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Analytical Solution of Fractional-Order Hyperbolic Telegraph Equation, Using Natural Transform Decomposition Method

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Abstract: In the current paper, fractional-order hyperbolic telegraph equations are considered for analytical solutions, using the decomposition method based on natural transformation. The fractional derivative is defined by the Caputo operator. The present technique is implemented for both fractional- and integer-order equations, showing that the current technique is an accurate analytical instrument for the solution of partial differential equations of fractional-order arising in all branches of applied sciences. For this purpose, several examples related to hyperbolic telegraph models are presented to explain the procedure of the suggested method. It is noted that the procedure of the present technique is simple, straightforward, accurate, and found to be a better mathematical technique to solve non-linear fractional partial differential equations.

Keywords: Natural transform; adomian decomposition method; caputo operator; hyperbolic telegraph equation

1. Introduction

Different physical developments in engineering and science can be effectively demonstrated by using the concept of fractional calculus. Fractional differential equations have attained greater attention because the fractional-order scheme converges to integer-order equations. In recent years, fractional differential equations have been used for computational analysis of real-world natural phenomena such as earthquake design, traffic models with fractional derivatives, evaluation of viscoelastic surface characteristics, fractional stochastic systems, control processing, diffusion processes, ecology and allometry in biology, etc. [1–3].

In this connection, communication systems are a significant feature in the universal community in this modern world. High-frequency communication schemes advance via important industry interests, due to the range of transmitter and microwave communication technologies. These schemes use the communication media to transmit the signal for delivering data from one node to another [4]. In the present article, a particular model, which is known as the hyperbolic telegraph equation, is considered for analysis in the form of its fractional- and integer-order equations. Such an equation is expressed as:

$$\frac{\partial^{\delta} u}{\partial t^{\delta}} + \rho u(x,t) + v\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + g(x,t), \qquad 0 < \delta \leq 2, \tag{1}$$

where ρ and v are known constants and $u(x,t)$ is an unknown function. For $\rho > 0$ and $v = 0$, the model denotes a damp wave equation. For $\rho > v > 0$, it is a telegraph equation. The model is widely used in signal analyses for the transmission and propagation of electrical impulses and in wave processes. The algorithm is utilized in the design of reaction diffusion in several fields of science and engineering. Even in medical sciences, researchers and scientists have discovered the model in a peristaltic flow system [5]. The various implementations of this type of model in aerospace and biomedical sciences have attracted researchers to solve the problem for the fractional time derivative. The integer-order linear PDEs are a specific situation of fractional-order PDEs because the previous may be collected by making the fractional-order derivative equal to the integer. The reaction of the fractional-order scheme finally converges to the outcome of the integer-order scheme.

Recently, many researchers have studied the hyperbolic telegraph equation in numerical and analytical methods such as the homotopy analysis method (HAM) [5], the Legendre polynomials and block-pulse-function [6], the radial basis function (RBF) [7], the Chebyshev tau method (CTM) [8], the Haar–inc collocation method [9], the Adomian decomposition method (ADM) [10], the Laplace decomposition method (LDM) [11], and the fictitious time integration method (FTIM) [12].

Among these techniques, the natural transformation decomposition method (NTDM) is a mixture of the well-known method of natural transformation and the method of Adomian decomposition. This new method is considered the best tool for some classes of coupled systems of non-linear partial differential equations in a simple and efficient way. This method gives a solution, either exact or approximate, in the form of a rapid convergence series. Hence, the NTDM is considered an excellent mathematical tool for a wide range of linear and non-linear PDEs [13,14]. A large number of physical problems have been studied by using NTDM, such as the study of the fractional telegraph equation [14], fractional-order PDE's with proportional delay [15] non-linear PDEs [16,17], the fractional uncertain flow of a system of polytropic gas [18], fractional physical models [19], fractional-order heat and wave equations [20] and fractional-order diffusion equations [21].

