds:

$$\int_{\Omega_1} v(x)\psi\left(\left|\frac{h_1(x)}{h_2(x)}\right|\right) d\mu_1(x) \leq \int_{\Omega_2} u(t)\psi\left(\left|\frac{g_1(t)}{g_2(t)}\right|\right) d\mu_2(t). \tag{2.4}$$

Theorem 2.3 ([7]) *Let $(\sum_1, \Omega_1, \mu_1)$ and $(\sum_2, \Omega_2, \mu_2)$ be measure spaces with positive σ-finite measures. Also let v be a weight function on Ω_1, let $k_1 : \Omega_1 \times \Omega_2 \to \mathbb{R}$ be a non-negative measurable function by* (2.2)*, let K_1 be defined on Ω_1 and $0 < p_1 \leq q_1 < \infty$. If $x \mapsto v(x)\frac{k_1(x,t)}{K_1(x)}$ is an integrable function on Ω_1 for each fixed $t \in \Omega_2$, then u is written as*

$$u(t) := \left[\int_{\Omega_1} v(x)\left(\frac{k_1(x,t)}{K_1(x)}\right)^{\frac{q_1}{p_1}} d\mu_1(x)\right]^{\frac{p_1}{q_1}} < \infty.$$

If the function ψ denotes a non-negative convex on the interval $I_1 \subseteq \mathbb{R}$, then the inequality

$$\left[\int_{\Omega_1} v(x)\left(\psi\left(A_{k_1}g(x)\right)\right)^{\frac{q_1}{p_1}} d\mu_1(x)\right]^{\frac{1}{q_1}} \leq \left[\int_{\Omega_2} u(t)\psi\left(g(t)\right) d\mu_2(t)\right]^{\frac{1}{p_1}} \tag{2.5}$$

holds for all measurable functions $g : \Omega_2 \to \mathbb{R}$ such that $\operatorname{Im} g \subseteq I_1$.

Theorem 2.4 ([7]) *Let $h_j \in U_1(g_j)$ for $j = 1, 2, 3$, with $h_2(x) > 0$ for all $x \in \Omega_1$. Let v be a weight function on Ω_1, and $k_1 : \Omega_1 \times \Omega_2 \to \mathbb{R}$ be a non-negative measurable function, then u is written as*

$$u(t) := g_2(t)\int_{\Omega_1} \frac{v(x)k_1(x,t)}{h_2(x)}\, d\mu_1(x) < \infty.$$

If the function $\psi : (0,\infty) \times (0,\infty) \to \mathbb{R}$ represents a convex and increasing function, then the following inequality holds:

$$\int_{\Omega_1} v(x)\psi\left(\left|\frac{h_1(x)}{h_2(x)}\right|, \left|\frac{h_3(x)}{h_2(x)}\right|\right) d\mu_1(x) \leq \int_{\Omega_2} u(t)\psi\left(\left|\frac{g_1(t)}{g_2(t)}\right|, \left|\frac{g_3(t)}{g_2(t)}\right|\right) d\mu_2(t). \tag{2.6}$$

3 Main results

Below, we show the definition of the new left Riemann fractional derivative, then we discuss Hardy-type inequalities for the new left Riemann fractional derivative.

According to [19], if $g \in H^1(a_1, b_1)$, $0 < a_1 < b_1 \leq \infty$, $\alpha \in (0,1)$, then the left new Riemann fractional derivative ${}_{a_1}^{\mathrm{CFR}}D^{\alpha}$ is defined by

$$\left({}_{a_1}^{\mathrm{CFR}}D^{\alpha}g\right)(x) = \frac{M(\alpha)}{1-\alpha}\frac{d}{dx}\int_{a_1}^{x} g(t)\exp\bigl(\lambda(x-t)\bigr)\,dt, \tag{3.1}$$

with $\lambda = \frac{-\alpha}{1-\alpha}$ and $x \geq a_1$. Here $M(\alpha)$ is a normalization constant depending on α.

Theorem 3.1 *Let $0 < \alpha < 1$, $p_1 > 1$ and $q_1 > 1$. Also, let ${}_{a_1}^{\mathrm{CFR}}D^{\alpha}$ be defined by (3.1). If $g' \in L^{q_1}(a_1, b_1)$, then the following inequality holds true*:

$$\int_{a_1}^{b_1}\left|\left({}_{a_1}^{\mathrm{CFR}}D^{\alpha}g\right)(x)\right|^{q_1}dx \leq C_1\int_{a_1}^{b_1}\left|g'(t)\right|^{q_1}dt, \tag{3.2}$$

where $\frac{1}{p_1} + \frac{1}{q_1} = 1$, $\lambda = \frac{-\alpha}{1-\alpha}$ and $C_1 = (\frac{M(\alpha)}{1-\alpha})^{q_1}(-\frac{1}{p_1\lambda})^{q_1/p_1}(b_1 - a_1)$.

Proof We have

$$\begin{aligned}
\left|\left({}_{a_1}^{\mathrm{CFR}}D^{\alpha}g\right)(x)\right| &= \left|\frac{M(\alpha)}{1-\alpha}\frac{d}{dx}\int_{a_1}^{x} g(t)\exp\bigl(\lambda(x-t)\bigr)\,dt\right| \\
&= \left|\frac{M(\alpha)}{1-\alpha}\frac{d}{dx}\bigl(g(x) * \exp(\lambda x)\bigr)\right| \\
&= \left|\frac{M(\alpha)}{1-\alpha}\left(\frac{dg}{dx}(x) * \exp(\lambda x)\right)\right| \\
&\leq \frac{M(\alpha)}{1-\alpha}\int_{a_1}^{x}\left|g'(t)\right|\exp\bigl(\lambda(x-t)\bigr)\,dt.
\end{aligned}$$

By using Hölder's inequality for $\{p_1, q_1\}$, we can write

$$\begin{aligned}
\left|\left({}_{a_1}^{\mathrm{CFR}}D^{\alpha}g\right)(x)\right| &\leq \frac{M(\alpha)}{1-\alpha}\left(\int_{a_1}^{x}\left|g'(t)\right|^{q_1}dt\right)^{\frac{1}{q_1}}\left(\int_{a_1}^{x}\exp\bigl(p_1\lambda(x-t)\bigr)\,dt\right)^{\frac{1}{p_1}} \\
&= \frac{M(\alpha)}{1-\alpha}\left(-\frac{1}{p_1\lambda} + \frac{\exp(p_1\lambda(x-a_1))}{p_1\lambda}\right)^{\frac{1}{p_1}}\left(\int_{a_1}^{x}\left|g'(t)\right|^{q_1}dt\right)^{\frac{1}{q_1}}.
\end{aligned}$$

Thus we get

$$\begin{aligned}
\left|\left({}_{a_1}^{\mathrm{CFR}}D^{\alpha}g\right)(x)\right|^{q_1} &\leq \left(\frac{M(\alpha)}{1-\alpha}\right)^{q_1}\left(-\frac{1}{p_1\lambda} + \frac{\exp(p_1\lambda(x-a_1))}{p_1\lambda}\right)^{\frac{q_1}{p_1}}\int_{a_1}^{x}\left|g'(t)\right|^{q_1}dt \\
&\leq \left(\frac{M(\alpha)}{1-\alpha}\right)^{q_1}\left(-\frac{1}{p_1\lambda}\right)^{\frac{q_1}{p_1}}\int_{a_1}^{b_1}\left|g'(t)\right|^{q_1}dt.
\end{aligned}$$

Integrating both sides from a_1 to b_1, we obtain the following inequality:

$$\int_{a_1}^{b_1}\left|\left({}_{a_1}^{\mathrm{CFR}}D^{\alpha}g\right)(x)\right|^{q_1}dx \leq \left(\frac{M(\alpha)}{1-\alpha}\right)^{q_1}\left(-\frac{1}{p_1\lambda}\right)^{\frac{q_1}{p_1}}(b_1 - a_1)\int_{a_1}^{b_1}\left|g'(t)\right|^{q_1}dt.$$