In the present research work, NTDM is used for the analytical solution of the fractional-order hyperbolic telegraph equation, which is given in Equation (1).

2. Mathematical Modeling

We consider an infinitesimal piece of telegraph cable wire as an electrical circuit, as shown in Figure 1, and suppose that the cable is insulated imperfectly, so that both capacitance and leakage to the floor are present [6,22].

C = capacitance to the ground,
x = distance from the cable end,
$u(x,t)$ = voltage at any time, on the cable,
G = inductance of the cable
$i(x,t)$ = current at any time, on the cable,
L = inductance of the cable, and
R = resistance of the cable.

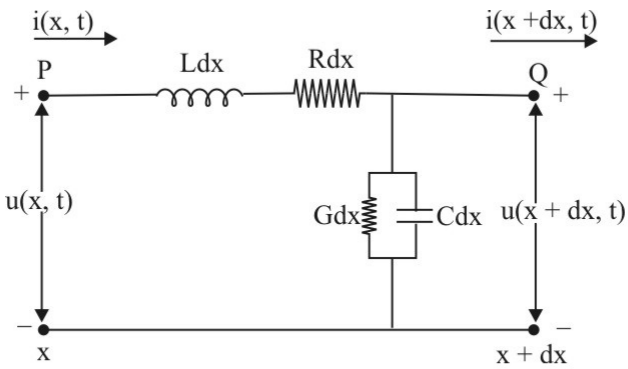

Figure 1. Schematic diagram of a telegraphic transmission line with leakage.

The voltage drop across the inductor is given as

$$u = L\frac{di}{dt} \tag{2}$$

The voltage drop across the capacitor is given by

$$u = \frac{1}{c}\int idt \tag{3}$$

The voltage at terminal Q is equal to the voltage at terminal P, minus the drop in voltage along the element PQ, so if Equations (2) and (3) are combined, this leads to the following result:

$$u(x + dx, t) = u(x, t) - [\bar{R}dx] - [\bar{L}dx]\frac{di}{dt} \tag{4}$$

and differentiating (4) with respect to x:

$$\frac{\partial u}{\partial x} = -\bar{R}i - \bar{L}\frac{di}{dt} \tag{5}$$

The current through the capacitor is given as

$$I_c = C\frac{\partial u}{\partial t} \tag{6}$$

The current through the inductor is given as

$$u = L\frac{\partial I_L}{\partial t} \tag{7}$$

Now, differentiating (4) with respect to t

$$c^2\frac{\partial^2 i}{\partial x^2} = \frac{\partial^2 i}{\partial t^2} + (\theta + \phi)\frac{\partial i}{\partial t} + \theta\phi i \tag{8}$$

Similarly, we get

$$c^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2} + (\theta + \phi)\frac{\partial u}{\partial t} + \theta\phi u \tag{9}$$

where $\theta = \frac{G}{C}$, $\phi = \frac{R}{L}$ and $c^2 = \frac{1}{LC}$. Equations (8) and (9) are called one-dimensional hyperbolic second-order telegraph equations.
If the fractional derivative model of Equations (8) and (9),

$$c^2 \frac{\partial^\delta i}{\partial x^\delta} = \frac{\partial^\beta i}{\partial t^\beta} + (\theta + \phi) \frac{\partial^\alpha i}{\partial t^\alpha} + \theta \phi i \tag{10}$$

and:

$$c^2 \frac{\partial^\delta u}{\partial x^\delta} = \frac{\partial^\beta u}{\partial t^\beta} + (\theta + \phi) \frac{\partial^\alpha u}{\partial t^\alpha} + \theta \phi u \qquad 1 < \beta, \delta \leq 2, \quad 0 < \alpha \leq 1 \tag{11}$$

3. Preliminaries

Definition 1. *The natural transform of $\bar{g}(t)$ is defined as [23,24]:*

$$\mathbb{N}^{+\cdot}[\bar{g}(t)] = W(s, \mu) = \frac{1}{\mu} \int_0^\infty e^{\frac{-st}{\mu}} \bar{g}(t) dt; \quad s, \mu > 0$$

where s and μ are the variables of transformation.