Let $C_1 = (\frac{M(\alpha)}{1-\alpha})^{q_1}(-\frac{1}{p_1\lambda})^{\frac{q_1}{p_1}}(b-a)$. Then we obtain (3.2). □

Corollary 3.1 *Let v be a weight function on (a_1, b_1), $0 < \alpha < 1$ and $\lambda = \frac{-\alpha}{1-\alpha}$. Also, let ${}^{\mathrm{CFR}}_{a_1}D^{\alpha}$ be defined by (3.1), let $g \in H^1(a_1, b_1)$ and define u on (a_1, b_1) by*

$$u(t) = -\lambda \int_t^{b_1} \frac{v(x)\exp(\lambda(x-t))}{1-\exp(\lambda(x-a_1))}\,dx < \infty.$$

If the function $\psi : (0,\infty) \to \mathbb{R}$ represents a convex and increasing function, then the inequality

$$\int_{a_1}^{b_1} v(x)\psi\left(\frac{\lambda^2}{1-\exp(\lambda(x-a_1))}\left|\left({}^{\mathrm{CFR}}_{a_1}D^{\alpha}g\right)(x)\right|\right)dx \le \int_{a_1}^{b_1} u(t)\psi\left(\left|g'(t)\right|\right)dt \tag{3.3}$$

holds true.

Proof By applying Theorem 2.1 with $\Omega_1 = \Omega_2 = (a_1, b_1)$, $d\mu_1(x) = dx$, $d\mu_2(t) = dt$,

$$k_1(x,t) = \begin{cases} -\frac{\exp(\lambda(x-t))}{\lambda}, & a_1 \le t \le x, \\ 0, & x < t \le b_1, \end{cases}$$

then we find $K_1(x) = \frac{1-\exp(\lambda(x-a_1))}{\lambda^2}$. Also, if g is replaced by g' and h is taken as ${}^{\mathrm{CFR}}_{a_1}D^{\alpha}g$, we obtain (3.3). □

Remark 3.1 In Corollary 3.1, let $v(x) = 1 - \exp(\lambda(x-a_1))$ be a particular weight function on (a_1, b_1). Then we obtain the following inequality:

$$\begin{aligned} &\int_{a_1}^{b_1} \left(1 - exp\left(\lambda(x-a_1)\right)\right)\psi\left(\frac{\lambda^2}{1-\exp(\lambda(x-a_1))}\left|\left({}^{\mathrm{CFR}}_{a_1}D^{\alpha}g\right)(x)\right|dx\right) \\ &\quad \le \int_{a_1}^{b_1} \left(1-\exp\left(\lambda(b_1-t)\right)\right)\psi\left(\left|g'(t)\right|\right)dt. \end{aligned} \tag{3.4}$$

If the function $\psi : (0,\infty) \to \mathbb{R}$ is defined by $\psi(x) = x^{q_1}$ for $q_1 > 1$, then (3.4) reduces to the following inequality:

$$\begin{aligned} &\int_{a_1}^{b_1} \left(1-\exp\left(\lambda(x-a_1)\right)\right)\left(\frac{\lambda^2}{1-\exp(\lambda(x-a_1))}\left|\left({}^{\mathrm{CFR}}_{a_1}D^{\alpha}g\right)(x)\right|\right)^{q_1}dx \\ &\quad \le \int_{a_1}^{b_1} \left(1-\exp\left(\lambda(b_1-t)\right)\right)\left|g'(t)\right|^{q_1}dt. \end{aligned} \tag{3.5}$$

From $x \in (a_1, b_1)$ and $\lambda < 0$, then for the left-hand side of (3.5) holds the following inequality:

$$\begin{aligned} &\int_{a_1}^{b_1} \frac{\lambda^{2q_1}}{(1-\exp(\lambda(x-a_1)))^{q_1-1}}\left|\left({}^{\mathrm{CFR}}_{a_1}D^{\alpha}g\right)(x)\right|^{q_1}dx \\ &\quad \ge \frac{\lambda^{2q_1}}{(1-\exp(\lambda(b_1-a_1)))^{q_1-1}}\int_{a_1}^{b_1}\left|\left({}^{\mathrm{CFR}}_{a_1}D^{\alpha}g\right)(x)\right|^{q_1}dx. \end{aligned} \tag{3.6}$$

Also, the right-hand of (3.5) satisfies the following inequality:

$$\int_{a_1}^{b_1} \bigl(1-\exp\bigl(\lambda(b_1-t)\bigr)\bigr)\bigl|g'(t)\bigr|^{q_1}\,dt \le \bigl(1-\bigl(\lambda(b_1-a_1)\bigr)\bigr)\int_{a_1}^{b_1}\bigl|g'(t)\bigr|^{q_1}\,dt. \tag{3.7}$$

So, by using (3.6) and (3.7) in (3.5) we obtain

$$\begin{aligned}\frac{\lambda^{2q_1}}{(1-\exp(\lambda(b_1-a_1)))^{q_1-1}}\int_{a_1}^{b_1}\bigl|\bigl({}^{\mathrm{CFR}}_{a_1}D^{\alpha}g\bigr)(x)\bigr|^{q_1}\,dx& \\ \le \bigl(1-\exp\bigl(\lambda(b_1-a_1)\bigr)\bigr)\int_{a_1}^{b_1}\bigl|g'(t)\bigr|^{q_1}\,dt&.\end{aligned}$$

That is, we can write

$$\int_{a_1}^{b_1}\bigl|\bigl({}^{\mathrm{CFR}}_{a_1}D^{\alpha}g\bigr)(x)\bigr|^{q_1}\,dx \le \left(\frac{1-\exp(\lambda(b_1-a_1))}{\lambda^2}\right)^{q_1}\int_{a_1}^{b_1}\bigl|g'(t)\bigr|^{q_1}\,dt.$$

Taking the power $\frac{1}{q_1}$ on both sides, we get

$$\bigl\|{}^{\mathrm{CFR}}_{a_1}D^{\alpha}g\bigr\|_{q_1} \le \frac{1-\exp(\lambda(b_1-a_1))}{\lambda^2}\bigl\|g'\bigr\|_{q_1}.$$

Next, we obtain a special case of Theorem 2.2 for the left new Riemann fractional derivative.

Corollary 3.2 *Let v be a weight function on (a_1,b_1), $0<\alpha<1$ and $\lambda=\frac{-\alpha}{1-\alpha}$. Let ${}^{\mathrm{CFR}}_{a_1}D^{\alpha}$ be defined by (3.1) and define u on (a_1,b_1) by*

$$u(t) = -\frac{g_2'(t)}{\lambda}\int_t^{b_1}\frac{v(x)\exp(\lambda(x-t))}{({}^{\mathrm{CFR}}_{a_1}D^{\alpha}g_2)(x)}\,dx<\infty.$$

If the function $\psi:(0,\infty)\to\mathbb{R}$ denotes a convex and increasing function, then the inequality

$$\int_{a_1}^{b_1} v(x)\psi\left(\left|\frac{({}^{\mathrm{CFR}}_{a_1}D^{\alpha}g_1)(x)}{({}^{\mathrm{CFR}}_{a_1}D^{\alpha}g_2)(x)}\right|\right)dx \le \int_{a_1}^{b_1} u(t)\psi\left(\left|\frac{g_1'(t)}{g_2'(t)}\right|\right)dt \tag{3.8}$$

holds true for all $g_j\in H^1(a_1,b_1)$ $(j=1,2)$.