Definition 2. *The inverse of a function's natural transformation is described:*

$$\mathbb{N}^{-}[W(s, \mu)] = \bar{g}(t) = \frac{1}{2\pi i} \int_{b-i\infty}^{b+i\infty} e^{\frac{st}{\mu}} W(s, \mu) ds;$$

in which s and μ are the natural transform factor s and b is a real constant, and the integral of the complex plane is given along $s = b$, $s = x_1 + ix_2$.

Definition 3. *Natural n^{th} derivative transform:*
If $\bar{g}^n(t)$ is the n^{th} feature $\bar{g}(t)$ derivative provided by,

$$\mathbb{N}[\bar{g}^n(t)] = W_n(s, \mu) = \frac{s^n}{\mu^n} W(s, \mu) - \sum_{k=0}^{n-1} \frac{s^{n-(k+1)}}{\mu^{n-k}} \bar{g}^k(0), \quad n \geq 1.$$

Theorem 1. *If $L_1(s, \mu)$, $L_2(s, \mu)$ are the transform functions, respectively, $\hbar(t)$ and $\ell(t)$ are given by;*

$$\mathbb{N}[\ell_1 * \ell_2] = \mu L_1(s, \mu) L_2(s, \mu)$$

where $\ell_1 * \ell_2$ is the convolution of two functions ℓ_1 and ℓ_2.

Definition 4. *Riemann–Liouville fractional-order integral: [1]*

$$I_x^\delta \bar{g}(x) = \begin{cases} \bar{g}(x) & \text{if } \delta = 0 \\ \frac{1}{\Gamma(\delta)} \int_0^x (x - v)^{\delta - 1} \bar{g}(v) dv & \text{if } \delta > 0, \end{cases}$$

where Γ represents the concept of the gamma variable by,

$$\Gamma(\Phi) = \int_0^\infty e^{-x} x^{\Phi - 1} dx \qquad \Phi \in \mathbb{C},$$

Definition 5. *The following mathematical expression defines the Caputo operator of the fractional derivative of order δ for $k \in \mathbb{N}$, $x > 0$, $\bar{g} \in \mathbb{C}_t$, $t \geq -1$ [2,3].*

$$D^\delta \bar{g}(x) = \frac{\partial^\delta \bar{g}(x)}{\partial \bar{t}^\delta} = \begin{cases} I^{k-\delta} \left[\frac{\partial^\delta \bar{g}(x)}{\partial \bar{t}^\delta} \right], & \text{if } k - 1 < \delta \leq k, k \in \mathbb{N} \\ \frac{\partial^\delta \bar{g}(x)}{\partial \bar{t}^\delta}, & \end{cases}$$

4. The Procedure of NTDM

To explain the procedure of NTDM, we consider the solution of the fractional-order general form of PDEs, using the proposed method. That is:

$$D^{\delta}u(x,t) + Ł_1 u(x,t) + N_1 u(x,t) = q(x,t), \quad m-1 < \delta < m, \tag{12}$$

where $D^{\delta} = \frac{\partial^{\delta}}{\partial t^{\delta}}$ is the operator of Caputo $\delta, m \in \mathbb{N}$, in which $Ł_1$ and N_1 are operators and q is the source function, with initial condition:

$$u(x,0) = h(x), \qquad 0 < \delta \leq 1, \quad t > 0. \tag{13}$$

We get the natural transformation applied to Equation (12)

$$\mathbb{N}^{+}\left[D^{\delta}u(x,t)\right] + \mathbb{N}^{+}\left[Ł_1 u(x,t) + N_1 u(x,t)\right] = \mathbb{N}^{+}\left[q(x,t)\right], \tag{14}$$

and using the natural transform differentiation property:

$$\frac{s^{\delta}}{u^{\delta}}\mathbb{N}^{+}\left[u(x,t)\right] - \frac{s^{\delta-1}}{u^{\delta}}u(x,0) = \mathbb{N}^{+}\left[q(x,t)\right] - \mathbb{N}^{+}\left[Ł_1 u(x,t) + N_1 u(x,t)\right],$$

$$\mathbb{N}^{+}\left[u(x,t)\right] = \frac{1}{s}u(x,0) + \frac{u^{\delta}}{s^{\delta}}\mathbb{N}^{+}\left[q(x,t)\right] - \frac{u^{\delta}}{s^{\delta}}\mathbb{N}^{+}\left[Ł_1 u(x,t) + N_1 u(x,t)\right],$$

Now, $u(x,0) = h(x)$

$$\mathbb{N}^{+}\left[u(x,t)\right] = \frac{h(x)}{s} + \frac{u^{\delta}}{s^{\delta}}\mathbb{N}^{+}\left[q(x,t)\right] - \frac{u^{\delta}}{s^{\delta}}\mathbb{N}^{+}\left[Ł_1 u(x,t) + N_1 u(x,t)\right]. \tag{15}$$

The following infinite series shows the NTDM approach $u(x,t)$

$$u(x,t) = \sum_{\ell=0}^{\infty} u_{\ell}(x,t), \tag{16}$$

The nonlinear term in the problem can be expressed in terms of the Adomian polynomial as

$$N_1 u(x,t) = \sum_{\ell=0}^{\infty} A_{\ell}, \tag{17}$$

$$A_{\ell} = \frac{1}{\ell!}\left[\frac{d^{\ell}}{d\lambda^{\ell}}\left[N_1 \sum_{\ell=0}^{\infty}(\lambda^{\ell}u_{\ell})\right]\right]_{\lambda=0}, \qquad \ell = 0,1,2\ldots \tag{18}$$

and substituting Equations (16) and (17) in Equation (15), we get:

$$\mathbb{N}^{+}\left[\sum_{\ell=0}^{\infty} u_{\ell}(x,t)\right] = \frac{h(x)}{s} + \frac{u^{\delta}}{s^{\delta}}\mathbb{N}^{+}\left[q(x,t)\right] - \frac{u^{\delta}}{s^{\delta}}\mathbb{N}^{+}\left[Ł_1 \sum_{\ell=0}^{\infty} u_{\ell}(x,t) + \sum_{\ell=0}^{\infty} A_{\ell}\right]. \tag{19}$$

Applying the natural transformation linearity,

$$\mathbb{N}^{+}\left[u_0(x,t)\right] = \frac{h(x)}{s} + \frac{u^{\delta}}{s^{\delta}}\mathbb{N}^{+}\left[q(x,t)\right], \tag{20}$$

$$\mathbb{N}^{+}\left[u_1(x,t)\right] = -\frac{u^{\delta}}{s^{\delta}}\mathbb{N}^{+}\left[Ł_1 u_0(x,t) + A_0\right].$$

We can usually write,

$$\mathbb{N}^{+}\left[u_{\ell+1}(x,t)\right] = -\frac{u^{\delta}}{s^{\delta}}\mathbb{N}^{+}\left[Ł_1 u_{\ell}(x,t) + A_{\ell}\right], \quad \ell \geq 1. \tag{21}$$

Equations (20) and (21), applying the inverse natural transformation:

$$u_0(x,t) = h(x) + \mathbb{N}^{-}\left[\frac{u^{\delta}}{s^{\delta}}\mathbb{N}^{+}\left[q(x,t)\right]\right],$$

$$u_{\ell+1}(x,t) = -\mathbb{N}^{-}\left[\frac{u^{\delta}}{s^{\delta}}\mathbb{N}^{+}\left[Ł_1 u_{\ell}(x,t) + A_{\ell}\right]\right]. \tag{22}$$