Proof Using Theorem 2.2 with $\Omega_1=\Omega_2=(a_1,b_1)$, $d\mu_1(x)=dx$, $d\mu_2(t)=dt$ and we get

$$k_1(x,t) = \begin{cases} -\frac{\exp(\lambda(x-t))}{\lambda}, & a_1\le t\le x,\\ 0, & x<t\le b_1.\end{cases}$$

Also, if g_j is replaced by g_j' and h_j is taken as ${}^{\mathrm{CFR}}_{a_1}D^{\alpha}g_j$ for $j=1,2$, then we obtain the inequality (3.8). □

Corollary 3.3 *Let v be a weight function on (a_1,b_1), $0<p_1\le q_1<\infty$, $0<\alpha<1$ and $\lambda=\frac{-\alpha}{1-\alpha}$. Let ${}^{\mathrm{CFR}}_{a_1}D^{\alpha}$ be defined by (3.1) and define u on (a_1,b_1) by*

$$u(t)=\left[\int_t^{b_1} v(x)\left(\frac{-\lambda\exp(\lambda(x-t))}{(1-\exp(\lambda(x-a_1)))}\right)^{\frac{q_1}{p_1}}dx\right]^{\frac{p_1}{q_1}}<\infty.$$

If the function ψ denotes a convex and non-negative increasing on an interval $I_1\subseteq\mathbb{R}$, then the following inequality holds true:

$$\left[\int_{a_1}^{b_1} v(x)\left(\psi\left(\frac{\lambda^2}{1-\exp(\lambda(x-a_1))}\left({}^{\mathrm{CFR}}_{a_1}D^{\alpha}g\right)(x)\right)\right)^{\frac{q_1}{p_1}}dx\right]^{\frac{1}{q_1}} \le\left[\int_{a_1}^{b_1}u(t)\psi\left(g'(t)\right)dt\right]^{\frac{1}{p_1}}, \tag{3.9}$$

for all measurable functions $g':(a_1,b_1)\to\mathbb{R}$ such that $\operatorname{Im}g'\subseteq I_1$.

Proof By using Theorem 2.3 with $\Omega_1=\Omega_2=(a_1,b_1)$, $d\mu_1(x)=dx$, $d\mu_2(t)=dt$,

$$k_1(x,t)=\begin{cases}-\frac{\exp(\lambda(x-t))}{\lambda}, & a_1\le t\le x\\ 0, & x<t\le b_1,\end{cases}$$

then we find $K_1(x)=\frac{1-\exp(\lambda(x-a_1))}{\lambda^2}$. Also, if g is replaced by g' and h is taken as ${}^{\mathrm{CFR}}_{a_1}D^{\alpha}g$, we obtain (3.9). □

Corollary 3.4 *Let v be a weight function on (a_1,b_1), $0<\alpha<1$ and $\lambda=\frac{-\alpha}{1-\alpha}$. Let ${}^{\mathrm{CFR}}_{a_1}D^{\alpha}$ be defined by (3.1), and $g_j\in H^1(a_1,b_1)$ for $j=1,2,3$, where $g_2(x)>0$ for all $x\in(a_1,b_1)$. If $0<a_1<b_1<\infty$ and $x\mapsto-\frac{v(x)g_2'(t)\exp(\lambda(x-t))}{\lambda({}^{\mathrm{CFR}}_{a_1}D^{\alpha}g_2)(x)}$ is integrable function over (a_1,b_1), then $u(t)$ is defined as*

$$u(t)=-\frac{g_2'(t)}{\lambda}\int_t^{b_1}\frac{v(x)\exp(\lambda(x-t))}{({}^{\mathrm{CFR}}_{a_1}D^{\alpha}g_2)(x)}dx.$$

If the function $\psi:(0,\infty)\times(0,\infty)\to\mathbb{R}$ denotes a convex and increasing function, then the following inequality holds true:

$$\int_{a_1}^{b_1}v(x)\psi\left(\left|\frac{({}^{\mathrm{CFR}}_{a_1}D^{\alpha}g_1)(x)}{({}^{\mathrm{CFR}}_{a_1}D^{\alpha}g_2)(x)}\right|,\left|\frac{({}^{\mathrm{CFR}}_{a_1}D^{\alpha}g_3)(x)}{({}^{\mathrm{CFR}}_{a_1}D^{\alpha}g_2)(x)}\right|\right)dx \le\int_{a_1}^{b_1}u(t)\psi\left(\left|\frac{g_1'(t)}{g_2'(t)}\right|,\left|\frac{g_3'(t)}{g_2'(t)}\right|\right)dt. \tag{3.10}$$