5. Results

Example 1. *Consider the hyperbolic telegraph differential equation as follows:*

$$\frac{\partial^{\delta} u}{\partial t^{\delta}} = u - 2\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2}, \qquad 0 < \delta \leq 2, \tag{23}$$

with initial condition:

$$u(x,0) = e^{x} \qquad u_t(x,0) = -2e^{x} \tag{24}$$

Taking the natural transform of (23)

$$\frac{s^{\delta}}{u^{\delta}}\mathbb{N}^{+}\left[u(x,t)\right] - \frac{s^{\delta-1}}{u^{\delta}}u(x,0) - \frac{s^{\delta-2}}{u^{\delta-1}}u_t(x,0) = \mathbb{N}^{+}\left[u - 2\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2}\right].$$

Applying the inverse natural transform,

$$u(x,t) = \mathbb{N}^{-}\left[\frac{u(x,0)}{s} + \frac{u_t(x,0)u}{s^2} + \frac{u^{\delta}}{s^{\delta}}\mathbb{N}^{+}\left[u - 2\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2}\right]\right].$$

We use the ADM procedure,

$$u_0(x,t) = \mathbb{N}^{-}\left[\frac{u(x,0)}{s} + \frac{u_t(x,0)u}{s^2}\right] = \mathbb{N}^{-}\left[\frac{e^{x}}{s} - \frac{2e^{x}u}{s^2}\right]$$

$$u_0(x,t) = e^{x} - 2e^{x}t, \tag{25}$$

$$u_{\ell+1}(x,t) = \mathbb{N}^{-}\left[\frac{u^{\delta}}{s^{\delta}}\mathbb{N}^{+}\left[u_{\ell} - 2\frac{\partial u_{\ell}}{\partial t} - \frac{\partial^2 u_{\ell}}{\partial x^2}\right]\right], \qquad \ell = 0,1,2,\ldots$$

for $\ell = 0$:

$$u_1(x,t) = \mathbb{N}^{-}\left[\frac{u^{\delta}}{s^{\delta}}\mathbb{N}^{+}\left[u_0 - 2\frac{\partial u_0}{\partial t} - \frac{\partial^2 u_0}{\partial x^2}\right]\right]$$

$$u_1(x,t) = \mathbb{N}^{-}\left[\frac{4e^{x}u^{\delta}}{s^{\delta+1}}\right] = 4e^{x}\frac{t^{\delta}}{\Gamma(\delta+1)}. \tag{26}$$

The next terms are

$$u_2(x,t) = \mathbb{N}^-\left[\frac{u^\delta}{s^\delta}\mathbb{N}^+\left[u_1 - 2\frac{\partial u_1}{\partial t} - \frac{\partial^2 u_1}{\partial x^2}\right]\right] = -\frac{8e^x\delta\Gamma(\delta)t^{2\delta-1}}{\Gamma(\delta+1)\Gamma(2\delta)},$$
$$u_3(x,t) = \mathbb{N}^-\left[\frac{u^\delta}{s^\delta}\mathbb{N}^+\left[u_2 - 2\frac{\partial u_2}{\partial t} - \frac{\partial^2 u_2}{\partial x^2}\right]\right] = \frac{16e^x\delta\Gamma(\delta)(2\delta-1)\Gamma(2\delta-1)t^{3\delta-2}}{\Gamma(\delta+1)\Gamma(2\delta)\Gamma(3\delta-1)}, \tag{27}$$
$$\vdots$$

The NTDM solution is

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + u_4(x,t)\dots$$
$$u(x,t) = e^x - 2e^x t + 4e^x\frac{t^\delta}{\Gamma(\delta+1)} - \frac{8e^x\delta\Gamma(\delta)t^{2\delta-1}}{\Gamma(\delta+1)\Gamma(2\delta)} + \frac{16e^x\delta\Gamma(\delta)(2\delta-1)\Gamma(2\delta-1)t^{3\delta-2}}{\Gamma(\delta+1)\Gamma(2\delta)\Gamma(3\delta-1)} + \cdots$$

If $\delta = 2$, then the solution can be expressed as

$$u(x,t) = e^x\left(1 - 2t + \frac{2t^2}{2!} - \frac{8t^3}{3!} + \frac{16t^4}{4!}\cdots\right). \tag{28}$$

Example 1 exact result is

$$u(x,t) = e^{(x-2t)}$$

Example 2. *Consider the hyperbolic telegraph equation as given by:*

$$\frac{\partial^\delta u}{\partial t^\delta} = u - \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2}, \qquad 0 < \delta \le 2, \tag{29}$$

with initial condition is

$$u(x,0) = e^x \qquad u_t(x,0) = -e^x \tag{30}$$

Taking the natural transform of (29),

$$\frac{s^\delta}{u^\delta}\mathbb{N}^+[u(x,t)] - \frac{s^{\delta-1}}{u^\delta}u(x,0) - \frac{s^{\delta-2}}{u^{\delta-1}}u_t(x,0) = \mathbb{N}^+\left[u - \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2}\right].$$

Applying the inverse natural transform,

$$u(x,t) = \mathbb{N}^-\left[\frac{u(x,0)}{s} + \frac{u_t(x,0)u}{s^2} + \frac{u^\delta}{s^\delta}\mathbb{N}^+\left[u - \frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2}\right]\right].$$

We use the ADM procedure,

$$u_0(x,t) = \mathbb{N}^-\left[\frac{u(x,0)}{s} + \frac{u_t(x,0)u}{s^2}\right] = \mathbb{N}^-\left[\frac{e^x}{s} - \frac{e^x u}{s^2}\right]$$
$$u_0(x,t) = e^x - e^x t, \tag{31}$$
$$u_{\ell+1}(x,t) = \mathbb{N}^-\left[\frac{u^\delta}{s^\delta}\mathbb{N}^+\left[u_\ell - \frac{\partial u_\ell}{\partial t} - \frac{\partial^2 u_\ell}{\partial x^2}\right]\right], \qquad \ell = 0,1,2,\dots$$

for $\ell = 0$:

$$u_1(x,t) = \mathbb{N}^- \left[\frac{u^\delta}{s^\delta} \mathbb{N}^+ \left[u_0 - \frac{\partial u_0}{\partial t} - \frac{\partial^2 u_0}{\partial x^2} \right] \right]$$
$$u_1(x,t) = \mathbb{N}^- \left[\frac{e^x u^\delta}{s^{\delta+1}} \right] = e^x \frac{t^\delta}{\Gamma(\delta+1)}. \tag{32}$$

The next terms are,

$$u_2(x,t) = \mathbb{N}^- \left[\frac{u^\delta}{s^\delta} \mathbb{N}^+ \left[u_1 - \frac{\partial u_1}{\partial t} - \frac{\partial^2 u_1}{\partial x^2} \right] \right] = -\frac{e^x \delta \Gamma(\delta) t^{2\delta-1}}{\Gamma(\delta+1)\Gamma(2\delta)},$$
$$u_3(x,t) = \mathbb{N}^- \left[\frac{u^\delta}{s^\gamma} \mathbb{N}^+ \left[u_2 - \frac{\partial u_2}{\partial t} - \frac{\partial^2 u_2}{\partial x^2} \right] \right] = \frac{e^x \delta \Gamma(\delta)(2\delta-1)\Gamma(2\delta-1) t^{3\delta-2}}{\Gamma(\delta+1)\Gamma(2\delta)\Gamma(3\delta-1)}, \tag{33}$$
$$\vdots$$