Proof Using Theorem 2.4 with $\Omega_1=\Omega_2=(a_1,b_1)$, $d\mu_1(x)=dx$, $d\mu_2(t)=dt$, we get

$$k_1(x,t)=\begin{cases}-\frac{\exp(\lambda(x-t))}{\lambda}, & a_1\le t\le x,\\ 0, & x<t\le b_1,\end{cases}$$

Also, $K_1(x)=\frac{1-\exp(\lambda(x-a_1))}{\lambda^2}$. Also, if g_j is replaced by g_j' and h_j is taken as ${}^{\mathrm{CFR}}_{a_1}D^{\alpha}g_j$ for $j=1,2$, then we obtain the inequality (3.10). □

4 Examples

Below, we will show the application of our some of our main results with three examples.

Example 4.1 In Theorem 3.1, let $g(x) = \sin x$ and $(a_1, b_1) = (0, \frac{\pi}{2})$. Then we obtain

$$\begin{aligned}
\left|\left({}_{a_1}^{\mathrm{CFR}}D^{\alpha}\sin\right)(x)\right| &= \left|\frac{M(\alpha)}{1-\alpha}\frac{d}{dx}\int_0^x \sin t\exp\bigl(\lambda(x-t)\bigr)\,dt\right| \\
&= \left|\frac{M(\alpha)}{1-\alpha}\frac{d}{dx}\bigl(\sin x * \exp(\lambda x)\bigr)\right| \\
&= \left|\frac{M(\alpha)}{1-\alpha}\bigl(\cos x * \exp(\lambda x)\bigr)\right| \\
&\leq \frac{M(\alpha)}{1-\alpha}\int_0^x |\cos t|\exp\bigl(\lambda(x-t)\bigr)\,dt.
\end{aligned}$$

By using Hölder's inequality for $\{p_1, q_1\}$, we can write

$$\begin{aligned}
\left|\left({}_{a_1}^{\mathrm{CFR}}D^{\alpha}\sin\right)(x)\right| &\leq \frac{M(\alpha)}{1-\alpha}\left(\int_0^x |\cos t|^{q_1}\,dt\right)^{\frac{1}{q_1}}\left(\int_0^x \exp\bigl(p_1\lambda(x-t)\bigr)\,dt\right)^{\frac{1}{p_1}} \\
&\leq \frac{M(\alpha)}{1-\alpha}\left(-\frac{1}{p_1\lambda}\right)^{\frac{1}{p_1}}\left(\int_0^x |\cos t|^{q_1}\,dt\right)^{\frac{1}{q_1}}.
\end{aligned}$$

Thus, we have

$$\left|\left({}_{a_1}^{\mathrm{CFR}}D^{\alpha}\sin\right)(x)\right|^{q_1} \leq \left(\frac{M(\alpha)}{1-\alpha}\left(-\frac{1}{p_1\lambda}\right)^{\frac{1}{p_1}}\right)^{q_1}\int_0^x |\cos t|^{q_1}\,dt.$$

Integrating both sides from 0 to $\frac{\pi}{2}$, we find

$$\int_0^{\frac{\pi}{2}}\left|\left({}_{a_1}^{\mathrm{CFR}}D^{\alpha}\sin\right)(x)\right|^{q_1}\,dx \leq \frac{\pi}{2}\left(\frac{M(\alpha)}{1-\alpha}\left(-\frac{1}{p_1\lambda}\right)^{\frac{1}{p_1}}\right)^{q_1}\int_0^x |\cos t|^{q_1}\,dt.$$

So, $g(x) = \sin x$ satisfies the Hardy-type inequality.