The NTDM solution is,

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + u_4(x,t) \ldots$$
$$u(x,t) = e^x - e^x t + e^x \frac{t^\delta}{\Gamma(\delta+1)} - \frac{e^x \delta \Gamma(\delta) t^{2\delta-1}}{\Gamma(\delta+1)\Gamma(2\delta)} + \frac{e^x \delta \Gamma(\delta)(2\delta-1)\Gamma(2\delta-1) t^{3\delta-2}}{\Gamma(\delta+1)\Gamma(2\delta)\Gamma(3\delta-1)}$$

If $\delta = 2$, *then the solution can be expressed as*

$$u(x,t) = e^x \left(1 - t + \frac{t^2}{2} - \frac{t^3}{6} + \frac{t^4}{24} \ldots \right). \tag{34}$$

Example 2 exact result is

$$u(x,t) = e^{(x-t)}.$$

6. Results and Discussion

The numerical results obtained by the proposed method were analyzed through different graphs. The solution-graphs were plotted for both fractional and integer simulation. In Example 1, the unknown variable u represents the voltage at any time and distance point on the cable, where x represents the distance from sending end of the cable and t is the time required. Figure 2 shows the solution-graph of u at different fractional-order $\delta = 2, 1.8, 1.7$ and the exact solution. It was observed that at $t = 1$, the value of u increased exponentially. It was shown that the solution obtained at $\delta = 2$ was in good contact with the exact solution of Example 1.

Moreover, as the fractional-order changed from one to two, the solution graphs represented the different dynamical behavior of Example 1, which enabled us to choose the optimum fractional-order of the model, which can best describe the physical scenario of Example 1.

Figure 3 represents the solution graph of u with respect to the variables x and t of Example 2. Different surfaces of the solutions u are plotted against the order pair (x,t) at various fractional-orders δ. The different dynamical behavior was represented by the solutions of different fractional order problems of Example 2. Figure 4 shows the solution graph of u at $t = 0.8$, as well, and the solution of Example 2 is discussed at different fractional-order $\delta = 2, 1.8, 1.7, 1.5$.

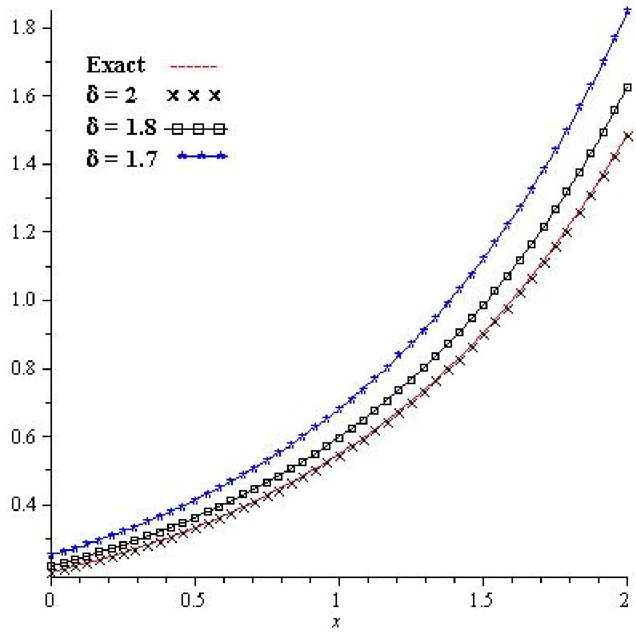

Figure 2. Natural transformation decomposition method (NTDM) solution of Example 1 at different fractional-order $\delta = 2, 1.8, 1.7$.