Example 4.2 In Corollary 3.2, let ${}_{a_1}^{\mathrm{CFR}}D^{\alpha}$ be the new Riemann fractional derivative and $\nu(x) = \exp(\lambda(x-a_1))(\lambda(x-a_1)+1)$ be a particular weight function. Also, let $\psi(x) = x^s$ be a convex function for $s \geq 1$, $x > 0$, and $g_j(x) = \exp(\lambda(x-a_1))$ be a function for $j = 1, 2$. Then we find

$$\begin{aligned}
\left({}_{a_1}^{\mathrm{CFR}}D^{\alpha}g_2\right)(x) &= \frac{M(\alpha)}{1-\alpha}\frac{d}{dx}\int_{a_1}^x \exp\bigl(\lambda(t-a_1)\bigr)\exp\bigl(\lambda(x-t)\bigr)\,dt \\
&= \frac{M(\alpha)}{1-\alpha}\bigl[\lambda\exp\bigl(\lambda(x-a_1)\bigr)(x-a_1) + \exp\bigl(\lambda(x-a_1)\bigr)\bigr] \\
&= \frac{M(\alpha)}{1-\alpha}\exp\bigl(\lambda(x-a_1)\bigr)\bigl[\lambda(x-a_1)+1\bigr].
\end{aligned}$$

So, we obtain

$$u(t) = -\frac{g_2'(t)}{\lambda}\int_t^{b_1}\frac{\nu(x)\exp(\lambda(x-t))}{\left({}_{a_1}^{\mathrm{CFR}}D^{\alpha}g_2\right)(x)}\,dx$$

$$= -\exp\bigl(\lambda(t-a_1)\bigr)\int_t^{b_1}\frac{\exp(\lambda(x-a_1))(\lambda(x-a_1)+1)\exp(\lambda(x-t))}{({}_{a_1}^{\mathrm{CFR}}D^{\alpha}g_2)(x)}\,dx$$
$$= -\frac{1-\alpha}{\lambda M(\alpha)}\bigl[\exp\bigl(\lambda(b_1-a_1)\bigr)-\exp\bigl(\lambda(t-a_1)\bigr)\bigr]<\infty.$$

Therefore, from (3.8) in Corollary 3.2, we can write

$$\int_{a_1}^{b_1}\exp\bigl(\lambda(x-a_1)\bigr)\bigl(\lambda(x-a_1)+1\bigr)\psi(1)\,dx$$
$$\leq \int_{a_1}^{b_1}-\frac{1-\alpha}{\lambda M(\alpha)}\bigl[\exp\bigl(\lambda(b_1-a_1)\bigr)-\exp\bigl(\lambda(t-a_1)\bigr)\bigr]\psi(1)\,dt.$$

After some calculation, we obtain

$$\lambda(b_1-a_1)^2\exp\bigl(\lambda(b_1-a_1)\bigr)\leq -\frac{1-\alpha}{\lambda M(\alpha)}\biggl[\exp\bigl(\lambda(b_1-a_1)\bigr)(b_1-a_1)-\frac{\exp(\lambda(b_1-a_1))-1}{\lambda}\biggr].$$

Example 4.3 In Corollary 3.3, let ${}_{a_1}^{\mathrm{CFR}}D^{\alpha}$ be the new Riemann fractional derivative and $\nu(x)=(1-\exp(\lambda(x-a_1)))^{\frac{q_1}{p_1}}$ be a particular weight function. Also, let $\psi(x)=x^s$ be a convex function for $s\geq 1$, $x>0$. Then we find

$$u(t)=\biggl[\int_t^{b_1}\nu(x)\biggl(-\frac{\lambda\exp(\lambda(x-t))}{1-\exp(\lambda(x-a_1))}\biggr)^{\frac{q_1}{p_1}}\,dx\biggr]^{\frac{p_1}{q_1}}$$
$$= -\lambda\biggl(\frac{p_1}{\lambda q_1}\biggr)^{\frac{p_1}{q_1}}\biggl[\exp\biggl(\frac{q_1}{p_1}\lambda(b_1-t)\biggr)-1\biggr]^{\frac{p_1}{q_1}}<\infty,$$