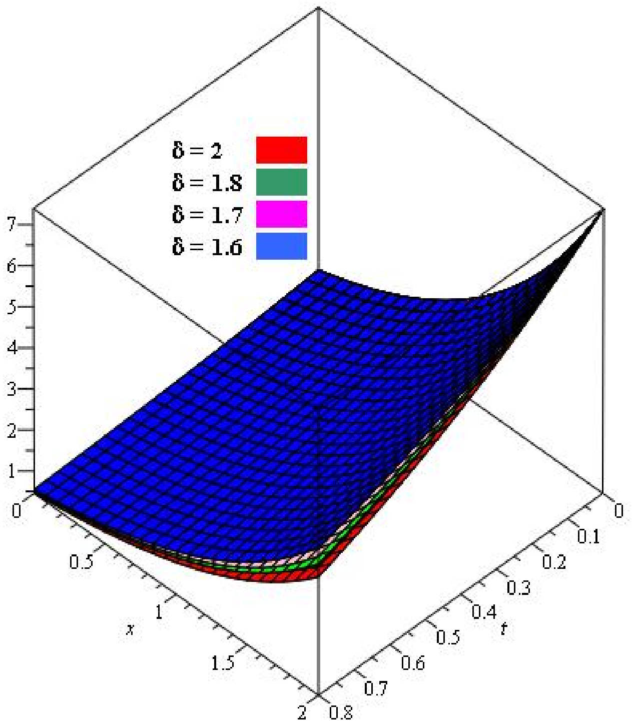

Figure 3. NTDM solution of Example 2 at different fractional-order $\delta = 2, 1.8, 1.7, 1.6$.

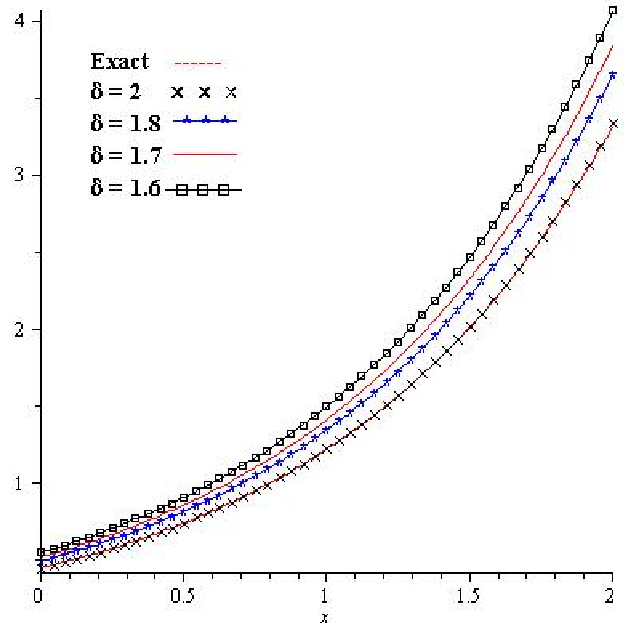

Figure 4. NTDM solution of Example 2 at different fractional-order $\delta = 2, 1.8, 1.7, 1.6$.

7. Physical Implementation

The physical interpretation of the above discussion can be stated as follows.

In the present research article, we derived the analytical solution of the telegraphic transmission line model, where since voltage drop across the induction and capacitor was discussed. The variable u represents the voltage at time t and distance x on the cable. Therefore, the objective of the current work was to obtain the proper voltage $u(x, t)$ at different points (x, t), which can be found by using any optimized fractional-order. This model represented the real-world phenomena in the best way compared to the conventional or integer-order model.

8. Conclusions

In this research paper, an efficient numerical technique called the natural transform decomposition method was used to obtain the analytical alternative of fractional- and integer-order equations. It was shown that the procedure of the suggested method was very simple, attractive, and straightforward. Some examples were regarded in this context to verify the validity and applicability of this technique. It was also observed through simulations that the results obtained by the proposed method for the fractional-orders of the problem approached the integer-order solutions as fractional-order alternatives to the integer-order of the problem. The solution graphs for each problem showed that the method had good agreement with the exact result of the problem. In conclusion, it was shown that the method had a very broad capacity of utilization towards an analytical solution of partial differential fractional equations and the scheme of partial differential fractional equations.

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