and from (3.9) we can write

$$\biggl[\int_{a_1}^{b_1}\bigl(1-\exp\bigl(\lambda(x-a_1)\bigr)\bigr)^{\frac{(1-s)q_1}{p_1}}\lambda^{2s}\bigl(\bigl({}_{a_1}^{\mathrm{CFR}}D^{\alpha}g\bigr)(x)\bigr)^{\frac{sq_1}{p_1}}\,dx\biggr]^{\frac{1}{q_1}}$$
$$\leq\biggl[\int_{a_1}^{b_1}(-\lambda)\biggl(\frac{p_1}{\lambda q_1}\biggr)^{\frac{p_1}{q_1}}\biggl(\exp\biggl(\frac{q_1}{p_1}\lambda(b_1-t)\biggr)-1\biggr)^{\frac{p_1}{q_1}}\bigl|g'(t)\bigr|^s\,dt\biggr]^{\frac{1}{p_1}}. \tag{4.1}$$

The left-hand side of (4.1) satisfies the following inequality:

$$\biggl[\int_{a_1}^{b_1}\bigl(1-\exp\bigl(\lambda(x-a_1)\bigr)\bigr)^{\frac{(1-s)q_1}{p_1}}\lambda^{2s}\bigl(\bigl({}_{a_1}^{\mathrm{CFR}}D^{\alpha}g\bigr)(x)\bigr)^{\frac{sq_1}{p_1}}\,dx\biggr]^{\frac{1}{q_1}}$$
$$\geq\lambda^{2s}\bigl(1-\exp\bigl(\lambda(b_1-a_1)\bigr)\bigr)^{\frac{1-s}{p_1}}\biggl(\int_{a_1}^{b_1}\bigl(\bigl({}_{a_1}^{\mathrm{CFR}}D^{\alpha}g\bigr)(x)\bigr)^{\frac{sq_1}{p_1}}\,dx\biggr)^{\frac{1}{q_1}}. \tag{4.2}$$

Also, the right-hand side of (4.1) satisfies the following inequality:

$$\biggl[\int_{a_1}^{b_1}(-\lambda)\biggl(\frac{p_1}{\lambda q_1}\biggr)^{\frac{p_1}{q_1}}\biggl[\exp\biggl(\frac{q_1}{p_1}\lambda(b_1-t)\biggr)-1\biggr]^{\frac{p_1}{q_1}}\bigl|g'(t)\bigr|^s\,dt\biggr]^{\frac{1}{p_1}}$$
$$\leq(-\lambda)^{\frac{1}{p_1}}\biggl(\frac{p_1}{\lambda q_1}\biggr)^{\frac{1}{q_1}}\biggl(\exp\biggl(\frac{q_1}{p_1}\lambda(b_1-a_1)\biggr)-1\biggr)^{\frac{1}{p_1}}\biggl(\int_{a_1}^{b_1}\bigl|g'(t)\bigr|^s\,dt\biggr)^{\frac{1}{p_1}}. \tag{4.3}$$

So, by using (4.2) and (4.3) in (4.1), we obtain

$$\left[\int_{a_1}^{b_1}\left(\left({}_{a_1}^{\mathrm{CFR}}D^{\alpha}g\right)(x)\right)^{\frac{sq_1}{p_1}}dx\right]^{\frac{1}{q_1}} \leq (-1)^{\frac{s}{p_1}}\lambda^{\frac{1}{p_1}-2s}\left(\frac{p_1}{\lambda q_1}\right)^{\frac{1}{q_1}}\left[\exp\left(\frac{q_1}{p_1}\lambda(b_1-a_1)\right)\right]^{\frac{s}{p_1}} \times\left(\int_{a_1}^{b_1}\left|g'(t)\right|^{s}dt\right)^{\frac{1}{p_1}}.$$

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